Bargmann transform and its applications to partial differential equations

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Abstract

This thesis is devoted to the fundamental properties and applications of the Bargmann transform and the Fock–Segal–Bargmann space. The fundamental properties include unitarity and invertibility of the transformation in L^2 spaces and embeddings of the Fock–Segal–Bargmann spaces in L^p for any $p > 0$. Applications include the linear partial differential equations such as the time-dependent Schrödinger equation in harmonic potential, the diffusion equation in self-similar variables, and the linearized Korteweg–de Vries equation and one nonlinear partial differential equation given by the Gross–Pitaevskii model for the rotating Bose–Einstein condensate. The main question considered in this work in the context of linear partial differential equation is whether the envelope of the Gaussian function remains bounded in the time evolution. We show that the answer to this question is positive for the diffusion equation, negative for the Schrödinger equation, and unknown for the Korteweg–de Vries equation. We also address the local and global well-posedness of the nonlocal evolution equation derived for the Bose–Einstein condensates at the lowest Landau level.

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Contents

Chapter 1

Introduction

1.1 Historical notes

In 1923, following discussions of Max Planck's and Albert Einstein's research on wave-particle duality in the context of photons, Louis de Broglie formulated an idea that the electrons could be described as waves. Attracted by this idea, Erwin Schrödinger wrote down in 1925 a relativistically invariant equation for the wave function of the electron, which is now known as *the Klein–Gordon equation*. The electron energy levels (which were known from experiments) were recovered from the new equation; however, the relativistic corrections to the energy levels of the hydrogen atom were inconsistent with the experimental measurements. As a result, Schrödinger shelved his relativistic equation and retreated to the nonrelativistic limit, where he derived what is now known as *the Schrödinger equation*.

Few years later, Vladimir Fock worked on the representation of eigenstates of many particles in the Schrödinger equation as products of eigenstates of a single particle in a Hilbert space [16]. This representation bears now the name of *the Fock space*, which is the direct sum of the symmetric or antisymmetric tensors in the tensor powers of a single-particle Hilbert space.

In 1961, Valentine Bargmann introduced the space of holomorphic functions square-integrable with respect to a Gaussian measure and the transformation between this space and a finite-dimensional Hilbert space [3]. Independently, Irving Segal [39] developed a similar concept in the setting of an infinite-dimensional Hilbert space. This space of holomorphic functions squared integrable with the Gaussian weight is now known as *the Segal–Bargmann space* and the transformation between the spaces is now known as *the Bargmann transform*.

It was shown by Segal [40] and Bargmann [4] in the setting of multi-particle wave functions that the Segal–Bargmann space is isomorphic to a bosonic Fock space. The

Schrödinger representation and the Fock representation are unitarily equivalent under the Bargmann transform. The review [41] gives a technically better description of the Segal-Bargmann space in infinitely many degrees of freedom and describes explicitly the corresponding transform.

Since then, the Bargmann transform and the Fock–Segal–Bargmann space have been widely used in many fields of mathematics [17, 27, 43]. An example of how the Bargmann transform is influential is given by the high number of citations of the pioneering paper [3], e.g. it was cited 22 times in 2020 according to MathSciNet.

Applications of the Bargmann transform in the quantized Yang–Mills theory on a space-time cylinder were developed in [12]. The Segal–Bargmann space was generalized to the group manifold of a compact Lie group [25, 26], which can be applied to the rotational degrees of freedom of a rigid body, where the configuration space is the compact Lie group. Recent paper [30] uses applications of the Bargmann transform to the imaginary time flow of a quadratic hyperbolic Hamiltonian on the symplectic plane.

Some special operators have been studied in the Fock space such as Toeplitz operators with generating symbols invariant under some group actions [24, 37]. Various properties of these operators such as boundedness, compactness, and eigenvalues have been studied by many authors, see [35, 44]. The C^*-algebra generated by such operators was explicitly described in [14] for the Fock space. Radial operators of polyanalytic or truepolyanalytic functions were recently analyzed in Fock spaces [32].

In the context of the nonlinear PDEs, the Bargmann transform was applied to study the Gross–Pitaevskii equation [1, 21] and Lagrangian systems in the semi-classical limit [31, 36]. Numerical aspects of the Bargmann transform for the time-dependent Schrödinger problems were discussed in the recent paper [38].

The purpose of this thesis is to review the basic properties of the Segal–Bargmann space and the Bargmann transform in applications to the partial differential equations such as the Schrödinger, diffusion, and Korteweg–de Vries equation. Although these equations are formulated in one spatial dimension, we will also review applications in the two-dimensional space in the context of the nonlinear Gross–Pitaevskii equation.

The following two sections of the introduction overview results obtained in the three different chapters of the thesis.

1.2 Properties of the Bargmann transform

In the simplest setting, the Bargmann transform is a transformation of a given real function of a single real variable to its complex-valued image in a single complex variable. To be precise, if $\varphi(y) : \mathbb{R} \mapsto \mathbb{R}$ is given, then the Bargmann transform of φ is

$$(\mathcal{B}\varphi)(z) := \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{4}} e^{\frac{\alpha}{2}z^2} \int_{-\infty}^{\infty} e^{-\alpha(z-y)^2}\varphi(y)dy, \quad z \in \mathbb{C}, \tag{1.1}$$

provided that the integral is finite. Here parameter $\alpha > 0$ is fixed arbitrarily.

Chapter 2 reviews properties of the Bargmann transform (1.1). We will show in Lemma 2.1 that $\mathcal{B}$ is a unitary transformation from $L^2(\mathbb{R})$ to $\mathcal{F} \subset L^2_\rho(\mathbb{C})$, where $\mathcal{F}$ is the Fock–Segal–Bargmann space given by

$$\mathcal{F} = \left\{ f \in L^2_\rho(\mathbb{C}) : \quad f(z) \text{ is entire in } z \in \mathbb{C} \right\} \tag{1.2}$$

and ρ is the weight in the weighted L^2 space given by

$$\rho(z) := \frac{\alpha}{\pi} e^{-\alpha|z|^2}. \tag{1.3}$$

The adjoint Bargmann transform is given by

$$(\mathcal{B}^* f)(y) = \frac{2^{\frac{1}{4}}\alpha^{\frac{5}{4}}}{\pi^{\frac{5}{4}}} \iint_{\mathbb{R}^2} e^{\frac{\alpha}{2}\bar{z}^2 - \alpha(y-\bar{z})^2 - \alpha|z|^2} f(z)dxd\xi, \tag{1.4}$$

We will show in Lemma 2.2 and Corollary 2.1 that $\mathcal{B}^* : L^2_\rho(\mathbb{C}) \mapsto L^2(\mathbb{R})$ and moreover, $\mathcal{B}^*$ is the left inverse of $\mathcal{B}$. On the other hand, $\mathcal{B}^* : \mathcal{F} \subset L^2_\rho(\mathbb{C}) \mapsto L^2(\mathbb{R})$ is also the right inverse of $\mathcal{B}$, which allows us to introduce the orthogonal projection operator

$$\Pi := \mathcal{B}\mathcal{B}^* : L^2_\rho(\mathbb{C}) \mapsto \mathcal{F} \subset L^2_\rho(\mathbb{C}). \tag{1.5}$$

Lemma 2.3 shows that Π is an identity in $\mathcal{F}$ and also gives a useful computational formula for the projection operator:

$$(\Pi f)(z) = \frac{\alpha}{\pi} \iint_{\mathbb{R}^2} f(z') e^{\alpha(z-z')\bar{z}'} dx' d\xi'. \tag{1.6}$$

Corollary 2.2 ensures that $\mathcal{B}^* : \mathcal{F} \subset L^2_\rho(\mathbb{C}) \mapsto L^2(\mathbb{R})$ is a unitary transformation.

Extending the Bargmann transform into $L^p_\rho(\mathbb{C})$ spaces for any $p > 0$ and the Fock–Segal–Bargmann space $\mathcal{F}_p$ defined similarly to (1.2), we show in Lemma 2.4 that if $f \in \mathcal{F}_p$, then for every $z \in \mathbb{C}$,

$$|f(z)| \leq \|f\|_{L^p_\rho} e^{\frac{1}{2}\alpha|z|^2}. \tag{1.7}$$

Due to this bound, Lemmas 2.5 and 2.6 establish the following sharp bound on the embedding of $\mathcal{F}_p$ into $\mathcal{F}_q$ for every $0 < p < q < \infty$:

$$\left(\frac{\alpha q}{2\pi}\right)^{\frac{1}{q}} \|u\|_q \leq \left(\frac{\alpha p}{2\pi}\right)^{\frac{1}{p}} \|u\|_p \tag{1.8}$$

where $u(z) = f(z)e^{-\frac{1}{2}\alpha|z|^2}$, $f \in \mathcal{F}_p$, and $\|u\|_p$ is the standard L^p norm on $\mathbb{C}$. The sharp bound (1.8) is very useful in applications of the Bargmann transform in the context of nonlinear partial differential equation.

1.3 Applications of the Bargmann transform

The main goal of this thesis is to study applications of the Bargmann transform to partial differential equations (PDEs). Chapter 3 explains applications to PDEs in one spatial dimension. Chapter 4 reviews applications to PDEs in two spatial dimensions.

1.3.1 PDEs in one spatial dimension

In the context of PDEs in one spatial dimension y and one temporal variable t, the Bargmann transform (1.1) maps a solution $\Phi(t,y) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ to an equivalent solution $F(t,z) : \mathbb{R} \times \mathbb{C} \mapsto \mathbb{C}$ which depends on artificial complex variable z. Since the time evolution is greatly simplified after the transform, the PDE for $F(t,z)$ can be analyzed and solved in a closed form. If $\Phi(t,\cdot)$ belongs to a subset of $L^2(\mathbb{R})$ space, then $F(t,\cdot)$ belongs to a subset of Fock–Segal–Bargmann space (1.2).

A basis in the Fock–Segal–Bargmann space $\mathcal{F}$ is reviewed in Section 3.2. The basis is given by monomials which represent Taylor series of holomorphic functions of a complex variable. By using the properties of the basis, it is then shown in Sections 3.3, 3.4, and 3.5 how the time evolution of the following three PDEs is simplified in the Fock–Segal–Bargmann space $\mathcal{F}$. The PDEs are given by the time-dependent Schrödinger equation with harmonic potential:

$$i\frac{\partial \Phi}{\partial t} = -\frac{\partial^2 \Phi}{\partial y^2} + (y^2 - 1)\Phi, \tag{1.9}$$

the diffusion equation with harmonic potential:

$$\frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial y^2} + (1 - y^2)\Phi, \tag{1.10}$$

and the linearized log–KdV (Korteweg–de Vries) equation:

$$\frac{\partial \Phi}{\partial t} = -\frac{\partial^3 \Phi}{\partial y^3} + (y^2 - 1)\frac{\partial \Phi}{\partial y} + 2y\Phi. \tag{1.11}$$

We show that the Bargmann transform maps the three evolution PDEs to their equivalent forms:

$$i\frac{\partial F}{\partial t} = 2z\frac{\partial F}{\partial z}, \tag{1.12}$$

$$\frac{\partial F}{\partial t} + 2z\frac{\partial F}{\partial z} = 0, \tag{1.13}$$

and

$$\frac{\partial F}{\partial t} = 2z\frac{\partial^2 F}{\partial z^2} + (2 - z^2)\frac{\partial F}{\partial z}. \tag{1.14}$$

The time-dependent Schrödinger equation (1.9) after Bargmann transform (1.12) can be solved in the general form $F(t,z) = f(ze^{-2it})$, where $f \in \mathcal{F}$ represents the initial data $F(0,z) = f(z)$. The diffusion equation (1.10) after Bargmann transform (1.13) can be solved in the general form $F(t,z) = f(ze^{-2t})$, where $f \in \mathcal{F}$ also represents the initial data $F(0,z) = f(z)$. Finally, solvability of the Cauchy problem for the linearized log–KdV equation (1.11) after Bargmann transform (1.14) is shown in Lemma 3.15 by using methods from [34].

The main motivation to study the three physically important PDEs (1.9), (1.10), and (1.11) was to answer the following general question formulated in [7]:

> Let $\Phi(t,y) = H(t,y)e^{-\frac{1}{2}y^2}$ be a solution to the time evolution problem with the initial condition $\varphi(y) = h(y)e^{-\frac{1}{2}y^2}$. If $h \in L^\infty(\mathbb{R})$, does $H(t,\cdot)$ remain in $L^\infty(\mathbb{R})$ for $t > 0$?

We show in Lemma 3.9 that the answer to this main question is *negative* for the Schrödinger equation (1.9). We construct a counter-example with a specific $h \in L^\infty(\mathbb{R})$ for which $H(t,\cdot) \notin L^\infty(\mathbb{R})$ for all $t > 0$. On the other hand, we show in Lemma 3.11 that the answer is *positive* for the diffusion equation (1.10). In the context of the linearized log–KdV equation (1.11), the answer to this main question is still *unknown*.

Among other interesting applications considered in Chapter 3, we mention the following. In Lemma 3.8, we verify the following representation of solution $\Phi(t,y)$ to the time-independent Schrödinger equation (1.9):

$$\Phi(t,y) = \int_{\mathbb{R}} K_t(y,y')\varphi(y')dy', \quad K_t(y,y') := \frac{1}{\sqrt{\pi(1-e^{-4it})}} e^{-\frac{1}{2}y^2 + \frac{1}{2}(y')^2 - \frac{(y'-ye^{-2it})^2}{1-e^{-4it}}}, \tag{1.15}$$

where $\Phi(0,y) = \varphi(y)$.

A similar representation is also derived for the diffusion equation (1.10). We discuss applications of this representation to analysis of the nonlinear diffusion equation with the power nonlinearity written in self-similar variables, which were applied for study of blowup in [22, 23] (see also recent work [33]).

In Lemma 3.12, we show applications of the explicit solution of the diffusion equation (1.10) in analysis of another diffusion equation with the logarithmic nonlinearity:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u\log(u). \tag{1.16}$$

Blow-up versus global decay in the solutions of the nonlinear diffusion equation (1.16) was investigated recently by Alfaro & Carles [2]. We show how some results can be recovered by using the diffusion equation (1.10) arising after the applications of self-similar variables.

Finally, the linearized log–KdV equation (1.11) is related to the linearization of the log–KdV equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} u \log(u) + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1.17}$$

at the Gaussian solution $u_0(x) = e^{\frac{1}{2} - \frac{1}{2}x^2}$. The log–KdV equation (1.17) was derived in [29] and justified in [13] in the context of the granular chains with nearly harmonic interactions. Analysis of the time evolution of the log–KdV equation (1.17) was reported in [7] and [34].

Although the time evolution of the linearized log–KdV equation (1.17) is well-posed, as we show in Lemma 3.15, this useful feature is special for the class of linearized log–KdV equations. In the final Lemma 3.16, we construct another example of the linear equation of the same class:

$$\frac{\partial \Phi}{\partial t} = -\frac{\partial^3 \Phi}{\partial y^3} + y \frac{\partial^2 \Phi}{\partial y^2} + (y^2 - 1) \frac{\partial \Phi}{\partial y} + y(3 - y^2)\Phi, \tag{1.18}$$

for which the Bargmann transform gives the linear transport equation

$$\frac{\partial F}{\partial t} = -2z^2 \frac{\partial F}{\partial z}. \tag{1.19}$$

It is clear from the exact solution $F(t,z) = f(\frac{z}{1+2tz})$, where $F(0,z) = f(z)$ that the time evolution of the linear equation (1.18) is not well-posed in a subset of $L^2(\mathbb{R})$.

1.3.2 PDEs in two spatial dimensions

In the context of PDEs in two spatial dimensions (x, y) and one temporal variable t, one can introduce the complex variable $z = x + iy$ from (x, y) and interpret the solution $u(t, x, y)$ of the PDE in complex variables as $u(t, z)$. In this case, the adjoint Bargmann transform (1.4) would map a complex-valued functions $f \in \mathcal{F}$ to an artificial real variable. This artificial variable does not have many applications but the projection operator written in terms of $\mathcal{B}$ and $\mathcal{B}^*$ in (1.5) and (1.6) plays the central role in analysis of the nonlinear PDEs in two spatial dimensions.

The content of Chapter 4 refer to just one example of such PDEs, given by the Gross–Pitaevskii equation for rotating Bose–Einstein condensates:

$$i\frac{\partial u}{\partial t} = -\Delta u + 2i(x\partial_y - y\partial_x)u + (x^2 + y^2)u + |u|^2 u - 2u, \tag{1.20}$$

The rotation takes place at the critical rotational frequency, for which the model is simplified at the lowest Landau level [5, 7]. We review the recent study of [20, 21], where the Gross–Pitaevskii equation was reduced to the nonlocal evolution equation

$$i\frac{\partial u}{\partial t} = \widehat{\Pi}(|u|^2 u), \tag{1.21}$$

where $\widehat{\Pi}$ is an analogue of Π in (1.5) in the sense that it acts on general functions in $L^p(\mathbb{C})$ spaces and return functions of the class $u(z) = f(z)e^{-\frac{1}{2}|z|^2}$, where $f \in \mathcal{F}_p \subset L^p_\rho(\mathbb{C})$. Justification of the reduction is well elaborated in the recent works [15, 19].

In Lemmas 4.1 and 4.2, we show that the time evolution of the nonlocal equation (1.21) is locally well-posed in $L^p(\mathbb{C})$ for any $p \geq 1$ and is globally well-posed in $L^2(\mathbb{C})$.

1.4 Further directions

Although the main results of this thesis are based on review of literature and reconstruction of the classical proofs and results, we believe that the methods of the Bargmann transform and Fock–Segal–Bargmann spaces will be useful for future analysis of many partial differential equations. The linear and nonlinear Schrödinger equations (including the Gross–Pitaevskii model) and the linear and nonlinear diffusion equations are natural examples where these methods have already been widely useful.

The new applications include the log–KdV equation and related to models, which were much less studied in the literature. It remains to be seen if the methods will help to solve the open problems for uniqueness of solutions and orbital stability of Gaussian solitons discussed in [7] and [34]. Although the linear evolution is greatly simplified after the Bargmann transform and the basis of Hermite functions is replaced by the basis of monomials for Taylor series, we were still unable to answer positively or negatively on the main question whether the envelope of the Gaussian function remains bounded in the time evolution if it is bounded initially. Further work in this direction is needed.

We also hope that the Bargmann transform can find its way in the analysis of new Gross–Pitaevskii models with the logarithmic nonlinearities, such as the model discussed in [8]. Similarly to applications to the lowest Landau level model in [21] and in the semi-classical limit [1], we can anticipate new inspiring applications of the Bargmann transform to problems of Bose–Einstein condensation.

Finally, Bargmann constraints, Bargmann maps, and Bargmann systems are known in the theory of integrable systems [6], e.g. when the Schrödinger eigenvalue problem arises in the Lax system for the integrable Korteweg–de Vries equation. The class of reflectionless potentials can be expressed as the sum of squared eigenfunctions of the Schrödinger spectral problem, which inspired many new studies in integrable systems, e.g. [11]. This is another area of applied mathematics, where the properties of the Bargmann transform reviewed in this thesis can be useful in future applications.

Chapter 2

Bargmann transform

2.1 Introduction

Let $z = x - i\xi \in \mathbb{C}$ be an extension of $x \in \mathbb{R}$ to $z \in \mathbb{C}$. Let $\alpha > 0$ be fixed arbitrarily and define $L^2_\rho(\mathbb{C})$ by its weight

$$\rho(z) := \frac{\alpha}{\pi} e^{-\alpha|z|^2} \tag{2.1}$$

and the standard inner product

$$\langle f, g \rangle_{L^2_\rho(\mathbb{C})} := \frac{\alpha}{\pi} \iint_{\mathbb{R}^2} f(z)\overline{g(z)} e^{-\alpha|z|^2} dx d\xi. \tag{2.2}$$

The induced squared norm in $L^2_\rho(\mathbb{C})$ is defined by $\|f\|^2_{L^2_\rho(\mathbb{C})} := \langle f, f \rangle_{L^2_\rho(\mathbb{C})}$.

For a given function $\varphi(y) : \mathbb{R} \mapsto \mathbb{R}$, the Bargmann transform of φ is given by

$$(\mathcal{B}\varphi)(z) := \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{4}} e^{\frac{\alpha}{2}z^2} \int_{-\infty}^{\infty} e^{-\alpha(z-y)^2} \varphi(y) dy, \quad z \in \mathbb{C}, \tag{2.3}$$

provided that the integral is finite.

Transformation (2.3) was introduced by Bargmann in [3]. Different normalizations of the complex variable z and the fixed parameter α were used in the literature. Analysis of the Bose–Einstein condensation in the semi-classical limit in [1] uses $\alpha = 1/\sqrt{h}$ with the small parameter $h > 0$. Introduction of the Bargmann transform in harmonic analysis in [17, Section 1.6] uses $\alpha = 1/\sqrt{\pi}$. Recent work on the lowest Landau level in [21] uses $\alpha = 1$. Monograph on Fock spaces [43] covers many properties of the Bargmann transform in its general normalization with $\alpha > 0$.

The L^2-based Fock space denoted by $\mathcal{F}$ is the space of all entire functions in $L^2_\rho(\mathbb{C})$:

$$\mathcal{F} = \left\{ f \in L^2_\rho(\mathbb{C}) : \quad f(z) \text{ is entire in } z \in \mathbb{C} \right\} \tag{2.4}$$

To make notations easier, we use notation $f(z)$ for all complex-valued functions in $L^2_\rho(\mathbb{C})$, independently whether they are holomorphic or not.

Fock spaces can be extended to the L^p-Lebesgue spaces. We use the following norm in $L^p_\rho(\mathbb{C})$ space:

$$\|f\|_{L^p_\rho} := \left(\frac{p\alpha}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{1}{2}p\alpha|z|^2} dz \right)^{\frac{1}{p}}, \tag{2.5}$$

which coincides with the definition of $\|f\|_{L^2_\rho}$ for $p = 2$. Similarly to (2.4), we define the L^p-based Fock space denoted by $\mathcal{F}_p$ by

$$\mathcal{F}_p = \left\{ f \in L^p_\rho(\mathbb{C}) : \quad f(z) \text{ is entire in } z \in \mathbb{C} \right\}. \tag{2.6}$$

Thus, $\mathcal{F} \equiv \mathcal{F}_{p=2}$.

Section 2.2 reviews properties of the Bargmann transform in the L^2-based Fock space $\mathcal{F} \subset L^2_\rho(\mathbb{C})$. Section 2.3 studies embeddings of a Fock space in $L^p_\rho(\mathbb{C})$ into a Fock space in $L^q_\rho(\mathbb{C})$, where $0 < p \leq q \leq \infty$.

2.2 Properties of the Bargmann transform

The following lemma shows that the Bargmann transform $\mathcal{B}$ is well defined as a unitary transformation from $L^2(\mathbb{R})$ to $\mathcal{F} \subset L^2_\rho(\mathbb{C})$.

Lemma 2.1 $\mathcal{B} : L^2(\mathbb{R}) \mapsto \mathcal{F} \subset L^2_\rho(\mathbb{C})$ *is a unitary transformation.*

Proof. We shall prove that the range of $\mathcal{B}$ is in $L^2_\rho(\mathbb{C})$ and, moreover,

$$||\mathcal{B}\varphi||_{L^2_\rho(\mathbb{C})} = ||\varphi||_{L^2(\mathbb{R})} \quad \text{for every} \quad \varphi \in L^2(\mathbb{R}). \tag{2.7}$$

It follows formally that

$$\begin{aligned}
\|\mathcal{B}\varphi\|^2_{L^2_\rho(\mathbb{C})} &= \frac{\alpha}{\pi} \iint_{\mathbb{R}^2} |\mathcal{B}\varphi(z)|^2 e^{-\alpha|z|^2} dx d\xi \\
&= \frac{2^{\frac{1}{2}}\alpha^{\frac{3}{2}}}{\pi^{\frac{3}{2}}} \iiiint_{\mathbb{R}^4} e^{-2\alpha\xi^2 - \alpha(x-i\xi-y)^2 - \alpha(x+i\xi-y')^2} \varphi(y)\overline{\varphi(y')} dx d\xi dy dy' \\
&= \frac{2^{\frac{1}{2}}\alpha^{\frac{3}{2}}}{\pi^{\frac{3}{2}}} \iiiint_{\mathbb{R}^4} e^{-2\alpha\left(x-\frac{1}{2}(y+y')\right)^2 - \frac{\alpha}{2}(y-y')^2 + 2\alpha i\xi(y'-y)} \varphi(y)\overline{\varphi(y')} dx d\xi dy dy' \\
&= \frac{2^{\frac{1}{2}}\alpha^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \iint_{\mathbb{R}} e^{-2\alpha(x-y)^2} |\varphi(y)|^2 dx dy \\
&= \|\varphi\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

where we have used the following property for Dirac δ distribution:

$$\int_{\mathbb{R}} e^{2\alpha i\xi(y-y')}d\xi = \frac{\pi}{\alpha}\,\delta(y-y'). \tag{2.8}$$

In order to verify the previous formal computation, the interchange of integrations under the four integrals needs to be justified. This can be done by approximating the δ distribution with the Gaussian kernel, applying Fubini's theorem for absolutely integrable functions, and taking the limit. The result ensures that $\mathcal{B}\varphi \in L^2_\rho(\mathbb{C})$ with the isometry property (2.7). Furthermore, since $(\mathcal{B}\varphi)(z)$ does not depend on $\bar{z}$ for every $z \in \mathbb{C}$, $(\mathcal{B}\varphi)(z)$ is entire in $\mathbb{C}$. Thus, $\mathcal{B}\varphi \in \mathcal{F}$. ∎

The following lemma introduces the adjoint Bargmann transform $\mathcal{B}^*$.

Lemma 2.2 *The adjoint transform of* $\mathcal{B} : L^2(\mathbb{R}) \mapsto \mathcal{F} \subset L^2_\rho(\mathbb{C})$ *is given by*

$$(\mathcal{B}^*f)(y) = \frac{2^{\frac{1}{4}}\alpha^{\frac{5}{4}}}{\pi^{\frac{5}{4}}}\iint_{\mathbb{R}^2} e^{\frac{\alpha}{2}\bar{z}^2-\alpha(y-\bar{z})^2-\alpha|z|^2} f(z)dxd\xi, \tag{2.9}$$

where $\mathcal{B}^* : L^2_\rho(\mathbb{C}) \mapsto L^2(\mathbb{R})$.

Proof. First, we prove that $\mathcal{B}^*f \in L^2(\mathbb{R})$ for every $f \in L^2_\rho(\mathbb{C})$. We compute formally:

$$\begin{aligned}
\|\mathcal{B}^*f\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} |(\mathcal{B}^*f)(y)|^2 dy \\
&= \frac{2^{\frac{1}{2}}\alpha^{\frac{5}{2}}}{\pi^{\frac{5}{2}}}\int_{\mathbb{R}}\iiiint_{\mathbb{R}^4} g(z)\overline{g(z')}e^{-\alpha(y-x)^2-\alpha(y-x')^2+2\alpha i(\xi-\xi')y+\alpha i(x'\xi'-x\xi)}dxd\xi dx'd\xi'dy \\
&= \frac{2^{\frac{1}{2}}\alpha^{\frac{5}{2}}}{\pi^{\frac{5}{2}}}\int_{\mathbb{R}}\iiiint_{\mathbb{R}^4} g(z)\overline{g(z')}e^{-2\alpha(y-\frac{x+x'}{2}-i\frac{\xi-\xi'}{2})^2-\frac{\alpha}{2}(x-x')^2-\frac{\alpha}{2}(\xi-\xi')^2+\alpha i(x'\xi-x\xi')}dxd\xi dx'd\xi'dy \\
&= \frac{\alpha^2}{\pi^2}\iiiint_{\mathbb{R}^4} g(z)\overline{g(z')}e^{-\frac{\alpha}{2}(x-x')^2-\frac{\alpha}{2}(\xi-\xi')^2+\alpha i(x'\xi-x\xi')}dxd\xi dx'd\xi',
\end{aligned}$$

where $g(z) = f(z)e^{-\frac{\alpha}{2}|z|^2}$ is defined in $L^2(\mathbb{C})$. Let us define

$$G(z) := \iint_{\mathbb{R}^2} \overline{g(z')}e^{-\frac{\alpha}{2}(x-x')^2-\frac{\alpha}{2}(\xi-\xi')^2+\alpha i(x'\xi-x\xi')}dx'd\xi',$$

so that

$$|G(z)| \le \iint_{\mathbb{R}^2} |g(z')|e^{-\frac{\alpha}{2}(x-x')^2-\frac{\alpha}{2}(\xi-\xi')^2}dx'd\xi' =: (h * |g|)(z)$$

is the double convolution operator with $h(z) := e^{-\frac{\alpha}{2}|z|^2}$. By using the Cauchy–Schwarz inequality and Young's convolution inequalities, we obtain

$$
\begin{aligned}
\|\mathcal{B}^* f\|^2_{L^2(\mathbb{R})} &\le \frac{\alpha^2}{\pi^2}\|gG\|_{L^1(\mathbb{C})} \\
&\le \frac{\alpha^2}{\pi^2}\|g\|_{L^2(\mathbb{C})}\|h * |g|\|_{L^2(\mathbb{C})} \\
&\le \frac{\alpha^2}{\pi^2}\|h\|_{L^1(\mathbb{C})}\|g\|^2_{L^2(\mathbb{C})} \\
&\le \frac{2\alpha}{\pi}\|g\|^2_{L^2(\mathbb{C})} \\
&= 2\|f\|^2_{L^2_\rho(\mathbb{C})},
\end{aligned}
$$

hence $\mathcal{B}^* f \in L^2(\mathbb{R})$ if $f \in L^2_\rho(\mathbb{C})$.

It remains to prove that

$$\langle f, \mathcal{B}\varphi\rangle_{L^2_\rho(\mathbb{C})} = \langle \mathcal{B}^* f, \varphi\rangle_{L^2(\mathbb{R})}, \quad \text{for every} \quad f \in L^2_\rho(\mathbb{C}), \;\; \varphi \in L^2(\mathbb{R}).$$

This follows by combining all integrations into the triple integral

$$\langle f, \mathcal{B}\varphi\rangle_{L^2_\rho(\mathbb{C})} = \frac{2^{\frac{1}{4}}\alpha^{\frac{5}{4}}}{\pi^{\frac{5}{4}}} \iiint_{\mathbb{R}^3} f(z) e^{\frac{\alpha}{2}\bar{z}^2 - \alpha(\bar{z}-y)^2 - \alpha|z|^2}\overline{\varphi(y)}\,dy dx d\xi,$$

which coincides with the triple integral obtained from $\langle \mathcal{B}^* f, \varphi\rangle_{L^2(\mathbb{R})}$. All formal computations are justified because Fubini's theorem can be used to interchange integrations. ∎

The following corollary is deduced from Lemmas 2.1 and 2.2.

Corollary 2.1 $\mathcal{B}^* : L^2_\rho(\mathbb{C}) \mapsto L^2(\mathbb{R})$ *is the left inverse of* $\mathcal{B} : L^2(\mathbb{R}) \mapsto \mathcal{F} \subset L^2_\rho(\mathbb{C})$*, so that* $\mathcal{B}^*\mathcal{B}\varphi = \varphi$ *for every* $\varphi \in L^2(\mathbb{R})$*.*

Proof. By Lemma 2.2, we have for every $\varphi \in L^2(\mathbb{R})$:

$$\|\mathcal{B}\varphi\|^2_{L^2_\rho(\mathbb{C})} = \langle \mathcal{B}\varphi, \mathcal{B}\varphi\rangle_{L^2_\rho(\mathbb{C})} = \langle \mathcal{B}^*\mathcal{B}\varphi, \varphi\rangle_{L^2(\mathbb{R})}.$$

By Lemma 2.1, it follows that $\|\mathcal{B}\varphi\|^2_{L^2_\rho(\mathbb{C})} = \|\varphi\|^2_{L^2(\mathbb{R})}$, hence

$$\langle \mathcal{B}^*\mathcal{B}\varphi, \varphi\rangle_{L^2(\mathbb{R})} = \langle \varphi, \varphi\rangle_{L^2(\mathbb{R})}, \quad \text{for every} \quad \varphi \in L^2(\mathbb{R}).$$

This implies that $\mathcal{B}^*\mathcal{B}\varphi = \varphi$ for every $\varphi \in L^2(\mathbb{R})$. ∎

Remark 2.1 The adjoint Bargmann transform $\mathcal{B}^*$ is not a unitary transformation from $L^2_\rho(\mathbb{C})$ to $L^2(\mathbb{R})$ in the sense that there exists $f \in L^2_\rho(\mathbb{C})$ such that $\|\mathcal{B}^* f\|^2_{L^2(\mathbb{R})} \neq \|f\|^2_{L^2_\rho(\mathbb{C})}$. Indeed, $\mathcal{B}^*\bar{z} = 0$ for nonzero $\bar{z} \in L^2_\rho(\mathbb{C})$:

$$\begin{aligned}
\mathcal{B}^*\bar{z} &= \frac{2^{\frac{1}{4}}\alpha^{\frac{5}{4}}}{\pi^{\frac{5}{4}}} \iint_{\mathbb{R}^2} (x+i\xi) e^{-\frac{3}{2}\alpha x^2 - \frac{1}{2}\alpha\xi^2 - \alpha y^2 + 2\alpha xy + i\alpha\xi(2y-x)} dx d\xi \\
&= \frac{2^{\frac{1}{4}}\alpha^{\frac{5}{4}}}{\pi^{\frac{5}{4}}} \iint_{\mathbb{R}^2} (x+i\xi) e^{-2\alpha(x-y)^2 - \alpha y^2 - \frac{1}{2}\alpha(\xi + i(x-2y))^2} dx d\xi \\
&= \frac{2^{\frac{5}{4}}\alpha^{\frac{3}{4}}}{\pi^{\frac{3}{4}}} \int_{\mathbb{R}} (x-y) e^{-2\alpha(x-y)^2 - \alpha y^2} dx \\
&= 0.
\end{aligned}$$

Hence, $\mathcal{B}^* : L^2_\rho(\mathbb{C}) \mapsto L^2(\mathbb{R})$ is not the right inverse of $\mathcal{B} : L^2(\mathbb{R}) \mapsto \mathcal{F} \subset L^2_\rho(\mathbb{C})$.

The following lemma shows that the following operator

$$\Pi := \mathcal{B}\mathcal{B}^* : L^2_\rho(\mathbb{C}) \mapsto \mathcal{F} \subset L^2_\rho(\mathbb{C}) \tag{2.10}$$

is an identity on $\mathcal{F} \subset L^2_\rho(\mathbb{C})$. This implies that $\mathcal{B}^* : \mathcal{F} \subset L^2_\rho(\mathbb{C}) \mapsto L^2(\mathbb{R})$ is the right inverse of $\mathcal{B} : L^2(\mathbb{R}) \mapsto \mathcal{F} \subset L^2_\rho(\mathbb{C})$.

Lemma 2.3 *If* $f \in \mathcal{F} \subset L^2_\rho(\mathbb{C})$, *then* $\Pi f = f$.

Proof. Let us first deduce the explicit formula for Πf if $f \in L^2_\rho(\mathbb{C})$:

$$\begin{aligned}
(\Pi f)(z) &= \frac{2^{\frac{1}{2}}\alpha^{\frac{3}{2}}}{\pi^{\frac{3}{2}}} e^{\frac{1}{2}\alpha z^2} \int_{\mathbb{R}} e^{-\alpha(z-y)^2} \iint_{\mathbb{R}^2} e^{\frac{1}{2}\alpha\bar{z}'^2 - \alpha(y-\bar{z}')^2 - \alpha|z'|^2} f(z') dx' d\xi' dy \\
&= \frac{2^{\frac{1}{2}}\alpha^{\frac{3}{2}}}{\pi^{\frac{3}{2}}} e^{-\frac{1}{2}\alpha z^2} \iint_{\mathbb{R}^2} f(z') e^{-\frac{3}{2}\alpha x'^2 - \frac{1}{2}\alpha\xi'^2 - i\alpha x'\xi'} \left[\int_{\mathbb{R}} e^{-2\alpha y^2 + 2\alpha y(x-i\xi) + 2\alpha y(x'+i\xi')} dy \right] dx' d\xi' \\
&= \frac{\alpha}{\pi} \iint_{\mathbb{R}^2} f(z') e^{\alpha(z-z')\bar{z}'} dx' d\xi'.
\end{aligned}$$

Next, we show that if $f \in \mathcal{F} \subset L^2_\rho(\mathbb{C})$, then $\Pi f = f$. We shall use the distributional property for every $z = x - i\xi \in \mathbb{C}$ and $z_0 = x_0 - i\xi_0 \in \mathbb{C}$:

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z - z_0} = \pi \delta(x - x_0)\delta(\xi - \xi_0),$$

with δ being one-dimensional Dirac distribution and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x - i\partial_\xi)$. Let us rewrite the integral on $\mathbb{R}^2$ as the limit of the integral on the compact domain $D_R \backslash D_\epsilon$:

$$\begin{aligned}
&\frac{\alpha}{\pi} \iint_{\mathbb{R}^2} f(z') e^{\alpha(z-z')\bar{z}'} dx' d\xi' \\
&= \frac{1}{\pi} \lim_{\epsilon \to 0, R \to \infty} \iint_{D_R \backslash D_\epsilon} \left[\frac{\partial}{\partial \bar{z}'} \frac{f(z') e^{\alpha(z-z')\bar{z}'}}{z - z'} + \pi f(z') e^{\alpha(z-z')\bar{z}'} \delta(x' - x) \delta(\xi' - \xi) \right] dx' d\xi',
\end{aligned}$$

where we have used that $f \in \mathcal{F}$ is entire and introduced the square $D_R := [-R, R] \times [-R, R]$ and the hole $D_\epsilon := \{z' \in \mathbb{C} : \ |z' - z| \leq \epsilon\}$. Since

$$\lim_{\epsilon \to 0} \iint_{D_\epsilon} f(z') e^{\alpha(z-z')\bar{z}'} dx' d\xi' = 0,$$

we have

$$\begin{aligned}
&\frac{1}{\pi} \lim_{\epsilon \to 0} \iint_{D_\epsilon} \frac{\partial}{\partial \bar{z}'} \frac{f(z') e^{\alpha(z-z')\bar{z}'}}{z - z'} dx' d\xi' \\
&= -\lim_{\epsilon \to 0} \iint_{D_\epsilon} f(z') e^{\alpha(z-z')\bar{z}'} \delta(x' - x) \delta(\xi' - \xi) dx' d\xi' \\
&= -f(z).
\end{aligned}$$

On the other hand, we claim that for every $\epsilon > 0$:

$$\begin{aligned}
&\lim_{R \to \infty} \iint_{D_R \backslash D_\epsilon} \left[\frac{\partial}{\partial \bar{z}'} \frac{f(z') e^{\alpha(z-z')\bar{z}'}}{z - z'} + \pi f(z') e^{\alpha(z-z')\bar{z}'} \delta(x' - x) \delta(\xi' - \xi) \right] dx' d\xi' \\
&= -\iint_{D_\epsilon} \frac{\partial}{\partial \bar{z}'} \frac{f(z') e^{\alpha(z-z')\bar{z}'}}{z - z'} dx' d\xi', \qquad (2.11)
\end{aligned}$$

which yields

$$\begin{aligned}
(\Pi f)(z) &= \frac{\alpha}{\pi} \iint_{\mathbb{R}^2} f(z') e^{\alpha(z-z')\bar{z}'} dx' d\xi' \\
&= -\frac{1}{\pi} \lim_{\epsilon \to 0} \iint_{D_\epsilon} \frac{\partial}{\partial \bar{z}'} \frac{f(z') e^{\alpha(z-z')\bar{z}'}}{z - z'} dx' d\xi' \\
&= f(z),
\end{aligned}$$

as stated above. In order to prove (2.11), we note that the second term with δ distribution is identically zero for every $\epsilon > 0$ because the support of δ-distribution is outside $D_R \backslash D_\epsilon$.

For the first term, we integrate by parts,

$$\begin{aligned}\iint_{D_R\setminus D_\epsilon} \frac{\partial}{\partial \bar{z}'} F(z,z')dx'd\xi' = \quad & \frac{1}{2}\int_{-R}^{R} \left[F(z,R-i\xi') - F(z,-R-i\xi')\right] d\xi' \\ & + \frac{i}{2}\int_{-R}^{R} \left[F(z,x'+iR) - F(z,x'-iR)\right] dx' \\ & - \iint_{D_\epsilon} \frac{\partial}{\partial \bar{z}'} F(z,z')dx'd\xi',\end{aligned}$$

where

$$F(z,z') := \frac{f(z')e^{\alpha(z-z')\bar{z}'}}{z-z'}.$$

It remains to prove that

$$\lim_{R\to\infty} \int_{-R}^{R} F(z,\pm R - i\xi')d\xi' = \lim_{R\to\infty} \int_{-R}^{R} F(z,x'\pm iR)dx' = 0. \tag{2.12}$$

We prove the second limit, whereas the proof of the first limit is analogous. To do so, we use the representation

$$\begin{aligned}F(z,x'\pm iR) = \quad & \frac{f(x'\pm iR)}{(x-x') - i(\xi \pm R)} e^{-i\alpha(\xi x' \mp Rx) - \alpha(x'^2 + R^2 - xx' \mp R\xi)} \\ = \quad & \frac{f(x'\pm iR)}{(x-x') - i(\xi \pm R)} e^{-i\alpha(\xi x' \mp Rx) - \frac{1}{2}\alpha(x'-x)^2 - \frac{1}{2}\alpha(\xi \mp R)^2 - \frac{1}{2}\alpha(x'^2+R^2) + \frac{1}{2}\alpha(x^2+\xi^2)},\end{aligned}$$

so that for every fixed $z \in \mathbb{C}$, there exists an R-independent positive constant C_z such that for every large $R > 0$, we have

$$|F(z,x'\pm iR)| \leq CR^{-1}|f(x'\pm iR)|e^{-\frac{1}{2}\alpha(x'^2+R^2)}.$$

This implies that

$$\begin{aligned}\left|\int_{-R}^{R} F(z,x'\pm iR)dx'\right| \quad \leq \quad & CR^{-1}\int_{-R}^{R} |f(x'\pm iR)|e^{-\frac{1}{2}\alpha(x'^2+R^2)}dx' \\ \leq \quad & \sqrt{2}CR^{-1/2}\left(\int_{-\infty}^{\infty} |f(x'\pm iR)|^2 e^{-\alpha(x'^2+R^2)}dx'\right)^{1/2}.\end{aligned}$$

However, since $f \in L^2_\rho(\mathbb{C})$, we have

$$\int_{-\infty}^{\infty} I(R)dR < \infty, \quad \text{where } \; I(R) := \int_{-\infty}^{\infty} |f(x'\pm iR)|^2 e^{-\alpha(x'^2+R^2)}dx'.$$

This implies that $I(R) \to 0$ as $R \to \infty$ since $I(R)$ is smooth and positive for every $R \in \mathbb{R}$ if $f \in \mathcal{F} \subset L^2_\rho(\mathbb{C})$. Since $R^{-1/2} \to 0$ as $R \to \infty$, the estimate above justifies the second limit in (2.12). ∎

The following corollary is deduced from Lemma 2.3.

Corollary 2.2 $\mathcal{B}^* : \mathcal{F} \subset L^2_\rho(\mathbb{C}) \mapsto L^2(\mathbb{R})$ *is a unitary transformation.*

Proof. We show that $\|\mathcal{B}^* f\|^2_{L^2(\mathbb{R})} = \|f\|^2_{L^2_\rho(\mathbb{C})}$ for every $f \in \mathcal{F} \subset L^2_\rho(\mathbb{C})$. Indeed, by Lemma 2.2, we have

$$\|\mathcal{B}^* f\|^2_{L^2(\mathbb{R})} = \langle \mathcal{B}^* f, \mathcal{B}^* f\rangle_{L^2(\mathbb{R})} = \langle f, \mathcal{B}\mathcal{B}^* f\rangle_{L^2_\rho(\mathbb{C})}.$$

By Lemma 2.3, it follows for every $f \in \mathcal{F} \subset L^2_\rho(\mathbb{C})$ that

$$\langle f, \mathcal{B}\mathcal{B}^* f\rangle_{L^2_\rho(\mathbb{C})} = \langle f, f\rangle_{L^2_\rho(\mathbb{C})} = \|f\|^2_{L^2_\rho(\mathbb{C})},$$

and the assertion is proven. ■

Remark 2.2 The operator $\Pi : L^2_\rho(\mathbb{C}) \mapsto \mathcal{F} \subset L^2_\rho(\mathbb{C})$ is an orthogonal projection, since for every $f \in L^2_\rho(\mathbb{C})$, we have

$$\begin{aligned}
\langle (I-\Pi)f, \Pi f\rangle_{L^2_\rho(\mathbb{C})} &= \langle (I-\mathcal{B}\mathcal{B}^*)f, \mathcal{B}\mathcal{B}^* f\rangle_{L^2_\rho(\mathbb{C})} \\
&= \langle f, \mathcal{B}\mathcal{B}^* f\rangle_{L^2_\rho(\mathbb{C})} - \langle \mathcal{B}\mathcal{B}^* f, \mathcal{B}\mathcal{B}^* f\rangle_{L^2_\rho(\mathbb{C})} \\
&= \langle f, \mathcal{B}\mathcal{B}^* f\rangle_{L^2_\rho(\mathbb{C})} - \langle \mathcal{B}^*\mathcal{B}\mathcal{B}^* f, \mathcal{B}^* f\rangle_{L^2(\mathbb{R})} \\
&= \langle f, \mathcal{B}\mathcal{B}^* f\rangle_{L^2_\rho(\mathbb{C})} - \langle \mathcal{B}^* f, \mathcal{B}^* f\rangle_{L^2(\mathbb{R})} \\
&= \langle f, \mathcal{B}\mathcal{B}^* f\rangle_{L^2_\rho(\mathbb{C})} - \langle f, \mathcal{B}\mathcal{B}^* f\rangle_{L^2_\rho(\mathbb{C})} \\
&= 0,
\end{aligned}$$

where the results of Lemmas 2.2 and Corollary 2.1 have been used.

Remark 2.3 The bound $\|\mathcal{B}^* f\|^2_{L^2(\mathbb{R})} \leq 2\|f\|^2_{L^2_\rho(\mathbb{C})}$ for every $f \in L^2_\rho(\mathbb{C})$ was obtained in the proof of Lemma 2.2. The isometry result $\|\mathcal{B}^* f\|^2_{L^2(\mathbb{R})} = \|f\|^2_{L^2_\rho(\mathbb{C})}$ in Corollary 2.2 shows that the upper bound is not sharp on $\mathcal{F} \subset L^2_\rho(\mathbb{C})$. The following examples suggest that the upper bound is not sharp everywhere on $L^2_\rho(\mathbb{C})$.

Example 2.1 Consider $f(z) = e^{\frac{1}{2}a|z|^2}$, with $a \in \mathbb{C}$ satisfying $\mathrm{Re}(a) \in (-\infty, \alpha)$, for which $f \in L^2_\rho(\mathbb{C})$. We compute

$$\begin{aligned}
\mathcal{B}^* f &= \frac{2^{\frac{1}{4}}\alpha^{\frac{5}{4}}}{\pi^{\frac{5}{4}}} \iint_{\mathbb{R}^2} e^{\frac{1}{2}a(x^2+\xi^2)-2\alpha(x-y)^2-\alpha y^2-\frac{1}{2}\alpha(\xi+i(x-2y))^2} dx d\xi \\
&= \frac{2^{\frac{1}{4}}\alpha^{\frac{5}{4}}}{\pi^{\frac{5}{4}}} \iint_{\mathbb{R}^2} e^{-\frac{1}{2(\alpha-a)}[(2\alpha-a)x-2\alpha y]^2-\alpha y^2-\frac{1}{2}(\alpha-a)[\xi+\frac{i\alpha}{\alpha-a}(x-2y)]^2} dx d\xi \\
&= \frac{2^{\frac{3}{4}}\alpha^{\frac{5}{4}}}{\pi^{\frac{3}{4}}\sqrt{\alpha-a}} \int_{\mathbb{R}} e^{-\frac{1}{2(\alpha-a)}[(2\alpha-a)x-2\alpha y]^2-\alpha y^2} dx d\xi \\
&= \frac{2^{\frac{5}{4}}\alpha^{\frac{5}{4}}}{\pi^{\frac{1}{4}}(2\alpha-a)} e^{-\alpha y^2}.
\end{aligned}$$

If $a = a_R + ia_I$ with $a_R, a_I \in \mathbb{R}$, then we obtain:

$$\|\mathcal{B}^* f\|^2_{L^2(\mathbb{R})} = \frac{4\alpha^2}{(2\alpha - a_R)^2 + a_I^2}, \quad \|f\|^2_{L^2_\rho(\mathbb{C})} = \frac{\alpha}{\alpha - a_R}.$$

Hence, we have

$$\frac{\|\mathcal{B}^* f\|^2_{L^2(\mathbb{R})}}{\|f\|^2_{L^2_\rho(\mathbb{C})}} = \frac{4\alpha(\alpha - a_R)}{(2\alpha - a_R)^2 + a_I^2} = 1 - \frac{(a_R^2 + a_I^2)}{(2\alpha - a_R)^2 + a_I^2} \leq 1.$$

The quotient is exactly 1 if and only $a = 0$ when $f(z) = 1$ is entire.

Example 2.2 Consider $f(z) = e^{az+b\bar{z}}$, with $a, b \in \mathbb{C}$. Then, we compute

$$\begin{aligned}
\mathcal{B}^* f &= \frac{2^{\frac{1}{4}}\alpha^{\frac{5}{4}}}{\pi^{\frac{5}{4}}} \iint_{\mathbb{R}^2} e^{a(x-i\xi)+b(x+i\xi)-2\alpha(x-y)^2-\alpha y^2-\frac{1}{2}\alpha(\xi+i(x-2y))^2} dx d\xi \\
&= \frac{2^{\frac{1}{4}}\alpha^{\frac{5}{4}}}{\pi^{\frac{5}{4}}} \iint_{\mathbb{R}^2} e^{-2\alpha(x-y)^2-\frac{1}{2\alpha}(a-b)^2+2ay+2b(x-y)-\alpha y^2-\frac{1}{2}\alpha(\xi+i(x-2y)+i\alpha^{-1}(a-b))^2} dx d\xi \\
&= \frac{2^{\frac{3}{4}}\alpha^{\frac{3}{4}}}{\pi^{\frac{3}{4}}} \int_{\mathbb{R}} e^{-2\alpha(x-y-\frac{b}{2\alpha})^2-\frac{1}{2\alpha}(a^2-2ab)+2ay-\alpha y^2} dx \\
&= \frac{2^{\frac{1}{4}}\alpha^{\frac{1}{4}}}{\pi^{\frac{1}{4}}} e^{-\frac{1}{2\alpha}(a^2-2ab)+2ay-\alpha y^2},
\end{aligned}$$

from which we obtain

$$\|\mathcal{B}^* f\|^2_{L^2(\mathbb{R})} = e^{\frac{1}{\alpha}(|a|^2+2\mathrm{Re}(ab))}$$

and

$$\|f\|^2_{L^2_\rho(\mathbb{C})} = e^{\frac{1}{\alpha}(|a|^2+2\mathrm{Re}(ab)+|b|^2)},$$

so that

$$\frac{\|\mathcal{B}^* f\|^2_{L^2(\mathbb{R})}}{\|f\|^2_{L^2_\rho(\mathbb{C})}} = e^{-\frac{1}{\alpha}|b|^2} \leq 1.$$

The quotient is exactly 1 if and only $b = 0$ when $f(z) = e^{az}$ is entire.

2.3 Embedding of Fock spaces

Here we discuss embedding of a Fock space in $L^p_\rho(\mathbb{C})$ into a Fock space in $L^q_\rho(\mathbb{C})$, where $0 < p \leq q \leq \infty$ and ρ is the same weight as in (2.1). The presentation is based on [10] and Section 2.1 in [43].

The following lemma gives pointwise embedding of the L^p-based Fock space $\mathcal{F}_p$.

Lemma 2.4 *Fix $0 < p < \infty$. For every $z \in \mathbb{C}$ and every $f \in \mathcal{F}_p$, the following is true:*

$$|f(z)| \leq \|f\|_{L^p_\rho} e^{\frac{1}{2}\alpha|z|^2}. \tag{2.13}$$

Proof. First, we show that if $f \in \mathcal{F}_p$, then $|f|^p$ is subharmonic for every $0 < p < \infty$. Indeed, let $f = u + iv$, then $|f|^p = (u^2 + v^2)^{\frac{p}{2}}$, and we obtain

$$\begin{aligned}
\nabla|f|^p &= p(u^2+v^2)^{\frac{p-2}{2}}(u\nabla u + v\nabla v) \\
&= p|f|^{p-2}(u\nabla u + v\nabla v), \qquad (2.14)\\
\Delta|f|^p &= p(u^2+v^2)^{\frac{p-2}{2}}(u\Delta u + |\nabla u|^2 + v\Delta v + |\nabla v|^2) \\
&\quad + p(p-2)(u^2+v^2)^{\frac{p-4}{2}}|u\nabla u + v\nabla v|^2 \\
&= p^2|f|^{p-4}|u\nabla u + v\nabla v|^2, \qquad (2.15)
\end{aligned}$$

where we have used $\Delta u = \Delta v = 0$ and the Cauchy–Riemann equations implying $|\nabla u|^2 = |\nabla v|^2$ and $\nabla u \cdot \nabla v = 0$. Hence, we have $\Delta|f|^p \geq 0$ for every $0 < p < \infty$.

Next, by using the mean value property for subharmonic functions, we have

$$2\pi r|f(0)|^p \leq \int_0^{2\pi} |f(z)|_{|z|=r}|^p r\, d\theta,$$

which yields

$$|f(0)|^p \int_0^\infty 2\pi r e^{-\frac{1}{2}\alpha p r^2} dr \leq \int_0^\infty \int_0^{2\pi} |f(z)|_{|z|=r}|^p e^{-\frac{1}{2}\alpha p r^2} r\, dr\, d\theta,$$

where θ is the angle in the parametrization $z = re^{i\theta}$ at fixed r. After integration, we obtain

$$|f(0)|^p \leq \frac{\alpha p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{1}{2}\alpha p|z|^2} dz,$$

hence $|f(0)| \leq \|f\|_{L^p_\rho}$ for every $f \in \mathcal{F}_p$.

In order to extend this result for every $z \in \mathbb{C}$, we set $F(\omega) := f(z-\omega)e^{\alpha\omega\bar{z} - \frac{1}{2}\alpha|z|^2}$ and use $|F(0)|^p \leq \|F\|_{L^p_\rho}$, which yields

$$\begin{aligned}
|f(z)|^p e^{-\frac{1}{2}\alpha p|z|^2} &\leq \frac{p\alpha}{2\pi} \int_{\mathbb{C}} |f(z-\omega)|^p e^{\alpha p \mathrm{Re}(\bar{z}\omega) - \frac{1}{2}\alpha p|z|^2 - \frac{1}{2}\alpha p|\omega|^2} d\omega \\
&= \frac{p\alpha}{2\pi} \int_{\mathbb{C}} |f(z-\omega)|^p e^{-\frac{1}{2}\alpha p|z-\omega|^2} d\omega \\
&= \frac{p\alpha}{2\pi} \int_{\mathbb{C}} |f(\zeta)|^p e^{-\frac{1}{2}\alpha p|\zeta|^2} d\zeta \\
&= \|f\|^p_{L^p_\rho}.
\end{aligned}$$

Extracting the p-root yields the inequality (2.13). ■

The following lemma gives an embedding of $\mathcal{F}_p$ to $\mathcal{F}_q$ for $0 < p < q < \infty$.

Lemma 2.5 *Fix $0 < p < q < \infty$. Then, $\mathcal{F}_p \subsetneq \mathcal{F}_q$ and the inclusion is continuous.*

Proof. By using bound (2.13), we obtain for every $f \in \mathcal{F}_p$:

$$\begin{aligned}
\|f\|^q_{L^q_\rho} &= \frac{q\alpha}{2\pi}\int_{\mathbb{C}} |f(z)|^q e^{-\frac{1}{2}q\alpha|z|^2} dz \\
&= \frac{q\alpha}{2\pi}\int_{\mathbb{C}} |f(z)|^p |f(z)|^{q-p} e^{-\frac{1}{2}q\alpha|z|^2} dz \\
&\leq \frac{q\alpha}{2\pi}||f||^{q-p}_{L^p_\rho}\int_{\mathbb{C}} |f(z)|^p e^{-\frac{1}{2}p\alpha|z|^2} dz \\
&= \frac{q}{p}||f||^q_{L^p_\rho}
\end{aligned}$$

Extracting the q-th root yields the bound

$$\|f\|_{L^q_\rho} \leq \left(\frac{q}{p}\right)^{\frac{1}{q}} \|f\|_{L^p_\rho}, \tag{2.16}$$

hence $\mathcal{F}_p \subseteq \mathcal{F}_q$ is a continuous embedding. It remains to show that $\mathcal{F}_p \neq \mathcal{F}_q$.

Let us assume that $\mathcal{F}_p = \mathcal{F}_q$ and obtain a contradiction. If $\mathcal{F}_p = \mathcal{F}_q$, then the identity map $I : \mathcal{F}_p \mapsto \mathcal{F}_q$ is bounded and bijection. By the open mapping theorem, there exists C such that

$$C^{-1}||f||_{L^p_\rho} \leq ||f||_{L^q_\rho} \leq C||f||_{L^p_\rho}, \text{ for every } f \in \mathcal{F}_p. \tag{2.17}$$

Consider $f(z) = z^n$, $n \in \mathbb{N}$ and compute

$$\begin{aligned}
||z^n||^p_{L^p_\rho} &= \frac{p\alpha}{2\pi}\int_0^{2\pi}\int_0^\infty r^n p e^{-\frac{p\alpha}{2}r^2} r dr d\theta \\
&= p\alpha \int_0^\infty r^{np+1} e^{-\frac{p\alpha}{2}r^2} dr \\
&= \frac{1}{(p\alpha)^{\frac{np}{2}}}\int_0^\infty s^{\frac{np}{2}} e^{-s} ds \\
&= \frac{1}{(p\alpha)^{\frac{np}{2}}}\Gamma\left(\frac{np}{2}+1\right),
\end{aligned}$$

where Γ is the Gamma function. By Stirling's formula, we have

$$\Gamma\left(\frac{np}{2}+1\right) \sim \left(\frac{np}{2e}\right)^{\frac{np}{2}} \sqrt{2\pi n} \quad \text{as} \quad n \to \infty,$$

hence

$$\|z^n\|_{L^p_\rho} \sim \left(\frac{n}{2\alpha e}\right)^{\frac{n}{2}} (2\pi n)^{\frac{1}{2p}} \quad \text{as} \quad n \to \infty.$$

Since $\|z^n\|_{L^q_\rho}$ grows slower for $q > p$ as $n \to \infty$, there is no positive constant C in the lower bound (2.17) for large $n \in \mathbb{N}$. Hence, $\mathcal{F}_p \neq \mathcal{F}_q$. ∎

The bound (2.16) is not sharp. The following lemma provides the sharp bound for the embedding of $\mathcal{F}_p$ into $\mathcal{F}_q$.

Lemma 2.6 *Fix $0 < p < q < \infty$. For every $f \in \mathcal{F}_p$, it is true that*

$$\left(\frac{\alpha q}{2\pi}\right)^{\frac{1}{q}} \|u\|_q \leq \left(\frac{\alpha p}{2\pi}\right)^{\frac{1}{p}} \|u\|_p \tag{2.18}$$

where $u(z) = |f(z)|e^{-\frac{1}{2}\alpha|z|^2}$ and $\|u\|_p$ is the standard L^p norm on $\mathbb{C}$.

Proof. First, we note the following elementary identity for every $s > 0$:

$$\int_{\mathbb{C}} |\nabla u^{\frac{s}{2}}(z)|^2 dz = \frac{\alpha s}{2} \int_{\mathbb{C}} u^s(z) dz \tag{2.19}$$

Indeed, we have

$$\begin{aligned}
\int_{\mathbb{C}} |\nabla u^{\frac{s}{2}}|^2 dz &= \int_{\mathbb{C}} \left(|\nabla |f|^{\frac{s}{2}}|^2 + \frac{\alpha^2 s^2}{4} |z|^2 |f|^s - \frac{\alpha s}{2} z \cdot \nabla |f|^s \right) e^{-\frac{1}{2}\alpha s|z|^2} dz \\
&= \int_{\mathbb{C}} \left(|\nabla |f|^{\frac{s}{2}}|^2 + \frac{1}{4}|f|^s \Delta + \frac{\alpha s}{2}|f|^s + \frac{1}{2}(\nabla |f|^s) \cdot \nabla \right) e^{-\frac{1}{2}\alpha s|z|^2} dz \\
&= \frac{\alpha s}{2} \int_{\mathbb{C}} |f|^s e^{-\frac{1}{2}\alpha s|z|^2} dz + \int_{\mathbb{C}} \left(|\nabla |f|^{\frac{s}{2}}|^2 - \frac{1}{4}\Delta |f|^s \right) e^{-\frac{1}{2}\alpha s|z|^2} dz,
\end{aligned}$$

where the Green identity has been used with the zero boundary terms at infinity. It follows from (2.14) and (2.15) that the second term is identically zero, and we obtain (2.19).

Next, it follows from Lemma 2.4 that if $f \in \mathcal{F}_p$, then $u \in L^\infty(\mathbb{C})$ and $|u(z)| \leq \|f\|_{L^p_\rho}$. Since $u \in L^p(\mathbb{C})$, we have $u \in L^s(\mathbb{C})$ for every $p \leq s \leq \infty$ by interpolation. Let us set the function

$$g(s) := \frac{\alpha}{2\pi} \|u\|_s^s \tag{2.20}$$

for the given $u \in L^s(\mathbb{C})$. The function g is continuously differentiable in s on (p, ∞) with the derivative given by

$$s\frac{d}{ds} g(s) = \frac{\alpha}{2\pi} \iint_{\mathbb{R}^2} u^s \ln u^s dx dy. \tag{2.21}$$

We use the logarithmic Sobolev inequality from [9]:

$$\int_{\mathbb{C}} \rho \ln \rho dz \leq \frac{1}{\pi t} \int_{\mathbb{C}} |\nabla \rho^{\frac{1}{2}}|^2 dz + \ln t - 2, \tag{2.22}$$

where $t > 0$ is parameter and ρ is a positive function on $\mathbb{C}$ satisfying $\rho^{\frac{1}{2}} \in H^1(\mathbb{C})$ and the normalization $\int_{\mathbb{C}} \rho dz = 1$. To satisfy the constraint, we choose $\rho := \frac{u^s}{\|u\|_s^s}$ and obtain from (2.22) that

$$\int_{\mathbb{C}} u^s \ln \left(\frac{u^s}{\|u\|_s^s} \right) dz \leq \frac{1}{\pi t} \int_{\mathbb{C}} \left| \nabla u^{\frac{s}{2}} \right|^2 dz + (\ln t - 2) \|u\|_s^s.$$

By using identity (2.19), the bound can be rewritten in the form:

$$\int_{\mathbb{C}} u^s \ln u^s dz \leq \left(\ln t + \ln \|u\|_s^s + \frac{\alpha s}{2\pi t} - 2 \right) \|u\|_s^s,$$

Setting $t := \frac{\alpha s}{2\pi}$ and using (2.20) and (2.21) yields a closed inequality for $g(s)$:

$$s \frac{d}{ds} g(s) \leq (\ln s + \ln g(s) - 1)\, g(s). \tag{2.23}$$

When the inequality (2.23) is replaced by the equality, there exists the exact solution given by $G(s) = \frac{1}{s}$. Substitution $g(s) := \frac{h(s)}{s}$ reduces the inequality (2.23) to the form

$$\frac{d}{ds} h(s) \leq \frac{1}{s} h(s) \ln h(s). \tag{2.24}$$

If $h(p) = 1$ and $s > p$, then the comparison principle for differential equations gives $h(s) \leq 1$ and $\ln h(s) \leq 0$. Therefore, we obtain for every $q > p$:

$$qg(q) = h(q) \leq 1 = h(p) = pq(p),$$

which recovers (2.18) in view of (2.20). ∎

Example 2.3 *If $f(z) = 1$ and $u(z) = e^{-\frac{1}{2}\alpha |z|^2}$, then*

$$\|u\|_q^q = \int_{\mathbb{C}} e^{-\frac{1}{2}\alpha q |z|^2} dz = 2\pi \int_0^\infty e^{-\frac{1}{2}\alpha q r^2} r dr = \frac{2\pi}{\alpha q}.$$

Hence, the inequality (2.18) becomes the equality for the Gaussian functions. In other words, the inequality (2.18) is sharp.

Chapter 3

Applications to partial differential equations in one dimension

3.1 Introduction

The Bargmann transform $\mathcal{B}$ defined by (2.3) is useful for applications to the Schrödinger operator with the harmonic potential in one dimension. We fix $\alpha = \frac{1}{2}$ everywhere in this chapter and consider the Schrödinger operator $L : D(L) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ defined by the differential expression

$$L := -\partial_y^2 + y^2 - 1 \tag{3.1}$$

and the domain

$$D(L) := \left\{ \varphi \in L^2(\mathbb{R}) : \quad \partial_y^2 \varphi \in L^2(\mathbb{R}), \quad y^2 \varphi \in L^2(\mathbb{R}) \right\}. \tag{3.2}$$

Review of properties of this operator can be found in [28].

Section 3.2 reviews orthonormal basis in the Fock space $\mathcal{F} \subset L^2_\rho(\mathbb{C})$ related to the orthonormal basis of Gauss–Hermite functions, which are eigenfunctions of the operator L in $L^2(\mathbb{R})$. Sections 3.3, 3.4, and 3.5 study applications of the Bargmann transform to the linearized Schrödinger, heat, and Korteweg–de Vries equations, respectively.

3.2 Orthonormal basis in the Fock space

Recall the orthonormal basis $\{u_n\}_{n \in \mathbb{N}_0}$ in $L^2(\mathbb{R})$ given by the Gauss–Hermite functions

$$u_n(y) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(y) e^{-\frac{1}{2}y^2}, \quad n \in \mathbb{N}_0, \tag{3.3}$$

where H_n is the Hermite polynomial of degree n and $\mathbb{N}_0 := \{0, 1, 2, \dots\}$.

The following lemma computes the image of the orthonormal basis $\{u_n\}_{n\in\mathbb{N}_0}$ under the Bargmann transform $\mathcal{B} : L^2(\mathbb{R}) \mapsto \mathcal{F} \subset L^2_\rho(\mathbb{C})$ given by (2.3) with $\alpha = \frac{1}{2}$.

Lemma 3.1 *Let u_n be given by (3.3) and $\alpha = \frac{1}{2}$. If $f_n := \mathcal{B}u_n$, then*

$$f_n(z) = \frac{z^n}{\sqrt{2^n n!}}, \quad n \in \mathbb{N}_0. \tag{3.4}$$

Proof. Substituting (3.3) into (2.3) yields

$$\mathcal{B}u_n = \frac{1}{\sqrt{2^n \pi n!}} \int_{\mathbb{R}} H_n(y) e^{-\left(\frac{1}{2}z-y\right)^2} dy.$$

By using the generating formula for Hermite polynomials,

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2},$$

we evaluate the integrals for every $n \in \mathbb{N}_0$ by integrating by parts n times thanks to the fast decay of the integrand at infinity:

$$\begin{aligned}
\mathcal{B}u_n &= \frac{1}{\sqrt{2^n \pi n!}} (-1)^n e^{-\frac{1}{4}z^2} \int_{\mathbb{R}} e^{yz} \frac{d^n}{dy^n} e^{-y^2} dy \\
&= \frac{1}{\sqrt{2^n \pi n!}} z^n \int_{\mathbb{R}} e^{-\left(\frac{1}{2}z-y\right)^2} dy \\
&= \frac{1}{\sqrt{2^n n!}} z^n,
\end{aligned}$$

where the last equality is justified for a fixed $z \in \mathbb{C}$ since e^{-y^2} is extended as the entire function in a complex plane with fast decay as $\mathrm{Re}(y) \to \pm\infty$. ∎

By Lemma 2.1, the transformation $\mathcal{B} : L^2(\mathbb{R}) \mapsto \mathcal{F} \subset L^2_\rho(\mathbb{C})$ is unitary, which suggests that the set of monomials $\{f_n\}_{n\in\mathbb{N}_0}$ given by (3.4) is an orthonormal basis in $\mathcal{F} \subset L^2_\rho(\mathbb{C})$. Indeed, this can be checked directly.

Lemma 3.2 *Let f_n be given by (3.4). Then, we have*

$$\langle f_n, f_m \rangle_{L^2_\rho(\mathbb{C})} = \delta_{n,m}, \tag{3.5}$$

where $\delta_{n,m}$ is the Kronecker's symbol and the inner product in $L^2_\rho(\mathbb{C})$ is normalized by (2.2).

Proof. We use the polar coordinates for (x,ξ) in $\mathbb{R}^2$:

$$\begin{aligned}\langle f_n, f_m\rangle_{L^2_\rho(\mathbb{C})} &= \frac{1}{2\pi\sqrt{2^n 2^m n! m!}}\int_{\mathbb{R}^2} z^n \bar{z}^m e^{-\frac{1}{2}|z|^2} dx d\xi \\ &= \frac{1}{2\pi\sqrt{2^n 2^m n! m!}}\left(\int_0^{2\pi} e^{i(n-m)\theta} d\theta\right)\left(\int_0^{\infty} r^{n+m+1} e^{-\frac{1}{2}r^2} dr\right).\end{aligned}$$

This yields zero for $n \neq m$. For $n = m$, we further compute

$$\begin{aligned}\langle f_n, f_n\rangle_{L^2_\rho(\mathbb{C})} &= \frac{1}{2^n n!}\left(\int_0^{\infty} r^{2n+1} e^{-\frac{1}{2}r^2} dr\right) \\ &= \frac{1}{2^n n!}\left(\int_0^{\infty} (2t)^n e^{-t} dt\right),\end{aligned}$$

which yields the normalization $\langle f_n, f_n\rangle_{L^2_\rho(\mathbb{C})} = 1$. ∎

As an application of the Bargmann transform $\mathcal{B}$, we can establish the orthogonality and normalization conditions for the Hermite polynomials $\{H_n\}_{n\in\mathbb{N}_0}$, which is well-known in the properties of the Hermite polynomials.

Corollary 3.1 *The Hermite polynomials $\{H_n\}_{n\in\mathbb{N}_0}$ satisfy the orthogonality and normalization conditions:*

$$\int_{\mathbb{R}} H_n(y) H_m(y) e^{-y^2} dy = 2^n n! \sqrt{\pi}\delta_{n,m}. \tag{3.6}$$

Proof. This follows by the transformation of Corollary 2.1 with the account of (3.4) and (3.5):

$$\delta_{n,m} = \langle f_n, f_m\rangle_{L^2_\rho(\mathbb{C})} = \langle \mathcal{B}u_n, \mathcal{B}u_m\rangle_{L^2_\rho(\mathbb{C})} = \langle \mathcal{B}^*\mathcal{B}u_n, u_m\rangle_{L^2(\mathbb{R})} = \langle u_n, u_m\rangle_{L^2(\mathbb{R})}.$$

Substitution of (3.3) yields (3.6). ∎

Remark 3.1 Every entire function f can be represented by the Taylor series which converges uniformly and absolutely for every $z \in \mathbb{C}$:

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) z^n. \tag{3.7}$$

If $f \in \mathcal{F} \subset L^2_\rho(\mathbb{C})$, this Taylor series becomes the decomposition over the orthonormal basis of monomials $\{f_n\}_{n\in\mathbb{N}_0}$ given by (3.4):

$$f(z) = \sum_{n=0}^{\infty} \langle f, f_n\rangle_{L^2_\rho(\mathbb{C})} f_n(z). \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$f^{(n)}(0) = \frac{1}{2^{n+1}\pi} \iint_{\mathbb{R}^2} f(z)\bar{z}^n e^{-\frac{1}{2}|z|^2} dx d\xi, \quad n \in \mathbb{N}_0.$$

In particular, for every $f \in \mathcal{F}$, it is true that

$$f(0) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} f(z) e^{-\frac{1}{2}|z|^2} dx d\xi,$$

which is a remarkable identity in complex analysis.

Remark 3.2 If $\varphi = \mathcal{B}^* f$ with f given by (3.8), then we obtain a decomposition of a function in $L^2(\mathbb{R})$ over the Gauss–Hermite functions given by (3.3):

$$\varphi(y) = \sum_{n=0}^{\infty} \langle \varphi, u_n \rangle_{L^2(\mathbb{R})} u_n(y). \tag{3.9}$$

Indeed, using transformations in Corollary 2.1 and the correspondence (3.4), we obtain:

$$\langle \varphi, u_n \rangle_{L^2(\mathbb{R})} = \langle \mathcal{B}^* f, u_n \rangle_{L^2(\mathbb{R})} = \langle f, \mathcal{B} u_n \rangle_{L^2_\rho(\mathbb{C})} = \langle f, f_n \rangle_{L^2_\rho(\mathbb{C})}. \tag{3.10}$$

The following two lemmas explore the orthonormal basis in $\mathcal{F} \subset L^2_\rho(\mathbb{C})$.

Lemma 3.3 *Let $f \in \mathcal{F}$. Then for every $z \in \mathbb{C}$, it is true that*

$$|f(z)| \leq e^{\frac{1}{4}|z|^2} \|f\|_{L^2_\rho(\mathbb{C})}. \tag{3.11}$$

Proof. By Parseval's equality, we have

$$\|f\|^2_{L^2_\rho(\mathbb{C})} = \sum_{n \in \mathbb{N}_0} \langle f, f_n \rangle^2_{L^2_\rho(\mathbb{C})}.$$

This gives by the Schwarz inequality for sequences

$$|f(z)| \leq \sum_{n \in \mathbb{N}_0} |\langle f, f_n \rangle_{L^2_\rho(\mathbb{C})}| |f_n(z)| \leq \left(\sum_{n \in \mathbb{N}_0} |f_n(z)|^2 \right)^{1/2} \|f\|_{L^2_\rho(\mathbb{C})}.$$

The proof is completed with the explicit computation:

$$\sum_{n \in \mathbb{N}_0} |f_n(z)|^2 = \sum_{n \in \mathbb{N}_0} \frac{|z|^{2n}}{2^n n! \pi} = e^{\frac{1}{2}|z|^2}.$$

Taking square root yields (3.11). ∎

Remark 3.3 Bound (3.11) coincides with the bound (2.13) for $p = 2$ and $\alpha = \frac{1}{2}$.

Remark 3.4 The uniform bound (3.11) can be used in the proof of Lemma 2.3 to control

$$|F(z, x' \pm iR)| \leq CR^{-1}|f(x' \pm iR)|e^{-\frac{1}{4}(x'^2+R^2)} \leq CR^{-1}\|f\|_{L^2_\rho(\mathbb{C})}.$$

However, this is still insufficient for the proof of (2.12).

Lemma 3.4 *Let $f \in \mathcal{F}$. The Taylor decomposition over the basis in (3.7) can be written in the form of representation*

$$f(z) = \langle f, E_z \rangle_{L^2_\rho(\mathbb{C})}, \tag{3.12}$$

where $E_z \in \mathcal{F}$ is given by $E_z(z') := e^{\frac{1}{2}\bar{z}z'}$.

Proof. Formal interchange of integration and summation in the representation (3.8) yields

$$\overline{E_z(z')} := \sum_{n\in\mathbb{N}_0} f_n(z)\overline{f_n(z')} = \sum_{n\in\mathbb{N}_0} \frac{z^n(\bar{z}')^n}{2^n n!} = e^{\frac{1}{2}z\bar{z}'},$$

from which $E_z(z')$ in (3.12) follows by complex conjugation. ∎

Remark 3.5 Comparison of (3.11) and (3.12) suggests that $\|E_z\|_{L^2_\rho(\mathbb{C})} = e^{\frac{1}{4}|z|^2}$, which can be confirmed by an explicit computation. The representation formula (3.12) is in agreement with the explicit formula:

$$(\Pi f)(z) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} f(z')e^{\frac{1}{2}(z-z')\bar{z}'}dx'd\xi' \tag{3.13}$$

obtained in Lemma 2.3 for the projection operator Π in (2.10).

Next, we obtain the explicit formulas of projections of monomials $z^m\bar{z}^k$ to the Fock space $\mathcal{F}$ by using the projection operator $\Pi = \mathcal{B}\mathcal{B}^*$ defined by (2.10). By Lemma 2.3, we have $\Pi z^m = z^m$ for every $m \in \mathbb{N}_0$. The following lemma gives us the general case.

Lemma 3.5 *For every $m, k \in \mathbb{N}$ with $k \leq m$, it is true that*

$$\Pi(z^m\bar{z}^k) = \frac{(2m)!!}{(2m-2k)!!}z^{m-k} = \frac{2^k m!}{(m-k)!}z^{m-k}, \tag{3.14}$$

whereas $\Pi(z^m\bar{z}^k) = 0$ if $k > m$.

Proof. The following computational formula

$$\partial_z \Pi f = \frac{1}{2} \Pi \bar{z} f(z) \tag{3.15}$$

follows by the formal differentiation of (3.13). Using (3.15) with $f(z) = z^m$ and $f(z) = z^m \bar{z}$, we obtain $\Pi(z^m \bar{z}) = 2m z^{m-1}$ and $\Pi(z^m \bar{z}^2) = (2m)(2m-2) z^{m-2}$ respectively. By induction, we obtain (3.14) for every $k \leq m$.

For $k = m$, we have $\Pi(|z|^{2m}) = C_m$ with constant $C_m = \frac{(2m)!!}{(2m-2k)!!}$, hence further iterations of (3.15) yield $\Pi(z^m \bar{z}^k) = 0$ for $k > m$. ∎

3.3 Evolution of the linear Schrödinger equation

The Gauss–Hermite functions (3.3) are eigenfunctions of the Schrödinger operator L given by (3.1) and (3.2). To be precise, we know that

$$L u_n = (2n) u_n, \quad n \in \mathbb{N}_0. \tag{3.16}$$

Since $\{u_n\}_{n \in \mathbb{N}_0}$ is an orthonormal basis in $L^2(\mathbb{R})$, it diagonalizes operator L in $L^2(\mathbb{R})$.

The following lemma gives the transformation of the operator L defined in $L^2(\mathbb{R})$ to the operator $\mathcal{B} L \mathcal{B}^*$ defined in the Fock space $\mathcal{F} \subset L^2_\rho(\mathbb{C})$.

Lemma 3.6 *If* $\mathcal{L} := \mathcal{B} L \mathcal{B}^* : D(\mathcal{L}) \subset \mathcal{F} \mapsto \mathcal{F}$, *then*

$$(\mathcal{L} f)(z) = 2z \frac{df}{dz}, \quad f \in D(\mathcal{L}), \tag{3.17}$$

with

$$D(\mathcal{L}) = \left\{ f \in L^2_\rho(\mathbb{C}) : \quad z \partial_z f \in L^2_\rho(\mathbb{C}) \right\}. \tag{3.18}$$

Proof. By using the decomposition over the orthonormal basis for every $\varphi \in L^2(\mathbb{R})$ given by (3.9), we can write

$$L\varphi = \sum_{n \in \mathbb{N}_0} 2n a_n u_n, \quad a_n := \langle \varphi, u_n \rangle_{L^2(\mathbb{R})}.$$

Let $f := \mathcal{B}\varphi \in \mathcal{F}$, so that $\varphi = \mathcal{B}^* f$. Then,

$$\mathcal{L} f = \sum_{n \in \mathbb{N}_0} 2n a_n f_n, \quad a_n = \langle f, f_n \rangle_{L^2_\rho(\mathbb{C})},$$

where the last equation is due to (3.10). Since f_n are monomials given by (3.4), we confirm that $(\mathcal{L} f)(z) = 2z f'(z)$ for every $f \in \mathcal{F}$. ∎

Remark 3.6 The representation (3.17) implies $\mathcal{L}f_n = 2nf_n$ for the monomials in (3.4), in agreement with $Lu_n = 2nu_n$ for the Hermite–Gauss functions in (3.3).

Corollary 3.2 *The inverse transformation to (3.17) is* $L = \mathcal{B}^*\mathcal{L}\mathcal{B} : D(L) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$*, where*

$$L\varphi = \mathcal{B}^*(|z|^2 - 2)\mathcal{B}\varphi, \quad \varphi \in D(L). \tag{3.19}$$

Proof. This follows from the computational formula (3.15) applied to $f \in \mathcal{F}$: $2f'(z) = \Pi(\bar{z}f(z))$, from which

$$2zf'(z) = 2\frac{d}{dz}[zf(z)] - 2f(z) = \Pi(|z|^2 f) - 2f = \mathcal{B}\mathcal{B}^*(|z|^2 - 2)\mathcal{B}\mathcal{B}^* f.$$

By using the inverse operators, thanks to Lemmas 2.3 and 3.6, we obtain

$$L = \mathcal{B}^*\mathcal{L}\mathcal{B} = \mathcal{B}^*(|z|^2 - 2)\mathcal{B},$$

which gives the assertion. ∎

Remark 3.7 The representation (3.19) implies that the Schrödinger operator L in $L^2(\mathbb{R})$ is equivalent to the nonlocal operator $\Pi(|z|^2 - 2)$ in the Fock space $\mathcal{F} \subset L^2_\rho(\mathbb{C})$.

We shall now express the general solution of the Cauchy problem for the time-dependent Schrödinger equation:

$$\begin{cases} i\dot{\Phi} = L\Phi \\ \Phi|_{t=0} = \varphi \in L^2(\mathbb{R}), \end{cases} \tag{3.20}$$

where the dot denotes the derivative with respect to time. Solutions to the Cauchy problem (3.20) can be obtained in a closed integral form by reducing L under the Bargmann transform in the Fock space $\mathcal{F}$.

After the Bargmann transform $\mathcal{B}$, the Cauchy problem (3.20) is rewritten in the form:

$$\begin{cases} i\dot{F} = \mathcal{L}F \\ F|_{t=0} = f \in \mathcal{F}, \end{cases} \tag{3.21}$$

where $f = \mathcal{B}\varphi$ and $F(t, \cdot) = \mathcal{B}\Phi(t, \cdot)$. The following lemma gives the unique solution of the Cauchy problem (3.21) in a closed analytical form.

Lemma 3.7 *For every* $f \in \mathcal{F}$*, there exists the unique solution to the Cauchy problem (3.21) which can be written in the form:*

$$F(t, z) = f\left(ze^{-2it}\right). \tag{3.22}$$

Proof. The partial differential equation

$$i\frac{\partial F}{\partial t} = 2z\frac{\partial F}{\partial z} \tag{3.23}$$

is nothing but the transport equation after a transformation of variables $z \to \log z$ and $t \to 2it$. Hence, its general solution is given by

$$F(t, z) := G(\log z - 2it),$$

where the function G is found from the initial condition $G(\log z) = f(z)$. Expressing G yields (3.22). The solution to the Cauchy problem (3.21) is conjugate to the solution to the Cauchy problem (3.20) since $\mathcal{B}$ is a unitary transformation. Therefore, the solution to (3.21) is unique because the solution to (3.20) is unique. ∎

Remark 3.8 Decomposition of $f \in \mathcal{F}$ via the monomials $\{f_n\}_{n\in\mathbb{N}_0}$ yields the solution to the Cauchy problem (3.21) for $F(t, z)$ written in the series form

$$F(t, z) = \sum_{n\in\mathbb{N}_0} \langle f, f_n\rangle_{L^2_\rho(\mathbb{C})} f_n(z)e^{-2int}. \tag{3.24}$$

By using $\mathcal{B}^*$ and $\mathcal{B}$, the corresponding solution to the Cauchy problem (3.20) can be written in the series form:

$$\Phi(t, y) = \sum_{n\in\mathbb{N}_0} \langle \varphi, u_n\rangle_{L^2_\rho(\mathbb{C})} u_n(y)e^{-2int}, \tag{3.25}$$

which is the decomposition of $\Phi \in L^2(\mathbb{R})$ over the Hermite–Gauss functions $\{u_n\}_{n\in\mathbb{N}_0}$.

Lemma 3.8 *The unique solution to the Cauchy problem (3.20) can be expressed in the integral form:*

$$\Phi(t, y) = \int_{\mathbb{R}} K_t(y, y')\varphi(y')dy', \tag{3.26}$$

where

$$K_t(y, y') := \frac{1}{\sqrt{\pi(1 - e^{-4it})}} e^{-\frac{1}{2}y^2 + \frac{1}{2}(y')^2 - \frac{(y' - ye^{-2it})^2}{1-e^{-4it}}}. \tag{3.27}$$

Proof. By using the series form (3.24), the unique solution to the Cauchy problem (3.21) can be rewritten in the integral form: in the Green's function form:

$$F(t, z) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} G_t(z, z')f(z')e^{-\frac{1}{2}|z'|^2}dx'd\xi', \tag{3.28}$$

where

$$G_t(z,z') := \sum_{n\in\mathbb{N}_0} f_n(z)\overline{f_n(z')}e^{-2nit} = e^{\frac{1}{2}z\bar{z}'e^{-2it}} = \overline{E_{ze^{-2it}}(z')}. \tag{3.29}$$

As follows from Example 2.2 with $a = \frac{1}{2}\bar{z}'e^{-2it}$, $b = 0$, and $\alpha = \frac{1}{2}$, we have

$$\mathcal{B}^*\left(e^{\frac{1}{2}z\bar{z}'e^{-2it}}\right) = \frac{1}{\pi^{\frac{1}{4}}}e^{-\frac{1}{4}(\bar{z}')^2e^{-4it}+\bar{z}'e^{-2it}-\frac{1}{2}y^2}.$$

By using $\mathcal{B}^*$ and $\mathcal{B}$, the corresponding solution to the Cauchy problem (3.20) can be rewritten in the integral form (3.26) with

$$K_t(y,y') := \frac{1}{2\pi^{\frac{3}{2}}}\iint_{\mathbb{R}^2} e^{-\frac{1}{2}(y-\bar{z}e^{-2it})^2-\frac{1}{2}(y'-z)^2+\frac{1}{4}\bar{z}^2e^{-4it}+\frac{1}{4}z^2-\frac{1}{2}|z|^2}dxd\xi, \tag{3.30}$$

where we replaced z' by z. The kernel $K_t(y,y')$ is the π-periodic function in t. In the limits $t\to 0$ and $t\to\pi/2$, we obtain

$$\lim_{t\to 0} K_t(y,y') = \frac{1}{2\pi^{\frac{3}{2}}}\iint_{\mathbb{R}^2} e^{-\frac{1}{2}(y^2+(y')^2)+(y+y')x+i(y-y')\xi-x^2}dxd\xi = \delta(y-y')$$

and

$$\lim_{t\to \frac{\pi}{2}} K_t(y,y') = \frac{1}{2\pi^{\frac{3}{2}}}\iint_{\mathbb{R}^2} e^{-\frac{1}{2}(y^2+(y')^2)+(y'-y)x-i(y+y')\xi-x^2}dxd\xi = \delta(y+y'),$$

where the property (2.8) is used for Dirac δ distribution. By factorizing the argument of the exponential function in $K_t(y,y')$ as

$$\begin{aligned}&\frac{1}{2}(y-\bar{z}e^{-2it})^2+\frac{1}{2}(y'-z)^2-\frac{1}{4}\bar{z}^2e^{-4it}-\frac{1}{4}z^2+\frac{1}{2}|z|^2\\&=\frac{1}{4}(1-e^{-4it})\left[\xi-ix+2i\frac{y'-ye^{-2it}}{1-e^{-4it}}\right]^2+(x-y')^2+\frac{1}{2}(y^2-(y')^2)+\frac{(y'-ye^{-2it})^2}{1-e^{-4it}}\end{aligned}$$

and performing integrations first in ξ and then in x, we obtain a closed-form expression (3.27). ■

Next, we investigate *the main question of this thesis* on the evolution of the envelope to the Gaussian function. If $\Phi(t,y) = H(t,y)e^{-\frac{1}{2}y^2}$ and $\varphi(y) = h(y)e^{-\frac{1}{2}y^2}$, then the envelope function $H(t,y)$ is a solution to the Cauchy problem:

$$\begin{cases} i\dot{H} = -\partial_y^2 H + 2y\partial_y H,\\ H|_{t=0} = h.\end{cases} \tag{3.31}$$

We ask the following: *If $h\in L^\infty(\mathbb{R})$, does $H(t,\cdot)$ remain in $L^\infty(\mathbb{R})$ for $t>0$?* The following lemma gives a negative answer to this question.

Lemma 3.9 *There exists $h \in L^\infty(\mathbb{R})$ and $t_0 > 0$ such that the unique solution to the Cauchy problem (3.31) satisfies $H(t,\cdot) \notin L^\infty(\mathbb{R})$ for every $t \in (0, t_0)$.*

Proof. Instead of solving (3.31) directly, we use the transformation

$$H(t,y) = e^{\frac{1}{2}y^2}\mathcal{B}^* F(t,\cdot),$$

where $F(t,z)$ is the unique solution to the Cauchy problem (3.21) given by Lemma 3.7. It suffices to construct one example of such h for which the assertion is true.

Let us consider $f(z) = \sin(z)$, then the unique solution to the Cauchy problem (3.21) is given by (3.22) rewritten in the explicit form $F(t,z) = \sin(ze^{-2it})$. By using the result of Example 2.2 with $a = \pm ie^{-2it}$, $b = 0$, and $\alpha = \frac{1}{2}$ as well as the linear superposition principle, we obtain

$$H(t,y) = \frac{1}{\pi^{\frac{1}{4}}} e^{\cos(4t) - i\sin(4t)} \left[\sin(2y\cos(2t))\cosh(2y\sin(2t)) - i\cos(2y\cos(2t))\sinh(2y\sin(2t))\right].$$

We confirm that $h \in L^\infty(\mathbb{R})$ by computing

$$h(y) = H(0,y) = \frac{1}{\pi^{\frac{1}{4}}} e^1 \sin(2y).$$

However, we have

$$|H(t,y)|^2 = \frac{1}{2\pi^{\frac{1}{2}}} e^{2\cos(4t)} \left[\cosh(4y\sin(2t)) - \cos(4y\cos(2t))\right],$$

hence $H(t,y)$ is unbounded as $|y| \to \infty$ for every $t \in (0, \frac{\pi}{2})$. ∎

Lemma 3.10 *The unique solution to the Cauchy problem (3.31) can be obtained in the integral form:*

$$H(t,y) = \frac{1}{\sqrt{\pi(1-e^{-4it})}} \int_{\mathbb{R}} e^{-\frac{(y'-ye^{-2it})^2}{1-e^{-4it}}} h(y') dy'. \tag{3.32}$$

Proof. The integral form follows from (3.26) and (3.27). Alternatively, the same solution can be derived by reducing the evolution equation

$$i\frac{\partial H}{\partial t} = -\frac{\partial^2 H}{\partial y^2} + 2y\frac{\partial H}{\partial y}$$

by using the similarity reduction. Indeed, let us define $H(t,y) = \tilde{H}(\tau = \alpha(t), \eta = \beta(t)y)$ for some $\alpha(t)$ and $\beta(t)$. If we pick them from solutions to differential equations

$$\alpha'(t) = \beta^2(t), \quad i\beta'(t) = 2\beta(t), \tag{3.33}$$

then $\tilde{H}(\tau,\eta)$ satisfies the linear Schrödinger equation $i\tilde{H}_\tau = -\tilde{H}_{\eta\eta}$ with the exact solution

$$\tilde{H}(\tau,\eta) = \frac{1}{\sqrt{4\pi i\tau}} \int_{\mathbb{R}} \tilde{H}(0,\eta')e^{-\frac{(\eta'-\eta)^2}{4i\tau}} d\eta'. \tag{3.34}$$

Solving differential equations (3.33) by

$$\alpha(t) = \alpha_0 - \frac{1}{4i}\beta_0^2 e^{-4it}, \quad \beta(t) = \beta_0 e^{-2it} \tag{3.35}$$

and picking integration constants $\alpha_0 = \frac{1}{4i}\beta_0^2$ and $\beta_0 = 1$ from the initial conditions $\alpha(0) = 0$ and $\beta(0) = 1$, we verify that the fundamental solution (3.34) recovers (3.32) with $H(t,y) = \tilde{H}(\tau = \alpha(t), \eta = \beta(t)y)$. ■

Example 3.1 Let $h(y) = e^{-\frac{1}{2}a^2y^2}$ with $a > 0$. The convolution integral in (3.32) can be evaluated after the solution is rewritten in the explicit form:

$$\begin{aligned} H(t,y) &= \frac{e^{it}}{\sqrt{2\pi i \sin(2t)}} \int_{\mathbb{R}} e^{-\frac{1}{2}a^2(y')^2 - \frac{1}{2i\sin(2t)}(e^{it}y' - e^{-it}y)^2} dy' \\ &= \frac{1}{\sqrt{1 + ia^2\sin(2t)e^{-2it}}} e^{-\frac{1}{2}\frac{a^2y^2e^{-4it}}{1+ia^2\sin(2t)e^{-2it}}}. \end{aligned} \tag{3.36}$$

It is clear that $H(t,y)$ becomes unbounded in a finite time. Indeed,

$$H\left(\frac{\pi}{4}, y\right) = \frac{1}{\sqrt{1+a^2}} e^{\frac{1}{2}\frac{a^2y^2}{1+a^2}}$$

is unbounded and so is $H(t,y)$ for $t \in (t_0, \frac{\pi}{2} - t_0)$, where $t_0 \in (0, \frac{\pi}{8})$ is given by the root of

$$\mathrm{Re}\frac{e^{-4it}}{1 + ia^2\sin(2t)e^{-2it}} = 0,$$

or equivalently, by the root of $\cos(4t_0) = a^2/(2+a^2)$.

Remark 3.9 The exact solution in Example 3.1 is related to the reduction of the time-dependent Schrödinger equation $i\dot{\Phi} = L\Phi$ for the Gaussian solutions

$$\Phi(t,y) = A(t)e^{-\frac{1}{2}B(t)y^2}. \tag{3.37}$$

to a system of differential equations for $A(t)$ and $B(t)$. Indeed, direct substitution yields

$$i\dot{A} = A(B-1), \quad i\dot{B} = 2(B^2 - 1),$$

which have exact solutions

$$A(t) = \frac{1}{\sqrt{1 + ia^2 \sin(2t) e^{-2it}}}, \qquad B(t) = 1 + \frac{a^2 e^{-4it}}{1 + ia^2 \sin(2t) e^{-2it}}. \tag{3.38}$$

Substituting (3.38) to (3.37) and extracting $H(t,y) = \Phi(t,y) e^{\frac{1}{2}y^2}$ yields the exact formula (3.36). It is easy to verify that $\mathrm{Re}B(t) > 0$ for all $t \in \mathbb{R}$, which implies that $\Phi(t,\cdot) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$.

Example 3.2 Let $h(y) = \frac{\sin(\pi y)}{\pi y}$. By using the Fourier transform

$$\hat{h}(\xi) = \int_{\mathbb{R}} h(y) e^{iy\xi} dy = 1_{[-\pi,\pi]}$$

we can rewrite the convolution integral in (3.32) in the following equivalent form:

$$\begin{aligned} H(t,y) &= \frac{1}{2\pi\sqrt{\pi(1-e^{-4it})}} \int_{\mathbb{R}} \int_{-\pi}^{\pi} e^{-\frac{(y'-ye^{-2it})^2}{1-e^{-4it}}} e^{-i\xi y'} d\xi dy' \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\xi y e^{-2it} - \frac{1}{4}\xi^2(1-e^{-4it})} d\xi. \end{aligned}$$

For every $t \in (0, \frac{\pi}{2})$, the solution can be rewritten in the form

$$H(t,y) = \frac{1}{2\pi y \sin(2t)} \int_{-\pi y \sin(2t)}^{\pi y \sin(2t)} e^{-iz\cot(2t) - z - \frac{iz^2}{2y^2}\cot(2t) - \frac{z^2}{2y^2}} dz,$$

from which we prove hereafter that $|H(t,y)| \to \infty$ as $|y| \to \infty$. For definiteness, we work for $t \in (0, \frac{\pi}{4})$, a similar result for $t \in (\frac{\pi}{4}, \frac{\pi}{2})$ can be obtained by symmetry, whereas the estimates for $t = \frac{\pi}{4}$ can be obtained directly from

$$H\left(\frac{\pi}{4}, y\right) = \frac{1}{2\pi y} \int_{-\pi y}^{\pi y} e^{-z - \frac{z^2}{2y^2}} dz.$$

By using the transformation

$$u = z + z^2/(2y^2), \quad z = -y^2 + y^2\sqrt{1 + 2u/y^2},$$

we write

$$\mathrm{Im}H(t,y) = -\frac{1}{2\pi y \sin(2t)} \int_{-f(t,y)}^{f(t,y)} \frac{e^{-u}\sin[u\cot(2t)]}{\sqrt{1+2u/y^2}} du,$$

where

$$f(t,y) := \pi y \sin(2t) \left[1 - \pi \frac{\sin(2t)}{2y}\right]$$

and $t \in (0, \frac{\pi}{4})$ is fixed. Define a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that

$$y_n \left[1 - \frac{\pi \sin(2t)}{2y_n}\right] = \frac{2n}{\cos(2t)}, \quad n \in \mathbb{N}.$$

We have $y_n \to +\infty$ as $n \to \infty$. Fix $N \in \mathbb{N}$ such that for every $n \geq N$, we have the following estimate:

$$e^{\pi \tan(2t)} > \left(1 - \frac{2\pi(2n+1)}{\cot(2t)y_n^2}.\right)^{-1/2}.$$

Let us use the following anti-derivative:

$$\int_\alpha^\beta e^{-u}\sin(au)du = \frac{1}{1+a^2}\left[e^{-\alpha}\sin(a\alpha) - e^{-\beta}\sin(a\beta) + ae^{-\alpha}\cos(a\alpha) - ae^{-\beta}\cos(a\beta)\right].$$

Since $\sin[u\cot(2t)] \geq 0$ for $u \in [-\pi(2n+2)\tan(2t), -\pi(2n+1)\tan(2t)]$ and $\sin[u\cot(2t)] \leq 0$ for $u \in [-\pi(2n+1)\tan(2t), -\pi(2n)\tan(2t)]$ for $t \in (0, \frac{\pi}{4})$, we obtain the following estimate:

$$\begin{aligned}
&\int_{-f(t,y_{n+1})}^{-f(t,y_n)} \frac{e^{-u}\sin[u\cot(2t)]}{\sqrt{1+2u/y_n^2}}du \\
\geq &\int_{-\pi(2n+2)\tan(2t)}^{-\pi(2n+1)\tan(2t)} e^{-u}\sin[u\cot(2t)]du \\
&+ \frac{1}{\sqrt{1-\frac{2\pi(2n+1)}{\cot(2t)y_n^2}}}\int_{-\pi(2n+1)\tan(2t)}^{-\pi(2n)\tan(2t)} e^{-u}\sin[u\cot(2t)]du \\
\geq &\sin(2t)\cos(2t)(e^{\pi\tan(2t)}+1)e^{2\pi n\tan(2t)}\left[e^{\pi\tan(2t)} - \left(1 - \frac{2\pi(2n+1)}{\cot(2t)y_n^2}.\right)^{-1/2}\right].
\end{aligned}$$

The lower bound is posivite and diverges exponentially fast as $n \to \infty$. Since

$$\left|\int_{-f(t,y_N)}^{\infty} \frac{e^{-u}\sin[u\cot(2t)]}{\sqrt{1+2u/y^2}}du\right| < \infty,$$

for the fixed $N \in \mathbb{N}$ and the previous lower bound is positive and diverges exponentially fast, we confirm that $|\mathrm{Im}H(t,y_n)| \to \infty$ as $n \to \infty$.

3.4 Evolution of the linear diffusion equation

Here we consider the linear diffusion equation

$$\frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial y^2} + (1-y^2)\Phi. \tag{3.39}$$

Bargmann transform with $F(t,\cdot) = \mathcal{B}\Phi(t,\cdot)$ maps it to the following transport equation

$$\frac{\partial F}{\partial t} + 2z\frac{\partial F}{\partial z} = 0. \tag{3.40}$$

With a simple modification of computations of Section 3.3 (or by formal change of $t \mapsto -it$), the Cauchy problem associated with the transport equation (3.40) is solved in the form

$$F(t,z) = f(ze^{-2t}). \tag{3.41}$$

Similarly, the Cauchy problem associated with the diffusion equation (3.39) is solved in the Green's function form (3.26) with the kernel $K_t(y,y')$ in the form

$$K_t(y,y') := \frac{1}{\sqrt{\pi(1-e^{-4t})}} e^{-\frac{1}{2}y^2 + \frac{1}{2}(y')^2 - \frac{(y'-ye^{-2t})^2}{1-e^{-4t}}}. \tag{3.42}$$

If $\Phi(t,y) = H(t,y)e^{-\frac{1}{2}y^2}$, then the envelope of the Gaussian function $H(t,y)$ satisfies the linear diffusion equation in self-similar variables:

$$\frac{\partial H}{\partial t} = \frac{\partial^2 H}{\partial y^2} - 2y\frac{\partial H}{\partial y}. \tag{3.43}$$

It follows from (3.42) that the unique solution of the Cauchy problem associated with the diffusion equation (3.43) is given in the integral form:

$$H(t,y) = \frac{1}{\sqrt{\pi(1-e^{-4t})}} \int_{\mathbb{R}} e^{-\frac{(y'-ye^{-2t})^2}{1-e^{-4t}}} h(y')dy'. \tag{3.44}$$

This solution can also be found by using the similarity reduction

$$H(t,y) = \tilde{H}(\tau = \alpha(t), \eta = \beta(t)y),$$

where $\alpha'(t) = \beta^2(t)$, $\beta'(t) = -2\beta(t)$, and $\tilde{H}_\tau = \tilde{H}_{\eta\eta}$.

Referring to the main question on the evolution of the envelope of the Gaussian function, the following lemma gives a positive answer to the question.

Lemma 3.11 *For every $h \in L^\infty(\mathbb{R})$, the unique solution to the diffusion equation (3.43) with $H(0,y) = h(y)$ satisfies $H(t,\cdot) \in L^\infty(\mathbb{R})$ for every $t \in \mathbb{R}_+$.*

Proof. By the change of variables, we can rewrite the exact solution (3.44) as the convolution integral

$$H(t,y) = \int_{\mathbb{R}} G_t(y-y')h(y'e^{-2t})dy', \quad G_t(y) := \frac{1}{\sqrt{\pi(e^{4t}-1)}} e^{-\frac{y^2}{e^{4t}-1}},$$

hence we obtain by standard estimates that

$$\|H(t,\cdot)\|_{L^\infty} \leq \|G_t\|_{L^1} \|h(\cdot e^{-2t})\|_{L^\infty} \leq \|h\|_{L^\infty}. \tag{3.45}$$

Hence, if $h \in L^\infty(\mathbb{R})$, then $H(t,\cdot) \in L^\infty(\mathbb{R})$ for every $t > 0$. ∎

Remark 3.10 The bound (3.45) is equivalent to the weak maximum principle for the diffusion equation (3.43).

Remark 3.11 More general convolution estimates give for $1 \leq q \leq \infty$:

$$\|G_t\|_{L^q} = \frac{1}{(q\pi^{q-1})^{\frac{1}{2q}}} \frac{e^{-2(1-\frac{1}{q})t}}{(1-e^{-4t})^{\frac{q-1}{2q}}}$$

and for $1 \leq p, q \leq \infty$:

$$\|H(t,\cdot)\|_{L^p} = \frac{1}{(q\pi^{q-1})^{\frac{1}{2q}}} \frac{e^{\frac{2}{p}t}}{(1-e^{-4t})^{\frac{q-1}{2q}}} \|h\|_{L^r},$$

where $r = \frac{1}{1+p^{-1}-q^{-1}}$. No singularity appears in the estimates at $t = 0$ if $r = p$ (and $q = 1$) but the upper bound grows exponentially as $t \to \infty$ if $1 \leq p < \infty$. On the other hand, there is no exponential growth at infinity if $p = \infty$ (and $r = \frac{q}{q-1}$) but the upper bound is singular at $t = 0$ if $1 \leq r < \infty$ (and $1 < q \leq \infty$).

Example 3.3 Let $h(y) = e^{-a^2y^2}$ with $a > 0$. The convolution integral in (3.44) can be evaluated explicitly:

$$\begin{aligned} H(t,y) &= \frac{1}{\sqrt{\pi(1-e^{-4t})}} \int_{\mathbb{R}} e^{-a^2(y')^2 - \frac{(y'-ye^{-2t})^2}{1-e^{-4t}}} dy'. \\ &= \frac{1}{\sqrt{1+a^2(1-e^{-4t})}} e^{-\frac{a^2y^2e^{-4t}}{1+a^2(1-e^{-4t})}}. \end{aligned} \tag{3.46}$$

It is clear that $H(t,y)$ is bounded for every $t > 0$ and although it is decaying to zero at infinity for $t > 0$, it becomes delocalized as $t \to +\infty$:

$$\lim_{t\to+\infty} H(t,y) = \frac{1}{\sqrt{1+a^2}}, \quad y \in \mathbb{R}.$$

Remark 3.12 The exact solution in the previous example is related to the reduction of the linear diffusion equation $\dot{\Phi} + L\Phi = 0$ for the Gaussian solutions

$$\Phi(t,y) = A(t)e^{-\frac{1}{2}B(t)y^2}. \tag{3.47}$$

to the following system of differential equations for $A(t)$ and $B(t)$:

$$\dot{A} = A(1-B), \quad \dot{B} = 2(1-B^2).$$

This system admits the exact solutions

$$A(t) = \frac{1}{\sqrt{1+a^2(1-e^{-4t})}}, \qquad B(t) = 1 + \frac{2a^2 e^{-4t}}{1+a^2(1-e^{-4t})}, \tag{3.48}$$

which can be verified by a substitution. Substituting (3.48) to (3.47) and extracting $H(t,y) = \Phi(t,y)e^{\frac{1}{2}y^2}$ yields the exact formula (3.46). Since $B(t) \geq 1$, we have $\Phi(t,\cdot) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$ including the limit $t \to +\infty$.

We give two examples on how the linear diffusion equation (3.43) arises in the context of the nonlinear diffusion equations.

The first example is the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + |u|^{p-1}u, \tag{3.49}$$

where $p > 1$. The space-independent solutions of the differential equation

$$\dot{x} = |x|^{p-1}x$$

blows up in a finite time if $x(0) > 0$. The self-similar blow-up solutions to the nonlinear diffusion equation (3.49) were studied in [22, 23] (see also recent work [33]). By using the substitution

$$u(t,x) = (T-t)^{-\frac{1}{p-1}}w(s,y), \quad y = \frac{x}{2\sqrt{T-t}}, \quad s = -\frac{1}{4}\log(T-t), \tag{3.50}$$

for a fixed blow-up time $T > 0$, we obtain the nonlinear diffusion equation in self-similar variables

$$\frac{\partial w}{\partial s} = \frac{\partial^2 w}{\partial y^2} - 2y\frac{\partial w}{\partial y} + 4\left(|w|^{p-1} - \frac{1}{p-1}\right)w. \tag{3.51}$$

Linearizing at the constant solution $w_0 := (p-1)^{-\frac{1}{p-1}}$ and using the exponential transformation

$$w(s,y) = w_0 + H(s,y)e^{4s}$$

yields the linear diffusion equation (3.43) with $t = s$. Hence, the result of Lemma 3.11 is relevant for analysis of the linearization of the nonlinear diffusion equation (3.51).

The second example is the logarithmic diffusion (log-diffusion) equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u\log(u). \tag{3.52}$$

This nonlinear equation was investigated in many details by Alfaro & Carles [2].

Compared to the nonlinear diffusion equation (3.49), unique positive solutions with bounded continuous derivatives up to the second order exist globally and no blow-up may occur in a finite time [2]. This is already clear from the space-independent solutions of the differential equation

$$\dot{x} = 2x\log(x)$$

with the exact solution

$$x(t) = (x_0)^{e^{2t}}, \quad t > 0$$

for $x_0 > 0$. If $x_0 \in (0,1)$, the solution decays to zero super-exponentially fast: $x(t) \to 0$ as $t \to +\infty$. If $x_0 \in (1,\infty)$, the solution blows up to infinity super-exponentially fast: $x(t) \to +\infty$ as $t \to +\infty$. Both decay and growth occur in the infinite time intervals.

The authors of [2] has shown that the same conclusion remains true if the initial condition $u_0(x)$ to the log-diffusion equation (3.52) satisfies either $u_0(x) \leq 1-\epsilon$ (decay) or $u_0(x) \geq 1+\epsilon$ (blow-up) for $\epsilon > 0$ and all $x \in \mathbb{R}$. They also obtained a similar conclusion for the Gaussian solutions

$$u(t,x) = B(t)e^{-\frac{1}{2}A(t)x^2}, \tag{3.53}$$

where $A(t)$ and $B(t)$ satisfy the following system of differential equations:

$$\dot{A} = 2A(1-A) \quad \dot{B} = 2B\log(B) - AB.$$

The system admits the exact solution

$$A(t) = \frac{a_0 e^{2t}}{1 + a_0(e^{2t}-1)} \tag{3.54}$$

and

$$\log B(t) = e^{2t}\left[\log b_0 + \frac{a_0}{2(1-a_0)}\log(a_0 + (1-a_0)e^{-2t})\right], \tag{3.55}$$

where $A(0) = a_0$ and $B(0) = b_0$. If $\log b_0 + \frac{a_0\log(a_0)}{2(1-a_0)} < 0$, the solution decays super-exponentially fast as $t \to \infty$, whereas if $\log b_0 + \frac{a_0\log(a_0)}{2(1-a_0)} > 0$, the solution grows super-exponentially fast as $t \to \infty$.

By using the substitution

$$u(t,x) = \gamma(t)w(y,s), \quad y = \frac{x}{2\sqrt{T-t}}, \quad s = -\frac{1}{4}\log(T-t), \tag{3.56}$$

with $\gamma(t)$ satisfying the differential equation $\dot{\gamma} = 2\gamma \log \gamma$, for a fixed "blow-up" time $T > 0$, we obtain the log–diffusion equation in self-similar variables

$$\frac{\partial w}{\partial s} = \frac{\partial^2 w}{\partial y^2} - 2y\frac{\partial w}{\partial y} + 2e^{-4s} w \log(w), \tag{3.57}$$

which is non-autonomous compared to the nonlinear diffusion equation in self-similar variables given by (3.51).

The following lemma describes bounded solutions of the log-diffusion equation (3.57).

Lemma 3.12 *For every $\epsilon > 0$ (small), there exists $\delta > 0$ such that if $w_0 \in L^\infty(\mathbb{R})$ satisfies $\|w_0 - 1\|_{L^\infty} \leq \delta$, then the unique solution of the log–difusion equation (3.57) with the initial condition $w(0, y) = w_0(y)$ satisfies $W(s, \cdot) \in L^\infty(\mathbb{R})$ for $s \in \mathbb{R}_+$ and*

$$\|W(s, \cdot) - 1\|_{L^\infty} \leq \epsilon, \quad s > 0. \tag{3.58}$$

Proof. Let $N(w, s) := 2e^{-4s} w \log(w)$. The variation of parameter formula that extends the solution given by (3.44) is given by

$$\begin{aligned} w(s, y) &= \frac{1}{\sqrt{\pi(1 - e^{-4s})}} \int_{\mathbb{R}} e^{-\frac{(y' - ye^{-2s})^2}{1 - e^{-4s}}} w_0(y') dy' \\ &+ \int_0^s \frac{ds'}{\sqrt{\pi(e^{-4s'} - e^{-4s})}} \int_{\mathbb{R}} e^{-\frac{(y' - ye^{-2s})^2}{e^{-4s'} - e^{-4s}}} N(w(s', y'), s') dy', \end{aligned}$$

where $w_0(y) := w(0, y)$ is the initial condition. The previous formula can be rewritten as a weighted convolution in the form:

$$\begin{aligned} w(s, y) &= \frac{1}{\sqrt{\pi(e^{4s} - 1)}} \int_{\mathbb{R}} e^{-\frac{(y' - y)^2}{e^{4s} - 1}} w_0(y' e^{-2s}) dy' \\ &+ \int_0^s \frac{ds'}{\sqrt{\pi(e^{4(s - s')} - 1)}} \int_{\mathbb{R}} e^{-\frac{(y' - y)^2}{e^{4(s - s')} - 1}} N(w(s', y' e^{2s}), s') dy'. \end{aligned}$$

If $w = 1 + W$ with $\|W(s, \cdot)\|_{L^\infty} \leq \epsilon \ll 1$ for all $s > 0$, then the nonlinearity is expanded in powers as

$$N(1 + W, s) = 2e^{-4s} \left[W + \frac{1}{2} W^2 + \mathcal{O}(W^3) \right],$$

with the obvious bound as long as $\|W(s, \cdot)\|_{L^\infty} \leq \epsilon \ll 1$:

$$\|N(1 + W(s, \cdot), s)\|_{L^\infty} \leq 4e^{-4s} \|W(s, \cdot)\|_{L^\infty}. \tag{3.59}$$

As a result, the bound on $\|W(s,\dot{)}\|_{L^\infty}$ can be controlled globally over $s \geq 0$. Indeed, it follows from the variation of parameter formula with the same convolution estimate as in the proof of Lemma 3.11 that

$$\|W(s,\cdot)\|_{L^\infty} \leq \|W_0\|_{L^\infty} + 4\int_0^s e^{-4s'}\|W(s',\cdot)\|_{L^\infty} ds', \tag{3.60}$$

where the bound (3.59) has been used. By Gronwall's inequality, this yields

$$\|W(s,\cdot)\|_{L^\infty} \leq \|W_0\|_{L^\infty} e^{4\int_0^s e^{-4s'}ds'} \leq \|W_0\|_{L^\infty} e^1 \leq \epsilon, \tag{3.61}$$

provided that $\|W_0\|_{L^\infty} \leq \epsilon e^{-1} =: \delta$. Hence, the bound (3.61) yields (3.58). ∎

Remark 3.13 The global solution obtained in Lemma 3.12 gives

$$1 - \epsilon \leq \|w(s,\cdot)\|_{L^\infty} \leq 1 + \epsilon,$$

for every $s > 0$. However, this only corresponds to the finite interval $t \in [0, T]$ due to the self-similar reduction (3.56). Therefore, the result of Lemma 3.12 is weaker than the result of Corollary 3.10 in [2].

Remark 3.14 Consider a modified transformation in self-similar variables given by

$$u(t,x) = \gamma(t) w(y,s), \quad y = \frac{x}{2\sqrt{T+t}}, \quad s = \frac{1}{4}\log(T+t), \tag{3.62}$$

with $\gamma(t)$ satisfying the same differential equation $\dot{\gamma} = 2\gamma \log \gamma$. Here $T > 0$ is the parameter and the solution (3.62) is formally defined for all $t > 0$. The function $w(y,s)$ satisfies the log–diffusion equation in self-similar variables

$$\frac{\partial w}{\partial s} = \frac{\partial^2 w}{\partial y^2} + 2y\frac{\partial w}{\partial y} + 2e^{4s} w \log(w), \tag{3.63}$$

which has exponentially growing term compared to (3.57). The solution can be written as a weighted convolution in the form:

$$\begin{aligned} w(s,y) \;=\; & \frac{1}{\sqrt{\pi(1-e^{-4s})}} \int_{\mathbb{R}} e^{-\frac{(y'-y)^2}{1-e^{-4s}}} w_0(y' e^{2s}) dy' \\ & + \int_0^s \frac{ds'}{\sqrt{\pi(1-e^{-4(s-s')})}} \int_{\mathbb{R}} e^{-\frac{(y'-y)^2}{1-e^{-4(s-s')}}} N(w(s', y' e^{2s}), s') dy', \end{aligned}$$

where $N(w,s) := 2e^{4s} w \log(w)$ is the nonlinear term and $w_0(y) := w(0,y)$ is the initial condition. Expansion near $w = 1$ with the estimate like in (3.59) leads to the linear integral inequality with exponentially growing kernel compared to the inequality (3.60) with the exponentially decaying kernel. As a result, the global bound like (3.61) cannot be obtained in this case and $\|W(s,\cdot)\|_{L^\infty}$ is expected to grow in time. With the decaying $\gamma(t) = \gamma_0^{e^{2t}}$ double-exponentially if $\gamma_0 \in (0,1)$ and growing $\|W(s,\cdot)\|_{L^\infty}$, the decomposition $w(s,y) = 1 + W(s,y)$ is no longer useful.

3.5 Evolution of the linear KdV equation

Here we consider the Cauchy problem for the linear Korteweg–de Vries (KdV) equation:

$$\begin{cases} \dot{\Phi} = \partial_y L\Phi, \\ \Phi|_{t=0} = \varphi \in L^2(\mathbb{R}). \end{cases} \tag{3.64}$$

In order to reformulate the Cauchy problem (3.64) in the Fock space $\mathcal{F}$, we need to see how the Bargmann transform $\mathcal{B}$ is applied to the derivative ∂_y and to the multiplication by y. The following lemma gives the transformation formula.

Lemma 3.13 *Fix $f \in \mathcal{F}$. Then,*

$$(\mathcal{B}\partial_y\mathcal{B}^* f)(z) = f'(z) - \frac{1}{2}zf(z) \tag{3.65}$$

and

$$(\mathcal{B}y\mathcal{B}^* f)(z) = f'(z) + \frac{1}{2}zf(z). \tag{3.66}$$

Proof. It follows from integration by parts for every $\varphi \in H^1(\mathbb{R})$ that

$$\begin{aligned} (\mathcal{B}\partial_y\varphi)(z) &= \frac{1}{\pi^{\frac{1}{4}}}e^{\frac{1}{4}z^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-y)^2}\varphi'(y)dy \\ &= \frac{1}{\pi^{\frac{1}{4}}}e^{\frac{1}{4}z^2}\int_{-\infty}^{\infty} (y-z)e^{-\frac{1}{2}(z-y)^2}\varphi(y)dy \\ &= \frac{1}{\pi^{\frac{1}{4}}}e^{\frac{1}{4}z^2}\frac{d}{dz}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-y)^2}\varphi(y)dy \\ &= \frac{d}{dz}(\mathcal{B}\varphi)(z) - \frac{1}{2}z(\mathcal{B}\varphi)(z), \end{aligned}$$

which yields the formula (3.65) for $f = \mathcal{B}\varphi$ with $\varphi = \mathcal{B}^* f$. Similarly, for every $\varphi \in L^2(\mathbb{R})$ with $y\varphi \in L^2(\mathbb{R})$, we obtain

$$\begin{aligned} (\mathcal{B}y\varphi)(z) &= \frac{1}{\pi^{\frac{1}{4}}}e^{\frac{1}{4}z^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-y)^2}y\varphi(y)dy \\ &= \frac{1}{\pi^{\frac{1}{4}}}e^{\frac{1}{4}z^2}\frac{d}{dz}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-y)^2}\varphi(y)dy + z(\mathcal{B}\varphi)(z) \\ &= \frac{d}{dz}(\mathcal{B}\varphi)(z) + \frac{1}{2}z(\mathcal{B}\varphi)(z), \end{aligned}$$

which yields the formula (3.66) for $f = \mathcal{B}\varphi$ with $\varphi = \mathcal{B}^* f$. ∎

Remark 3.15 *Transformation (3.17) follows from (3.65) and (3.66) because*

$$L = -\partial_y^2 + y^2 - 1 = (-\partial_y + y)(\partial_y + y)$$

and

$$(\mathcal{B}(\partial_y + y)\mathcal{B}^* f)(z) = 2\frac{df}{dz}, \qquad (\mathcal{B}(-\partial_y + y)\mathcal{B}^* f)(z) = zf(z).$$

By using (3.17) and (3.65), we rewrite the Cauchy problem (3.64) in the equivalent form:

$$\begin{cases} \dot{F} = 2\partial_z z \partial_z F - z^2 \partial_z F, \\ F|_{t=0} = f \in \mathcal{F}, \end{cases} \tag{3.67}$$

where $f = \mathcal{B}\varphi$ and $F(t, \cdot) = \mathcal{B}\Phi(t, \cdot)$. Compared to (3.23), the evolution equation is now generated by the second-order differential operator. Using the following decomposition over the monomials in (3.4)

$$F(t, z) = \sum_{n=0}^{\infty} b_n(t) f_n(z) = \sum_{n=0}^{\infty} a_n(t) z^n, \tag{3.68}$$

one can derive the system of differential equations:

$$\frac{da_n}{dt} = 2(n+1)^2 a_{n+1} - (n-1)a_{n-1}, \tag{3.69}$$

or equivalently,

$$\frac{db_n}{dt} = \sqrt{2}\left[(n+1)\sqrt{n+1}b_{n+1} - (n-1)\sqrt{n}b_{n-1}\right]. \tag{3.70}$$

System (3.69) is closed for $\mathfrak{a} := \{a_n\}_{n\in\mathbb{N}}$, whereas $a_0(t)$ is obtained from the decoupled equation $a_0'(t) = 2a_1(t)$. By using the substitution

$$b_n(t) = \frac{i^n c_n(t)}{\sqrt{n}}, \qquad n \in \mathbb{N} \tag{3.71}$$

one can rewrite system (3.70) in the form

$$\frac{dc_n}{dt} = \sqrt{2}i(J\mathfrak{c})_n, \quad (J\mathfrak{c})_n := (n+1)\sqrt{n}c_{n+1} + n\sqrt{n-1}c_{n-1}, \tag{3.72}$$

where $\mathfrak{c} := \{c_n\}_{n\in\mathbb{N}}$ and J is the Jacobi operator.

The Jacobi operator is said to have a limit circle at infinity if a solution $\mathfrak{c}$ of $J\mathfrak{c} = z\mathfrak{c}$ with $c_1 = 1$ belongs to $\ell^2(\mathbb{N})$ for some $z \in \mathbb{C}$ [42]. This property is justified J in (3.72).

Lemma 3.14 *Consider $J\mathfrak{c} = 0$ with $c_1 = 1$. Then, $\mathfrak{c} \in \ell^2(\mathbb{N})$.*

Proof. The solution $\mathfrak{c}$ of $J\mathfrak{c} = 0$ satisfies the recurrency relation

$$c_{n+1} = \frac{\sqrt{n(n-1)}}{n+1} c_{n-1}, \quad n \in \mathbb{N}, \tag{3.73}$$

from $c_n = 0$ for even n and $c_n \neq 0$ for odd n with the explicit solution given by

$$c_{2m+1} = (-1)^m \prod_{k=1}^{m} \sqrt{2k(2k-1)} 2k+1, \quad m \in \mathbb{N}. \tag{3.74}$$

In order to analyze the decay $|c_{2m+1}| \to 0$ as $m \to \infty$, we rewrite

$$\begin{aligned} |c_{2m+1}| &= \exp\left[\frac{1}{2}\sum_{k=1}^{m} \log(1-\frac{1}{2k}) - \sum_{k=1}^{m}\log(1+\frac{1}{2k})\right] \\ &= \exp\left[-\frac{3}{4}\sum_{k=1}^{m}\frac{1}{k} + \mathcal{O}(\frac{1}{m})\right], \quad \text{as } m \to \infty. \end{aligned}$$

Hence, there is $C > 0$ such that

$$|c_{2m+1}| \leq \frac{C}{m^{3/4}}, \quad m \in \mathbb{N} \tag{3.75}$$

so that $\mathfrak{c} \in \ell^2(\mathbb{N})$. ∎

As a consequence of Lemma 3.14, there exists a self-adjoint extension of the Jacobi operator J subject to a boundary condition at infinity, so that the spectrum of J in $\ell^2(\mathbb{N})$ consists of a countable set of simple real isolated eigenvalues [42]. The existence of the self-adjoint extension of J guarantees local well-posedness of the Cauchy problem (3.67) in a subspace of $\mathcal{F}$.

Lemma 3.15 *Assume that $f \in \mathcal{F}$ is given by the decomposition (3.68) with*

$$\sum_{n=0}^{\infty} n|b_n(0)|^2 < \infty.$$

There exists the unique solution of the Cauchy problem (3.67) with $F(t,\cdot)$ given by the decomposition (3.68) for every $t > 0$ with

$$\sum_{n=0}^{\infty} n|b_n(t)|^2 = \sum_{n=0}^{\infty} n|b_n(0)|^2 < \infty.$$

Proof. Since J has a self-adjoint extension in $\ell^2(\mathbb{N})$ subject to a boundary condition at infinity [42], the evolution operator $e^{\sqrt{2}iJ} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is unitary. Hence, there exists the unique solution to the evolution problem (3.72) in the form $\mathfrak{c}(t) = e^{\sqrt{2}itJ}\mathfrak{c}(\mathfrak{o})$ satisfying for every $t > 0$:

$$\sum_{n=0}^{\infty} n|b_n(t)|^2 = \sum_{n\in\mathbb{N}} |c_n(t)|^2 = \sum_{n\in\mathbb{N}} |c_n(0)|^2 = \sum_{n=0}^{\infty} n|b_n(0)|^2.$$

The unique solution defines the unique solution of the Cauchy problem (3.67) with $F(t,\cdot)$ given by the decomposition (3.68) for every $t > 0$. ■

Remark 3.16 The results of Lemmas 3.13, 3.14, and 3.15 are very similar to the results of [34] for the following Cauchy problem:

$$\begin{cases} \dot{\Phi} = \partial_y(L-2)\Phi, \\ \Phi|_{t=0} = \varphi \in L^2(\mathbb{R}). \end{cases} \tag{3.76}$$

The Cauchy problem (3.76) arises in the linearization of the logarithmic KdV (log-KdV) equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} u \log(u) + \frac{\partial^3 u}{\partial x^3} = 0$$

at the Gaussian solutions

$$u_0(x) = e^{\frac{1}{2}-\frac{1}{2}x^2}.$$

By using the travelling reference frame in the form $\Phi(t,y) \mapsto \Phi(t, y-2t)$, the shifted problem (3.76) can be cast in the form (3.64). This explains why both problems share the same properties.

Let us modify the Cauchy problem (3.64) as follows:

$$\begin{cases} \dot{\Phi} = (\partial_y - y)L\Phi, \\ \Phi|_{t=0} = \varphi \in L^2(\mathbb{R}). \end{cases} \tag{3.77}$$

This toy problem does not arise in modeling of physical processes. Nevertheless, this example shows that the Cauchy problem associated with the linear KdV equation may be ill-posed in a subspace of $L^2(\mathbb{R})$.

The Cauchy problem (3.77) can be easily solved with the Bargmann transform. By Lemmas 3.6 and 3.13, we rewrite the Cauchy problem (3.77) in the equivalent form:

$$\begin{cases} \dot{F} = -2z^2\partial_z F, \\ F|_{t=0} = f \in \mathcal{F}, \end{cases} \tag{3.78}$$

where $f = \mathcal{B}\varphi$ and $F(t,\cdot) = \mathcal{B}\Phi(t,\cdot)$. The following lemma shows that the Cauchy problem (3.78) is ill-posed.

Lemma 3.16 *The only $f \in \mathcal{F}$, for which the Cauchy problem (3.78) gives the unique solution $F(t, \cdot) \in \mathcal{F}$ for $t \neq 0$, is the constant function.*

Proof. The partial differential equation

$$\frac{\partial F}{\partial t} = -2z^2 \frac{\partial F}{\partial z} \tag{3.79}$$

is the transport equation after a transformation of variables $z \to z^{-1}$ and $t \to 2t$. Hence, its general solution is given by

$$F(t, z) := G(z^{-1} + 2t),$$

where the function G is found from the initial condition $G(z^{-1}) = f(z)$. Expressing G yields the unique solution to the Cauchy problem (3.78):

$$F(t, z) = f\left(\frac{z}{1 + 2tz}\right). \tag{3.80}$$

Since f is entire and $\frac{z}{1+2tz}$ has a pole at $z = -\frac{1}{2t}$ for every $t \neq 0$, $F(t, \cdot)$ is entire if and only if $|f(\infty)| < \infty$. However, by Liouville's theorem, the only entire and bounded function is the constant function. ∎

Corollary 3.3 *The Cauchy problem (3.78) is ill-posed in $L^2(\mathbb{R})$.*

Proof. This follows from the conjugacy of the Cauchy problem (3.77) in $L^2(\mathbb{R})$ and the Cauchy problem (3.78) in $\mathcal{F}$ since $\mathcal{B}$ is a unitary transformation. ∎

Remark 3.17 Since

$$(\partial_y - y)L = -\partial_y^3 + y\partial_y^2 + (y^2 - 1)\partial_y + y(3 - y^2),$$

ill-posedness of the Cauchy problem (3.77) in Lemma 3.16 is related to ill-posedness of the diffusion equation with operator $y\partial_y^2$ which defines forward diffusion for $y > 0$ and backward diffusion for $y < 0$. This term is absent in the Cauchy problem (3.64) since

$$\partial_y L = -\partial_y^3 + (y^2 - 1)\partial_y + 2y,$$

for which well-posedness in a subset of $L^2(\mathbb{R})$ follows from Lemma 3.15.

Chapter 4

Applications to partial differential equations in two dimensions

4.1 Introduction

Compared to applications of the Bargmann transform in one dimension, two-dimensional problems in space (x, y) can be formulated by using the complex variable $z = x + iy$. Therefore, holomorphic functions in $\mathcal{F} \subset L^2_\rho(\mathbb{C})$ will be defined on the physical two-dimensional space (x, y). The adjoint Bargmann transform $\mathcal{B}^*$ transforms these functions to a spectral space in $L^2(\mathbb{R})$, which is no longer the physical space.

This formalism is opposite to applications of the Bargmann transform in one dimension where $\mathcal{B}$ transforms functions defined in the physical space $L^2(\mathbb{R})$ to the holomorphic functions in the spectral space $z = x - i\xi$.

Do to these differences, we fix $\alpha = 1$ everywhere in this chapter and use $z = x + iy$. The projection operator Π is rewritten in the form:

$$(\Pi f)(z) = \frac{1}{\pi} \iint_{\mathbb{R}^2} f(z') e^{(z-z')\bar{z}'} dx' dy'. \tag{4.1}$$

The inner product in $L^2_\rho(\mathbb{C})$ (with new definition of $\rho(z) = \frac{1}{\pi} e^{-|z|^2}$) is given by

$$\langle f, g \rangle_{L^2_\rho(\mathbb{C})} = \frac{1}{\pi} \iint_{\mathbb{R}^2} f(z) \bar{g}(\bar{z}) e^{-|z|^2} dx dy. \tag{4.2}$$

Section 4.2 reviews applications of the Bargmann transform to the Gross–Pitaevskii equation for the rotating Bose–Einstein condensates at the lowest Landau level.

4.2 Lowest Landau Level Equation

The lowest Landau level (LLL) equation arises when one considers rotating Bose–Einstein condensates at the critical rotational frequency [5]. The main model is the following Gross–Pitaevskii equation:

$$i\frac{\partial u}{\partial t} = -\Delta u + 2i(x\partial_y - y\partial_x)u + (x^2+y^2)u + |u|^2u - 2u, \tag{4.3}$$

where the frequency shift (the last term in the right-hand side) is added for convenience and the rotational term (the second term in the right-hand side) has a specific coefficient of the critical rotational frequency [18].

Let $z = x + iy$ and consider the functions in the form

$$u(z) = f(z)e^{-\frac{1}{2}|z|^2}, \tag{4.4}$$

where $f \in \mathcal{F} \subset L^2_\rho(\mathbb{C})$ and $L^2_\rho(\mathbb{C})$ is defined from (4.2). The projection operator acting on functions of the form $u(z) = f(z)e^{-\frac{1}{2}|z|^2}$ will be denoted by $\widehat{\Pi}$. It is given by

$$(\widehat{\Pi}u)(z) = \frac{1}{\pi}e^{-\frac{|z|^2}{2}}\iint_{\mathbb{R}^2} e^{z\bar{z}'-\frac{|z'|^2}{2}}u(z')dx'dy' = (\Pi u e^{\frac{1}{2}|z|^2})(z)e^{-\frac{1}{2}|z|^2}. \tag{4.5}$$

With the use of complex variable $z = x + iy$, the Gross–Pitaevskii equation (4.3) can be rewritten in the form

$$i\frac{\partial u}{\partial t} = \left(\bar{z} - 2\frac{\partial}{\partial z}\right)\left(z + 2\frac{\partial}{\partial \bar{z}}\right)u + |u|^2u \tag{4.6}$$

For the functions u in the form (4.4) with $f \in \mathcal{F}$, we have $\left(z + 2\frac{\partial}{\partial z}\right)u = 0$. However, the evolution of a local equation

$$i\frac{\partial u}{\partial t} = u|u|^2 \tag{4.7}$$

is not closed in the space of functions in the form (4.4) because $|u|^2u = |f|^2fe^{-\frac{3}{2}|z|^2}$ is not in the form $\tilde{f}(z)e^{-\frac{1}{2}|z|^2}$ with $\tilde{f} \in \mathcal{F} \subset L^2_\rho(\mathbb{C})$. The way to get around this issue is to decompose u into the sum of two terms: the leading-order term satisfying the following closed initial-value problem

$$\begin{cases} i\frac{\partial u}{\partial t} = \widehat{\Pi}(|u|^2u) \\ u(0,z) = u_0(z). \end{cases} \tag{4.8}$$

and the error term which satisfies the residual equation. As is shown in many works (see, e.g., [15, 19]), the approximation error can be controlled to be small in some norm during the time evolution on a finite time interval. The approximation result justifies the validity of the leading-order approximation defined by solutions of the initial-value problem (4.8).

We refer to the nonlocal equation in (4.8) as the lowest Landau level (LLL) equation. Well-posedness of the initial-value problem (4.8) has been obtained in [21] (see also earlier works in [1, 20]). The review of these results is given next.

By Lemma 2.6 and Example 2.3, for every $0 < p < q < \infty$ and every u in the form (4.4), the following embedding property is true:

$$\left(\frac{q}{2\pi}\right)^{\frac{1}{q}} \|u\|_{L^q(\mathbb{C})} \leq \left(\frac{p}{2\pi}\right)^{\frac{1}{p}} \|u\|_{L^p(\mathbb{C})}, \tag{4.9}$$

where the inequality becomes the equality if and only if $u(z)$ is constant proportional to a translation of $e^{-\frac{1}{2}|z|^2}$.

The following lemma states the local well-posedness of the initial-value problem (4.8) in the L^p spaces.

Lemma 4.1 *The initial-value problem (4.8) is locally well-posed in $L^p(\mathbb{C})$ for any $p \geq 1$.*

Proof. It follows from (4.5) that

$$|(\widehat{\Pi}u)(z)| \leq \frac{1}{\pi} \iint_{\mathbb{R}^2} e^{-\frac{1}{2}(x-x')^2 - \frac{1}{2}(y-y')^2} |u(z')| dx'dy'.$$

By the generalized Young's inequality for convolution integrals, the projection operator $\widehat{\Pi} : L^p(\mathbb{C}) \mapsto L^p(\mathbb{C})$, $1 \leq p \leq \infty$ is bounded with the p-independent bound:

$$\|\widehat{\Pi}u\|_{L^p(\mathbb{C})} \leq 2\|u\|_{L^p(\mathbb{C})}, \tag{4.10}$$

due to the exact value:

$$\iint_{\mathbb{R}^2} e^{-\frac{1}{2}(x-x')^2 - \frac{1}{2}(y-y')^2} dx'dy' = 2\pi.$$

Let us rewrite the evolution problem (4.8) in the integral form $u = A(u)$, where

$$A(u)(t, \cdot) := u_0 - i \int_0^t \widehat{\Pi}(|u(s, \cdot)|^2 u(s, \cdot)) ds, \quad t > 0.$$

We show that there exists a sufficiently small $t_0 > 0$ such that the operator A is a closed contraction operator in the ball B_δ defined by $\sup_{t\in[0,t_0]} \|u(t, \cdot)\|_{L^p(\mathbb{C})} \leq \delta$ with $\delta := 2\|u_0\|_{L^p(\mathbb{C})}$.

It follows from (4.9) and (4.10) that

$$\begin{aligned}\sup_{t\in[0,t_0]} \|A(u)(t,\cdot)\|_{L^p(\mathbb{C})} &\leq \|u_0\|_{L^p(\mathbb{C})} + \int_0^{t_0} \|\widehat{\Pi}(|u(s,\cdot)|^2 u(s,\cdot))\|_{L^p(\mathbb{C})} ds \\ &\leq \|u_0\|_{L^p(\mathbb{C})} + 2\int_0^{t_0} \||u(s,\cdot)|^2 u(s,\cdot)\|_{L^p(\mathbb{C})} ds \\ &\leq \|u_0\|_{L^p(\mathbb{C})} + 2\int_0^{t_0} \|u(s,\cdot)\|_{L^\infty(\mathbb{C})} \|u(s,\cdot)\|_{L^p(\mathbb{C})} ds \\ &\leq \|u_0\|_{L^p(\mathbb{C})} + 2\left(\frac{p}{2\pi}\right)^{\frac{2}{p}} \int_0^{t_0} \|u(s,\cdot)\|^3_{L^p(\mathbb{C})} ds,\end{aligned}$$

so that $\sup_{t\in[0,t_0]} \|A(u)(t,\cdot)\|_{L^p(\mathbb{C})} \leq \delta = 2\|u_0\|_{L^p(\mathbb{C})}$ if

$$4\left(\frac{p}{2\pi}\right)^{\frac{2}{p}} t_0 \delta^2 \leq 1.$$

Hence, the operator is closed in the ball B_δ. It is also a contraction in the ball B_δ because

$$\begin{aligned}\sup_{t\in[0,t_0]} \|A(u)(t,\cdot) - A(v)(t,\cdot)\|_{L^p(\mathbb{C})} &\leq \int_0^{t_0} \|\widehat{\Pi}(|u|^2 u - |v|^2 v)\|_{L^p(\mathbb{C})} ds \\ &\leq 2\int_0^{t_0} \left(\|u\|_{L^\infty(\mathbb{C})} + \|v\|_{L^\infty(\mathbb{C})}\right)^2 \|u - v\|_{L^p(\mathbb{C})} ds \\ &\leq 2\left(\frac{p}{2\pi}\right)^{\frac{2}{p}} \int_0^{t_0} \left(\|u\|_{L^p(\mathbb{C})} + \|v\|_{L^p(\mathbb{C})}\right)^2 \|u - v\|_{L^p(\mathbb{C})} ds \\ &\leq 8\left(\frac{p}{2\pi}\right)^{\frac{2}{p}} t_0 \delta^2 \sup_{t\in[0,t_0]} \|u - v\|_{L^p(\mathbb{C})} ds,\end{aligned}$$

where we have used the elementary inequality

$$||u|^2 u - |v|^2 v| = |u^2(\bar{u} - \bar{v}) + (u+v)\bar{v}(u - v)| \leq (|u| + |v|)^2 |u - v|.$$

Thus, A is a contraction in the ball B_δ if

$$8\left(\frac{p}{2\pi}\right)^{\frac{2}{p}} t_0 \delta^2 < 1.$$

For any $\delta > 0$, there is a sufficiently small $t_0 > 0$ satisfying both the inequalities above such that the operator A has a unique fixed point in B_δ by the Banach fixed-point theorem. The proof of continuous dependence on the initial data $u_0 \in L^p(\mathbb{C})$ is standard and follows from the contraction mapping property. ∎

Remark 4.1 *Bound (4.10) is not sharp for* $p = 2$ *because Lemma 2.1 and Corollary 2.2 imply for the function* u *in the form (4.4) with* $f \in \mathcal{F} \subset L^2_\rho(\mathbb{C})$ *that*

$$\begin{aligned}
\frac{1}{\pi}\|\widehat{\Pi}u\|^2_{L^2(\mathbb{C})} &= \|\Pi(ue^{\frac{1}{2}|z|^2})\|^2_{L^2_\rho(\mathbb{C})} \\
&= \|\mathcal{B}\mathcal{B}^*(ue^{\frac{1}{2}|z|^2})\|^2_{L^2_\rho(\mathbb{C})} \\
&= \|\mathcal{B}^*(ue^{\frac{1}{2}|z|^2})\|^2_{L^2(\mathbb{R})} \\
&= \|ue^{\frac{1}{2}|z|^2}\|^2_{L^2_\rho(\mathbb{C})} \\
&= \frac{1}{\pi}\|u\|^2_{L^2(\mathbb{C})},
\end{aligned}$$

so that $\|\widehat{\Pi}u\|_{L^2(\mathbb{C})} = \|u\|_{L^2(\mathbb{C})}$.

The following lemma gives global well-posedness of the initial-value problem (4.8) in the L^2 space.

Lemma 4.2 *The initial-value problem (4.8) is globally well-posed in* $L^2(\mathbb{C})$ *if* $u_0 \in L^2(\mathbb{C})$*. Moreover, for every* $t \in \mathbb{R}$*, the following two quantities are conserved in time:*

$$\iint_{\mathbb{R}^2} |u(t,z)|^2 dxdy = \iint_{\mathbb{R}^2} |u_0(z)|^2 dxdy \tag{4.11}$$

and

$$\iint_{\mathbb{R}^2} |u(t,z)|^4 dxdy = \iint_{\mathbb{R}^2} |u_0(z)|^4 dxdy \tag{4.12}$$

Proof. By Lemma 4.1, if $u_0 \in L^2(\mathbb{C})$, there exists a local solution to the initial-value problem (4.8) in $L^2(\mathbb{C})$. By the embedding properties, $u(t,\cdot) \in L^p(\mathbb{C})$, $t \in [0, t_0]$ for every $2 \le q \le \infty$ including $q = 4$.

Next, we show the L^2-conservation property (4.11). Since $\widehat{\Pi}$ is a bounded operator by (4.10) and the solution $u(t,\cdot) \in B_\delta \subset L^2(\mathbb{C})$ belongs to a bounded set closed with its boundary, we have $u \in C^1((0,t_0), L^2(\mathbb{C}))$. Hence, it is allowed to differentiate the L^2 norm and to use the time evolution equation in (4.8):

$$\begin{aligned}
\frac{d}{dt}\iint_{\mathbb{R}^2} |u(t,z)|^2 dxdy &= i\langle u, \bar{\widehat{\Pi}}(|u|^2u)\rangle_{L^2(\mathbb{C})} - i\langle \widehat{\Pi}(|u|^2u), u\rangle_{L^2(\mathbb{C})} \\
&= i\langle \widehat{\Pi}u, |u|^2u\rangle_{L^2(\mathbb{C})} - i\langle |u|^2u, \bar{\widehat{\Pi}}u\rangle_{L^2(\mathbb{C})} \\
&= i\langle u, |u|^2u\rangle_{L^2(\mathbb{C})} - i\langle |u|^2u, u\rangle_{L^2(\mathbb{C})} \\
&= 0,
\end{aligned}$$

where we have used the fact that $\Pi : L^2_\rho(\mathbb{C}) \mapsto L^2_\rho(\mathbb{C})$ is an orthogonal projection (Remark 2.2), which implies that $\widehat{\Pi} : L^2(\mathbb{C}) \mapsto L^2(\mathbb{C})$ is also an orthogonal projection due to the relation (4.5). Due to the L^2 conservation, the local solution $u(t, \cdot) \in B_\delta \subset L^2(\mathbb{C})$, $t \in [0, t_0]$ is continuously extended to the global solution $u(t, \cdot) \in B_\delta \subset L^2(\mathbb{C})$, $t \in \mathbb{R}$.

Finally, we show the L^4-conservation property (4.12). By continuous embedding (4.9), it follows that if $u \in C^1(\mathbb{R}, L^2(\mathbb{C}))$, then $u \in C^1(\mathbb{R}, L^4(\mathbb{C}))$. Hence, it is allowed to differentiate the L^4 norm and to use the time evolution equation in (4.8):

$$\begin{aligned} \frac{d}{dt} \iint_{\mathbb{R}^2} |u(t,z)|^4 dxdy &= 2i\langle |u|^2 u, \bar{\widehat{\Pi}}(|u|^2 u)\rangle_{L^2(\mathbb{C})} - 2i\langle \widehat{\Pi}(|u|^2 u), |u|^2 u\rangle_{L^2(\mathbb{C})} \\ &= 0, \end{aligned}$$

by the same property that $\widehat{\Pi} : L^2(\mathbb{C}) \mapsto L^2(\mathbb{C})$ is an orthogonal projection. ■

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