Completeness of squared eigenfunctions of the Zakharov-Shabat spectral problem

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Introduction

We consider Nonlinear Schrödinger equation for $u(x,t) : \mathbb{R}^2 \to \mathbb{C}$:

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{1}$$

which is integrable.

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Key property of integrability is the existence of the following pair for $v(x,t) : \mathbb{R}^2 \to \mathbb{C}^2$ with spectral parameter k:

$$v_x = \begin{bmatrix} -ik & u \\ -\overline{u} & ik \end{bmatrix} v,$$

$$v_t = \begin{bmatrix} -2ik^2 + i|u|^2 & iu_x + 2ku \\ i\overline{u}_x - 2k\overline{u} & 2ik^2 - i|u|^2 \end{bmatrix} v.$$

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Compatibility condition: $v_{xt} = v_{tx}$ if u is a solution of (1).

Inverse Scattering Transform

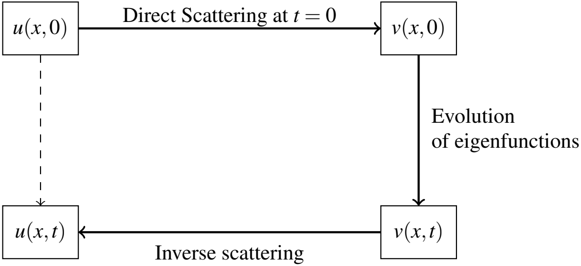

Figure: Inverse Scattering Transform scheme.

Motivation

Exact solution to (1) on the zero background ($u \to 0$ as $x \to \pm\infty$) :

$$u(x,t) = u_0 \operatorname{sech}\left[u_0(x - 2p_0 t)\right] e^{i\left[p_0 x + (u_0^2 - p_0^2)t\right]},$$

where u_0 is constant amplitude and p_0 is a simple shift of carrier-wave wave number.

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Let us perturb this solution with $\delta u(x,t)$ (variation of potential):

$$u(x,t) + \delta u(x,t) \quad \longrightarrow \quad i(\delta u)_t + (\delta u)_{xx} + 2u^2 \delta\overline{u} + 4|u|^2 \delta u = 0.$$

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- The main purpose is to solve Initial Value Problem for $\delta u(x,t)$ in terms of the squared eigenfunctions of the Lax Pair.
- If we build a basis of orthogonal squared eigenfunctions in $L^2(\mathbb{R})$, then the solution $\delta u(x,t)$ can be decomposed as a superposition of squared eigenfunctions.
- To do this, I reviewed proof of completeness of squared eigenfunctions summarized by Jianke Yang in the book "Nonlinear Waves in Integrable and Nonintegrable systems".

Outline

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Direct Scattering

Rewriting the linear system as below:

$$\begin{cases} v_x = -ik\sigma_3 v + Q(u)v, \\ v_t = -2ik^2\sigma_3 v + R(u)v, \end{cases}$$

where

$$Q(u) = \begin{bmatrix} 0 & u \\ -\overline{u} & 0 \end{bmatrix}, \quad R(u) = \begin{bmatrix} i|u|^2 & 2ku + iu_x \\ -2k\overline{u} + i\overline{u}_x & -i|u|^2 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

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We can represent the solutions in the matrix form below

$$V(x,t) = J(x,t)e^{(-ikx-2ik^2t)\sigma_3}.$$

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$$V(x,t) = J(x,t)e^{(-ikx-2ik^2t)\sigma_3}.$$

We define two fundamental solutions J_- and J_+ with the following boundary conditions

$$J_\pm(x) \to I, \text{ as } x \to \pm\infty.$$

Analyticity of $J_\pm$

Define the matrices $J_\pm$ as follows

$$J_- = \begin{bmatrix} M & \widehat{M} \end{bmatrix} \to I \quad \text{as } x \to -\infty, \qquad J_+ = \begin{bmatrix} \widehat{N} & N \end{bmatrix} \to I \quad \text{as } x \to +\infty$$

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Lemma

If $u \in L^1(\mathbb{R})$, then for every $k \in \mathbb{R}$ there exist unique bounded solutions $M, \widehat{M}, N, \widehat{N}$. Moreover M, N are analytic functions in k for $Im(k) > 0$ and continuous for $Im(k) \geq 0$, while $\widehat{M}, \widehat{N}$ are analytic functions in k for $Im(k) < 0$, and continuous for $Im(k) \leq 0$.

Remark

In the respective planes of analyticity, Jost solutions satisfy $J_\pm(x) \to I$ as $|k| \to \infty$, so then

$$M \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \widehat{M} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \widehat{N} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad N \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad as\ |k| \to \infty.$$

Scattering data

Since we set $t = 0$ and dropped from the list of arguments we have

$$V(x) = J_{\pm}(x)e^{-ikx\sigma_3} = J_{\pm}(x)E.$$

Now solutions of the spectral problem can be written as:

$$\Phi = \begin{bmatrix} \phi & \widehat{\phi} \end{bmatrix} = J_{-}(x)E = \begin{bmatrix} M & \widehat{M} \end{bmatrix} E = \begin{bmatrix} e^{-ikx}M & e^{ikx}\widehat{M} \end{bmatrix},$$
$$\Psi = \begin{bmatrix} \widehat{\psi} & \psi \end{bmatrix} = J_{+}(x)E = \begin{bmatrix} \widehat{N} & N \end{bmatrix} E = \begin{bmatrix} e^{-ikx}\widehat{N} & e^{ikx}N \end{bmatrix}.$$

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Since Φ, Ψ are solutions to linear equation, they are linearly related:

$$\Phi = \Psi S, \qquad S = \begin{bmatrix}a & -\overline{b} \\ b & \overline{a}\end{bmatrix}.$$

Lemma

If $u \in L^1(\mathbb{R})$, then a is analytic in $\mathbb{C}_+$ and b is only defined for $k \in \mathbb{R}$.

Adjoint spectral problem

Lemma

Let J satisfy the spectral problem below

$$J_x = -ik[\sigma_3, J] + Q(u)J,$$

where $[\sigma_3, J] = \sigma_3 J - J\sigma_3$*. Then, the adjoint spectral problem is given by*

$$K_x = -ik[\sigma_3, K] - KQ(u),$$

where the adjoint equation is defined with respect to the inner product (without complex conjugation)

$$f, g \in L^2(\mathbb{R}) : \langle f, g \rangle := \int_{\mathbb{R}} f(x)g(x)dx.$$

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Corollary

Solution to the adjoint spectral problem can be expressed as $K = J^{-1}$ (up to constant multiplication).

Analyticity of $J_\pm^{-1}$

$$J_-^{-1} := \begin{bmatrix} M^* \\ \widehat{M}^* \end{bmatrix} \to I \quad \text{as } x \to -\infty, \qquad J_+^{-1} := \begin{bmatrix} \widehat{N}^* \\ N^* \end{bmatrix} \to I \quad \text{as } x \to +\infty$$

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If $u \in L^1(\mathbb{R})$, then for every $k \in \mathbb{R}$ there exist unique bounded solutions $M^*, \widehat{M}^*, \widehat{N}^*, N^*$. Moreover M^*, N^* are analytic functions for $Im(k) < 0$ and continuous for $Im(k) \leq 0$, while $\widehat{M}^*, \widehat{N}^*$ are analytic functions for $Im(k) > 0$ and continuous for $Im(k) \geq 0$.

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In the respective planes of analyticity, Jost solutions satisfy $J_{\pm}^{-1}(x) \to I$ as $|k| \to \infty$, so then

$$M^* \to \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \widehat{M}^* \to \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \widehat{N}^* \to \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad N^* \to \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Inverse Scattering

Riemann-Hilbert problem : $\phi_+(k) - \phi_-(k) = f(k), \quad k \in \mathbb{R}.$

$\phi_\pm \to 0$ as $|k| \to \infty$.

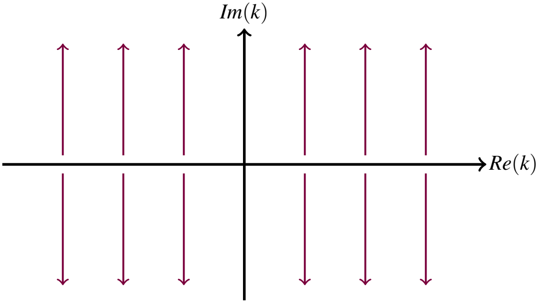

Figure: Riemann-Hilbert problem for $\phi(k)$.

Regular Riemann-Hilbert problem

New matrices $P^{\pm}$

$$P^{-} := \begin{bmatrix} M^{*} \\ N^{*} \end{bmatrix}, \quad P^{+} := \begin{bmatrix} M & N \end{bmatrix},$$

where P^{-} is analytic in $\mathbb{C}_{-}$, and P^{+} is analytic in $\mathbb{C}_{+}$.

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where P^- is analytic in $\mathbb{C}_-$, and P^+ is analytic in $\mathbb{C}_+$.

Riemann-Hilbert problem then can be formulated as follows:

$$P^-P^+ = \begin{bmatrix} M^*M & M^*N \\ N^*M & N^*N \end{bmatrix} = \begin{bmatrix} 1 & \overline{b}e^{-2ikx} \\ be^{2ikx} & 1 \end{bmatrix} = I + \begin{bmatrix} 0 & \overline{b}e^{-2ikx} \\ be^{2ikx} & 0 \end{bmatrix} = I + \Delta$$

It can be rewritten as

$$(P^+ - I) - ((P^-)^{-1} - I) = (P^-)^{-1}\Delta.$$

Properties of $P^{\pm}$:

$$\det P^- = \overline{a}, \quad \det P^+ = a, \quad P^{\pm} \to I, \text{ as } |k| \to \infty.$$

If $a, \overline{a} \neq 0$, then it is called regular Riemann-Hilbert problem

Solution to regular Riemann-Hilbert problem

Lemma

Regular Riemann-Hilbert problem has a unique solution subject to the boundary conditions $P^{\pm} \to I$ as $|k| \to \infty$ in their domains of analyticity

Using Plemelj formula we get solutions below

$$\begin{cases} (P^-)^{-1} - I = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{(P^-)^{-1}\Delta}{\xi-(k-i0)} d\xi, \\ P^+ - I = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{(P^-)^{-1}\Delta}{\xi-(k+i0)} d\xi. \end{cases}$$

Outline

Completeness of squared eigenfunctions

Theorem

The set of squared eigenfunctions $\{Z^-, Z^+\}$ is complete, i.e. every $f \in L^2(\mathbb{R})$ can be written as follows:

$$f(x) = \int_{\mathbb{R}} \left[\tilde{C}(k) Z^-(x,k) + \tilde{D}(k) Z^+(x,k)\right] dk,$$

where

$$\tilde{C}(k) = -\frac{1}{\pi \bar{a}^2(k)} \int_{\mathbb{R}} \Omega^-(y,k) f(y) dy, \qquad \tilde{D}(k) = -\frac{1}{\pi a^2(k)} \int_{\mathbb{R}} \Omega^+(y,k) f(y) dy.$$

Squared eigenfunctions $Z^{\pm}$ and adjoint squared eigenfunctions $\Omega^{\pm}$ are constructed as follows

$$Z^+ = \begin{bmatrix} -\phi_1^2 \\ \phi_2^2 \end{bmatrix}, \quad Z^- = \begin{bmatrix} \widehat{\phi}_1^2 \\ -\widehat{\phi}_2^2 \end{bmatrix}, \quad \Omega^+ = -\begin{bmatrix} (\widehat{\psi}_1^*)^2 \\ (\widehat{\psi}_2^*)^2 \end{bmatrix}, \quad \Omega^- = -\begin{bmatrix} (\psi_1^*)^2 \\ (\psi_2^*)^2 \end{bmatrix}.$$

Sketch of the proof

To prove our main theorem, we follow the procedure below:

- Express the variation of scattering data in terms of the variation of potential to get the adjoint squared eigenfunctions
- Express the variation of potential in terms of the variation of scattering data to get squared eigenfunctions
- Obtain completeness and orthogonality relations

Linearized spectral problem

Consider the following perturbation

$$u(x,t) + \delta u(x,t),$$

where $u(x,t)$ is a solution to NLS equation and $\delta u(x,t)$ is a variation of potential.

$$\Phi_x = -ik\sigma_3\Phi + Q(u)\Phi \quad \longrightarrow \quad (\delta\Phi)_x = -ik\sigma_3\delta\Phi + Q\delta\Phi + (\delta Q)\Phi,$$

where

$$\delta Q = \begin{bmatrix} 0 & \delta u \\ -\delta\overline{u} & 0 \end{bmatrix}.$$

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Solving for $\delta\Phi$ yeilds in

$$\delta\Phi(x) = \Phi(x)\int_{-\infty}^{x} \Phi^{-1}(y)\delta Q(y)\Phi(y)dy.$$

Variation of scattering data

Using boundary conditions of $\Phi(x)$ as $x \to +\infty$ we get

$$\delta S = \int_{-\infty}^{+\infty} \Psi^{-1}(x)\delta Q(x)\Phi(x)dx,$$

from where we can easily get expressions for entries of the scattering matrix S:

$$\delta a = \int_{-\infty}^{+\infty} \widehat{\psi}^* \delta Q \phi dx, \qquad \delta \overline{a} = \int_{-\infty}^{+\infty} \psi^* \delta Q \widehat{\phi} dx,$$
$$-\delta \overline{b} = \int_{-\infty}^{+\infty} \widehat{\psi}^* \delta Q \widehat{\phi} dx, \qquad \delta b = \int_{-\infty}^{+\infty} \psi^* \delta Q \phi dx.$$

Note that $\delta\overline{b}, \delta b$ are related to eigenfunctions that are analytic in opposite half planes.

New scattering data

To solve the problem of analyticity, we introduce new scattering data:

$$\rho = \frac{b}{\bar{a}}, \qquad \tilde{\rho} = \frac{\bar{b}}{a}.$$

By taking variation we finally have a relation between variation of scattering data and variation of potential

$$\delta\rho = \frac{1}{\bar{a}^2}\int_{-\infty}^{+\infty} \psi^* \delta Q \widehat{\psi} dx, \qquad \delta\tilde{\rho} = -\frac{1}{a^2}\int_{-\infty}^{+\infty} \widehat{\psi}^* \delta Q \psi dx.$$

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We can rewrite them as follows

$$\delta\rho = \frac{1}{\bar{a}^2}\left\langle \underbrace{\begin{bmatrix} \psi_1^*\widehat{\psi}_2 \\ -\psi_2^*\widehat{\psi}_1 \end{bmatrix}}_{=\Omega^-}, \begin{bmatrix} \delta u \\ \delta\bar{u} \end{bmatrix} \right\rangle, \qquad \delta\tilde{\rho} = \frac{1}{a^2}\left\langle \underbrace{\begin{bmatrix} -\widehat{\psi}_1^*\psi_2 \\ \widehat{\psi}_2^*\psi_1 \end{bmatrix}}_{=\Omega^+}, \begin{bmatrix} \delta u \\ \delta\bar{u} \end{bmatrix} \right\rangle.$$

Adjoint squared eigenfunctions

We can relate eigenfunctions and adjoint eigenfunctions in the following way:

$$\Psi^{-1} = \begin{bmatrix} \widehat{\psi}_1 & \psi_1 \\ \widehat{\psi}_2 & \psi_2 \end{bmatrix}^{-1} = \frac{1}{det\Psi} \begin{bmatrix} \psi_2 & -\psi_1 \\ -\widehat{\psi}_2 & \widehat{\psi}_1 \end{bmatrix} = \begin{bmatrix} \psi_2 & -\psi_1 \\ -\widehat{\psi}_2 & \widehat{\psi}_1 \end{bmatrix} = \begin{bmatrix} \widehat{\psi}_1^* & \widehat{\psi}_2^* \\ \psi_1^* & \psi_2^* \end{bmatrix},$$

which allows us to write $\Omega^{\pm}$ in the following shape

$$\Omega^- = -\begin{bmatrix} \widehat{\psi}_2^2 \\ \widehat{\psi}_1^2 \end{bmatrix} = -\begin{bmatrix} (\psi_1^*)^2 \\ (\psi_2^*)^2 \end{bmatrix}, \qquad \Omega^+ = -\begin{bmatrix} \psi_2^2 \\ \psi_1^2 \end{bmatrix} = -\begin{bmatrix} (\widehat{\psi}_1^*)^2 \\ (\widehat{\psi}_2^*)^2 \end{bmatrix}.$$

Riemann-Hilbert problem

Now we want to express $\begin{bmatrix} \delta u \\ \delta \bar{u} \end{bmatrix}$ in terms of $\delta\rho, \delta\tilde{\rho}$.

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Define new matrices (to switch to $\delta\rho, \delta\tilde{\rho}$):

$$F^+ = P^+ \begin{bmatrix}1 & 0 \\ 0 & 1/a\end{bmatrix}, \qquad F^- = (P^-)^{-1} \begin{bmatrix}1 & 0 \\ 0 & \overline{a}\end{bmatrix}.$$

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Proposition

Let $\delta F^{\pm}$ be variations of $F^{\pm}$, then

$$\left(\delta F^{\pm}(F^{\pm})^{-1}\right)(x) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\Pi(x,\xi)}{\xi - k}\,d\xi,$$

where

$$\Pi(x,\xi) = \Phi\begin{bmatrix}0 & \delta\tilde{\rho} \\ \delta\rho & 0\end{bmatrix}\Phi^{-1}.$$

Expansions for P^+, F^+

From solution to Riemann Hilbert problem for P^+ we can can expand P^+ as follows

$$P^+ = I + \frac{1}{k} P_1^+(x) + O\left(\frac{1}{k^2}\right) \tag{1}$$

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Lemma

Let P^+ be expanded as in (1), then

$$P_1^+ = \frac{1}{2i}\begin{bmatrix} \int_{-\infty}^{x} |u(y)|^2 dy & u \\ \overline{u} & \int_{x}^{+\infty} |u(y)|^2 dy \end{bmatrix}.$$

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Lemma

F^+ can be expanded as

$$F^+ = \begin{bmatrix} 1 + \frac{1}{2ik} \int_{-\infty}^{x} |u|^2 dy + O(\frac{1}{k^2}) & \frac{u}{2ik} + O(\frac{1}{k^2}) \\ \frac{\overline{u}}{2ik} + O(\frac{1}{k^2}) & 1 + \frac{1}{2ik} \int_{x}^{+\infty} |u|^2 dy + O(\frac{1}{k^2}) \end{bmatrix}$$

Squared eigenfunctions

Lemma

Variation of potentials at $O(\frac{1}{k})$ *are*

$$\begin{cases} \delta u = -\frac{1}{\pi}\int_{\mathbb{R}} \Pi_{12}(x,\xi)d\xi \\ \delta \overline{u} = -\frac{1}{\pi}\int_{\mathbb{R}} \Pi_{21}(x,\xi)d\xi. \end{cases}$$

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Lemma

Perturbation δu is expressed in terms of $\delta\rho$ as follows

$$\begin{bmatrix}\delta u\\ \delta\overline{u}\end{bmatrix} = \frac{1}{\pi}\int_{\mathbb{R}}\Big(Z^-(x,\xi)\delta\rho(\xi) + Z^+(x,\xi)\delta\tilde{\rho}(\xi)\Big)d\xi,$$

where

$$Z^- = \begin{bmatrix}\widehat{\phi}_1^2\\ -\widehat{\phi}_2^2\end{bmatrix}, \qquad Z^+ = \begin{bmatrix}-\phi_1^2\\ \phi_2^2\end{bmatrix}.$$

Completeness relation

Lemma

The sets $\{\Omega^+, \Omega^-\}$ *and* $\{Z^+, Z^-\}$ *satisfy the following completeness relation if* $a, \overline{a} \neq 0$*:*

$$\delta(x-y)I = \frac{1}{\pi}\int_{\mathbb{R}} \left[\frac{1}{\overline{a}^2(\xi)} Z^-(x,\xi)\Omega^-(y,\xi) + \frac{1}{a^2(\xi)} Z^+(x,\xi)\Omega^+(y,\xi) \right] d\xi.$$

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Idea of the proof is to combine relations below

$$\delta\rho = \frac{1}{\overline{a}^2}\left\langle \Omega^-, \begin{bmatrix} \delta u \\ \delta\overline{u} \end{bmatrix} \right\rangle, \quad \delta\tilde{\rho} = \frac{1}{a^2}\left\langle \Omega^+, \begin{bmatrix} \delta u \\ \delta\overline{u} \end{bmatrix} \right\rangle,$$

$$\begin{bmatrix} \delta u \\ \delta\overline{u} \end{bmatrix} = \frac{1}{\pi}\int_{\mathbb{R}} \Big(Z^-(x,\xi)\delta\rho(\xi) + Z^+(x,\xi)\delta\tilde{\rho}(\xi) \Big) d\xi.$$

Orthogonality relation

Combining equalities from previous slide in opposite way results in the following lemma

Lemma

The squared eigenfunctions $Z^{\pm}$ and the adjoint squared eigenfunctions $\Omega^{\pm}$ satisfy the following orthogonality relations:

$$\begin{aligned}
\left\langle \Omega^{-}(x,\xi), Z^{-}(x,\xi') \right\rangle &= \pi \bar{a}^2(\xi)\delta(\xi-\xi'),\\
\left\langle \Omega^{+}(x,\xi), Z^{+}(x,\xi') \right\rangle &= \pi a^2(\xi)\delta(\xi-\xi'),\\
\left\langle \Omega^{-}(x,\xi), Z^{+}(x,\xi') \right\rangle &= 0,\\
\left\langle \Omega^{+}(x,\xi), Z^{-}(x,\xi') \right\rangle &= 0.
\end{aligned}$$

Completeness of squared eigenfunctions

Theorem

The set of squared eigenfunctions $\{Z^-, Z^+\}$ is complete, i.e. every $f \in L^2(\mathbb{R})$ can be written as follows:

$$f(x) = \int_{\mathbb{R}} \left[\tilde{C}(k) Z^-(x,k) + \tilde{D}(k) Z^+(x,k) \right] dk,$$

where

$$\tilde{C}(k) = -\frac{1}{\pi \bar{a}^2(k)} \int_{\mathbb{R}} \Omega^-(y,k) f(y) dy, \qquad \tilde{D}(k) = -\frac{1}{\pi a^2(k)} \int_{\mathbb{R}} \Omega^+(y,k) f(y) dy.$$

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$$\begin{bmatrix} \delta u \\ \delta \bar{u} \end{bmatrix} = \frac{1}{\pi} \int_{\mathbb{R}} \left(Z^-(x,\xi) \delta\rho(\xi) + Z^+(x,\xi) \delta\tilde{\rho}(\xi) \right) d\xi.$$

$$\delta\rho = \frac{1}{\bar{a}^2} \left\langle \Omega^-, \begin{bmatrix} \delta u \\ \delta \bar{u} \end{bmatrix} \right\rangle, \quad \delta\tilde{\rho} = \frac{1}{a^2} \left\langle \Omega^+, \begin{bmatrix} \delta u \\ \delta \bar{u} \end{bmatrix} \right\rangle,$$

Outline

Linearized NLS equation

Since in all previous computations, we have set $t=0$ and omitted t from arguments of the fundamental solutions Φ, Φ^{-1}. Let us now augment all expressions by explicitly writing dependence on t:

$$Z^{-(t)} = \begin{bmatrix} (\widehat{\phi}_1^{(t)})^2 \\ -(\widehat{\phi}_2^{(t)})^2 \end{bmatrix}, \qquad Z^{+(t)} = \begin{bmatrix} -(\phi_1^{(t)})^2 \\ (\phi_2^{(t)})^2 \end{bmatrix},$$

$$\Omega^{-(t)} = -\begin{bmatrix} (\widehat{\psi}_2^{(t)})^2 \\ (\widehat{\psi}_1^{(t)})^2 \end{bmatrix}, \qquad \Omega^{+(t)} = -\begin{bmatrix} (\psi_2^{(t)})^2 \\ (\psi_1^{(t)})^2 \end{bmatrix}.$$

Linearized NLS equation

Proposition

Let u be a solution of the NLS equation (1). *Then, variation* $(\delta u, \delta \overline{u})$ *are solution of the following linearized NLS equation:*

$$\mathcal{L}\begin{bmatrix}\delta u(x,t)\\ \delta \overline{u}(x,t)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

where

$$\mathcal{L} = \begin{bmatrix} i\partial_t + \partial_{xx} + 4|u|^2 & 2u^2 \\ -2\overline{u}^2 & i\partial_t - \partial_{xx} - 4|u|^2 \end{bmatrix}$$

is the linearization operator.

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is the linearization operator.

Theorem

The time-dependent squared eigenfunctions $Z^{-(t)}, Z^{+(t)}$ *satisfy the linearized NLS equation:*

$$\mathcal{L}Z^{-(t)}(x,k,t) = \mathcal{L}Z^{+(t)}(x,k,t) = 0.$$

Adjoint linearized NLS equation

Theorem

The time-dependent adjoint squared eigenfunctions $\Omega^{-(t)}, \Omega^{+(t)}$ satisfy the adjoint linearized NLS equation:

$$\mathcal{L}^* \Omega^{-(t)}(x,k,t) = \mathcal{L}^* \Omega^{+(t)}(x,k,t) = 0,$$

where

$$\mathcal{L}^* = \begin{bmatrix} -i\partial_t + \partial_{xx} + 4|u|^2 & -2\bar{u}^2 \\ 2u^2 & -i\partial_t - \partial_{xx} - 4|u|^2 \end{bmatrix}.$$

Outline

Future Directions

Throughout the whole thesis we assumed that $a, \overline{a}$ are nonzero for all $k \in \mathbb{C}$. If we allow $a, \overline{a}$ to have zeros, then we have the case of Nonregular Riemann-Hilbert problem.

$$(P^{+} - I) - ((P^{-})^{-1} - I) = (P^{+} - I) - \left(\begin{bmatrix} M_1^* & M_2^* \\ N_1^* & N_2^* \end{bmatrix}^{-1} - I \right)$$
$$= (P^{+} - I) - \frac{1}{\overline{a}} \begin{bmatrix} N_2^* & -M_2^* \\ -N_1^* & M_1^* \end{bmatrix} = (P^{-})^{-1} \Delta.$$

Thus, if $\overline{a} = 0$ we cannot extend analytically in the lower half plane.

Solutions to NLS on nonzero background

Soliton solutions to NLS equation on nonzero background $(u \to -e^{it/2})$:

- Akhmediev breather

$$u(x,t) = \left[-1 + \frac{2k^2 \cosh(\lambda k t) + 2i\lambda k \sinh(\lambda k t)}{\cosh(\lambda k t) - \lambda \cos(2kx)}\right] e^{it/2},$$

where $k = \sqrt{1-\lambda^2}$ and $\lambda \in (0,1)$ is a free parameter.

- Kuznetsov-Ma breather

$$u(x,t) = \left[-1 + \frac{2\beta^2 \cos(\lambda\beta t) + 2i\lambda\beta \sin(\lambda\beta t)}{\lambda \cosh(2\beta x) - \cos(\lambda\beta t)}\right] e^{it/2},$$

where $\beta = \sqrt{\lambda^2 - 1}$ and $\lambda \in (1, +\infty)$ is a free parameter.

- Peregrine's Rogue Wave

$$u(x,t) = \left[-1 + \frac{4(1+it)}{1+4x^2+t^2}\right] e^{it/2}$$

The End

Thank you!