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ASYMPTOTIC STABILITY OF SMALL BOUND STATES IN THE DISCRETE NONLINEAR SCHRÖDINGER EQUATION*

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Abstract. Asymptotic stability of small bound states in one dimension is proved in the framework of a discrete nonlinear Schrödinger equation with septic and higher power-law nonlinearities and an external potential supporting a simple isolated eigenvalue. The analysis relies on the dispersive decay estimates from Pelinovsky and Stefanov [*J. Math. Phys.*, 49 (2008), 113501] and the arguments of Mizumachi [*J. Math. Kyoto Univ.*, 48 (2008), pp. 471–497] for a continuous nonlinear Schrödinger equation in one dimension. Numerical simulations suggest that the actual decay rate of perturbations near the asymptotically stable bound states is higher than the one used in the analysis.

Key words. discrete nonlinear Schrödinger equations, bound states, asymptotic stability, Strichartz estimates

AMS subject classifications. 35Q55, 37J20, 39A11

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1. Introduction.

Asymptotic stability of solitary waves in the context of continuous nonlinear Schrödinger equations in one, two, and three spatial dimensions was considered in a number of recent works (see Cuccagna [4] for a review of literature). Little is known, however, about asymptotic stability of solitary waves in the context of discrete nonlinear Schrödinger (DNLS) equations.

Orbital stability of a global energy minimizer under a fixed mass constraint was proved by Weinstein [27] for the DNLS equation with power nonlinearity

$$i\dot{u}_n + \Delta_d u_n + |u_n|^{2p} u_n = 0, \quad n \in \mathbb{Z}^d,$$

where Δ_d is a discrete Laplacian in d dimensions and $p > 0$. For $p < \frac{2}{d}$ (subcritical case), it is proved that the ground state of an arbitrary energy exists, whereas for $p \geq \frac{2}{d}$ (critical and supercritical cases), there is an energy threshold below which the ground state does not exist.

Ground states of the DNLS equation with power-law nonlinearity correspond to single-humped solitons, which are excited in numerical and physical experiments by single-site initial data with sufficiently large amplitude [11]. Such experiments have been physically realized in optical settings with both focusing [7] and defocusing [15] nonlinearities. We would like to consider long-time dynamics of the ground states and prove their asymptotic stability under some assumptions on the spectrum of the linearized DNLS equation. From the beginning, we will work in the space of one spatial dimension ($d = 1$) and in the presence of an external potential V. These

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specifications are motivated by physical applications (see, e.g., the recent work of [18] and references therein for a relevant discussion). We hence write the main model in the form

$$i\dot{u}_n = (-\Delta + V_n)u_n + \gamma|u_n|^{2p}u_n, \quad n \in \mathbb{Z}, \tag{1}$$

where $\Delta u_n := u_{n+1} - 2u_n + u_{n-1}$ and $\gamma = 1$ ($\gamma = -1$) for defocusing (focusing) nonlinearity. Besides physical applications, the role of potential V in our work can be explained by looking at the differences between the recent works of Mizumachi [17] and Cuccagna [5] for a continuous nonlinear Schrödinger equation in one dimension. Using an external potential, Mizumachi proved asymptotic stability of small bound states bifurcating from the lowest eigenvalue of the Schrödinger operator $H_0 = -\partial_x^2 + V$ under some assumptions on the spectrum of H_0. He needed only spectral theory of the self-adjoint operator H_0 in L^2 since spectral projections and small nonlinear terms were controlled in the corresponding norm. Pioneering works along the same lines are attributed to Soffer and Weinstein [23, 24, 25], Pillet and Wayne [22], and Yau and Tsai [28, 29, 30]. Compared to this approach, Cuccagna proved asymptotic stability of nonlinear space-symmetric bound states in energy space of the continuous nonlinear Schrödinger equation with $V \equiv 0$. He had to invoke the spectral theory of non–self-adjoint operators arising in the linearization of the nonlinear Schrödinger equation, following earlier works of Buslaev and Perelman [1, 2], Buslaev and Sulem [3], and Gang and Sigal [8, 9].

Since our work is novel in the context of the DNLS equation, we would like to simplify the spectral formalism and to focus on nonlinear analysis of asymptotic stability. This is the main reason why we work with small bound states bifurcating from the lowest eigenvalue of the discrete Schrödinger operator $H = -\Delta + V$. We will make use of the dispersive decay estimates obtained recently for operator H by Stefanov and Kevrekidis [26] (for $V \equiv 0$), Komech, Kopylova, and Kunze [14] (for compact V), and Pelinovsky and Stefanov [21] (for decaying V). With more effort and more elaborate analysis, our results can be generalized to large bound states with or without potential V under some restrictions on the spectrum of the non–self-adjoint operator associated with linearization of the DNLS equation.

From a technical point of view, many previous works on asymptotic stability of solitary waves in continuous nonlinear Schrödinger equations address critical and supercritical cases, which in $d = 1$ correspond to $p \geq 2$. Subcritical nonlinearities were addressed in this context only recently by Kirr and Zarnescu [12] and Kirr and Mizrak [13]. Because the dispersive decay in the l^∞ norm is slower for the DNLS equation, the critical power appears at $p = 3$ and the proof of asymptotic stability of discrete solitons can be developed for $p \geq 3$. The most interesting case of the cubic DNLS equation for $p = 1$ is excluded from our consideration. To prove asymptotic stability of discrete solitons for $p \geq 3$, we extend the pointwise dispersive decay estimates from [21] to Strichartz estimates, which allow us better control of the dispersive parts of the solution. The nonlinear analysis follows the steps in the proof of asymptotic stability of small bound states in the continuous nonlinear Schrödinger equation [17].

In addition to analytical results, we also approximate time evolution of small bound states numerically in the DNLS equation (1) with $p = 1, 2, 3$. We not only confirm the asymptotic stability of small bound states in all the cases but also find that the actual decay rate of perturbations near the small bound state is faster than the one used in our analytical arguments.

The article is organized as follows. The main result for $p \geq 3$ is formulated in section 2. Linear estimates are derived in section 3. The proof of the main theorem is developed in section 4. Numerical illustrations for $p = 1, 2, 3$ are discussed in section 5. Appendix A gives proofs of technical formulas used in section 3.

2. Preliminaries and the main result. In what follows, we use boldface notation for vectors in discrete spaces l^1_s and l^2_s on $\mathbb{Z}$ defined by their norms

$$\|\mathbf{u}\|_{l^1_s} := \sum_{n\in\mathbb{Z}} (1+n^2)^{s/2}|u_n|, \quad \|\mathbf{u}\|_{l^2_s} := \left(\sum_{n\in\mathbb{Z}} (1+n^2)^s |u_n|^2\right)^{1/2}.$$

Components of $\mathbf{u}$ are denoted by regular font, e.g., u_n for $n \in \mathbb{Z}$.

We shall make the following assumptions on the external potential $\mathbf{V}$ and on the spectrum of the self-adjoint operator $H = -\Delta + \mathbf{V}$ in l^2.

(V1) $\mathbf{V} \in l^1_{2\sigma}$ for a fixed $\sigma > \frac{5}{2}$.

(V2) $\mathbf{V}$ is generic in the sense that no solution $\boldsymbol{\psi}_0$ of equation $H\boldsymbol{\psi}_0 = 0$ exists in $l^2_{-\sigma}$ for $\frac{1}{2} < \sigma \leq \frac{3}{2}$.

(V3) $\mathbf{V}$ supports exactly one negative simple eigenvalue $\omega_0 < 0$ of H with an eigenvector $\boldsymbol{\psi}_0 \in l^2$ and no eigenvalues above 4.

Assumptions (V1)–(V3) are satisfied for the single-node potential with $V_n = -\delta_{n,0}$ for any $n \in \mathbb{Z}$. This potential is known (see Appendix A in [14]) to have only one negative simple eigenvalue at $\omega_0 < 0$, the continuous spectrum at $[0, 4]$, and no resonances at 0 and 4. Explicit computations show that the eigenvalue exists at $\omega_0 = 2 - \sqrt{5}$ with the corresponding eigenvector

$$\psi_{0,n} = e^{-\kappa|n|}, \quad n \in \mathbb{Z}, \quad \kappa = \operatorname{arcsinh}(2^{-1}).$$

The first two assumptions (V1) and (V2) are needed for the dispersive decay estimates developed in [21]. The last assumption (V3) is needed for existence of a family $\boldsymbol{\phi}(\omega)$ of real-valued decaying solutions of the stationary DNLS equation

$$(-\Delta + V_n)\phi_n(\omega) + \gamma\phi_n^{2p+1}(\omega) = \omega\phi_n(\omega), \quad n \in \mathbb{Z}, \tag{2}$$

near $\omega = \omega_0 < 0$. This is a standard local bifurcation of decaying solutions in a system of infinitely many algebraic equations.

LEMMA 1 (local bifurcation of stationary solutions). *Assume that $\mathbf{V} \in l^\infty$ and that H has a simple eigenvalue $\omega_0 < 0$ with a normalized eigenvector $\boldsymbol{\psi}_0 \in l^2$ such that $\|\boldsymbol{\psi}_0\|_{l^2} = 1$. Let $\epsilon := \omega - \omega_0$, $\gamma = +1$, and $p \geq \frac{1}{2}$. There exist $\epsilon_0 > 0$ and $C > 0$ such that for all $\omega \in [\omega_0, \omega_0 + \epsilon_0)$, there exists a unique real-valued solution $\boldsymbol{\phi}(\omega) \in C([\omega_0, \omega_0 + \epsilon_0), l^2)$ of the stationary DNLS equation (2) satisfying*

$$\left\| \boldsymbol{\phi}(\omega) - \frac{\epsilon^{\frac{1}{2p}}\boldsymbol{\psi}_0}{\|\boldsymbol{\psi}_0\|^{1+\frac{1}{p}}_{l^{2p+2}}} \right\|_{l^2} \leq C\epsilon^{1+\frac{1}{2p}}.$$

Moreover, $\boldsymbol{\phi}(\omega) \in C^2((\omega_0, \omega_0 + \epsilon_0), l^2)$ and $\boldsymbol{\phi}(\omega)$ decays exponentially to zero as $|n| \to \infty$.

Proof. Existence and uniqueness of the real-valued solution $\boldsymbol{\phi}(\omega) \in C([\omega_0, \omega_0 + \epsilon_0), l^2)$ follows by the standard method of Lyapunov–Schmidt reductions [16]. C^2 smoothness follows from the Implicit Function Theorem as the nonlinear vector field $\{\phi_n^{2p+1}\}_{n\in\mathbb{Z}}$ is C^2 near $\boldsymbol{\phi} = \mathbf{0}$ for $p \geq \frac{1}{2}$. Exponential decay follows from the variational method [20] (where $p = 2$ is specified without loss of generality). ∎

Remark 1. Because of the exponential decay of $\boldsymbol{\phi}(\omega)$ as $|n| \to \infty$, the solution $\boldsymbol{\phi}(\omega)$ exists in l^2_σ for all $\sigma \geq 0$. In addition, since $\|\boldsymbol{\phi}\|_{l^1} \leq C_\sigma \|\boldsymbol{\phi}\|_{l^2_\sigma}$, for any $\sigma > \frac{1}{2}$, the solution $\boldsymbol{\phi}(\omega)$ also exists in l^1.

To work with solutions of the DNLS equation (1) for all $t \in \mathbb{R}_+$ starting with some initial data at $t = 0$, we need global well-posedness of the Cauchy problem for (1).

LEMMA 2 (global well-posedness). *Fix $\sigma \geq 0$. For any $\mathbf{u}_0 \in l^2_\sigma$, there exists a unique solution $\mathbf{u}(t) \in C^1(\mathbb{R}_+, l^2_\sigma)$ such that $\mathbf{u}(0) = \mathbf{u}_0$ and $\mathbf{u}(t)$ depends continuously on $\mathbf{u}_0$.*

Proof. The proof is based on the contraction mapping arguments and Gronwall's inequality [19] because H is a bounded operator from l^2_σ to l^2_σ for any $\sigma \geq 0$ and the flux conservation equation

$$i\frac{d}{dt}|u_n|^2 = u_n(\bar{u}_{n+1} + \bar{u}_{n-1}) - \bar{u}_n(u_{n+1} + u_{n-1}) \tag{3}$$

gives the bounds on $\|\mathbf{u}(t)\|_{l^2_\sigma}$ for all $t \in \mathbb{R}_+$. $\square$

Remark 2. Global well-posedness holds also on $\mathbb{R}_-$ (and thus on $\mathbb{R}$) since the DNLS equation (1) is a reversible dynamical system. We shall work in the positive time intervals only.

Equipped with the results above, we decompose a solution to the DNLS equation (1) into a family of stationary solutions with time varying parameters and a radiation part using the substitution

$$\mathbf{u}(t) = e^{-i\theta(t)}\left(\boldsymbol{\phi}(\omega(t)) + \mathbf{z}(t)\right), \tag{4}$$

where $(\omega, \theta) \in \mathbb{R}^2$ represents a two-dimensional orbit of stationary solutions $\mathbf{u}(t) = e^{-i\theta - i\omega t}\boldsymbol{\phi}(\omega)$ (their time evolution will be specified later) and $\mathbf{z}(t) \in C^1(\mathbb{R}_+, l^2_\sigma)$ solves the time-evolution equation in the form

$$i\dot{\mathbf{z}} = (H - \omega)\mathbf{z} - (\dot{\theta} - \omega)(\boldsymbol{\phi}(\omega) + \mathbf{z}) - i\dot{\omega}\partial_\omega\boldsymbol{\phi}(\omega) + \mathbf{N}(\boldsymbol{\phi}(\omega) + \mathbf{z}) - \mathbf{N}(\boldsymbol{\phi}(\omega)), \tag{5}$$

where $H = -\Delta + \mathbf{V}$, $[\mathbf{N}(\boldsymbol{\psi})]_n = \gamma|\psi_n|^{2p}\psi_n$, and $\partial_\omega\boldsymbol{\phi}(\omega) \in l^2$ exists thanks to Lemma 1. The linearized time evolution at the stationary solution $\boldsymbol{\phi}(\omega)$ involves operators

$$L_- = H - \omega + \mathbf{W}, \quad L_+ = H - \omega + (2p+1)\mathbf{W},$$

where $W_n = \gamma\phi_n^{2p}(\omega)$ and $\mathbf{W}$ decays exponentially as $|n| \to \infty$ thanks to Lemma 1. The linearized time evolution in variables $\mathbf{v} = \mathrm{Re}(\mathbf{z})$ and $\mathbf{w} = \mathrm{Im}(\mathbf{z})$ can be characterized by the non–self-adjoint eigenvalue problem

$$L_+\mathbf{v} = -\lambda\mathbf{w}, \quad L_-\mathbf{w} = \lambda\mathbf{v}. \tag{6}$$

Using Lemma 1, we derive the following result.

LEMMA 3 (double null subspace). *For any $\epsilon \in (0, \epsilon_0)$, the linearized eigenvalue problem (6) admits a double zero eigenvalue with a one-dimensional kernel, isolated from the rest of the spectrum. The generalized kernel is spanned by vectors $(\mathbf{0}, \boldsymbol{\phi}(\omega)), (-\partial_\omega\boldsymbol{\phi}(\omega), \mathbf{0}) \in l^2$ satisfying*

$$L_-\boldsymbol{\phi}(\omega) = \mathbf{0}, \qquad L_+\partial_\omega\boldsymbol{\phi}(\omega) = \boldsymbol{\phi}(\omega).$$

Proof. By Lemma 1 in [21], operator H has the essential spectrum on $[0, 4]$. Because of the exponential decay of $\mathbf{W}$ as $|n| \to \infty$, the essential spectrum of L_+

and L_- is shifted by $-\omega > 0$, so that the zero point in the spectrum of the linearized eigenvalue problem (6) is isolated from the continuous spectrum and other isolated eigenvalues for small $\epsilon \in (0, \epsilon_0)$. The geometric kernel of the linearized operator $L = \mathrm{diag}(L_+, L_-)$ is one-dimensional since $L_- \boldsymbol{\phi}(\omega) = \mathbf{0}$ is nothing but the stationary DNLS equation (2), whereas L_+ has an empty kernel thanks to the perturbation theory and Lemma 1. Indeed, for small $\epsilon \in (0, \epsilon_0)$, we have

$$\langle \boldsymbol{\psi}_0, L_+ \boldsymbol{\psi}_0 \rangle = 2p\gamma\epsilon + \mathcal{O}(\epsilon^2) \neq 0.$$

By the perturbation theory, a simple zero eigenvalue of L_+ for $\epsilon = 0$ becomes a positive eigenvalue for $\epsilon > 0$ (if $\gamma = +1$). The second (generalized) eigenvector $(-\partial_\omega \boldsymbol{\phi}(\omega), \mathbf{0})$ is found by direct computation thanks to Lemma 1. It remains to show that the third (generalized) eigenvector does not exist. If it does, it would satisfy the equation

$$L_- \mathbf{w}_0 = -\partial_\omega \boldsymbol{\phi}(\omega).$$

However, by Lemma 1, we obtain

$$\langle \boldsymbol{\phi}(\omega), \partial_\omega \boldsymbol{\phi}(\omega) \rangle = \frac{1}{2} \frac{d}{d\omega} \|\boldsymbol{\phi}(\omega)\|_{l^2}^2 = \frac{\epsilon^{\frac{1}{p}-1}}{2p \|\boldsymbol{\psi}_0\|_{l^{2p+2}}^{2+\frac{2}{p}}} (1 + \mathcal{O}(\epsilon)) > 0 \tag{7}$$

for $\epsilon \in (0, \epsilon_0)$. Therefore, no $\mathbf{w}_0 \in l^2$ exists. $\square$

We say that $(\mathbf{v}, \mathbf{w}) \in l^2$ is symplectically orthogonal to the eigenvectors of the generalized kernel if

$$\langle \mathbf{v}, \boldsymbol{\phi}(\omega) \rangle = 0, \quad \langle \mathbf{w}, \partial_\omega \boldsymbol{\phi}(\omega) \rangle = 0,$$

where $\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{n \in \mathbb{Z}} u_n \bar{w}_n$. Under this condition, $(\mathbf{v}, \mathbf{w}) \in l^2$ belongs to the invariant subspace of the linearized problem (6) that complements its two-dimensional null space.

To determine the time evolution of varying parameters (ω, θ) in the evolution equation (5), we shall add the condition that $\mathbf{z}(t)$ is symplectically orthogonal to the two-dimensional null subspace of the linearized problem (6). To normalize the eigenvectors uniquely, we set

$$\boldsymbol{\psi}_1 = \frac{\boldsymbol{\phi}(\omega)}{\|\boldsymbol{\phi}(\omega)\|_{l^2}}, \quad \boldsymbol{\psi}_2 = \frac{\partial_\omega \boldsymbol{\phi}(\omega)}{\|\partial_\omega \boldsymbol{\phi}(\omega)\|_{l^2}} \tag{8}$$

and require that

$$\langle \mathrm{Re}\mathbf{z}(t), \boldsymbol{\psi}_1 \rangle = \langle \mathrm{Im}\mathbf{z}(t), \boldsymbol{\psi}_2 \rangle = 0. \tag{9}$$

By Lemma 1, both eigenvectors $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ are locally close to $\boldsymbol{\psi}_0$, the eigenvector of H for eigenvalue ω_0, in any norm; that is, for any $\epsilon \in (0, \epsilon_0)$, there exists $C > 0$ such that

$$\|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_0\|_{l^2} + \|\boldsymbol{\psi}_2 - \boldsymbol{\psi}_0\|_{l^2} \leq C\epsilon. \tag{10}$$

Although the vector field of the time-evolution problem (5) does not lie in the orthogonal complement of $\boldsymbol{\psi}_0$, that is, in the absolutely continuous spectrum of H, the difference is small for small $\epsilon > 0$. We shall prove that the conditions (9) define a unique decomposition (4).

LEMMA 4 (decomposition). *Fix $\epsilon \in (0, \epsilon_0)$. There exists $\delta > 0$ such that any $\mathbf{u} \in l^2$ satisfying*

$$\|\mathbf{u} - \boldsymbol{\phi}(\omega_0 + \epsilon)\|_{l^2} \le \delta \epsilon^{\frac{1}{2p}} \tag{11}$$

can be uniquely decomposed by (4) and (9) with $(\omega, \theta) \in \mathbb{R}^2$ and $\mathbf{z} \in l^2$. Moreover, there exists $C > 0$ such that

$$|\omega - \omega_0 - \epsilon| \le C\delta\epsilon, \quad |\theta| \le C\delta, \quad \|\mathbf{z}\|_{l^2} \le C\delta\epsilon^{\frac{1}{2p}}. \tag{12}$$

The mapping $\mathbf{u} \mapsto (\omega, \theta, \mathbf{z})$ is a C^1 diffeomorphism.

Proof. We write the decomposition (4) in the form

$$\mathbf{z} = e^{i\theta} \left(\mathbf{u} - \boldsymbol{\phi}(\omega_0 + \epsilon)\right) + \left(e^{i\theta} \boldsymbol{\phi}(\omega_0 + \epsilon) - \boldsymbol{\phi}(\omega)\right). \tag{13}$$

First, we show that the constraints (9) give unique values of (ω, θ) satisfying bounds (12) provided the bound (11) holds. To do so, we write $\omega = \omega_0 + \epsilon + \epsilon\Omega$ with a new parameter Ω and rewrite (9) and (13) as a fixed-point problem $\mathbf{F}(\Omega, \theta) = \mathbf{0}$, where $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is given by

$$\mathbf{F}_1(\Omega, \theta) = \epsilon^{-\frac{1}{2p}} \begin{bmatrix} \langle \boldsymbol{\phi}^{(0)} \cos\theta - \boldsymbol{\phi}^{(\Omega)}, \boldsymbol{\psi}_1 \rangle \\ \langle \boldsymbol{\phi}^{(0)} \sin\theta, \boldsymbol{\psi}_2 \rangle \end{bmatrix},$$

$$\mathbf{F}_2(\Omega, \theta) = \epsilon^{-\frac{1}{2p}} \begin{bmatrix} \langle \mathrm{Re}(\mathbf{u} - \boldsymbol{\phi}^{(0)}) e^{i\theta}, \boldsymbol{\psi}_1 \rangle \\ \langle \mathrm{Im}(\mathbf{u} - \boldsymbol{\phi}^{(0)}) e^{i\theta}, \boldsymbol{\psi}_2 \rangle \end{bmatrix}.$$

Here $\boldsymbol{\phi}^{(0)} := \boldsymbol{\phi}(\omega_0 + \epsilon)$, $\boldsymbol{\phi}^{(\Omega)} := \boldsymbol{\phi}(\omega_0 + \epsilon + \epsilon\Omega)$, and the factor $\epsilon^{-\frac{1}{2p}}$ is included for convenience. We note that $\mathbf{F}$ is C^1 in (Ω, θ) thanks to Lemma 1, $\mathbf{F}_1(0, 0) = \mathbf{0}$, and the Jacobian $D_{(\Omega,\theta)} \mathbf{F}_1(0, 0)$ is given by

$$\mathbf{D} = \begin{bmatrix} -\epsilon^{-\frac{1}{2p}+1} \langle \partial_\omega \boldsymbol{\phi}^{(0)}, \boldsymbol{\psi}_1^{(0)} \rangle & 0 \\ 0 & \epsilon^{-\frac{1}{2p}} \langle \boldsymbol{\phi}^{(0)}, \boldsymbol{\psi}_2^{(0)} \rangle \end{bmatrix},$$

where $\boldsymbol{\psi}_{1,2}^{(0)} = \boldsymbol{\psi}_{1,2}|_{\omega = \omega_0 + \epsilon}$ and $\partial_\omega \boldsymbol{\phi}^{(0)} = \partial_\omega \boldsymbol{\phi}(\omega_0 + \epsilon)$. Thanks to the bound (11) and the normalization of $\boldsymbol{\psi}_{1,2}$, there exists an (ϵ, δ)-independent constant $C_0 > 0$ such that

$$\|\mathbf{F}_2(\Omega, \theta)\| \le C_0 \delta \quad \forall (\Omega, \theta) \in \mathbb{R}^2.$$

On the other hand, there exist ϵ-independent constants $C_1, C_2 > 0$ such that

$$|\mathbf{D}_{11}| = \epsilon^{-\frac{1}{2p}+1} \partial_\omega \|\boldsymbol{\phi}^{(0)}\|_{l^2} \ge C_1,$$

$$|\mathbf{D}_{22}| = \epsilon^{-\frac{1}{2p}} \frac{\|\boldsymbol{\phi}^{(0)}\|_{l^2} \partial_\omega \|\boldsymbol{\phi}^{(0)}\|_{l^2}}{\|\partial_\omega \boldsymbol{\phi}^{(0)}\|_{l^2}} \ge C_2.$$

As a result, $\mathbf{D}$ is invertible for small $\epsilon > 0$ independently of δ. By the Implicit Function Theorem, there exists a unique root of $\mathbf{F}(\Omega, \theta) = \mathbf{0}$ near $(0, 0)$ for any $\mathbf{u}$ satisfying (11) such that the first two bounds (12) are satisfied. Existence of a unique $\mathbf{z} \in l^2$ and the third bound (12) follows from the representation (13) and the triangle inequality. Since $\mathbf{F}(\Omega, \theta)$ depends linearly on $\mathbf{u}$, the fixed point of $\mathbf{F}(\Omega, \theta)$ is C^1 with respect to $\mathbf{u}$, so that the mapping $\mathbf{u} \mapsto (\omega, \theta, \mathbf{z})$ is a C^1 diffeomorphism. ∎

Assuming $(\omega,\theta) \in C^1(\mathbb{R}_+,\mathbb{R}^2)$ and using the decomposition (4), we define the time evolution of (ω,θ) from the projections of the time-evolution equation (5) with the symplectic orthogonality conditions (9). The resulting system is written in the matrix-vector form

$$\mathbf{A}(\omega,\mathbf{z}) \left[\begin{array}{c} \dot{\omega} \\ \dot{\theta}-\omega \end{array} \right] = \mathbf{f}(\omega,\mathbf{z}), \tag{14}$$

where

$$\mathbf{A}(\omega,\mathbf{z}) = \left[\begin{array}{cc} \langle \partial_\omega \boldsymbol{\phi}(\omega), \boldsymbol{\psi}_1\rangle - \langle \mathrm{Re}\mathbf{z}, \partial_\omega \boldsymbol{\psi}_1\rangle & \langle \mathrm{Im}\mathbf{z}, \boldsymbol{\psi}_1\rangle \\ \langle \mathrm{Im}\mathbf{z}, \partial_\omega \boldsymbol{\psi}_2\rangle & \langle \boldsymbol{\phi}(\omega) + \mathrm{Re}\mathbf{z}, \boldsymbol{\psi}_2\rangle \end{array} \right]$$

and

$$\mathbf{f}(\omega,\mathbf{z}) = \left[\begin{array}{l} \langle \mathrm{Im}\mathbf{N}(\boldsymbol{\phi}+\mathbf{z}) - \mathbf{W}\mathrm{Im}\mathbf{z}, \boldsymbol{\psi}_1\rangle \\ \langle \mathrm{Re}\mathbf{N}(\boldsymbol{\phi}+\mathbf{z}) - \mathbf{N}(\boldsymbol{\phi}) - (2p+1)\mathbf{W}\mathrm{Re}\mathbf{z}, \boldsymbol{\psi}_2\rangle \end{array} \right].$$

Using an elementary property for power functions, there exists $C_p > 0$ such that

$$||a+b|^{2p}(a+b) - |a|^{2p}a| \leq C_p(|a|^{2p}|b| + |b|^{2p+1}) \quad \forall a,b \in \mathbb{C}.$$

As a result, we bound the vector fields of (5) and (14) by

$$\|\mathbf{N}(\boldsymbol{\phi}(\omega)+\mathbf{z}) - \mathbf{N}(\boldsymbol{\phi}(\omega))\|_{l^2} \leq C\left(\||\boldsymbol{\phi}(\omega)|^{2p}|\mathbf{z}|\|_{l^2} + \|\mathbf{z}\|_{l^2}^{2p+1}\right) \tag{15}$$

and

$$\|\mathbf{f}(\omega,\mathbf{z})\| \leq C\sum_{j=1}^{2}\left(\||\boldsymbol{\phi}(\omega)|^{2p-1}|\boldsymbol{\psi}_j||\mathbf{z}|^2\|_{l^1} + \||\boldsymbol{\psi}_j||\mathbf{z}|^{2p+1}\|_{l^1}\right) \tag{16}$$

for some $C > 0$, where the pointwise multiplication of vectors on $\mathbb{Z}$ is understood in the sense of $(|\boldsymbol{\phi}||\boldsymbol{\psi}|)_n = \phi_n\psi_n$. By Lemmas 1 and 4, $\mathbf{A}(\omega,\mathbf{z})$ is invertible for a small $\mathbf{z} \in l^2$ and a small $\epsilon \in (0,\epsilon_0)$ so that solutions of system (14) enjoy the estimates

$$|\dot{\omega}| \leq C\epsilon^{2-\frac{1}{p}}\left(\||\boldsymbol{\psi}_1||\mathbf{z}|^2\|_{l^1} + \||\boldsymbol{\psi}_2||\mathbf{z}|^2\|_{l^1}\right), \tag{17}$$

$$|\dot{\theta}-\omega| \leq C\epsilon^{1-\frac{1}{p}}\left(\||\boldsymbol{\psi}_1||\mathbf{z}|^2\|_{l^1} + \||\boldsymbol{\psi}_2||\mathbf{z}|^2\|_{l^1}\right) \tag{18}$$

for some $C > 0$ uniformly in $\|\mathbf{z}\|_{l^2} \leq C_0\epsilon^{\frac{1}{2p}}$ for some $C_0 > 0$.

Remark 3. The estimates (17) and (18) show that if $\|\mathbf{z}\|_{l^2} \leq C\delta\epsilon^{\frac{1}{2p}}$ for some $C > 0$, then

$$|\omega(t)-\omega(0)| \leq C\delta^2\epsilon^2, \quad \left|\theta(t) - \int_0^t \omega(t')dt'\right| \leq C\delta^2\epsilon \quad \forall t \in [0,T], \tag{19}$$

for any fixed $T > 0$. These bounds are tighter than the bounds (12) of Lemma 4. They become comparable with bounds (12) for larger time intervals $[0,T]$, where $T \leq \frac{C_0}{\delta\epsilon}$ for some $C_0 > 0$. Our main task is to extend bounds (19) globally to $T = \infty$.

Thanks to estimate (7), the stationary solution $e^{-i\omega t}\boldsymbol{\phi}(\omega)$ is orbitally stable for a fixed ω near ω_0 [27], so that a trajectory $\mathbf{u}(t)$ of the DNLS equation (1) originating from a point in a local neighborhood of solution $\boldsymbol{\phi}(\omega(0))$ in Lemma 1 remains in a local neighborhood of the solution orbit $e^{-i\theta(t)}\boldsymbol{\phi}(\omega(t))$ for all $t \in \mathbb{R}_+$, where time evolution of $(\omega(t), \theta(t))$ obeys system (14) and the remainder term $\mathbf{z}(t) = e^{i\theta(t)}\mathbf{u}(t) - \boldsymbol{\phi}(\omega(t))$ satisfies system (5). To prove the main result on asymptotic stability, we need to show that $\omega(t)$ approaches some $\omega_\infty \in (\omega_0, \omega_0 + \epsilon_0)$ as $t \to \infty$, whereas the remainder term $\mathbf{z}(t)$ decays to zero in l^∞ norm as $t \to \infty$. Our main result is formulated as follows.

Theorem 1 (asymptotic stability in the energy space). *Assume* (V1)–(V3)*, and fix $\gamma = +1$ and $p \geq 3$. Let $\epsilon > 0$ and $\delta > 0$ be sufficiently small, and assume that $\theta(0) = 0$, $\omega(0) = \omega_0 + \epsilon$, and*

$$\|\mathbf{u}(0) - \boldsymbol{\phi}(\omega_0 + \epsilon)\|_{l^2} \leq \delta \epsilon^{\frac{1}{2p}}.$$

Then, there exist $\theta_\infty \in \mathbb{R}$, $\omega_\infty \in (\omega_0, \omega_0 + \epsilon_0)$, $(\omega, \theta) \in C^1(\mathbb{R}_+, \mathbb{R}^2)$, and $\mathbf{y}(t) = \mathbf{u}(t) - e^{-i\theta(t)}\boldsymbol{\phi}(\omega(t)) \in X := C^1(\mathbb{R}_+, l^2) \cap L^6(\mathbb{R}_+, l^\infty)$ such that $\mathbf{u}(t)$ solves the DNLS equation (1) *and*

$$\lim_{t\to\infty}\left(\theta(t) - \int_0^t \omega(s)ds\right) = \theta_\infty, \quad \lim_{t\to\infty}\omega(t) = \omega_\infty,$$
$$\lim_{t\to\infty}\|\mathbf{u}(t) - e^{-i\theta(t)}\boldsymbol{\phi}(\omega(t))\|_{l^\infty} = 0.$$

Remark 4. The result remains true for the focusing case $\gamma = -1$ with the only difference being that the local bifurcation of Lemma 1 occurs in the domain $(\omega_0 - \epsilon_0, \omega_0]$. Without loss of generality, we shall develop an analysis for the defocusing case $\gamma = +1$ only.

Theorem 1 is proved in section 4. To bound solutions of the time-evolution problem (5) in the space X (intersected with some other spaces of technical nature), we need some linear estimates, which are described in section 3.

3. Linear estimates. We need several types of linear estimates; each is designed to control different nonlinear terms of the vector field of the evolution equation (5). For notational convenience, we shall use L^p_t and l^q_n to denote L^p space on $t \in [0, T]$ and l^q space on $n \in \mathbb{Z}$, where $T > 0$ is an arbitrary time including $T = \infty$. We will use spaces $L^p_t l^q_n$ and $l^q_n L^p_t$ with the norm

$$\|\mathbf{f}\|_{L^p_t l^q_n} = \left(\int_0^T \|\mathbf{f}(t)\|^p_{l^q_n} dt\right)^{1/p}, \quad \|\mathbf{f}\|_{l^q_n L^p_t} = \left(\sum_{n\in\mathbb{Z}} \|f_n\|^q_{L^p_t}\right)^{1/q}.$$

The notation $\langle n \rangle = (1 + n^2)^{1/2}$ is used for the weights in l^q_n norms. The constant $C > 0$ is a generic constant, which may change from one line to another line.

3.1. Decay and Strichartz estimates. Under assumptions (V1)–(V2) on the potential, the following result was proved in [21].

Lemma 5 (dispersive decay estimates). *Fix $\sigma > \frac{5}{2}$ and assume* (V1)–(V2). *There exists a constant $C > 0$ depending on $\mathbf{V}$ such that*

$$\left\|\langle n \rangle^{-\sigma} e^{-itH} P_{a.c.}(H)\mathbf{f}\right\|_{l^2_n} \leq C(1+t)^{-3/2}\|\langle n \rangle^\sigma \mathbf{f}\|_{l^2_n}, \tag{20}$$

$$\left\|e^{-itH} P_{a.c.}(H)\mathbf{f}\right\|_{l^\infty_n} \leq C(1+t)^{-1/3}\|\mathbf{f}\|_{l^1_n} \tag{21}$$

for all $t \in \mathbb{R}_+$, where $P_{a.c.}(H)$ is the projection to the absolutely continuous spectrum of H.

Remark 5. Unlike the continuous case, the upper bound (21) is nonsingular as $t \to 0$ because the discrete case always enjoys an estimate $\|\mathbf{f}\|_{l^\infty_n} \le \|\mathbf{f}\|_{l^2_n} \le \|\mathbf{f}\|_{l^1_n}$.

Using Lemma 5 and Theorem 1.2 of Keel and Tao [10], the following corollary transfers pointwise decay estimates into Strichartz estimates.

COROLLARY 1 (discrete Strichartz estimates). *There exists a constant $C > 0$ such that*

$$\left\| e^{-itH} P_{a.c.}(H)\mathbf{f} \right\|_{L^6_t l^\infty_n \cap L^\infty_t l^2_n} \le C\|\mathbf{f}\|_{l^2_n}, \tag{22}$$

$$\left\| \int_0^t e^{-i(t-s)H} P_{a.c.}(H)\mathbf{g}(s)ds \right\|_{L^6_t l^\infty_n \cap L^\infty_t l^2_n} \le C\|\mathbf{g}\|_{L^1_t l^2_n}. \tag{23}$$

We say that (r, w) is a Strichartz pair if $2 \le r, w \le \infty$, and $\frac{6}{r} + \frac{2}{w} \le 1$. For any Strichartz pair (r, w), there exists $2 \le \tilde{w} \le w$, so that $\frac{6}{r} + \frac{2}{\tilde{w}} = 1$. It follows that there is $\tau \in [0, 1]$ such that

$$\left(\frac{1}{r}, \frac{1}{\tilde{w}}\right) = \tau \left(\frac{1}{6}, 0\right) + (1-\tau)\left(0, \frac{1}{2}\right).$$

By the embedding $l^{\tilde{w}} \hookrightarrow l^w$, the Gagliardo–Nirenberg inequality (also known as the log-convexity of the l^p norms), and Young's inequality, we have for all Strichartz pairs

$$\|\mathbf{f}\|_{L^r_t l^w_n} \le \|\mathbf{f}\|_{L^r_t l^{\tilde{w}}_n} \le \|\mathbf{f}\|^{\tau}_{L^6_t l^\infty_n} \|\mathbf{f}\|^{1-\tau}_{L^\infty_t l^2_n} \le \|\mathbf{f}\|_{L^6_t l^\infty_n} + \|\mathbf{f}\|_{L^\infty_t l^2_n} =: \|\mathbf{f}\|_{L^6_t l^\infty_n \cap L^\infty_t l^2_n}. \tag{24}$$

3.2. Time averaged estimates. To control the evolution of the varying parameters (ω, θ), we derive additional time averaged estimates. Similar to the continuous case, these estimates are needed only in one dimension, because the time decay provided by the Strichartz estimates is insufficient to guarantee time integrability of $\dot{\omega}(t)$ and $\dot{\theta}(t) - \omega(t)$ bounded from above by the estimates (17) and (18). Without the time integrability of these quantities, the arguments on the decay of various norms of $\mathbf{z}(t)$ satisfying the time-evolution problem (5) cannot be closed.

LEMMA 6. *Fix $\sigma > \frac{5}{2}$ and assume* (V1) *and* (V2). *There exists a constant $C > 0$ depending on $\mathbf{V}$ such that*

$$\|\langle n\rangle^{-3/2} e^{-itH} P_{a.c.}(H)\mathbf{f}\|_{l^\infty_n L^2_t} \le C\|\mathbf{f}\|_{l^2_n}, \tag{25}$$

$$\left\| \int_{\mathbb{R}_+} e^{-itH} P_{a.c.}(H)\mathbf{F}(t)dt \right\|_{l^2_n} \le C\|\langle n\rangle^{3/2}\mathbf{F}\|_{l^1_n L^2_t}, \tag{26}$$

$$\left\| \langle n\rangle^{-\sigma} \int_0^t e^{-i(t-s)H} P_{a.c.}(H)\mathbf{F}(s)ds \right\|_{l^\infty_n L^2_t} \le C\|\langle n\rangle^{\sigma}\mathbf{F}\|_{l^1_n L^2_t}, \tag{27}$$

$$\left\| \langle n\rangle^{-\sigma} \int_0^t e^{-i(t-s)H} P_{a.c.}(H)\mathbf{F}(s)ds \right\|_{l^\infty_n L^2_t} \le C\|\mathbf{F}\|_{L^1_t l^2_n}, \tag{28}$$

$$\left\| \int_0^t e^{-i(t-s)H} P_{a.c.}(H)\mathbf{F}(s)ds \right\|_{L^6_t l^\infty_n \cap L^\infty_t l^2_n} \le C\|\langle n\rangle^3\mathbf{F}\|_{L^2_t l^2_n}. \tag{29}$$

To proceed with the proof, let us set up a few notations. First, introduce the perturbed resolvent $R_V(\lambda) := (H - \lambda)^{-1}$ for $\lambda \in \mathbb{C}\backslash[0, 4]$. We proved in [21, Theorem 1] that for any fixed $\omega \in (0, 4)$, there exists $R^{\pm}_V(\omega) = \lim_{\epsilon\downarrow 0} R(\omega \pm i\epsilon)$ in the norm of $B(\sigma, -\sigma)$ for any $\sigma > \frac{1}{2}$, where $B(\sigma, -\sigma)$ denotes the space of bounded operators from l^2_σ to $l^2_{-\sigma}$.

Next, we recall the Cauchy formula for e^{itH},

$$e^{-itH}P_{a.c.}(H)=\frac{1}{\pi}\int_0^4 e^{-it\omega}\mathrm{Im}R_V(\omega)d\omega=\frac{1}{2\pi i}\int_0^4 e^{-it\omega}\left[R^+(\omega)-R^-(\omega)\right]d\omega, \tag{30}$$

where the integral is understood in norm $B(\sigma,-\sigma)$. We shall parameterize the interval $[0,4]$ by $\omega=2-2\cos(\theta)$ for $\theta\in[-\pi,\pi]$.

Let $\chi_0,\chi\in C_0^\infty:\ \chi_0+\chi=1$ for all $\theta\in[-\pi,\pi]$, so that

$$\mathrm{supp}\chi_0\subset[-\theta_0,\theta_0]\cup(-\pi,-\pi+\theta_0)\cup(\pi-\theta_0,\pi)$$

and

$$\mathrm{supp}\chi\subset[\theta_0/2,\pi-\theta_0/2]\cup[-\pi+\theta_0/2,-\theta_0/2],$$

where $0<\theta_0\le\frac{\pi}{4}$. Note that the support of χ stays away from both 0 and π. Following Mizumachi [17], the proof of Lemma 6 relies on the following technical lemma.

LEMMA 7. *Assume* (V1) *and* (V2). *There exists a constant* $C>0$ *such that*

$$\sup_{n\in\mathbb{Z}}\|\chi R_V^{\pm}(\omega)\mathbf{f}\|_{L^2_\omega(0,4)}\le C\|\mathbf{f}\|_{l^2_n}, \tag{31}$$

$$\sup_{n\in\mathbb{Z}}\|\langle n\rangle^{-3/2}\chi_0 R_V^{\pm}(\omega)\mathbf{f}\|_{L^2_\omega(0,4)}\le C\|\mathbf{f}\|_{l^2_n}. \tag{32}$$

The proof of Lemma 7 is developed in Appendix A. Using Lemma 7, we can now prove Lemma 6.

Proof of Lemma 6. Let us first show (27), since it can be deduced from (20); however, it can also be viewed (and proved) as a dual of (25) as well. Indeed, (27) is equivalent to

$$\left\|\langle n\rangle^{-\sigma}\int_0^t e^{-i(t-s)H}P_{a.c.}(H)\langle n\rangle^{-\sigma}\mathbf{G}(s)ds\right\|_{l^\infty_n L^2_t}\le\|\mathbf{G}\|_{l^1_n L^2_t}.$$

Let $\mathbf{G}=\{g_n(t)\}_n\in l^1_nL^2_t$. In order to separate the n variable from the t variable, write $g_n(s)=\sum_{n_0}\delta_{n,n_0}g_{n_0}(s)$. By Minkowski's inequality, the embedding $l^2\hookrightarrow l^\infty$, and the dispersive decay estimate (20) for any $\sigma>\frac{5}{2}$, we have

$$\begin{aligned}
&\left\|\langle n\rangle^{-\sigma}\int_0^t e^{-i(t-s)H}P_{a.c.}(H)\langle n\rangle^{-\sigma}\sum_{n_0}\delta_{n,n_0}g_{n_0}(s)\,ds\right\|_{l^\infty_n L^2_t}\\
&\le C\sum_{n_0}\left\|\int_0^t\|\langle n\rangle^{-\sigma}e^{-i(t-s)H}P_{a.c.}(H)\langle n\rangle^{-\sigma}\delta_{n,n_0}\|_{l^2_n}|g_{n_0}(s)|\,ds\right\|_{L^2_t}\\
&\le C\sum_{n_0}\left\|\int_0^t\frac{|g_{n_0}(s)|\,ds}{(1+t-s)^{3/2}}\right\|_{L^2_t}\le C\sum_{n_0}\|g_{n_0}\|_{L^2_t}=C\|G\|_{l^1_nL^2_t},
\end{aligned}$$

where, in the last step, we have used the Hausdorff–Young inequality $L^1*L^2\hookrightarrow L^2$.

We show next that (26), (28), and (29) follow from (25). Indeed, (26) is simply a dual of (25) and (26) is hence equivalent to (25). For (28), we apply the so-called averaging principle, which tells us that to prove (28), it is sufficient to show it for

$\mathbf{F}(t) = \delta(t - t_0)\mathbf{f}$, where $\mathbf{f} \in l^2_n$ and $\delta(t-t_0)$ is Dirac's delta-function. Therefore, we obtain

$$\begin{aligned}
&\left\| \langle n\rangle^{-\sigma} \int_0^t e^{-i(t-s)H}\delta(s-t_0)P_{a.c.}(H)\mathbf{f}ds \right\|_{l^\infty_n L^2_t} \\
&\quad = \|\langle n\rangle^{-\sigma} e^{-i(t-t_0)H}P_{a.c.}(H)\mathbf{f}\|_{l^\infty_n L^2_t} \\
&\quad \le \|\langle n\rangle^{-3/2} e^{-i(t-t_0)H}P_{a.c.}(H)\mathbf{f}\|_{l^\infty_n L^2_t} \\
&\quad \le C\|\mathbf{f}\|_{l^2_n},
\end{aligned}$$

where, in the last step, we have used (25).

For (29), we argue as follows. Define

$$\begin{aligned}
T\mathbf{F}(t) &= \int_{\mathbb{R}} e^{-i(t-s)H}P_{a.c.}(H)\mathbf{F}(s)ds \\
&= e^{-itH}P_{a.c.}(H)\left(\int_{\mathbb{R}} e^{-isH}P_{a.c.}(H)\mathbf{F}(s)ds\right) \\
&= e^{-itH}P_{a.c.}(H)\mathbf{f},
\end{aligned}$$

where $\mathbf{f} = \int_{\mathbb{R}} e^{-isH}P_{a.c.}(H)\mathbf{F}(s)ds$. By an application of the Strichartz estimate (22) and subsequently (26), we obtain

$$\begin{aligned}
\|T\mathbf{F}\|_{L^6_t l^\infty_n \cap L^\infty_t l^2_n} &\le C\|\mathbf{f}\|_{l^2_n} \le \|\langle n\rangle^{3/2}\mathbf{F}\|_{l^1_n L^2_t} \\
&\le C\|\langle n\rangle^3 \mathbf{F}\|_{l^2_n L^2_t} = C\|\langle n\rangle^3\mathbf{F}\|_{L^2_t l^2_n},
\end{aligned}$$

where, in the last two steps, we have used Hölder's inequality and the fact that l^2_n and L^2_t commute. Now, by the Christ–Kiselev lemma (e.g., Theorem 1.2 in [10]), we conclude that the estimate (29) applies to $\int_0^t e^{-i(t-s)H}P_{a.c.}(H)\mathbf{F}(s)ds$, similar to $T\mathbf{F}(t)$. To complete the proof of Lemma 7, it remains only to prove (25). Let us write

$$e^{-itH}P_{a.c.}(H) = \chi e^{-itH}P_{a.c.}(H) + \chi_0 e^{-itH}P_{a.c.}(H).$$

Take a test function $\mathbf{g}(t)$ such that $\|\mathbf{g}\|_{l^1_n L^2_t} = 1$ and obtain

$$\begin{aligned}
\left|\left\langle \chi e^{-itH}P_{a.c.}(H)\mathbf{f}, \mathbf{g}(t)\right\rangle_{n,t}\right| &= \frac{1}{\pi}\left|\int_0^4 \left\langle \chi \mathrm{Im}R_V(\omega)\mathbf{f}, \int_{\mathbb{R}} e^{-it\omega}\mathbf{g}(t)dt\right\rangle_n d\omega\right| \\
&\le C\int_0^4 \langle |\chi R_V(\omega)\mathbf{f}|, |\hat{\mathbf{g}}(\omega)|\rangle_n\, d\omega \\
&\le C\|\chi R_V^{\pm}(\omega)\mathbf{f}\|_{l^\infty_n L^2_\omega(0,4)}\|\hat{\mathbf{g}}\|_{l^1_n L^2_\omega(0,4)}.
\end{aligned}$$

By Plancherel's theorem, $\|\hat{\mathbf{g}}\|_{l^1_n L^2_\omega(0,4)} \le \|\hat{\mathbf{g}}\|_{l^1_n L^2_\omega(\mathbb{R})} \le \|\mathbf{g}\|_{l^1_n L^2_t} = 1$. Using (31), we obtain

$$\left\|\chi e^{-itH}P_{a.c.}(H)\mathbf{f}\right\|_{l^\infty_n L^2_t} = \sup_{\|\mathbf{g}\|_{l^1_n L^2_t}=1}\left|\left\langle \chi e^{-itH}P_{a.c.}(H)\mathbf{f}, \mathbf{g}(t)\right\rangle_{n,t}\right| \le C\|\mathbf{f}\|_{l^2_n}.$$

Similarly, using (32) instead of (31), one concludes

$$\begin{aligned}
\left\|\langle n\rangle^{-3/2}\chi_0 e^{-itH}P_{a.c.}(H)\mathbf{f}\right\|_{l^\infty_n L^2_t} &= \sup_{\|\langle n\rangle^{3/2}\mathbf{g}\|_{l^1_n L^2_t}=1}\left|\left\langle \chi_0 e^{-itH}P_{a.c.}(H)\mathbf{f}, \mathbf{g}(t)\right\rangle_{n,t}\right| \\
&\le C\|\mathbf{f}\|_{l^2_n}.
\end{aligned}$$

Combining the two estimates, we obtain (25). ∎

4. Proof of Theorem 1. Let $\mathbf{y}(t) = e^{-i\theta(t)}\mathbf{z}(t)$ and write the time-evolution problem for $\mathbf{y}(t)$ in the form

$$i\dot{\mathbf{y}} = H\mathbf{y} + \mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}_3,$$

where

$$\mathbf{g}_1 = \left(\mathbf{N}(\boldsymbol{\phi} + \mathbf{y}e^{i\theta}) - \mathbf{N}(\boldsymbol{\phi})\right)e^{-i\theta}, \;\; \mathbf{g}_2 = -(\dot{\theta} - \omega)\boldsymbol{\phi}e^{-i\theta}, \;\; \mathbf{g}_3 = -i\dot{\omega}\partial_\omega\boldsymbol{\phi}(\omega)e^{-i\theta}.$$

Let $P_0 = \langle\cdot, \boldsymbol{\psi}_0\rangle\boldsymbol{\psi}_0$, $Q = (I - P_0) \equiv P_{a.c.}(H)$, and decompose the solution $\mathbf{y}(t)$ into two orthogonal parts

$$\mathbf{y}(t) = a(t)\boldsymbol{\psi}_0 + \boldsymbol{\eta}(t),$$

where $\langle\boldsymbol{\eta}(t), \boldsymbol{\psi}_0\rangle = 0$ and $a(t) = \langle\mathbf{y}(t), \boldsymbol{\psi}_0\rangle$. The new coordinates $a(t)$ and $\boldsymbol{\eta}(t)$ satisfy the time-evolution problem

$$\begin{cases} i\dot{a} &= \omega_0 a + \langle\mathbf{g}, \boldsymbol{\psi}_0\rangle, \\ i\dot{\boldsymbol{\eta}} &= H\boldsymbol{\eta} + Q\mathbf{g}, \end{cases}$$

where $\mathbf{g} = \sum_{j=1}^{3}\mathbf{g}_j$. The time-evolution problem for $\boldsymbol{\eta}(t)$ can be written in the integral form

$$\boldsymbol{\eta}(t) = e^{-itH}Q\boldsymbol{\eta}(0) - i\int_0^t e^{-i(t-s)H}Q\mathbf{g}(s)ds. \tag{33}$$

Fix $\sigma > \frac{5}{2}$, $p \geq 3$, and introduce the norms

$$M_1 = \|\boldsymbol{\eta}\|_{L_t^6 l_n^\infty}, \quad M_2 = \|\boldsymbol{\eta}\|_{L_t^\infty l_n^2}, \quad M_3 = \|\langle n\rangle^{-\sigma}\boldsymbol{\eta}\|_{l_n^\infty L_t^2},$$
$$M_4 = \|a\|_{L_t^2}, \quad M_5 = \|a\|_{L_t^\infty}, \quad M_6 = \|\omega - \omega(0)\|_{L_t^\infty}, \quad M_7 = \left\|\theta - \int_0^t \omega(s)ds\right\|_{L_t^\infty},$$

where the integration in time t is performed on an interval $[0, T]$ for any $T > 0$ including $T = \infty$. Our goal is to show that $\dot{\omega}$ and $\dot{\theta} - \omega$ are in L_t^1, while the norms above satisfy an estimate of the form

$$\begin{aligned} M_1 + M_2 + M_3 &\leq C(\|\mathbf{y}(0)\|_{l_n^2} + \epsilon^{1-\frac{1}{p}}(M_3 + M_4)^2) \\ &+ C\left((M_3 + M_4)(\epsilon + M_6) + M_4^2 M_5^{2p-1} + (M_1 + M_2)^{2p+1}\right), \end{aligned} \tag{34}$$

$$M_4 + M_5 \leq C(M_2 + M_3 + M_4 + M_5)(\epsilon + M_6), \tag{35}$$

$$M_6 \leq C\epsilon^{2-\frac{1}{p}}(M_3 + M_4)^2, \tag{36}$$

$$M_7 \leq C\epsilon^{1-\frac{1}{p}}(M_3 + M_4)^2, \tag{37}$$

where $C > 0$ is T-independent and (ϵ, δ) are fixed by the initial conditions

$$\theta(0) = 0, \quad \omega(0) = \omega_0 + \epsilon, \quad \text{and} \quad \|\mathbf{y}(0)\|_{l_n^2} \leq \delta\epsilon^{\frac{1}{2p}}.$$

The estimates (34), (35), and (36) allow us to conclude, by elementary continuation arguments, that

$$M_1 + M_2 + M_3 + M_4 + M_5 \leq C\|\mathbf{y}(0)\|_{l_n^2} \leq C\delta\epsilon^{\frac{1}{2p}}$$

and

$$\|\omega - \omega_0 - \epsilon\|_{L_t^\infty} \le C\delta^2\epsilon^2, \quad \left\|\theta - \int_0^t \omega(s)ds\right\|_{L_t^\infty} \le C\delta^2\epsilon$$

for any $T \in (0,\infty)$. By interpolation, $a \in L_t^6$ so that $\mathbf{y}(t) \in L^6([0,T], l_n^\infty)$. Note that the second bounds agree with (19) of Remark 3.

Let us now take the opportunity to discuss the continuity of $t \to (\omega(t), \theta(t), \mathbf{z}(t))$. First, by Lemma 2, we have that $\mathbf{u}(t) \in C^1(\mathbb{R}_+, l^2)$. Hence, by the decomposition (4) we infer that $\mathbf{z}(t) \in L^\infty(\mathbb{R}_+, l^2)$. Once that is established, we have by (14) that $(\dot\omega, \dot\theta - \omega)$ are locally bounded functions; hence (ω, θ) are continuous. A re-examination of (4) yields that $\mathbf{z}(t)$ is continuous as well. Furthermore, by the integral formulation of system (14) and smallness of $\mathbf{z}(t)$, we thus have that $(\omega, \theta) \in C^1([0,T], \mathbb{R}^2)$. Since $\mathbf{u} = e^{-\theta(t)}\boldsymbol{\phi}(\omega(t)) + \mathbf{y}(t)$ and $\mathbf{u}(t) \in C^1(\mathbb{R}_+, l^2)$, this implies that $\mathbf{y}(t) \in C^1([0,T], l^2)$ by uniqueness of the decomposition in Lemma 4.

Theorem 1 holds for $T = \infty$. In particular, since $\dot\omega(t) \in L_t^1$ and $\|\omega - \omega_0 - \epsilon\|_{L_t^\infty} \le C\delta^2\epsilon^2$, there exists $\omega_\infty := \lim_{t\to\infty}\omega(t)$ such that $\omega_\infty \in (\omega_0, \omega_0 + \epsilon_0)$. Similarly, since $\dot\theta(t) - \omega(t) \in L_t^1$, there exists $\theta_\infty \in \mathbb{R}$ such that

$$\lim_{t\to\infty}\left(\theta(t) - \int_0^t \omega(s)ds\right) = \theta_\infty.$$

In addition, since $\mathbf{y}(t) \in L^6(\mathbb{R}_+, l_n^\infty)$ and $\mathbf{y} \in C(\mathbb{R}_+, l_n^\infty)$, then

$$\lim_{t\to\infty}\|\mathbf{u}(t) - e^{-i\theta(t)}\boldsymbol{\phi}(\omega(t))\|_{l_n^\infty} = \lim_{t\to\infty}\|\mathbf{y}(t)\|_{l_n^\infty} = 0.$$

Estimates for M_6 and M_7. By the estimate (17), we have

$$\begin{aligned}\int_0^T |\dot\omega| dt &\le C\epsilon^{2-\frac{1}{p}}\|\langle n\rangle^{-2\sigma}|\mathbf{y}|^2\|_{L_t^1 l_n^\infty}\left(\|\langle n\rangle^{2\sigma}\boldsymbol{\psi}_1\|_{L_t^\infty l_n^1} + \|\langle n\rangle^{2\sigma}\boldsymbol{\psi}_2\|_{L_t^\infty l_n^1}\right)\\ &\le C\epsilon^{2-\frac{1}{p}}\|\langle n\rangle^{-\sigma}\mathbf{y}\|^2_{l_n^\infty L_t^2}\\ &\le C\epsilon^{2-\frac{1}{p}}(M_3 + M_4)^2,\end{aligned}$$

where we have used the fact that $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ decay exponentially as $|n| \to \infty$ and that $\|\omega - \omega(0)\|_{L_t^\infty}$ is found to be small. As a result, we obtain

$$M_6 \le \int_0^T |\dot\omega| dt \le C\epsilon^{2-\frac{1}{p}}(M_3 + M_4)^2.$$

Similarly, we also obtain that

$$M_7 \le \int_0^T |\dot\theta - \omega| dt \le C\epsilon^{1-\frac{1}{p}}(M_3 + M_4)^2.$$

Estimates for M_4 and M_5. We use the projection formula $a = \langle \mathbf{y}, \boldsymbol{\psi}_0\rangle$ and recall the orthogonality relation (9), so that

$$\langle \mathbf{z}, \boldsymbol{\psi}_0\rangle = \langle \mathrm{Re}\mathbf{z}, \boldsymbol{\psi}_0 - \boldsymbol{\psi}_1\rangle + i\langle \mathrm{Im}\mathbf{z}, \boldsymbol{\psi}_0 - \boldsymbol{\psi}_2\rangle.$$

By Lemma 1 and definitions of $\boldsymbol{\psi}_{1,2}$ in (8), we have

$$\|\langle n\rangle^{2\sigma}(\boldsymbol{\psi}_0 - \boldsymbol{\psi}_{1,2})\|_{l_n^2} \le C|\omega - \omega_0|.$$

If $\sigma > \frac{1}{2}$, we obtain

$$\begin{aligned}
M_4 = \|\langle \mathbf{z}, \boldsymbol{\psi}_0\rangle\|_{L_t^2} &\le \|\langle \mathrm{Re}\mathbf{z}, \boldsymbol{\psi}_0 - \boldsymbol{\psi}_1\rangle\|_{L_t^2} + \|\langle \mathrm{Im}\mathbf{z}, \boldsymbol{\psi}_0 - \boldsymbol{\psi}_2\rangle\|_{L_t^2} \\
&\le \|\langle n\rangle^{-2\sigma}\mathbf{z}\|_{L_t^2 l_n^2}\left(\|\langle n\rangle^{2\sigma}(\boldsymbol{\psi}_0 - \boldsymbol{\psi}_1)\|_{L_t^\infty l_n^2} + \|\langle n\rangle^{2\sigma}(\boldsymbol{\psi}_0 - \boldsymbol{\psi}_2)\|_{L_t^\infty l_n^2}\right) \\
&\le C\|\langle n\rangle^{-\sigma}\mathbf{y}\|_{l_n^\infty L_t^2}\|\omega - \omega_0\|_{L_t^\infty} \le C(M_3 + M_4)(\epsilon + M_6)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
M_5 = \|\langle \mathbf{z}, \boldsymbol{\psi}_0\rangle\|_{L_t^\infty} &\le \|\langle \mathrm{Re}\mathbf{z}, \boldsymbol{\psi}_0 - \boldsymbol{\psi}_1\rangle\|_{L_t^\infty} + \|\langle \mathrm{Im}\mathbf{z}, \boldsymbol{\psi}_0 - \boldsymbol{\psi}_2\rangle\|_{L_t^\infty} \\
&\le \|\mathbf{y}\|_{L_t^\infty l_n^2}\left(\|(\boldsymbol{\psi}_0 - \boldsymbol{\psi}_1)\|_{L_t^\infty l_n^2} + \|(\boldsymbol{\psi}_0 - \boldsymbol{\psi}_2)\|_{L_t^\infty l_n^2}\right) \le C(M_2 + M_5)(\epsilon + M_6),
\end{aligned}$$

where we used the triangle inequality

$$\|\omega - \omega_0\|_{L_t^\infty} \le |\omega(0) - \omega_0| + \|\omega - \omega(0)\|_{L_t^\infty} = \epsilon + M_6.$$

Estimates for M_3. The free solution in the integral equation (33) is estimated by (25) for $\sigma > \frac{5}{2}$ as

$$\|\langle n\rangle^{-\sigma} e^{-itH} Q\boldsymbol{\eta}(0)\|_{l_n^\infty L_t^2} \le \|\langle n\rangle^{-3/2} e^{-itH} Q\boldsymbol{\eta}(0)\|_{l_n^\infty L_t^2} \le C\|\boldsymbol{\eta}(0)\|_{l_n^2}.$$

Since $\dot{\omega}$ and $\dot{\theta} - \omega$ are L_t^1 thanks to the estimates above, we treat the terms of the integral equation (33) with $\mathbf{g}_2$ and $\mathbf{g}_3$ similarly. By (28), we obtain

$$\begin{aligned}
&\left\|\langle n\rangle^{-\sigma}\int_0^t e^{-i(t-s)H} Q\mathbf{g}_{2,3}(s)\,ds\right\|_{l_n^\infty L_t^2} \\
&\quad\le C\left(\|\mathbf{g}_2\|_{L_t^1 l_n^2} + \|\mathbf{g}_3\|_{L_t^1 l_n^2}\right) \\
&\quad\le C\left(\|\dot{\theta} - \omega\|_{L_t^1}\|\boldsymbol{\phi}(\omega)\|_{L_t^\infty l_n^2} + \|\dot{\omega}\|_{L_t^1}\|\partial_\omega\boldsymbol{\phi}(\omega)\|_{L_t^\infty l_n^2}\right) \\
&\quad\le C\epsilon^{1-\frac{1}{2p}}(M_3 + M_4)^2.
\end{aligned}$$

On the other hand, using the bound (15) on the vector field $\mathbf{g}_1$, we estimate by (27) and (28)

$$\begin{aligned}
&\left\|\langle n\rangle^{-\sigma}\int_0^t e^{-i(t-s)H} Q\mathbf{g}_1(s)\,ds\right\|_{l_n^\infty L_t^2} \\
&\quad\le C(\|\langle n\rangle^\sigma|\boldsymbol{\phi}(\omega)|^{2p}|\mathbf{z}|\|_{l_n^1 L_t^2} + \||\mathbf{z}|^{2p+1}\|_{L_t^1 l_n^2}) \\
&\quad\le C\Bigg(\|\langle n\rangle^{-\sigma}\mathbf{y}\|_{l_n^\infty L_t^2}\|\langle n\rangle^\sigma|\boldsymbol{\phi}(\omega)|^{2p}\|_{L_t^\infty l_n^1} \\
&\qquad\qquad + \|a\|^{2p+1}_{L_t^{2p+1}}\|\boldsymbol{\psi}_0\|^{2p+1}_{l_n^{2(2p+1)}} + \|\boldsymbol{\eta}\|^{2p+1}_{L_t^{2p+1} l_n^{2(2p+1)}}\Bigg) \\
&\quad\le C\left((M_3 + M_4)(\epsilon + M_6) + M_4^2 M_5^{2p-1} + \|\boldsymbol{\eta}\|^{2p+1}_{L_t^{2p+1} l_n^{2(2p+1)}}\right),
\end{aligned}$$

where we have used the bounds

$$\|a\|^{2p+1}_{L_t^{2p+1}} \le \|a\|^{2p-1}_{L_t^\infty}\|a\|^2_{L_t^2}$$

and

$$\|\langle n\rangle^\sigma|\boldsymbol{\phi}(\omega)|^{2p}\|_{l_n^1} \le C\|\omega - \omega_0\|_{L_t^\infty}.$$

To deal with the last term in the estimate, we note that, if $p \geq 3$, then $((2p+1), 2(2p+1))$ is a Strichartz pair satisfying $\frac{6}{2p+1} + \frac{2}{2(2p+1)} \leq 1$. Bound (24) gives

$$\|\boldsymbol{\eta}\|_{L_t^{2p+1} l_n^{2(2p+1)}} \leq C \left(\|\boldsymbol{\eta}\|_{L_t^6 l_n^\infty} + \|\boldsymbol{\eta}\|_{L_t^\infty l_n^2} \right) = C(M_1 + M_2).$$

Combining all previous inequalities, we have

$$M_3 \leq C \Big(\|\boldsymbol{\eta}(0)\|_{l_n^2} + \epsilon^{1-\frac{1}{p}} (M_3 + M_4)^2 + (M_3 + M_4)(\epsilon + M_6) + M_4^2 M_5^{2p-1} + (M_1 + M_2)^{2p+1} \Big).$$

Estimates for M_1 and M_2. With the help of (22), the free solution is estimated by

$$\|e^{-itH} Q \boldsymbol{\eta}(0)\|_{L_t^6 l_n^\infty \cap L_t^\infty l_n^2} \leq C \|\boldsymbol{\eta}(0)\|_{l_n^2}.$$

With the help of (23), the nonlinear terms involving $\mathbf{g}_{2,3}$ are estimated by

$$\begin{aligned}\left\| \int_0^t e^{-i(t-s)H} Q \mathbf{g}_{2,3}(s) ds \right\|_{L_t^6 l_n^\infty \cap L_t^\infty l_n^2} &\leq C \left(\|\mathbf{g}_2\|_{L_t^1 l_n^2} + \|\mathbf{g}_3\|_{L_t^1 l_n^2} \right) \\ &\leq C \epsilon^{1-\frac{1}{2p}} (M_3 + M_4)^2.\end{aligned}$$

The nonlinear term involving $\mathbf{g}_1$ is estimated by the sum of two computations thanks to the bound (15). The first computation is completed with the help of (29),

$$\begin{aligned}&\left\| \int_0^t e^{-i(t-s)H} Q |\boldsymbol{\phi}(\omega)|^{2p} |\mathbf{y}| ds \right\|_{L_t^6 l_n^\infty \cap L_t^\infty l_n^2} \\ &\quad \leq C \| \langle n \rangle^3 |\boldsymbol{\phi}(\omega)|^{2p} |\mathbf{y}| \|_{L_t^2 l_n^2} \\ &\quad \leq \| \langle n \rangle^{3+\sigma} |\boldsymbol{\phi}(\omega)|^{2p} \|_{L_t^\infty l_n^2} \| \langle n \rangle^{-\sigma} \mathbf{y} \|_{l_n^\infty L_t^2} \\ &\quad \leq C(M_3 + M_4)(\epsilon + M_6),\end{aligned}$$

whereas the second computation is completed with the help of (23),

$$\begin{aligned}\left\| \int_0^t e^{-i(t-s)H} Q |\mathbf{y}|^{2p+1} ds \right\|_{L_t^6 l_n^\infty \cap L_t^\infty l_n^2} &\leq C \| |\mathbf{y}|^{2p+1} \|_{L_t^1 l_n^2} \leq C \|\mathbf{y}\|^{2p+1}_{L_t^{2p+1} l_n^{2(2p+1)}} \\ &\leq C \left(M_4^2 M_5^{2p-1} + (M_1 + M_2)^{2p+1} \right),\end{aligned}$$

if $p \geq 3$. We conclude that the estimates for M_1 and M_2 are the same as the one for M_3.

5. Numerical results. We now add some numerical computations which illustrate the asymptotic stability result of Theorem 1. In particular, we shall obtain numerically the rate at which the localized perturbations approach asymptotically to the small bound state. One advantage of numerical computations is that they are not limited to the case of $p \geq 3$ (which is the realm of our theoretical analysis above) but can be extended to arbitrary $p \geq 1$. In what follows, we illustrate the results for $p = 1$ (the cubic DNLS), $p = 2$ (the quintic DNLS), and $p = 3$ (the septic DNLS).

We use the same example of the single-node external potential with $V_n = -\delta_{n,0}$, $n \in \mathbb{Z}$, as in section 2. Solutions of the stationary DNLS equation (2) exist for ω in a

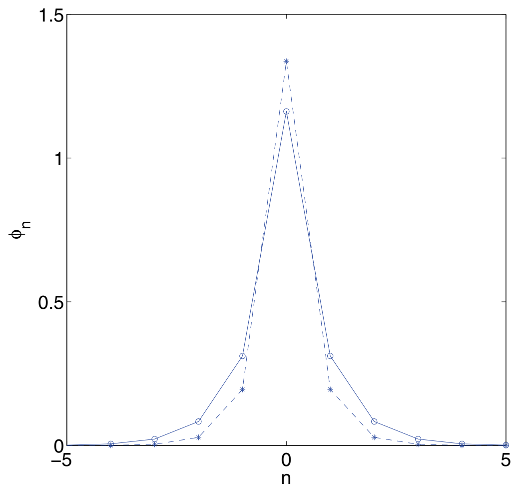

FIG. 1. *Two profiles of the bound state of the stationary DNLS equation* (2) *for* $V_n = -\delta_{n,0}$*,* $p = 1$*, and for* $\omega = -2$ *(solid line with circles) and* $\omega = -5$ *(dashed line with stars).*

local neighborhood of the negative eigenvalue $\omega_0 = 2 - \sqrt{5}$ of $H = -\Delta + \mathbf{V}$. We shall consider numerically the case $\gamma = -1$, for which the stationary solution bifurcates to the domain $\omega < \omega_0$ (Remark 4). Figure 1 illustrates the stationary solutions for $p = 1$ and two different values of ω, showcasing its increased localization (decreasing width and increasing amplitude), as ω deviates from ω_0 toward the negative domain.

In order to examine the dynamics of the DNLS equation (1), we consider single-node initial data $u_n = A\delta_{n,0}$ for any $n \in \mathbb{Z}$, with $A = 0.75$, and observe the temporal dynamics of the solution $\mathbf{u}(t)$. The resulting dynamics involves the asymptotic relaxation of the localized perturbation into a small bound state after shedding some "radiation." This dynamics was found to be typical for all values of $p = 1, 2, 3$. In Figure 2, upon suitable subtraction of the phase dynamics, we illustrate the approach of the wave profile to its asymptotic form in the l^∞ norm. The asymptotic form is obtained by running the numerical simulation for sufficiently long times, so that the profile has relaxed to the stationary state. Using a fixed-point algorithm, we identify the stationary state with the same l^2 norm (as the central portion of the lattice) and confirm that the result of further temporal dynamics is essentially identical to the stationary state. Subsequently the displayed l^∞ norm of the deviation from the asymptotic profile is computed (*locally*, in the central portion of the lattice), appropriately eliminating the phase by using the gauge invariance of the DNLS equation (1).

We have found from Figure 2 in the cases $p = 3$ (top panel), $p = 2$ (middle panel), and $p = 1$ (bottom panel) that the approach to the stationary state follows a power-law which is well approximated as $\propto t^{-3/2}$. The dashed line in all three figures represents such a decay in each of the cases. We note that the decay rate observed in numerical simulations of the DNLS equation (1) is faster than the decay rate $\propto t^{-1/6-\nu}$, $\nu > 0$, in Theorem 1.

Appendix A. Proof of Lemma 7. For the proof of Lemma 7, we will have to show both the "high frequency" estimate (31) and the "low frequency" estimate (32). To simplify notation, we drop the boldface font for vectors on $\mathbb{Z}$ in the appendix.

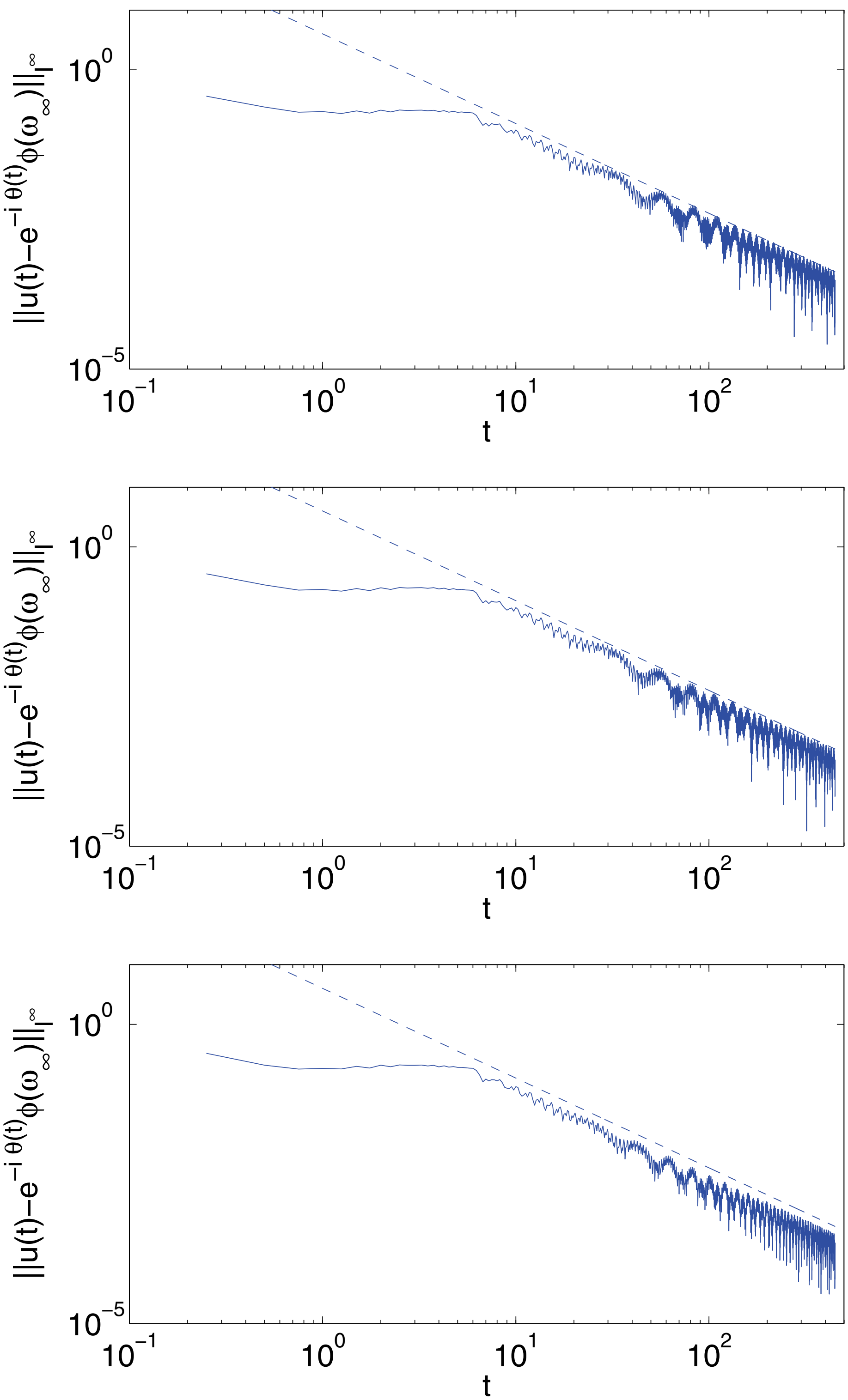

FIG. 2. *Evolution for $p = 3$ (top), 2 (middle), 1 (bottom) of $\|\mathbf{u}(t) - e^{-i\theta(t)}\boldsymbol{\phi}(\omega_\infty)\|$ (computed in the central portion of the lattice converging to the stationary state) as a function of time in a log-log scale (solid) and comparison with a $t^{-3/2}$ power law decay (dashed).*

A.1. Proof of (31). Recall the finite Born series representation of R_V,

$$R(\omega) = R_0(\omega) - R_0(\omega)VR_0(\omega) + R_0(\omega)VR(\omega)VR_0(\omega), \tag{38}$$

which is basically nothing but the resolvent identity iterated twice. We have shown in [21] that for the "sandwiched resolvent" $G_{U,W}(\omega) = UR_V(\omega)W$, we have the bounds (see estimate (33) in [21])

$$\sup_{\theta\in[-\pi,\pi]} \sum_m |G_m(\omega)| + \left|\frac{d}{d\theta}G_m(\omega)\right| \leq C\|U\|_{l^2_\sigma}\|W\|_{l^2_\sigma} \tag{39}$$

for any fixed $\sigma > \frac{5}{2}$, where $\omega = 2 - 2\cos(\theta)$.

For the three pieces arising from (38), similar arguments apply. Starting with the free resolvent term, we have

$$\begin{aligned}
\sup_{n\in\mathbb{Z}} \int_0^4 \chi|(R_0^\pm(\omega)f)_n|^2 d\omega &\leq C\sup_{n\in\mathbb{Z}}\int_{-\pi}^{\pi} \frac{\chi}{\sin(\theta)}\left|\sum_{m\in\mathbb{Z}} e^{i\theta|m-n|}f_m\right|^2 d\theta \\
&\leq C\sup_{n\in\mathbb{Z}}\int_{|\theta|\in[\theta_0/2,\pi-\theta_0/2]}\left(\left|\sum_{m\geq n} e^{i\theta m}f_m\right|^2 + \left|\sum_{m<n} e^{-i\theta m}f_m\right|^2\right)d\theta.
\end{aligned}$$

Introducing the sequence

$$(g^n)_m := \begin{cases} f_m, & m \geq n, \\ 0, & m < n, \end{cases}$$

we see that the last expression is simply $C(\|\widehat{g^n}\|^2_{L^2[\theta_0/2,\pi-\theta_0/2]} + \|\widehat{f-g^n}\|^2_{L^2[\theta_0/2,\pi-\theta_0/2]})$, which is equal by Plancherel's identity to

$$C\|g^n\|^2_{l^2} + \|f-g^n\|^2_{l^2} \leq 2C\|f\|^2_{l^2}.$$

For the second piece in (38), we use that $\|R_0^\pm(\omega)\|_{l^1\to l^\infty} \leq C/\sin(\theta)$ and $|\sin(\theta)| \geq C_0$ on $[\theta_0/2, \pi-\theta_0/2]$ for some $C_0 > 0$ to conclude

$$\begin{aligned}
\sup_{n\in\mathbb{Z}}\|\chi R_0^\pm(\omega)VR_0^\pm(\omega)f\|^2_{L^2_\lambda(0,4)} &\leq \int_{-\pi}^{\pi}\frac{\chi}{\sin^3(\theta)}\left(\sum_{n\in\mathbb{Z}}|V_n||R_0^\pm(\omega)f_n|\right)^2 d\theta \\
&\leq C\|V\|_{l^1}\sup_{n\in\mathbb{Z}}\int_{-\pi}^{\pi}\chi\left|(R_0^\pm(\omega)f)_n\right|^2 d\theta,
\end{aligned}$$

by the triangle inequality. At this point, we have reduced the estimate to the previous case, provided that $V \in l^1$.

For the third piece in (38), we make use of (39). We have, similar to the previous estimate,

$$\begin{aligned}
&\sup_{n\in\mathbb{Z}}\|\chi R_0^\pm(\omega)VR_V^\pm(\omega)VR_0^\pm(\omega)f\|^2_{L^2_\omega(0,4)} \\
&= \sup_{n\in\mathbb{Z}}\|\chi R_0^\pm(\omega)VR_V^\pm(\omega)|V|^{1/2}\mathrm{sgn}(V)V^{1/2}R_0^\pm(\omega)f\|^2_{L^2_\omega(0,4)} \\
&= \sup_{n\in\mathbb{Z}}\|\chi R_0^\pm(\omega)G_{V,|V|^{1/2}\mathrm{sgn}(V)}[|V|^{1/2}R_0^\pm(\omega)f]\|^2_{L^2_\omega(0,4)} \\
&\leq C\|V\|^2_{l^2_\sigma}\||V|^{1/2}\|^2_{l^2_\sigma}\||V|^{1/2}\|^2_{l^1}\sup_{n\in\mathbb{Z}}\int_{-\pi}^{\pi}\chi|(R_0^\pm(\omega)f)_n|^2 d\theta,
\end{aligned}$$

where, in the last inequality, we have again reduced the estimate to the first case.

A.2. Proof of (32). We only consider the interval $[-\theta_0, \theta_0]$ in the compact support of $\chi_0(\theta)$ since the arguments for other intervals are similar. Following the algorithm in [17] and the formalism in [21], we let $\psi^{\pm}(\theta)$ be two linearly independent solutions of

$$\psi_{n+1} + \psi_{n-1} + (\omega - 2)\psi_n = V_n \psi_n, \quad n \in \mathbb{Z}, \tag{40}$$

according to the boundary conditions $|\psi_n^{\pm} - e^{\mp in\theta}| \to 0$ as $n \to \pm\infty$. Let $\psi_n^{\pm}(\theta) = e^{\mp in\theta}\Psi_n^{\pm}(\theta)$ for all $n \in \mathbb{Z}$. Using the Green function representation, we obtain

$$\Psi_n^+(\theta) = 1 - \frac{i}{2\sin\theta} \sum_{m=n}^{\infty} \left(1 - e^{-2i\theta(m-n)}\right) V_m \Psi_m^+(\theta),$$
$$\Psi_n^-(\theta) = 1 - \frac{i}{2\sin\theta} \sum_{m=-\infty}^{n} \left(1 - e^{-2i\theta(n-m)}\right) V_m \Psi_m^-(\theta).$$

The discrete Green function for the resolvent operators $R^{\pm}(\omega)$ has the kernel

$$\left[R_V^{\pm}(\omega)\right]_{n,m} = \frac{1}{W(\theta_{\pm})} \begin{cases} \psi_n^+(\theta_{\pm})\psi_m^-(\theta_{\pm}) & \text{for } n \geq m, \\ \psi_m^+(\theta_{\pm})\psi_n^-(\theta_{\pm}) & \text{for } n < m, \end{cases}$$

where $\theta_- = -\theta_+$, $\theta_- \in [0, \pi]$ for $\omega \in [0, 4]$, and $W(\theta) = W[\psi^+, \psi^-] = \psi_n^+\psi_{n+1}^- - \psi_{n+1}^+\psi_n^-$ is the discrete Wronskian, which is independent of $n \in \mathbb{Z}$. We need to estimate

$$\|\chi_0 R_V^{\pm}(\omega) f\|^2_{L^2_\omega(0,4)}$$
$$= \int_{-\pi}^{\pi} \frac{2\chi_0^2 \sin\theta d\theta}{W^2(\theta)} \left(\sum_{m=-\infty}^{n-1} \psi_n^+(\theta)\psi_m^-(\theta) f_m + \sum_{m=n}^{\infty} \psi_n^-(\theta)\psi_m^+(\theta) f_m \right)^2.$$

We may assume that $n \geq 1$ for definiteness and split

$$\sum_{m=-\infty}^{n-1} \psi_m^-(\theta) f_m = \sum_{m=0}^{n-1} \psi_m^-(\theta) f_m + \sum_{m=-\infty}^{-1} e^{im\theta} f_m + \sum_{m=-\infty}^{-1} e^{im\theta}(\Psi_m^- - 1) f_m := I_1 + I_2 + I_3$$

and

$$\sum_{m=n}^{\infty} \psi_m^+(\theta) f_m = \sum_{m=n}^{\infty} e^{-im\theta} f_m + \sum_{m=n}^{\infty} e^{-im\theta}\left(\Psi_m^+(\theta) - 1\right) f_m := I_4 + I_5.$$

We are using the scattering theory from [21] to claim that

$$\sup_{\theta \in [-\theta_0, \theta_0]} \left(\|\Psi^{\pm}(\theta)\|_{l^\infty(\mathbb{Z}_\pm)} + \|\langle n \rangle^{-1} \Psi^{\pm}(\theta)\|_{l^\infty(\mathbb{Z}_\mp)} \right) < \infty, \tag{41}$$

where $\langle n \rangle = (1+n^2)^{1/2}$. Then, we have

$$|I_1| \leq \left(\sum_{m=0}^{n-1} |\Psi_m^-(\theta)|^2 \right)^{1/2} \left(\sum_{m=0}^{n-1} |f_m|^2 \right)^{1/2} \leq C_1 \langle n \rangle^{3/2} \|f\|_{l^2},$$

$$|I_3| \leq \left(\sum_{m=-\infty}^{-1} |\Psi_m^-(\theta) - 1|^2 \right)^{1/2} \left(\sum_{m=-\infty}^{-1} |f_m|^2 \right)^{1/2}$$

$$\leq C_3 \left\| \sum_{k=-\infty}^{m} |m-k||V_k| \right\|_{l^2_m(\mathbb{Z}_-)} \|f\|_{l^2},$$

$$|I_5| \leq \left(\sum_{m=n}^{\infty} |\Psi_m^+(\theta) - 1|^2 \right)^{1/2} \left(\sum_{m=n}^{\infty} |f_m|^2 \right)^{1/2} \leq C_5 \left\| \sum_{l=m}^{\infty} |k-m||V_k| \right\|_{l^2_m(\mathbb{Z}_+)} \|f\|_{l^2}$$

for some $C_1, C_3, C_5 > 0$. We note that

$$\left\| \sum_{k=-\infty}^{m} |m-k||V_k| \right\|_{l^2_m(\mathbb{Z}_-)} \leq \left\| \sum_{k=-\infty}^{m} |m-k||V_k| \right\|_{l^1_m(\mathbb{Z}_-)} \leq C_4 \|V\|_{l^1_2}$$

for some $C_4 > 0$. Therefore, the brackets in I_3 and I_5 are bounded if $V \in l^1_{2\sigma}$ for $\sigma > \frac{5}{2}$. Since I_2 and I_4 are given by the discrete Fourier transform, Parseval's equality implies that

$$\int_{-\pi}^{\pi} \left(I_2^2 + I_4^2 \right) d\theta \leq C_2 \|f\|_{l^2}^2$$

for some $C_2 > 0$. Using now the fact that $|W(\theta)| \geq W_0$ and $|\sin\theta| \leq C_0$ uniformly in $[-\theta_0, \theta_0]$, the support of $\chi_0(\theta)$, and using the property (41), we obtain

$$\|\chi_0 R_V^{\pm}(\omega) f\|_{L^2_\omega(0,4)}^2 \leq C \left(1 + \langle n \rangle^2 + \langle n \rangle^3\right) \|f\|_{l^2}^2,$$

which gives (32).

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