Bargmann transform and its applications to partial differential equations

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Outline of the thesis

The thesis consists of two main parts:

- Part 1: Fock space and Bargmann transform and some of its properties.
- Part 2: Some applications to partial differential equations:
    1. Evolution of the linear Schrödinger equation.
    2. Evolution of the linear diffusion equation.
    3. Evolution of the linear KdV equation.
    4. Well-posedness of the Gross–Pitaevskii equation at the Lowest Landau Level equation.

Fock space

Let $z = x - i\xi \in \mathbb{C}$ be an extension of $x \in \mathbb{R}$ to $z \in \mathbb{C}$. Let $\alpha > 0$ be fixed arbitrarily and define $L^2_\rho(\mathbb{C})$ by its weight

$$\rho(z) := \frac{\alpha}{\pi} e^{-\alpha |z|^2}$$

and the standard inner product

$$\langle f, g \rangle_{L^2_\rho(\mathbb{C})} := \frac{\alpha}{\pi} \iint_{\mathbb{R}^2} f(z)\overline{g(z)} e^{-\alpha |z|^2} dx d\xi.$$

The L^2-based Fock space denoted by $\mathcal{F}$ is the space of all entire functions in $L^2_\rho(\mathbb{C})$:

$$\mathcal{F} = \left\{ f \in L^2_\rho(\mathbb{C}) : \quad f(z) \text{ is entire in } z \in \mathbb{C} \right\}$$

Bargmann transform

Bargmann transform is defined by:

$$(\mathcal{B}\varphi)(z) := \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{4}} e^{\frac{\alpha}{2}z^2} \int_{-\infty}^{\infty} e^{-\alpha(z-y)^2}\varphi(y)dy, \quad z \in \mathbb{C},$$

provided that the integral is finite. It can be shown that if $\varphi \in L^2(\mathbb{R})$, then the integral is finite and $\mathcal{B}\varphi \in \mathcal{F}$, moreover, it is a unitary transformation which preserves the norm.

The adjoint Bargmann transform $\mathcal{B}^*$

- The adjoint transform of $\mathcal{B}$ is a transformation which satisfies:

  $$\langle f, \mathcal{B}\varphi \rangle_{L^2_\rho(\mathbb{C})} = \langle \mathcal{B}^* f, \varphi \rangle_{L^2(\mathbb{R})}, \quad \text{for every} \quad f \in L^2_\rho(\mathbb{C}), \;\; \varphi \in L^2(\mathbb{R}).$$

  After some computations we get that this transformation given by:

  $$(\mathcal{B}^* f)(y) = \frac{2^{\frac{1}{4}} \alpha^{\frac{5}{4}}}{\pi^{\frac{5}{4}}} \iint_{\mathbb{R}^2} e^{\frac{\alpha}{2} \bar{z}^2 - \alpha(y - \bar{z})^2 - \alpha |z|^2} f(z) dx d\xi,$$

  where $\mathcal{B}^* : L^2_\rho(\mathbb{C}) \mapsto L^2(\mathbb{R})$.
- $\mathcal{B}^*$ is the left inverse of $\mathcal{B}$, so that $\mathcal{B}^* \mathcal{B} \varphi = \varphi$ for every $\varphi \in L^2(\mathbb{R})$.
- $\mathcal{B}^*$ is not the right inverse of $\mathcal{B}$, since $\mathcal{B}^* \bar{z} = 0$ and by the same reason we get $\mathcal{B}^*$ is not an isometry and so is not a unitary transformation from $L^2_\rho(\mathbb{C})$ to $L^2(\mathbb{R})$.
- $\Pi := \mathcal{B}\mathcal{B}^* : L^2_\rho(\mathbb{C}) \mapsto \mathcal{F} \subset L^2_\rho(\mathbb{C})$

Embedding of Fock spaces

- Fock spaces can be extended to the L^p-Lebesgue spaces,

$$\mathcal{F}_p = \left\{ f \in L^p_\rho(\mathbb{C}) : \quad f(z) \text{ is entire in } z \in \mathbb{C} \right\}.$$

- Fix $0 < p < \infty$. For every $z \in \mathbb{C}$ and every $f \in \mathcal{F}_p$,

$$|f(z)| \leq \|f\|_{L^p_\rho} e^{\frac{1}{2}\alpha|z|^2}.$$

- Fix $0 < p < q < \infty$. Then, $\mathcal{F}_p \subsetneq \mathcal{F}_q$ and the inclusion is continuous,

$$\|f\|_{L^q_\rho} \leq \left(\frac{q}{p}\right)^{\frac{1}{q}} \|f\|_{L^p_\rho},$$

1. Applications to Schrödinger equation

We shall now express the Cauchy problem for the time-dependent Schrödinger equation:

$$\begin{cases} i\frac{\partial \Phi}{\partial t} = L\Phi \\ \Phi\big|_{t=0} = \varphi \in L^2(\mathbb{R}), \end{cases}$$

Where $L : D(L) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is the Schrödinger operator defined by:

$$L := -\partial_y^2 + y^2 - 1$$

Where the domain

$$D(L) := \{\varphi \in L^2(\mathbb{R}) : \quad \partial_y^2\varphi \in L^2(\mathbb{R}), \quad y^2\varphi \in L^2(\mathbb{R})\}.$$

Orthonormal basis in the Fock space

- Eigenfunctions of the Schrödinger operator are given by the Gauss–Hermite functions, which form orthonormal basis in $L^2(\mathbb{R})$.

$$u_n(y) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(y) e^{-\frac{1}{2}y^2}, \quad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\},$$

  To be precise, we know that

$$L u_n = (2n) u_n, \quad n \in \mathbb{N}_0.$$

- $\{f_n\}_{n\in\mathbb{N}_0}$ is an orthonormal basis in $\mathfrak{F} \subset L^2_\rho(\mathbb{C})$ where $f_n := \mathfrak{B} u_n = \frac{z^n}{\sqrt{2^n n!}}, \quad n \in \mathbb{N}_0, \, \alpha = \frac{1}{2}.$
- $f(z) = \sum_{n=0}^{\infty} \langle f, f_n \rangle_{L^2_\rho(\mathbb{C})} f_n(z).$

The transformation of L under the Bargmann transform

If $\mathcal{L} := \mathcal{B} L \mathcal{B}^* : D(\mathcal{L}) \subset \mathcal{F} \mapsto \mathcal{F}$, then

$$(\mathcal{L} f)(z) = 2z \frac{df}{dz}, \quad f \in D(\mathcal{L}),$$

After the Bargmann transform $\mathcal{B}$, the Cauchy problem is rewritten in the form:

$$\begin{cases} i\frac{\partial F}{\partial t} = \mathcal{L} F = 2z \frac{\partial F}{\partial z} \\ F\big|_{t=0} = f \in \mathcal{F}, \end{cases}$$

where $f = \mathcal{B}\varphi$ and $F(t, \cdot) = \mathcal{B}\Phi(t, \cdot)$.

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where $f = \mathcal{B}\varphi$ and $F(t, \cdot) = \mathcal{B}\Phi(t, \cdot)$.

For every $f \in \mathcal{F}$, there exists the unique solution to the $\mathcal{B}$–Cauchy problem which can be written in the form:

$$F(t, z) = f\left(ze^{-2it}\right).$$

The main question

If $h \in L^\infty(\mathbb{R})$, does $H(t,\cdot)$ remain in $L^\infty(\mathbb{R})$ for $t > 0$?

If $\Phi(t,y) = H(t,y)e^{-\frac{1}{2}y^2}$ and $\varphi(y) = h(y)e^{-\frac{1}{2}y^2}$, then the envelope function $H(t,y)$ is a solution to the Cauchy problem:

$$\begin{cases} i\frac{\partial H}{\partial t} = -\partial_y^2 H + 2y\partial_y H, \\ H\big|_{t=0} = h. \end{cases}$$

We have answered this question negatively according to the following lemma.

Lemma

There exists $h \in L^\infty(\mathbb{R})$ & $t_0 > 0$, s.t. $H(t,\cdot) \notin L^\infty(\mathbb{R})$ $\forall t \in (0,t_0)$.

2. Applications to linear diffusion equation

The linear diffusion equation:

$$\frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial y^2} + (1 - y^2)\Phi.$$

Bargmann transform with $F(t, \cdot) = \mathcal{B}\Phi(t, \cdot)$ maps it to the following transport equation

$$\frac{\partial F}{\partial t} + 2z\frac{\partial F}{\partial z} = 0.$$

the Cauchy problem associated with the transport equation is solved in the form

$$F(t, z) = f(ze^{-2t}).$$

The main question

If $h \in L^\infty(\mathbb{R})$, does $H(t,\cdot)$ remain in $L^\infty(\mathbb{R})$ for $t > 0$?

If $\Phi(t,y) = H(t,y)e^{-\frac{1}{2}y^2}$, then the envelope of the Gaussian function $H(t,y)$ satisfies the linear diffusion equation in self-similar variables:

$$\begin{cases} \frac{\partial H}{\partial t} = \partial_y^2 H - 2y\partial_y H, \\ H|_{t=0} = h. \end{cases}$$

The unique solution of this Cauchy problem is given in the integral form:

$$H(t,y) = \frac{1}{\sqrt{\pi(1-e^{-4t})}} \int_{\mathbb{R}} e^{-\frac{(y'-ye^{-2t})^2}{1-e^{-4t}}} h(y')dy'.$$

Lemma

For every $h \in L^\infty(\mathbb{R})$, the unique solution to the diffusion equation with $H(0,y) = h(y)$ satisfies $H(t,\cdot) \in L^\infty(\mathbb{R})$ for every $t \in \mathbb{R}_+$.

3. Applications to the linear KdV equation

This is a modification of the linear KdV equation:

$$\begin{cases} \frac{\partial \Phi}{\partial t} = (\partial_y - y)L\Phi, \\ \Phi\big|_{t=0} = \varphi \in L^2(\mathbb{R}). \end{cases}$$

In order to reformulate the Cauchy problem in $\mathcal{F}$, we need to see how the Bargmann transform $\mathcal{B}$ is applied to the derivative ∂_y and to the multiplication by y.

Lemma

Fix $f \in \mathcal{F}$. Then,

$$(\mathcal{B}\partial_y\mathcal{B}^* f)(z) = f'(z) - \frac{1}{2}zf(z)$$

and

$$(\mathcal{B}y\mathcal{B}^* f)(z) = f'(z) + \frac{1}{2}zf(z).$$

The solution of the Cauchy problem

We use the Bargmann transform and rewrite the Cauchy problem in the equivalent form:

$$\begin{cases} \frac{\partial F}{\partial t} = -2z^2 \partial_z F, \\ F|_{t=0} = f \in \mathfrak{F}, \end{cases}$$

This Cauchy problem is solved in the form:

$$F(t, z) = f\left(\frac{z}{1 + 2tz}\right).$$

Since f is entire and $\frac{z}{1+2tz}$ has a pole at $z = -\frac{1}{2t}$ for every $t \neq 0$, $F(t, \cdot)$ is entire if and only if $|f(\infty)| < \infty$. However, by Liouville's theorem, the only entire and bounded function is the constant function.

Lemma

The Cauchy problem is ill-posed in $L^2(\mathbb{R})$*.*

4. Applications to Gross–Pitaevskii equation

$$i\frac{\partial u}{\partial t} = -\Delta u + 2i(x\partial_y - y\partial_x)u + (x^2 + y^2)u + |u|^2 u - 2u,$$

With the use of complex variable $z = x + iy$, the Gross–Pitaevskii equation (15) can be rewritten in the form

$$i\frac{\partial u}{\partial t} = \left(\bar{z} - 2\frac{\partial}{\partial z}\right)\left(z + 2\frac{\partial}{\partial \bar{z}}\right)u + |u|^2 u$$

We will use the Bargmann transform particularly the projection operator acting on functions of the form $u(z) = f(z)e^{-\frac{1}{2}|z|^2}$ is given by:

$$(\widehat{\Pi} u)(z) = \frac{1}{\pi}e^{-\frac{|z|^2}{2}} \iint_{\mathbb{R}^2} e^{z\bar{z}' - \frac{|z'|^2}{2}} u(z')dx'dy' = (\Pi u e^{\frac{1}{2}|z|^2})(z)e^{-\frac{1}{2}|z|^2}$$

We decompose u into the sum of two terms: the leading-order term satisfying the following closed initial-value problem

$$\begin{cases} i\frac{\partial u}{\partial t} = \widehat{\Pi}(|u|^2 u) \\ u(0,z) = u_0(z). \end{cases} \tag{1}$$

and the error term which satisfies the residual equation. (1) is called the lowest Landau level (LLL) equation.

Lemma

The initial-value problem (1) is locally well-posed in $L^p(\mathbb{C})$ for any $p \geq 1$. Moreover it is globally well-posed in $L^2(\mathbb{C})$ if $u_0 \in L^2(\mathbb{C})$, and for every $t \in \mathbb{R}$, $||u(t,\cdot)||_{L^2} = ||u_0||_{L^2}$.

Summary

- The linear evolution is greatly simplified after the Bargmann transform, so that the three equations reduced to transport equation.
- We answered positively to the main question for diffusion equation and negatively for the Schrödinger equation.
- We considered a linear KdV equation with this method and we showed it has no solution unless it is constant.
- Bargmann transform is also useful in the well-posedness analysis of the Gross-Pitaevskii equation in two dimensions.

THANK YOU