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Nonlinearity management in higher dimensions

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Abstract

In the present paper, we revisit nonlinearity management of the time-periodic nonlinear Schrödinger equation and the related averaging procedure. By means of rigorous estimates, we show that the averaged nonlinear Schrödinger equation does not blow up in the higher dimensional case so long as the corresponding solution remains smooth. In particular, we show that the H^1 norm remains bounded, in contrast with the usual blow-up mechanism for the focusing Schrödinger equation. This conclusion agrees with earlier works in the case of strong nonlinearity management but contradicts those in the case of weak nonlinearity management. The apparent discrepancy is explained by the divergence of the averaging procedure in the limit of weak nonlinearity management.

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1. Introduction

In the past few years, there has been a large volume of literature regarding applications of the nonlinear Schrödinger equation (NLS) in the presence of the so-called *nonlinearity management* (often referred to also as Feshbach resonance management). The NLS is a prototypical dispersive nonlinear wave equation of the form

$$iu_t = -\Delta u + \Gamma(t)|u|^2 u + V(x)u, \tag{1}$$

where $u(x,t)$ is a complex envelope field, $V(x) \geqslant 0$ is an external potential, $\Gamma(t)$ is a time-periodic nonlinearity coefficient and Δ is the Laplacian operator with $x \in \mathbb{R}^d$, $d \geqslant 1$.

Nonlinearity management arises in optics for transverse beam propagation in layered optical media [1], as well as in atomic physics for the Feshbach resonance of the scattering length of inter-atomic interactions in Bose–Einstein condensates (BECs) [2]. In the latter case, the periodic variation of $\Gamma(t)$ through an external magnetic field has been used as a means of producing robust matter-wave breathers in quasi-one-dimensional BECs [3]. It has also been suggested that the nonlinearity management may prevent collapse-type phenomena in higher dimensions [4]. Theoretical studies of the collapse-type behaviour were performed with a reduction of the time-periodic PDE problem (1) to a time-periodic ODE problem using the variational method [5] and the method of moments [6]. Recent work [7] presented a rigorous proof that the NLS equation (1) with $\Gamma(t) < 0$ for all $t \geqslant 0$ may have a blow-up in three dimensions ($d = 3$) depending on parameters of the initial condition.

The physical relevance of the time-periodic NLS equation, as evidenced by the above works led to further developments in analysis of the evolution problem (1). As an example, the success of the averaging theory [8] for optical solitons in the presence of *strong dispersion management* led to an analogous development for *strong nonlinearity management* produced originally in [9] and systematized in [10]. Let ϵ be a small parameter that measures a short period and large variations of $\Gamma(t)$ in the form

$$\Gamma = \gamma_0 + \frac{1}{\epsilon}\gamma\left(\frac{t}{\epsilon}\right), \tag{2}$$

where $\gamma(\tau)$, $\tau = t/\epsilon$ has a unit period and zero mean. The time-periodic NLS equation (1) is averaged in the limit $\epsilon \to 0$. The averaging procedure is based on the transformation of solutions of the NLS equation (1),

$$u = \mathrm{e}^{\mathrm{i}\gamma_{-1}(\tau)|v|^2} v(x,t),$$

where $\gamma_{-1}(\tau)$ is the mean-zero anti-derivative of $\gamma(\tau)$, and subsequent expansion of $v(x,t)$ into an asymptotic series,

$$v = w(x,t) + \epsilon v_1(x,\tau,t) + O(\epsilon^2).$$

The secular growth of $v_1(x,\tau,t)$ in τ is removed if $w(x,t)$ satisfies the averaged NLS equation,

$$\mathrm{i}w_t = -\Delta w + \gamma_0|w|^2 w + V(x)w - \sigma^2(|\nabla|w|^2|^2 + 2|w|^2\Delta|w|^2)w, \tag{3}$$

where

$$\sigma^2 = \int_0^1 \gamma_{-1}^2(\tau)\,\mathrm{d}\tau.$$

Our paper addresses the question whether the averaged NLS equation (3) with $\gamma_0 < 0$ and $\sigma \neq 0$ arrests the blow-up of solutions of the NLS equation in all dimensions. We remind readers that the averaged NLS equation (3) with $\gamma_0 < 0$ and $\sigma = 0$ may have blow-up in $d \geqslant 2$ depending on the initial condition [11]. A similar question was addressed recently in [12] in the context of the cubic-quintic NLS equation in one dimension ($d = 1$).

Before we explain our results in more detail, let us take a moment to review the usual blow-up mechanism in the context of a semilinear focusing Schrödinger equation. That is, take $V(x) = 0$ and $\sigma = 0$ in the NLS equation (3). It is well known (see chapter 5 in [13]) that whenever $d \geqslant 2$ and $\gamma_0 < 0$, and for some (smooth) initial data $w_0(x)$ in H^1, there exists a time $T^* < \infty$, so that

$$\lim_{t\to T^*}\sup_{0<t<T^*}\int_{\mathbb{R}^d}|\nabla w(x,t)|^2\,\mathrm{d}x \to \infty \qquad \text{and} \qquad \lim_{t\to T^*}\sup_{0<t<T^*}\int_{\mathbb{R}^d}|w(x,t)|^4\,\mathrm{d}x \to \infty, \tag{4}$$

while the solution is sufficiently smooth in $t \in (0, T^*)$, and thus for $0 \leqslant t < T^*$,

$$H(w) = \int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla w(x,t)|^2 + \frac{\gamma_0}{4}|w(x,t)|^4\right)\mathrm{d}x = H(w_0).$$

In our work, we show that no such blow-up of solutions occurs in the averaged NLS equation (3) with $\gamma_0 < 0$ and $\sigma \neq 0$. That is, assuming that the solution exists and satisfies the conservation law $H(w) = H(w_0)$ and $P(w) = \|w\|^2_{L^2} = \|w_0\|^2_{L^2}$, one can in fact control *a priori* $\|w\|_{H^1}$ and $\|w\|_{L^4}$.

Unfortunately, our arguments do not allow to rigorously justify the global (and in fact even the local) existence of solutions, satisfying the physical conservation laws. In other words, the solutions may still blow up due to eventual loss of smoothness (and a subsequent failure to obey the conservation laws), although we consider that to be an unlikely scenario, especially for sufficiently smooth initial data $u_0(x)$.

We demonstrate our claims initially, in section 2, from the point of view of scaling arguments. Then using rigorous estimates in section 3, we show that the H^1 and the L^4 norm of the solution $w(x, t)$ are controlled by $H(w)$ and $P(w)$. In section 4, we compare the above conclusion and earlier works where blow-up of solutions of the full time-periodic NLS equation (1) have been reported. Section 5 summarizes our findings.

2. Formal scaling arguments

The averaged NLS equation (3) has a standard Hamiltonian form (see [10]) with the Hamiltonian functional:

$$H(w) = \int_{\mathbb{R}^d} \left(|\nabla w|^2 + \frac{\gamma_0}{2}|w|^4 + V(x)|w|^2 + \sigma^2|w|^2|\nabla|w|^2|^2\right) \mathrm{d}x. \tag{5}$$

Due to the gauge invariance, the averaged NLS equation (3) also conserves the squared L^2 norm:

$$P(w) = \int_{\mathbb{R}^d} |w|^2 \, \mathrm{d}x. \tag{6}$$

Solitary wave solutions of the averaged NLS equation (3) are critical points of $H(w)$ at the level set of fixed values of $P(w)$.

Using formal scaling arguments [14] (see also [11]), we consider a two-parameter family of dilatations:

$$w = bW(ax), \tag{7}$$

where (a, b) are parameters and $W(\xi)$ is a suitable function of $\xi = ax$. The squared L^2 norm (6) is preserved by the dilatations (7) whenever $b = a^{d/2}$. The Hamiltonian (5) at the dilatations (7) is scaled as a function of parameter $a > 0$:

$$H(a) = I_0(a) + a^2 I_1 + \gamma_0 a^d I_2 + \sigma^2 a^{2d+2} I_3, \tag{8}$$

where

$$I_1 = \int_{\mathbb{R}^d} |\nabla W|^2 \, \mathrm{d}\xi, \qquad I_2 = \frac{1}{2}\int_{\mathbb{R}^d} |W|^4 \, \mathrm{d}\xi, \qquad I_3 = \int_{\mathbb{R}^d} |W|^2|\nabla|W|^2|^2 \, \mathrm{d}\xi.$$

and

$$I_0(a) = \int_{\mathbb{R}^d} V\left(\frac{\xi}{a}\right) |W|^2 \, \mathrm{d}\xi.$$

Let us consider the case of no nonlinearity management and no external potential, when $\sigma^2 = 0$ and $V(x) = 0$. It follows from (8) that the Hamiltonian function $H(a)$ is positive definite in the defocusing case, when $\gamma_0 > 0$. In the focusing case, when $\gamma_0 < 0$, the Hamiltonian function $H(a)$ is bounded from below for $d = 1$ and $d = 2$, $\gamma_{\rm cr} < \gamma_0 < 0$ and is unbounded from below for $d = 2$, $\gamma_0 < \gamma_{\rm cr}$ and $d \geqslant 3$, where

$$\gamma_{\rm cr} = -\frac{I_1}{I_2}.$$

When $H(a)$ is unbounded from below as $a \to \infty$, the critical points of $H(w)$ at a fixed value of $P(w)$ (i.e., solitary wave solutions) cannot be stable for small width a^{-1} and instability of solitary waves implies a blow-up of localized initial data in the time evolution of the cubic NLS equation (see [11] for details).

When the nonlinearity management is applied, the last term in the decomposition (8) always dominates and it preserves the boundness of $H(a)$ from below for any $\sigma^2 > 0$. This indicates on the level of formal scaling arguments that the blow-up of solutions is arrested by the nonlinearity management term in the averaged NLS equation (3).

We note that the first term in the decomposition (8) does not change the conclusions above if $V(x)$ is a smooth non-negative potential, such that $I_0(a) \geqslant 0$. Typical examples of $V(x)$ are parabolic magnetic traps, when $V \sim x^2$, and periodic optical lattices, when $V \sim \sin^2(x)$.

3. On global solutions to the averaged equation

The rigorous analysis of local well-posedness of the averaged NLS equation (3) is not a trivial task. In fact, to the best of our knowledge, one cannot verify even the local existence and uniqueness of solutions to this problem. The problem has been considered in one dimension $d = 1$ by Poppenberg where existence and uniqueness of local solutions with initial data in H^∞ was established [15]. In higher dimensions $d \geqslant 2$, one needs to require [16] that the initial data be in $H^{s,m}$ for sufficiently large values of s, m, where

$$H^{s,m} = \left\{ f \in C^s(\mathbb{R}^d; \mathbb{C}) : \int_{\mathbb{R}^d} (1 + |x|)^{2m} (|f|^2 + |\partial^s f|^2)\,\mathrm{d}x < \infty \right\}.$$

In addition, one needs to assume a 'non-trapping' geometric condition on the symbol of the second-order operator, which depends on the profile of the initial data (see [16] for details).

In contrast, the local well-posedness of the time-periodic NLS equation (1) can be proved with the standard tool in the energy space $H^1 \equiv H^{1,0}$ (see section 2 in [17]). Therefore, one is tempted to assume that solutions of the averaged NLS equation (3), derived from the time-periodic NLS equation (1) with a regular asymptotic procedure, inherit local well-posedness. However, we cannot state a precise condition under which the averaged NLS equation (3) has a (local) solution that conserves constant values of P and H in time.

Thus, as we have explained in section 1, we will focus our attention on the following problem: assuming the existence of a local solution in space $H^1 \cap L^4 \cap \{w : \int |w|^2 |\nabla |w|^2|^2\,\mathrm{d}x < \infty\}$ and in a time interval $t \in (0, T)$, we will show that

$$\sup_{0<t<T} (\|\nabla u(\cdot, t)\|_{L^2} + \|u\|_{L^4}) < \infty.$$

In other words, the standard blow-up mechanism (4) for the cubic focusing Schrödinger equation does not occur.

To that end, we represent the Hamiltonian $H(w)$ in the form

$$H(w) = H_1(w) + \gamma_0 H_2(w), \tag{9}$$

where

$$H_1(w) = \int_{\mathbb{R}^d} (|\nabla w|^2 + V(x)|w|^2 + \sigma^2 |w|^2 |\nabla |w|^2|^2)\,\mathrm{d}x \geqslant 0$$

and

$$H_2(w) = \frac{1}{2} \int_{\mathbb{R}^d} |w|^4\,\mathrm{d}x \geqslant 0.$$

We consider the focusing case $\gamma_0 < 0$ and prove that $H_1(w)$ and $H_2(w)$ are bounded by the two conserved quantities $H(w)$ and $P(w)$.

First, we quote a variant of the Gagliardo–Nirenberg inequality (see section 1.1.16 on p 15 in [18]).

Lemma 1. *For all* $1 \leqslant p, q, r \leqslant \infty, \theta \in (0, 1)$ *and* $r^{-1} = \theta p^{-1} + (1-\theta)q^{-1}$, *it is true for every function* $f(x)$ *on* $x \in \mathbb{R}^d$ *that*

$$\|f\|_{L^r} \leqslant \|f\|_{L^p}^{\theta} \|f\|_{L^q}^{1-\theta}. \tag{10}$$

Next, we quote a modification of the Sobolev embedding theorem (see (19) on p 26 in [19]).

Lemma 2. *There exists a constant* $C_d > 0$, *which depends only on the dimension* $d \geqslant 1$, *so that it is true for every function* $f(x)$ *on* $x \in \mathbb{R}^d$ *that*

$$\|f\|_{L^2} \leqslant C_d(\|\nabla f\|_{L^2} + \|f\|_{L^1}). \tag{11}$$

The main result of our analysis is the following theorem.

Theorem 1. *There exist* $\mu(d) > 0$ *and* $C(\mu, d, \sigma) > 0$, *so that for every* $0 < \mu < \mu(d)$ *and every function* $\phi(x)$ *on* $x \in \mathbb{R}^d$, *it is true that*

$$\|\phi\|_{L^4}^4 \leqslant \mu H_1(\phi) + C(\mu, d, \sigma)\big(\|\phi\|_{L^2}^2 + \|\phi\|_{L^2}^4\big). \tag{12}$$

Proof. Let $f = h^{3/2}$ with $h(x) > 0$ in (11) and obtain

$$\int_{\mathbb{R}^d} h^3 \, dx \leqslant C_d^2 \left[\frac{3}{2} \left(\int_{\mathbb{R}^d} h|\nabla h|^2 \, dx \right)^{1/2} + \int_{\mathbb{R}^d} h^{3/2} \, dx \right]^2.$$

Next, we set $h = |\phi|^2$ and obtain

$$\begin{aligned} \int_{\mathbb{R}^d} |\phi|^6 \, dx &\leqslant C_d^2 \left[\frac{3}{2} \left(\int_{\mathbb{R}^d} |\phi|^2 |\nabla|\phi|^2|^2 \, dx \right)^{1/2} + \int_{\mathbb{R}^d} |\phi|^3 \, dx \right]^2 \\ &\leqslant C_d^2 \left[\frac{9}{2} \int_{\mathbb{R}^d} |\phi|^2 |\nabla|\phi|^2|^2 \, dx + 2 \left(\int_{\mathbb{R}^d} |\phi|^3 \, dx \right)^2 \right] \\ &\leqslant C_{d,\sigma}\big[H_1(\phi) + \|\phi\|_{L^3}^6\big], \end{aligned}$$

for some positive constant $C_{d,\sigma}$. We have used here that

$$(\sqrt{a} + b)^2 \leqslant 2(a + b^2).$$

By the Gagliardo–Nirenberg inequality (10), we have $\|\phi\|_{L^3} \leqslant \|\phi\|_{L^2}^{1/2} \|\phi\|_{L^6}^{1/2}$, such that the estimate for $\|\phi\|_{L^6}$ is rewritten in the form

$$\begin{aligned} \|\phi\|_{L^6}^6 &\leqslant C_{d,\sigma} H_1(\phi) + C_d^2 \|\phi\|_{L^6}^3 \|\phi\|_{L^2}^3 \\ &\leqslant C_{d,\sigma} H_1(\phi) + C_d^2 \left[\mu \|\phi\|_{L^6}^6 + \frac{1}{4\mu} \|\phi\|_{L^2}^6 \right], \end{aligned} \tag{13}$$

where we have used the Cauchy–Schwartz inequality:

$$\forall \mu > 0: \qquad ab \leqslant \mu a^2 + \frac{b^2}{4\mu}. \tag{14}$$

Let $\mu < \mu(d)$, where $2C_d^2 \mu(d) = 1$. The term $\|\phi\|_{L^6}^6$ can be estimated from the bound (13) as follows:

$$\|\phi\|_{L^6}^6 \leqslant \tilde{C}_{d,\sigma} H_1(\phi) + \tilde{C}_d \|\phi\|_{L^2}^6 \tag{15}$$

for some constants $\tilde{C}_{d,\sigma} > 0$ and $\tilde{C}_d > 0$. By the Gagliardo–Nirenberg inequality (10), we have $\|\phi\|_{L^4} \leqslant \|\phi\|_{L^6}^{3/4}\|\phi\|_{L^2}^{1/4}$, such that the upper bound for $\|\phi\|_{L^4}$ follows from (14) and (15):

$$\|\phi\|_{L^4}^4 \leqslant \|\phi\|_{L^2}\big[\hat{C}_{d,\sigma}\sqrt{H_1(\phi)} + \hat{C}_d\|\phi\|_{L^2}^3\big]$$
$$\leqslant \frac{\hat{C}_{d,\sigma}}{4\mu}\|\phi\|_{L^2}^2 + \hat{C}_d\|\phi\|_{L^2}^4 + \mu H_1(\phi),$$

which is the desired upper bound (12). □

As a corollary of the main theorem, we pick $\mu = \mu(d)/2$ and immediately obtain the following bounds on the two parts of the energy functionals (9).

Corollary 1. *There exist constants $C_1 > 0$ and $C_2 > 0$ that depend on d, γ_0, σ, so that*

$$H_1(w) \leqslant C_1(H(w) + P(w) + P^2(w)) \tag{16}$$

and

$$H_2(w) \leqslant C_2(H(w) + P(w) + P^2(w)). \tag{17}$$

Thus, if $H(w)$ and $P(w)$ are conserved by the time evolution, the Cauchy problem for the averaged NLS equation (3) has global solutions in H^1. Therefore, assuming that physically relevant solutions conserve Hamiltonian H and charge P, we conclude that the blow-up of solutions of the cubic NLS equation (3) with $\gamma_0 < 0$ in $d \geqslant 2$ is arrested by the nonlinearity management for any $\sigma \neq 0$.

4. Averaged equation versus full dynamics

As we have shown in section 3, solutions of the averaged NLS equation (3) with $\gamma_0 < 0$ and $\sigma \neq 0$, which preserve $H(w)$ and $P(w)$, do not blow up in higher dimensions. This result raises the question whether the blow-up of solutions is arrested in the time-periodic NLS equation (1) for any non-zero variance of the nonlinearity coefficient $\Gamma(t)$. We address this question within the ODE reduction of the time-periodic problem, which was considered recently through the variational method [4, 5] and the method of moments [6]. We note that the ODE reductions imply that the solution $u(x, t)$ remains infinitely smooth and the blow-up may only occur due to infinite increase in certain norms. Therefore, the blow-up in the ODE approach would correspond directly to the blow-up estimates obtained in section 3.

It follows from the method of moments [6] that the time evolution of the radially symmetric localized solutions of the full NLS equation (1) is approximated by a time-dependent, generalized Ermakov–Pinney [20] equation:

$$\ddot{R}(t) = \frac{Q_1}{R^3} + \Gamma(t)\frac{Q_2}{R^{d+1}}, \tag{18}$$

where $R(t) \geqslant 0$ is an effective width of a localized solution, while (Q_1, Q_2) are constants found from initial data, such that $Q_2 > 0$ (see section 3 in [6] for details). We shall consider the critical case $d = 2$ and rewrite the ODE (18) with the nonlinearity coefficient $\Gamma(t)$ in (2) in the explicit form

$$\ddot{R}(t) = \frac{\alpha + \beta\gamma(t/\epsilon)}{R^3}, \tag{19}$$

where $\alpha = Q_1 + \gamma_0 Q_2$, $\beta = Q_2/\epsilon > 0$ and $\gamma(\tau)$, $\tau = t/\epsilon$, is a periodic function with zero mean and unit period. Conditions for blow-up versus existence of bounded oscillations in solutions of the ODE (19) were found in [21] (see section 4 in [6]). The sufficient condition for the blow-up (when $R(t) \to 0$ in a finite time $t \to t_0$) is

$$\alpha + \beta \max_{0 \leqslant \tau \leqslant 1} (\gamma) < 0. \tag{20}$$

The necessary condition for the bounded oscillations (when $R(t) > 0$ for any $t \geqslant 0$) is

$$\alpha + \beta \max_{0 \leqslant \tau \leqslant 1} (\gamma) > 0. \tag{21}$$

Numerical simulations in the case $\alpha < 0$ showed that the condition (21) was also sufficient for the blow-up arrest for any $t \geqslant 0$ [6].

It is obvious from the definition of α and β that $\beta \gg |\alpha|$ in the asymptotic limit $\epsilon \to 0$. Therefore, the condition (21) is satisfied and solutions of the time-periodic ODE (19) do not collapse, in agreement with our results derived in the context of the averaged NLS equation (3).

Previous work [4] (see also the review in [10]) has also addressed the averaged NLS equation (3) in the limit of *weak* nonlinearity management, when $\gamma(\tau)$ and σ^2 are rescaled as $\gamma = \epsilon\tilde{\gamma}(\tau)$ and $\sigma^2 = \epsilon^2\tilde{\sigma}^2$ such that σ^2 is small in the limit $\epsilon \to 0$. Solutions of the averaged NLS equation (3) with $\sigma^2 = \epsilon^2\tilde{\sigma}^2$ blow up in a finite time for $\epsilon = 0$, but the small ϵ^2-terms formally stabilize the blow-up for any $\tilde{\sigma}^2 > 0$. Although the truncation of the critical NLS equation by neglecting terms of the order of ϵ^4 has been considered in many applications of nonlinear optics (see sections 4 and 5 in [22] and references therein), it is clearly insufficient for a correct identification of the domain, where the blow-up of solutions occurs. Indeed, while the averaged NLS equation (3) with small σ^2 predicts no blow-up of solutions, the full NLS equation (1) with $\Gamma(t) < 0$ may have the blow-up, according to recent results [7].

The weak nonlinearity management corresponds to the case $\beta \approx |\alpha|$. Both domains (20) and (21) for blow-up and bounded oscillations fit to the limit of weak nonlinearity management. However, we will show that the averaged equation cannot distinguish between these two different cases of global dynamics in solutions of the ODE problem (19). Following [5], we consider a simple time-periodic ODE:

$$\ddot{R}(t) = \frac{\alpha + \beta\sin(2\pi\tau)}{R^3}, \qquad \tau = \frac{t}{\epsilon}, \tag{22}$$

where $\alpha < 0$, $\beta > 0$ and (α, β) are order of $O(1)$ in the limit $\epsilon \to 0$. By using the formal asymptotic multi-scale expansion method (see [10] for details), we construct an asymptotic solution to the problem (22):

$$R = r(t) + \epsilon^2 R_2(\tau, r) + \epsilon^4 R_4(\tau, r) + O(\epsilon^6), \qquad \tau = \frac{t}{\epsilon}, \tag{23}$$

where R_2 and R_4 are recursively found from the set of linear inhomogeneous problems,

$$R_2 = -\frac{\beta}{(2\pi)^2 r^3}\sin(2\pi\tau), \qquad R_4 = -\frac{3\alpha\beta}{(2\pi)^4 r^7}\sin(2\pi\tau) + \frac{3\beta^2}{8(2\pi)^4 r^7}\cos(4\pi\tau).$$

The mean-value term $r(t)$ satisfies an extended dynamical equation that excludes secular growth of the correction terms of the series (23) in τ:

$$\ddot{r} = \frac{\alpha}{r^3} + \varepsilon^2\frac{3\beta^2}{2r^7} + \varepsilon^4\frac{15\alpha\beta^2}{2r^{11}} + O(\varepsilon^6), \qquad \varepsilon = \frac{\epsilon}{2\pi}. \tag{24}$$

The averaged ODE problem (24) is an equation of motion for an effective particle with a coordinate $r(t)$ in the potential field with an effective potential energy:

$$U(r) = \frac{\alpha}{2r^2} + \varepsilon^2\frac{\beta^2}{4r^6} + \varepsilon^4\frac{3\alpha\beta^2}{4r^{10}} + O(\varepsilon^6). \tag{25}$$

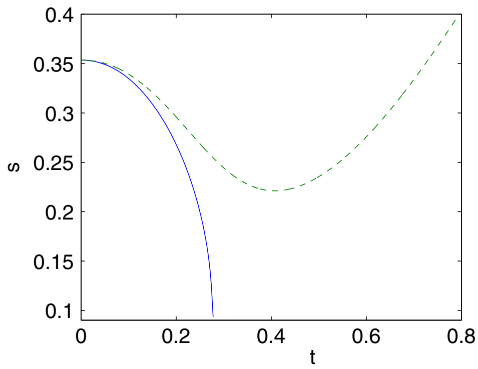

Figure 1. Numerical simulations of the full NLS equation (1) (solid curve) and the averaged NLS equation (3) (dashed curve) with a fourth order in time scheme, where spacings are $\mathrm{d}x = 0.075$ and $\mathrm{d}t = 10^{-5}$. The half-width of the wavefunction $s = \bar{x}/2$ is shown as a function of time t.

(This figure is in colour only in the electronic version)

When $\alpha < 0$ and $\varepsilon = 0$, the particle with $r(0) > 0$ reaches $r = 0$ in a finite time $t = t_0 < \infty$, that indicates the blow-up of a localized solution. When the next ε^2-term is taken into account (as in the approximation of weak nonlinearity management [4, 5]), the blow-up is arrested and the mean-value term $r(t)$ oscillates in an effective minimum of the potential energy $U(r)$, truncated at ε^2-terms. When the next ε^4-term is taken into account (beyond the approximation of weak nonlinearity management), the potential energy $U(r)$ with $\alpha < 0$ does not prevent the blow-up of the localized solution depending on the initial data $r(0)$. Existence versus non-existence of blow-up depends on the ratio of parameters (α, β) but the difference is detected within the averaging method only if convergence of the power series (25) is established in a closed analytical form.

Similarly, the averaged NLS equation (3) cannot be used in the limit of weak nonlinearity management for an accurate prediction of existence versus non-existence of blow-up of solutions. In order to illustrate this point, we have performed numerical simulations of the time-periodic NLS equation (1) in two dimensions ($d = 2$) with

$$\Gamma(t) = -20 + 8\sin(2\pi t),$$

where numerical values in $\Gamma(t)$ correspond to the sufficient condition for blow-up (20) to occur. We have observed numerically that collapse of radially symmetric Gaussian initial data does occur (see solid curve in figure 1) by monitoring the width of the one-dimensional slice along $y = 0$ of the wavefunction,

$$\bar{x} = \left(\frac{\int x^2 |u(x,0,t)|^2 \, \mathrm{d}x}{\int |u(x,0,t)|^2 \, \mathrm{d}x} \right)^{1/2},$$

until it becomes comparable to the lattice grid spacing used (at that scale collapse is arrested, since the numerical scheme cannot resolve scales below the grid spacing). On the other hand, numerical simulations of the averaged NLS equation (3) with the same parameters show that the width $\bar{x}$ never decreased below $\bar{x} < 0.45$ (see dashed curve in figure 1) indicating the absence of collapse in accordance with results of section 3. Similar outcomes of numerical simulations were observed for other numerical values in $\Gamma(t)$, as soon as the sufficient condition for blow-up (20) was satisfied.

We summarize that the averaged NLS equation (3) can only be used for modelling of the blow-up arrest in the limit of strong nonlinearity management of the full NLS equation (1). The averaged NLS equation cannot distinguish between the blow-up and no blow-up domains in the limit of weak nonlinearity management.

5. Conclusion

In conclusion, we have studied the question whether solutions of the averaged NLS equation blow up in finite time. The averaged NLS equation describes strong nonlinearity management of the time-periodic NLS equation. We have showed with formal scaling arguments and rigorous analysis that the blow-up of solutions in higher dimensions is arrested within the averaged NLS equation. We have also discussed the non-applicability of the averaged NLS equation to the weak nonlinearity management, where the blow-up of solutions can occur beyond the weak management limit.

It is an open problem to study conditions for blow-up in the time-periodic NLS equation, depending on parameters of the nonlinearity management and profile of initial data. Rigorous results on the latter problem are only available within the ODE approximation (18), when the PDE problem reduces to a dynamical system with 1 degree of freedom. It would be particularly interesting to study mathematically and to examine numerically whether the theoretical prediction from the method of moments provides an optimal bound for the full PDE problem with arbitrary initial data.

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