Soliton Interactions with Dispersive Wave Background

Ana Mucalica

Supervisor: Dmitry Pelinovsky

Department of Mathematics and Statistics, McMaster University

M.Sc. Defence, April 27, 2023

Table of Contents

Main Model

We are dealing with the canonical model for the shallow water waves, the Korteweg–de Vries (KdV) equation:

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

where t is the time evolution, x is the spatial coordinate for the wave propagation, and u is the fluid velocity.

Main Model

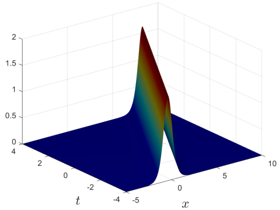
We are dealing with the canonical model for the shallow water waves, the Korteweg–de Vries (KdV) equation:

$$u_t + 6uu_x + u_{xxx} = 0, \qquad (1)$$

where t is the time evolution, x is the spatial coordinate for the wave propagation, and u is the fluid velocity.

One-soliton solution of the KdV equation (1)

$$u(x,t) = 2\mu^2 \operatorname{sech}^2\left(\mu(x - 4\mu^2 t - x_0)\right)$$



Tools Used for Construction of Interaction Solutions

Inverse Scattering Transform (IST):

$$\mathcal{L}v = \lambda v, \qquad \mathcal{L} := -\frac{\partial^2}{\partial x^2} - u \tag{2}$$

and

$$\frac{\partial v}{\partial t} = \mathcal{M}v, \qquad \mathcal{M} := -3u_x - 6u\frac{\partial}{\partial x} - 4\frac{\partial^3}{\partial x^3}, \tag{3}$$

λ is the time-independent spectral parameter.

Tools Used for Construction of Interaction Solutions

Inverse Scattering Transform (IST):

$$\mathcal{L}v = \lambda v, \qquad \mathcal{L} := -\frac{\partial^2}{\partial x^2} - u \tag{2}$$

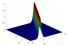
and

$$\frac{\partial v}{\partial t} = \mathcal{M}v, \qquad \mathcal{M} := -3u_x - 6u\frac{\partial}{\partial x} - 4\frac{\partial^3}{\partial x^3}, \tag{3}$$

λ is the time-independent spectral parameter.

⇨ (2) is the stationary Schrödinger equation

Tools Used for Construction of Interaction Solutions



Inverse Scattering Transform (IST):

$$\mathcal{L}v = \lambda v, \qquad \mathcal{L} := -\frac{\partial^2}{\partial x^2} - u \tag{2}$$

and

$$\frac{\partial v}{\partial t} = \mathcal{M}v, \qquad \mathcal{M} := -3u_x - 6u\frac{\partial}{\partial x} - 4\frac{\partial^3}{\partial x^3}, \tag{3}$$

λ is the time-independent spectral parameter.

- ⇨ (2) is the stationary Schrödinger equation
- ⇨ (3) represents the time evolution of the eigenfunctions

Tools Used for Construction of Interaction Solutions

Inverse Scattering Transform (IST):

$$\mathcal{L}v = \lambda v, \qquad \mathcal{L} := -\frac{\partial^2}{\partial x^2} - u \tag{2}$$

and

$$\frac{\partial v}{\partial t} = \mathcal{M}v, \qquad \mathcal{M} := -3u_x - 6u\frac{\partial}{\partial x} - 4\frac{\partial^3}{\partial x^3}, \tag{3}$$

λ is the time-independent spectral parameter.

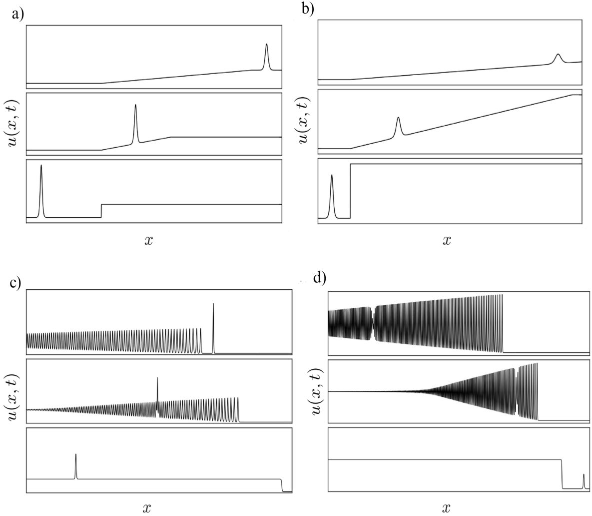
- ⇨ (2) is the stationary Schrödinger equation
- ⇨ (3) represents the time evolution of the eigenfunctions

Darboux Transformation:

$$\hat{u} := u + 2\frac{\partial^2}{\partial x^2}\log(v_0)$$

u is the known solution for the KdV equation (1).

Motivation for Soliton-Dispersive Wave Interactions



a) Soliton–RW tunneling.
b) Soliton–RW trapping.

c) Soliton–DSW tunneling.
d) Soliton–DSW trapping.

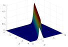
M. J. Ablowitz, J. T. Cole, M. A. Hoefer, et al. ArXiv 2211.14884v1, (2022).

Table of Contents

Introducing the Initial Value Problem

We consider the KdV equation (1) subject to the boundary conditions

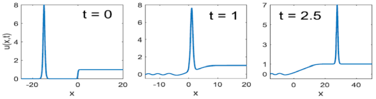
$$\lim_{x\to-\infty} u(t,x) = 0, \qquad \lim_{x\to+\infty} u(t,x) = c^2. \tag{BC}$$



The step-like initial data results in the appearance of a rarefaction wave (RW) for $t > 0$.

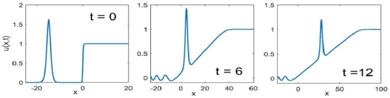
Summary of the Main Results

Transmitted Soliton:



⇨ Transmitted soliton corresponds to an isolated eigenvalue of $\mathcal{L}$.

Trapped Soliton :

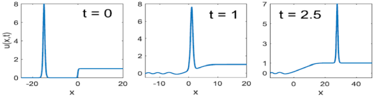


⇨ Trapped soliton corresponds to a “pseudo–embedded” eigenvalue inside the continuous spectrum of $\mathcal{L}$ in $[-c^2, 0]$

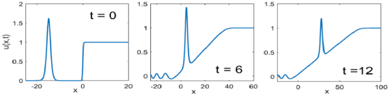
M. J. Ablowitz, X. D. Luo, and J. T. Cole, J. Math. Phys. **59** (2018), 091406

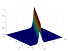
Summary of the Main Results

Transmitted Soliton:



Trapped Soliton :

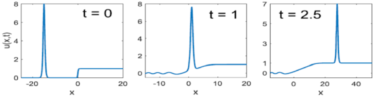


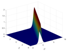


Corresponds to an isolated eigenvalue of $\mathcal{L}$.

Summary of the Main Results

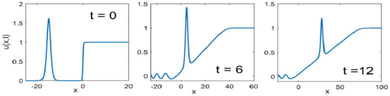
Transmitted Soliton:

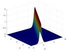




Corresponds to an isolated eigenvalue of $\mathcal{L}$.

Trapped Soliton :

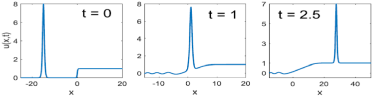




No eigenvalues. Related to resonant poles of $\mathcal{L}$.

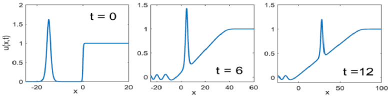
Summary of the Main Results

Transmitted Soliton:



- Corresponds to an isolated eigenvalue of $\mathcal{L}$.
- Can be generated by using DT.

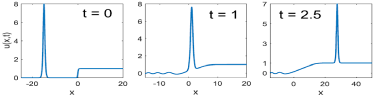
Trapped Soliton :



- No eigenvalues. Related to resonant poles of $\mathcal{L}$.

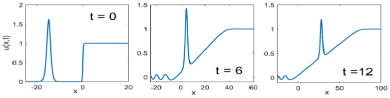
Summary of the Main Results

Transmitted Soliton:



- Corresponds to an isolated eigenvalue of $\mathcal{L}$.
- Can be generated by using DT.

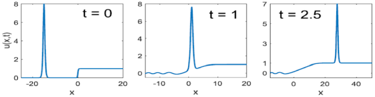
Trapped Soliton :



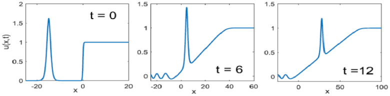
- No eigenvalues. Related to resonant poles of $\mathcal{L}$.
- DT produces unbounded solutions.

Summary of the Main Results

Transmitted Soliton:



Trapped Soliton :

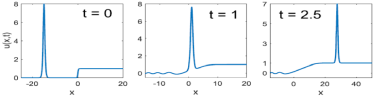


- Corresponds to an isolated eigenvalue of $\mathcal{L}$.
- Can be generated by using DT.
- The amplitude of transmitted soliton is determined by the initial amplitude.

- No eigenvalues. Related to resonant poles of $\mathcal{L}$.
- DT produces unbounded solutions.

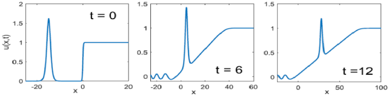
Summary of the Main Results

Transmitted Soliton:



- Corresponds to an isolated eigenvalue of $\mathcal{L}$.
- Can be generated by using DT.
- The amplitude of transmitted soliton is determined by the initial amplitude.

Trapped Soliton :



- No eigenvalues. Related to resonant poles of $\mathcal{L}$.
- DT produces unbounded solutions.
- The amplitude decays to the amplitude of the RW background.

Construction of Transmitted Soliton on Rarefaction Wave Background

Theorem

Let u be a bounded solution of the KdV equation (1) with the boundary conditions (BC) such that the spectrum of the Schrödinger equation (2) is purely continuous in $[-c^2, \infty)$. For every $\lambda_0 < -c^2$, there exists a choice of a smooth function v_0 such that the Darboux transformation (4) returns a bounded solution $\hat{u}$ of the KdV equation (1) such that the spectrum of the Schrödinger equation (2) consists of the purely continuous spectrum in $[-c^2, \infty)$ and a simple isolated eigenvalue λ_0.

Trapped Soliton

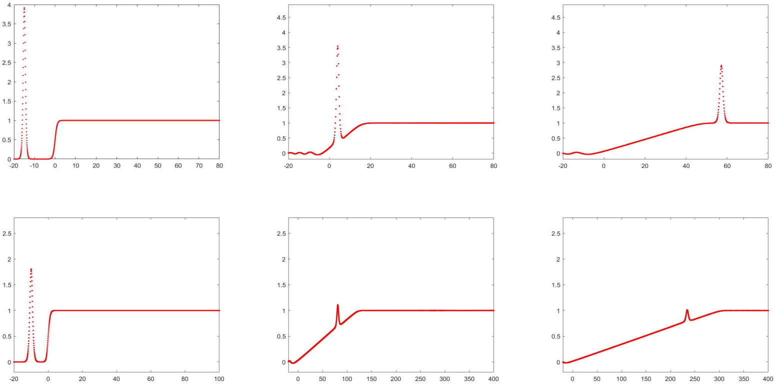
- An embedded eigenvalue $\lambda_0 \in (-c^2, 0)$ moves to a resonant pole off the imaginary axis. Resonant poles do not correspond eigenvalues.
- The eigenfunction of $\mathcal{L}$ is exponentially decaying for $x \to -\infty$, but exponentially growing for $x \to \infty$.

Numerical Experiments

We use Zabusky–Kruskal scheme to recover transmission of a large soliton over the RW background and trapping of a small soliton for initial data

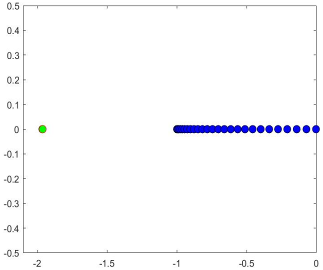
$$u_0(x) = 2\mu_0^2 \mathrm{sech}^2(\mu_0(x - x_0)) + \frac{1}{2}c^2 \left[1 + \tanh(\varepsilon x)\right], \tag{4}$$

where $x_0 < 0$ and $\varepsilon = 1$.



Numerical Experiments

Lax spectrum contains an **isolated eigenvalue** for the **transmitted soliton** but contains **no eigenvalues** for the **trapped soliton**.



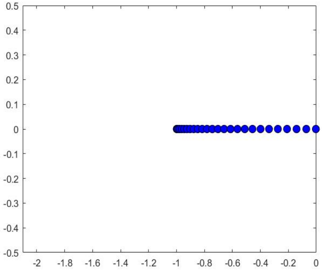


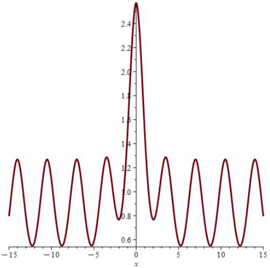
Table of Contents

Introducing the Soliton – Cnoidal Wave Problem

KdV equation (1) has a family of traveling periodic wave solutions

$$u(t,x) = 2k^2 \mathrm{cn}^2(x-ct;k), \qquad c = 4(2k^2-1).$$

Question: Can we construct and characterize solutions representing soliton-cnoidal wave interaction?



Remark Due to the unsteady, wavepacket-like character of the soliton-cnoidal wave interaction solutions, such wave patterns are referred to as breathers.

Lamé equation as the Spectral Problem

The spectral problem (2) with the normalized cnoidal wave potential is known as the Lamé equation

$$v''(x) - 2k^2 \mathrm{sn}^2(x,k)v(x) + \eta v(x) = 0, \quad \eta := \lambda + 2k^2 \tag{5}$$

where the single variable x stands for $x - c_0 t$.

Lamé equation as the Spectral Problem

The spectral problem (2) with the normalized cnoidal wave potential is known as the Lamé equation

$$v''(x) - 2k^2 \mathrm{sn}^2(x,k)v(x) + \eta v(x) = 0, \quad \eta := \lambda + 2k^2 \tag{5}$$

where the single variable x stands for $x - c_0 t$.

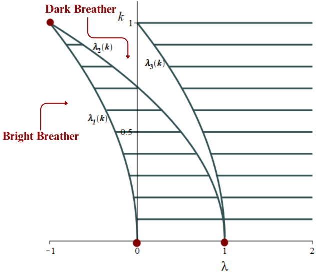


Figure: Floquet spectrum of the Lamé equation (5) with the band edges $\lambda_{1,2,3}(k)$ corresponding to three particular solutions $v_{1,2,3}(x)$.

The Eigenfunctions

Two linearly independent solutions of the Lamé equation (5) for $\lambda \neq \lambda_{1,2,3}(k)$ are given by the functions

$$v_{\pm}(x) = \frac{H(x \pm \alpha)}{\Theta(x)} e^{\mp x Z(\alpha)}, \tag{6}$$

where $\alpha \in \mathbb{C}$ is found from $\lambda \in \mathbb{R}$ by using the characteristic equation $\eta = k^2 + \mathrm{dn}^2(\alpha, k)$ and the Jacobi zeta function is $Z(\alpha) := \frac{\Theta'(\alpha)}{\Theta(\alpha)}$.

$$H(x) = \theta_1\left(\frac{\pi x}{2K(k)}\right), \quad \theta_1(u) = 2\sum_{n=1}^{\infty}(-1)^{n-1} q^{(n-\frac{1}{2})^2} \sin(2n-1)u$$

$$\Theta(x) = \theta_4\left(\frac{\pi x}{2K(k)}\right), \quad \theta_4(u) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos 2nu$$

Bright Breather on the Cnoidal Wave Background

Theorem

There exists an exact solution to the KdV equation (1) in the form

$$u(x,t) = 2\left[k^2 - 1 + \frac{E(k)}{K(k)}\right] + 2\partial_x^2 \log \tau(x,t), \tag{7}$$

where the τ-function is given by

$$\tau(x,t) := \Theta(x - c_0 t + \alpha_b)e^{\kappa_b(x - c_b t + x_0)} + \Theta(x - c_0 t - \alpha_b)e^{-\kappa_b(x - c_b t + x_0)}, \tag{8}$$

where $x_0 \in \mathbb{R}$ is arbitrary and $\alpha_b \in (0, K(k))$, $\kappa_b > 0$, and $c_b > c_0$ are uniquely defined from $\lambda \in (-\infty, \lambda_1(k))$.

Solution Surface for Bright Breather on Cnoidal Wave Background

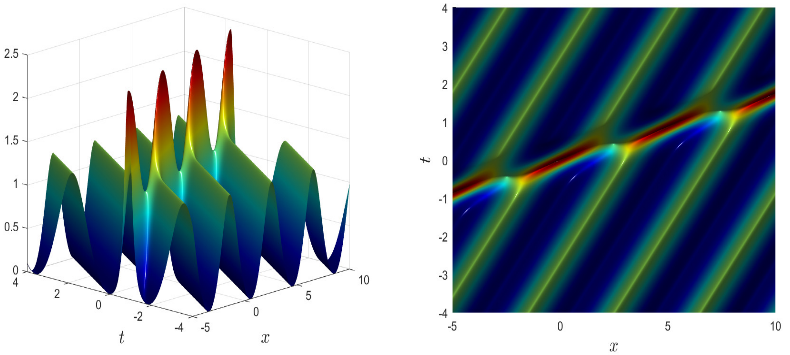


Figure: Bright breather on the cnoidal wave with $k = 0.8$ for $\lambda = -1.2$ and $x_0 = 0$.

Dark Breather on the Cnoidal Wave Background

Theorem

There exists an exact solution to the KdV equation (1) in the form

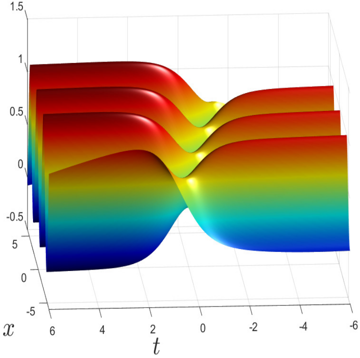
$$u(x,t) = 2\left[k^2 - 1 + \frac{E(k)}{K(k)}\right] + 2\partial_x^2 \log \tau(x,t), \tag{9}$$

where the τ-function is given by

$$\tau(x,t) := \Theta(x - c_0 t + \alpha_d)e^{-\kappa_d(x - c_d t + x_0)} + \Theta(x - c_0 t - \alpha_d)e^{\kappa_d(x - c_d t + x_0)}, \tag{10}$$

where $x_0 \in \mathbb{R}$ is arbitrary and $\alpha_d \in (0, K(k))$, $\kappa_d > 0$, and $c_d < c_0$ are uniquely defined from $\lambda \in (\lambda_2(k), \lambda_3(k))$.

Solution Surface for Dark Breather on Cnoidal Wave Background



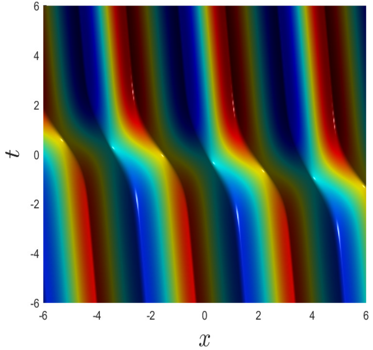
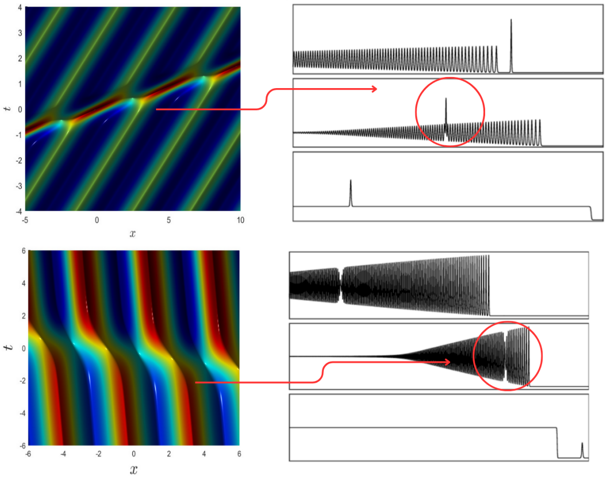


Figure: Dark breather on the cnoidal wave background with $k = 0.7$ for $\lambda = 0.265$ and $x_0 = 0$.

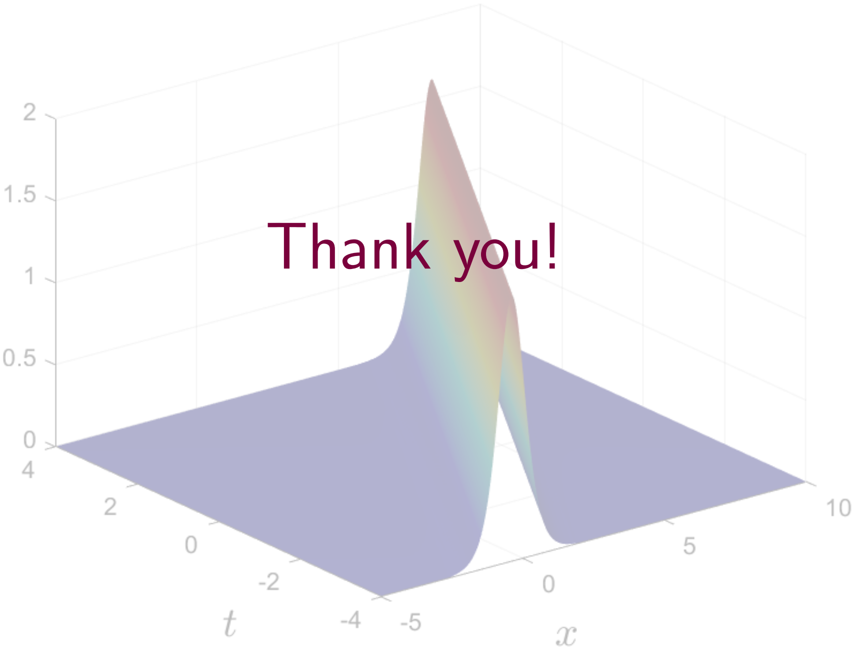
Direction of Breather Propagation



Published Work

A. Mucalica and D.E. Pelinovsky, “Solitons on the rarefaction wave background via the Darboux transformation”, Proc. R. Soc. A **478** (2022). DOI:10.1098/rspa.2022.0474

M. Hoefer, A. Mucalica and D.E. Pelinovsky, “KdV breathers on a cnoidal wave background”, J. Phys. A: Math. Theor. **56** 185701 (2023). DOI: 10.1088/1751-8121/acc6a8



Thank you!

Application of our Work

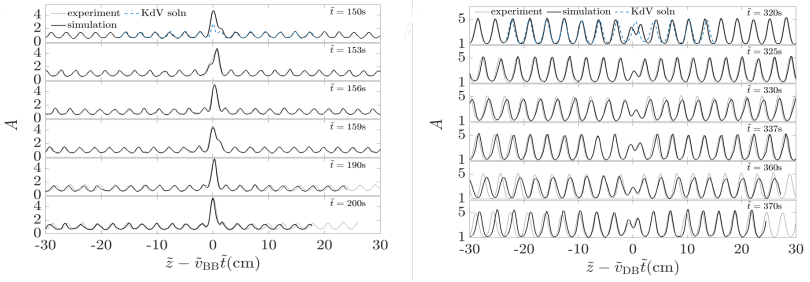


Figure: Experiment (light gray) compared with simulation of the conduit equation (black) with initial conditions from experiment, and the KdV breather solution (blue). Left: Bright Breather. Right: Dark Breather.

Y. Mao, M. A. Hoefer, et al. ArXiv: 2302.11161, (2023).