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Bifurcations of travelling wave solutions in the discrete NLS equations

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Abstract

We study discrete nonlinear Schrödinger (NLS) equations, which include the cubic NLS lattice with on-site interactions and the integrable Ablowitz–Ladik lattice. Standing wave solutions are known to exist in the discrete NLS equations outside of the finite spectral band. We study travelling wave solutions which have nonlinear resonances with unbounded linear spectrum. By using center manifold and normal form reductions, we show that a continuous NLS equation with the third-order derivative term is a canonical normal form for the discrete NLS equation near the zero-dispersion limit. Bifurcations of travelling wave solutions near the zero-dispersion limit are analyzed in the framework of the third-order derivative NLS equation.

We show that there exists a continuous two-parameter family of single-humped travelling wave solutions in the third-order derivative NLS equation, when it is derived from the integrable Ablowitz–Ladik lattice. On the contrary, there are no single-humped solutions in the third-order derivative NLS equation, when it is derived from the cubic NLS equation with on-site interactions. Nevertheless, we show that there exists an infinite discrete set of one-parameter families of double-humped travelling wave solutions in the latter case. Our results are valid in the neighborhood of the zero-dispersion point on the two-parameter plane of travelling wave solutions.

Keywords: Discrete NLS lattice; Ablowitz–Ladik lattice; Travelling waves; Bifurcations and persistence; Center manifold reductions; Normal forms; Asymptotic analysis

1. Introduction

Discretizations of the nonlinear Schrödinger (NLS) equation occur in many physical applications, including optical waveguides, periodic optical lattices, Fermi–Pasta–Ulum problems, and numerical finite difference schemes

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[22]. We address here the discrete NLS equation in the general form:

$$\mathrm{i}\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + \epsilon^2 f(u_n, u_{n+1}, u_{n-1}), \tag{1.1}$$

where $u_n = u_n(t) \in \mathbb{C}$, $t \in \mathbb{R}$, $n \in \mathbb{Z}$, ϵ is the step-size parameter, and $f : \mathbb{C}^3 \to \mathbb{C}$ is a nonlinear function, which satisfies the gauge symmetry:

$$\forall \alpha \in \mathbb{R} : \quad f(u_n\,\mathrm{e}^{\mathrm{i}\alpha}, u_{n+1}\,\mathrm{e}^{\mathrm{i}\alpha}, u_{n-1}\,\mathrm{e}^{\mathrm{i}\alpha}) = \mathrm{e}^{\mathrm{i}\alpha} f(u_n, u_{n+1}, u_{n-1}). \tag{1.2}$$

Cubic discrete NLS equations are of particular interest, since they are derived from the underlying physical problems by using asymptotic multi-scale expansions [9]. We shall apply our general theory to two most important discrete NLS equations with cubic nonlinearities: the discrete NLS (dNLS) lattice with on-site interactions [11,23],

$$f = |u_n|^2 u_n \tag{1.3}$$

and the Ablowitz–Ladik (AL) lattice [1],

$$f = |u_n|^2 (u_{n+1} + u_{n-1}). \tag{1.4}$$

Standing wave solutions of the form $\psi_n(t) = \phi_n\,\mathrm{e}^{-\mathrm{i}\omega t}$, where $n \in \mathbb{Z}$, $\phi_n \in \mathbb{R}$, and $\lim_{|n|\to\infty} \phi_n = 0$, exist for rather general nonlinearities $f(u_n, u_{n+1}, u_{n-1})$, provided that $\omega \in \mathbb{R}\backslash[-4, 0]$ [26,21]. We address here existence of the travelling wave solutions in the form:

$$u_n(t) = \phi(z)\,\mathrm{e}^{-\mathrm{i}\beta n - \mathrm{i}\omega t}, \qquad z = n - vt, \tag{1.5}$$

where ω, v, and β are real-valued parameters, while $z \in \mathbb{R}$, $\phi \in \mathbb{C}$, and $\lim_{|z|\to\infty} \phi(z) = 0$. Only two parameters (ω, v) are independent, since the term $\mathrm{e}^{-\mathrm{i}\beta n}$ can be included in the definition of $\phi(z)$. Two methods can be used to normalize parameter β uniquely. In one method, one can require [14] that the exponential decay of $\phi(z)$ be real-valued:

$$\lim_{|z|\to\infty} \mathrm{e}^{\kappa|z|}\,\phi(z) = \phi_\infty, \tag{1.6}$$

where $\kappa \in \mathbb{R}$ and $\phi_\infty \in \mathbb{C}$. In the other method, one can map a double eigenvalue of a linear operator to the origin. We explain these two methods in more details in Section 2.

The tail analysis of the travelling wave solutions (1.5) and (1.6) shows that parameters β and κ are uniquely defined in terms of ω and v (see relations (2.8) in Section 2). In particular, the travelling wave solutions (1.5) are exponentially decaying, if they exist, only if

$$(\omega + 2)^2 + v^2 \geq 4. \tag{1.7}$$

The main problem is to study if there exists any travelling wave solutions (1.5) and (1.6) in the domain (1.7). This problem has been recently attended in literature, where several contradictory results were obtained.

Feddersen and Duncan et al. [13,10] reported numerical results based on the trigonometric approximations and Newton–Raphson iterations. The numerics showed decaying solutions $\phi(z)$ with $v \neq 0$ in the dNLS lattice (1.3).

Flach et al. [15] developed an “inverse” method, which allowed them to compute the nonlinearity function $f(u_n, u_{n+1}, u_{n-1})$ from any existing travelling wave solution (1.5) with $\phi \in \mathbb{R}$. The method recovered the AL lattice (1.4), which exhibited the travelling wave solutions with real-valued $\phi(z)$ [1]. The method showed a contradiction for the dNLS lattice (1.3), where travelling wave solutions with real-valued $\phi(z)$ did not exist.

Numerical iterations for a minimization of a nonlinear functional were developed by Flach and Kladko [14]. The authors showed that there existed a critical point of the nonlinear functional, which corresponded to the travelling wave solution (1.5) in the dNLS lattice (1.3), but this point was not isolated from a dense set of other critical points. This numerical picture was very different in the case of standing wave solutions with $v = \beta = 0$, where a unique minimum of the nonlinear functional existed.

Existence of travelling wave solutions (1.5) in the dNLS lattice (1.3) was considered numerically by Ablowitz et al. [2,3] with the Petviashvili's iteration method and the discrete Fourier transform. Iterations of the Petviashvili's method converged to a decaying solution $\phi(z)$ in the dNLS lattice (1.3) with $v \neq 0$ but the limit to the continuous Fourier transform did not converge to a solution. The authors of [3] have conjectured that a true (continuous) travelling wave solution (1.5) does not exist in the dNLS lattice (1.3), contrary to the previous numerical results.

We analyze this problem near the zero-dispersion point on the boundary of the existence domain (1.7):

$$\omega = -2, \qquad v = 2. \tag{1.8}$$

We use the formalism of center manifold and normal form reductions [18], which was applied recently to the lattice Klein–Gordon equations by Iooss [17,19] and Mallet-Paret [27]. Methods of dynamical systems allow us to consider the bifurcations of travelling wave solutions when the nonlinearity term $\epsilon^2 f(u_n, u_{n+1}, u_{n-1})$ is small compared to the linear difference operator term in the general discrete NLS equation (1.1). We assume therefore that ϵ^2 is a small parameter of the problem. Contrary to these mathematical motivations, the recent physical applications are found for non-small values of ϵ^2, where the bifurcation analysis is not applicable [11,23].

It is natural to ask whether the bifurcation analysis can be developed near other general points on the boundary of the existence domain (1.7). Although the normal form can also be derived in the general case, it is integrable in all powers of ϵ^2, such that persistence or non-persistence of travelling wave solutions can only be studied by analyzing exponentially small in ϵ, beyond-all-orders correction terms. In the case of the special point (1.8), however, the persistence problem can be studied at the third order in ϵ, with the generally non-integrable polynomial normal form. We shall avoid dealing with exponentiall small, beyond-all-orders corrections terms in this paper.

When bifurcations of travelling wave solutions are considered in the neighborhood of the zero-dispersion point (1.8), it is natural to represent the point (ω, v) as follows:

$$\omega = -2 + \epsilon^2 \Omega, \qquad v = 2 + \epsilon^2 V. \tag{1.9}$$

The main result of this paper is the derivation and analysis of the normal form for bifurcations of travelling wave solutions near the zero-dispersion point (1.8). We show that the vector normal form reduces to the scalar third-order differential equation:

$$\frac{\mathrm{i}}{3\epsilon^2}\Phi''' - \mathrm{i}V\Phi' + \Omega\Phi = h(\Phi, \Phi', \Phi'', \Phi'''), \tag{1.10}$$

where $\Phi : \mathbb{R} \to \mathbb{C}$ and $h : \mathbb{C}^3 \to \mathbb{C}$, such that

$$\forall \alpha \in \mathbb{R} : \quad h(\Phi\, \mathrm{e}^{\mathrm{i}\alpha}, \Phi'\, \mathrm{e}^{\mathrm{i}\alpha}, \Phi''\, \mathrm{e}^{\mathrm{i}\alpha}, \Phi'''\, \mathrm{e}^{\mathrm{i}\alpha}) = \mathrm{e}^{\mathrm{i}\alpha}\, h(\Phi, \Phi', \Phi'', \Phi'''). \tag{1.11}$$

For the dNLS lattice (1.3), the nonlinearity function $h(\Phi, \Phi', \Phi'', \Phi''')$ takes the explicit form:

$$h = |\Phi|^2\Phi + \frac{1}{140}(6|\Phi|^2\Phi'' - 2\Phi^2\bar{\Phi}'' + (\Phi')^2\bar{\Phi} - 3|\Phi'|^2\Phi), \tag{1.12}$$

while for the integrable AL lattice (1.4), it takes the form:

$$h = -2\mathrm{i}|\Phi|^2\Phi' + \frac{\mathrm{i}}{100}(4\Phi\Phi''\bar{\Phi}' - 2\Phi\Phi'\bar{\Phi}'' - 2\Phi'\Phi''\bar{\Phi} + |\Phi|^2\Phi''' - \Phi^2\bar{\Phi}'''). \tag{1.13}$$

The differential equation (1.10) corresponds to the continuous NLS equation with the third-order derivative term (referred to as the third-order derivative NLS equation):

$$\mathrm{i}U_t + \frac{\mathrm{i}}{3\epsilon^2}U_{xxx} = h(U, U_x, U_{xx}, U_{xxx}), \tag{1.14}$$

when $U(x,t)$ is the travelling solitary wave solutions of the form:

$$U(x,t) = \Phi(z)\,\mathrm{e}^{-\mathrm{i}\Omega t}, \quad z = x - Vt. \tag{1.15}$$

Solitary wave solutions (1.15) of the third-order derivative NLS equation (1.14) are referred to as embedded solitons. When $h = |U|^2 U$, the embedded solitons of the third-order differential equation (1.10) were recently studied numerically and analytically [24,16,6,30]. When $h = -2\mathrm{i}|U|^2 U_x$, the third-order equation (1.10) is integrable and is referred to as the Hirota equation [20,12,29].

Embedded solitons (1.15) of the third-order derivative NLS equation (1.14) correspond to the travelling wave solutions (1.5) of the discrete NLS equation (1.1). Using the normal form equation (1.10), we show near the zero-dispersion point (1.8) that there exists a continuous two-parameter family of single-humped travelling wave solutions for the AL lattice (1.4) and no single-humped solutions in the dNLS lattice (1.3). On the other hand, we show near the same point that there exists an infinite discrete set of one-parameter families of double-humped travelling wave solutions in the dNLS lattice (1.3).

Our paper is structured as follows. We study the linear difference operator of the discrete NLS equation (1.1) in Section 2. The dynamical system formalism is reviewed in Section 3, in application to the discrete NLS equation (1.1). Center manifold reductions for the zero-dispersion point (1.8), which corresponds to the zero eigenvalue of algebraic multiplicity *six* and geometric multiplicity *two*, are described in Section 4. Vector normal forms for these center manifold reductions are derived in Section 5. Section 6 reports results on existence of travelling wave solutions in the dNLS lattice (1.3), while Section 7 reports similar results for the AL lattice (1.4). Open problems are discussed in Section 8. Appendix A describes a formal derivation of the third-order differential equation (1.10) from the discrete NLS equation (1.1).

2. Linear properties of the discrete NLS equations

The ansatz (1.5) for travelling wave solutions reduces the discrete NLS equation (1.1) to the differential advance–delay equation of the form:

$$\begin{aligned} -\mathrm{i}v\phi'(z) &= \phi(z+1)\,\mathrm{e}^{-\mathrm{i}\beta} + \phi(z-1)\,\mathrm{e}^{\mathrm{i}\beta} - (2+\omega)\phi(z) \\ &\quad + \epsilon^2 f(\phi(z), \phi(z+1)\,\mathrm{e}^{-\mathrm{i}\beta}, \phi(z-1)\,\mathrm{e}^{\mathrm{i}\beta}), \end{aligned} \tag{2.1}$$

where we have used the gauge symmetry (1.2). In this section, we set $\epsilon^2 = 0$ and study the linear properties of the problem (2.1), defined by the Fourier modes:

$$\phi(z) = \mathrm{e}^{\mathrm{i}kz}, \quad k \in \mathbb{R}, \tag{2.2}$$

where the wavenumber k is related to other parameters by the dispersion relation:

$$\omega = \omega(k) = -vk + 2(\cos(\beta - k) - 1). \tag{2.3}$$

The dispersion curve $\omega = \omega(k)$ is shown in Fig. 1 for (a) $v = 0$ and (b) $v = 0.5$, when $\beta = 0$. The wave spectrum resides on the segment $\omega \in [-4, 0]$ in the case $v = 0$ and on the line $\omega \in \mathbb{R}$ in the case $v \neq 0$. As a result, travelling wave solutions with $v \neq 0$ must have resonances with the wave spectrum (2.2) and (2.3).

Bifurcations of travelling wave solutions may occur from quadratic points of the dispersion relation $\omega = \omega(k)$ [17,19]. Directions of possible bifurcations are shown in Fig. 1 by vertical arrows. Choosing appropriate values of the parameter β, the quadratic points of $\omega = \omega(k)$ can be mapped to the zero roots $k = 0$. Therefore, we define the bifurcation point on the parameter plane (ω, v) by the two condition for the double root $k = 0$: $\omega = \omega(0)$ and $\omega'(0) = 0$. It follows from (2.3) that the bifurcation point (ω, v) is parameterized by β as follows:

$$\omega = 2(\cos\beta - 1), \qquad v = 2\sin\beta, \qquad \beta \in [0, 2\pi]. \tag{2.4}$$

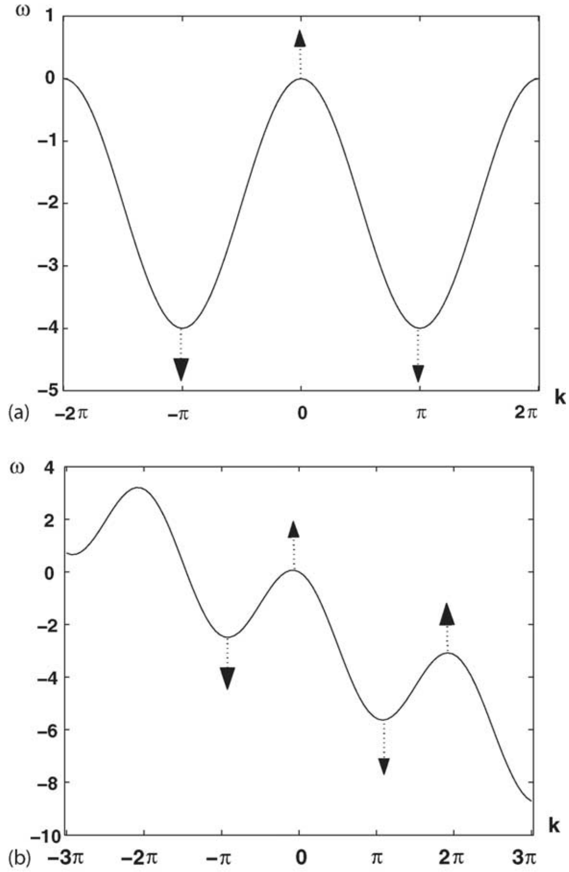

Fig. 1. Dispersion curves $\omega = \omega(k)$ for (a) $v = 0$ and (b) $v = 0.5$, when $\beta = 0$. The vertical arrows show directions of possible bifurcations of travelling wave solutions.

The circle (2.4) is nothing but the boundary of the existence domain (1.7). Due to the symmetry, it is sufficient to consider the half-plane $v \geq 0$ in the parameter space (ω, v), shown in Fig. 2. For $v \geq 0$, there are two bifurcation curves: (i) $\omega + 2 = \sqrt{4 - v^2}$, when $\beta \in [0, \pi/2]$ and (ii) $\omega + 2 = -\sqrt{4 - v^2}$, when $\beta \in [\pi/2, \pi]$. Bifurcations may occur outward the circle, away of the fundamental spectral band, shown by shaded area in Fig. 2. For instance, the standing wave solutions ($v = 0$) of the discrete NLS lattice (1.3) bifurcate to $\omega > 0$ for $\epsilon^2 > 0$ and to $\omega < -4$ for $\epsilon^2 < 0$ [21].

Let us define a set of roots $\{k_n\}_{n=1}^N$ as resonances of the travelling wave solutions (1.5) if $\omega(k_n) = \omega = \omega(0)$ and $k_n \neq 0$. The resonant roots $k = k_n$ are non-zero real-valued roots of the transcendental equation:

$$D(k) = \cos\beta + k \sin\beta - \cos(\beta - k) = 0, \quad \beta \in [0, 2\pi]. \tag{2.5}$$

The number N of resonant roots depends on the parameter β on the bifurcation curve (2.4). Fig. 2 shows the segments of the bifurcation curve, according to the number of resonant roots $N = 1$, 3, and 5. The number N is always odd and it increases unlimitedly as $v \to 0^+$. In the point $v = 0$, no resonances exist, such that $N = 0$.

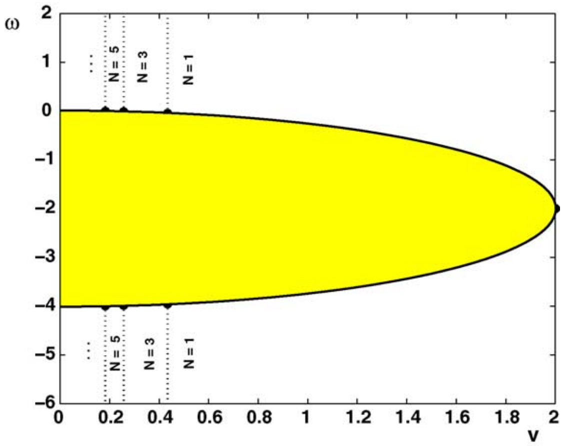

Fig. 2. The bifurcation curve on the parameter plane (ω, v). The shaded area shows the location of the fundamental spectral band, when travelling wave solutions do not exist. The vertical lines separate the parameter plane by the number N of resonance points.

The root $k = 0$ is a double root of $D(k) = 0$, when $\beta \neq \pi/2$. The special zero-dispersion point (1.8) corresponds to the particular value $\beta = \pi/2$, when the root $k = 0$ is a triple root of $D(k) = 0$:

$$\beta = \frac{\pi}{2}: \quad D(k) = k - \sin k = \frac{1}{6}k^3 + \mathrm{O}(k^5). \tag{2.6}$$

It is clear from (2.6) that no non-zero resonance roots exist for $\beta = \pi/2$, such that the only resonance root $k = k_1$ for $\beta \neq \pi/2$ degenerates into $k_1 = 0$ as $\beta \to \pi/2$.

In order to explain why the double root $k = 0$ gives the bifurcation point for travelling wave solutions (1.5), while the non-zero roots $k = k_n$ give resonances, we shall adopt the "tail analysis", developed in [14]. Assuming that the travelling wave solution $\phi(z)$ exists, we are looking for the exponentially decaying tails as $|z| \to \infty$:

$$\phi(z) = \phi_\infty \,\mathrm{e}^{-\kappa|z|}, \quad \kappa \in \mathbb{R}. \tag{2.7}$$

The tails satisfy the differential advance–delay equation (2.1) for $\epsilon^2 = 0$, provided that the four parameters $(\omega, v, \beta, \kappa)$ are related by the two equations:

$$\omega = 2\cos\beta\cosh\kappa - 2, \qquad v\kappa = 2\sin\beta\sinh\kappa. \tag{2.8}$$

The existence domain (1.7) follows from the parametrization (2.8). In the limit $\kappa \to 0$, the two relations (2.8) recover the bifurcation conditions (2.4), which is the boundary of the existence domain (1.7). In other words, bifurcations of the travelling wave solution $\phi(z)$ occur in the critical situation when the double root $k = 0$ for the Fourier mode (2.2) splits into two imaginary values $k = \pm i\kappa$ for the exponentially decaying tails (2.7). In parameter space (ω, v), this bifurcation occurs at the boundary between the linear wave spectrum (2.3) and the nonlinear wave spectrum (2.8), assuming that the travelling wave solutions $\phi(z)$ exist. If other roots $k = k_n$ are present for the same values of ω and v, this bifurcation becomes complicated by the fact that the resonant Fourier modes (2.2) with $k = k_n$ coexist with the tail solution (2.7) as $|z| \to \infty$.

In the main part of this paper, we derive and analyze the normal form for this bifurcation in the special case $\beta = \pi/2$. Normal forms for the general bifurcation in the case $\beta \neq \pi/2$ will be derived and analyzed elsewhere. Appendix A presents a formal asymptotic analysis that is useful to truncate the differential advance–delay equation (2.1) at the third-order ODE (1.10). Rigorous results based on the Center Manifold and Normal Form Theorems are reported in Sections 3–5.

3. Formalism of discrete dynamical systems

Let $\Omega = L^2(\mathbb{Z};\mathbb{C})$ denote the Hilbert space of square-summable bi-infinite complex-valued sequences, such that $u \in \Omega$ denotes the sequence $\{u_n\}_{n\in\mathbb{Z}}$. Let $\zeta(u) = \{u_{n+1}\}_{n\in\mathbb{Z}}$ denote the shift operator, $T(u) = \bar{u}(-t)$ denote the reversibility operator, and $R_\alpha(u) = \mathrm{e}^{\mathrm{i}\alpha}u$ denote the rotation by $\alpha \in \mathbb{R}$. The three operators are continuous, one-to-one operators that map Ω onto itself. Using these notations, the general discrete NLS equation (1.1) can be rewritten as an evolution equation in Ω:

$$\mathrm{i}\dot{u} = \zeta(u) - 2u + \zeta^{-1}(u) + \epsilon^2 f(u, \zeta(u), \zeta^{-1}(u)), \tag{3.1}$$

where $f : \Omega \to \Omega$ represents a continuous, shift, reversibility, and rotation invariant function, such that

$$f \circ \zeta = \zeta \circ f, \qquad f \circ T = T \circ f, \qquad f \circ R_\alpha = R_\alpha \circ f. \tag{3.2}$$

We follow Iooss [17,19] and Mallet-Paret [27] when we rewrite the differential advance–delay equation (2.1) as a matrix–vector evolution equation. Let $p \in [-1, 1]$ be a new independent variable and $\mathbf{u} = \mathbf{u}(z, p) = (u_1, u_2, u_3, u_4)^{\mathrm{T}}$ be the vector, defined by:

$$u_1 = \phi(z), \qquad u_2 = \phi(z + p), \qquad u_3 = \bar{\phi}(z), \qquad u_4 = \bar{\phi}(z + p). \tag{3.3}$$

Let $\delta^{\pm 1}$ be the difference operators, defined by

$$\delta^{\pm 1}\mathbf{u}(z, p) = \mathbf{u}(z, \pm 1). \tag{3.4}$$

Let $\mathcal{D}$ be the Banach space for the vector $\mathbf{u}(z, p)$:

$$\mathcal{D} = \{\mathbf{u} \in \mathbb{C}^4, \mathbf{u} \in C^1(\mathbb{R}, [-1, 1]) : u_2(z, 0) = u_1(z), u_4(z, 0) = u_3(z)\}. \tag{3.5}$$

The differential advance–delay equation (2.1) takes the matrix–vector evolution form in $\mathcal{D}$:

$$-\mathrm{i}v\mathcal{J}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}z} = \mathcal{L}\mathbf{u} + \epsilon^2\mathcal{M}(\mathbf{u}), \tag{3.6}$$

where $\mathcal{J} = \mathrm{diag}(1, 1, -1, -1)$, $\mathcal{L}$ is the linear operator,

$$\mathcal{L} = \begin{pmatrix} -(2+\omega) & \mathrm{e}^{-\mathrm{i}\beta}\,\delta^{+1} + \mathrm{e}^{\mathrm{i}\beta}\,\delta^{-1} & 0 & 0 \\ 0 & -\mathrm{i}v\frac{\partial}{\partial p} & 0 & 0 \\ 0 & 0 & -(2+\omega) & \mathrm{e}^{\mathrm{i}\beta}\,\delta^{+1} + \mathrm{e}^{-\mathrm{i}\beta}\,\delta^{-1} \\ 0 & 0 & 0 & \mathrm{i}v\frac{\partial}{\partial p} \end{pmatrix}, \tag{3.7}$$

and $\mathcal{M}(\mathbf{u})$ is the nonlinear operator,

$$\mathcal{M} = \begin{pmatrix} f(u_1, \mathrm{e}^{-\mathrm{i}\beta}\,\delta^{+1}u_2, \mathrm{e}^{\mathrm{i}\beta}\,\delta^{-1}u_2) \\ 0 \\ \bar{f}(u_3, \mathrm{e}^{\mathrm{i}\beta}\,\delta^{+1}u_4, \mathrm{e}^{-\mathrm{i}\beta}\,\delta^{-1}u_4) \\ 0 \end{pmatrix}. \tag{3.8}$$

Let $\mathcal{H}$ be the Banach space for the range of $\mathcal{L}$:

$$\mathcal{H} = \{\mathbf{F} \in \mathbb{C}^4,\ \mathbf{F} \in C^0(\mathbb{R}, [-1, 1])\}. \tag{3.9}$$

The linear operator $\mathcal{L}$ maps $\mathcal{D}$ into $\mathcal{H}$ continuously. Spectrum of the operator $\mathcal{L}$ is given by the explicit solution of the problem (3.6) with $\epsilon^2 = 0$:

$$\mathbf{u}(z,p) = (u_1, u_1\,\mathrm{e}^{\lambda p}, u_3, u_3\,\mathrm{e}^{\lambda p})^{\mathrm{T}}\mathrm{e}^{\lambda z}, \tag{3.10}$$

where λ is defined by roots of the characteristic equations:

$$N(\lambda) = 2 + \omega - \mathrm{i}v\lambda - 2\cos(\beta + \mathrm{i}\lambda) = 0 \tag{3.11}$$

$$\bar{N}(\lambda) = 2 + \omega + \mathrm{i}v\lambda - 2\cos(\beta - \mathrm{i}\lambda) = 0. \tag{3.12}$$

It is clear that if λ is the root of (3.11), then $\bar{\lambda}$ is the root of (3.12). There are infinitely many isolated eigenvalues $\lambda \in \mathbb{C}$, which are roots of analytic function $N(\lambda)$. We prove that all but finite number of roots are isolated away of the line $\mathrm{Re}(\lambda) = 0$, such that no accumulation points occur on $\mathrm{Re}(\lambda) = 0$.

We prove the statement by contradiction. Let us assume that there exists a sequence of roots $\{\lambda_n = p_n + \mathrm{i}q_n\}_{n=1}^{\infty}$ of the characteristic equations (3.11) and (3.12), such that $\lim_{n\to\infty} p_n = 0$. The values (p_n, q_n) satisfy the system of equations:

$$2 + \omega \pm vq_n - 2\cosh p_n \cos(\beta \mp q_n) = 0 \tag{3.13}$$

$$vp_n - 2\sinh p_n \sin(\beta \mp q_n) = 0, \tag{3.14}$$

where the plus-minus signs correspond to the roots of $N(\lambda)$ and $\bar{N}(\lambda)$, respectively. It follows from the system (3.13) and (3.14) that $|q_n|$ is bounded from above:

$$|q_n| \le \frac{|\omega| + 4\cosh^2(p_n/2)}{|v|},$$

such that the limit $\lim_{n\to\infty}\lambda_n = \lambda_* = \mathrm{i}q_*$ exists and gives an accumulation point on $\mathrm{Re}(\lambda) = 0$ and $|\mathrm{Im}(\lambda)| < \infty$. However, since $N(\lambda)$ is analytic on $\lambda \in \mathbb{C}$, zeros of $N(\lambda)$ cannot accumulate at $|\lambda_*| < \infty$, so that the contradiction holds.

Finitely many eigenvalues λ, which are located on $\mathrm{Re}(\lambda) = 0$, define the center manifold of the dynamical system (3.6) [18]. Since the conditions of the Center Manifold Theorem [5] are satisfied (the operator $\mathcal{L}$ forms a smooth strongly continuous semi-group and the cubic nonlinearity $\mathcal{M}(\mathbf{u})$ maps $\mathcal{D}$ to itself), the center manifold exists.

When parameters ω and v are defined on the bifurcation curve (2.4), then $N(\mathrm{i}k) = 2D(k)$, where $D(k)$ is given by (2.5). In this case, the center manifold of the problem (3.6) includes a composition of two double zero eigenvalues $\lambda = 0$ and N pairs of resonance eigenvalues $\lambda = \pm\mathrm{i}k_n$, where k_n are non-zero roots of $D(k)$. Dimension of the center manifold changes at different segments of the bifurcation curve (2.4), shown in Fig. 2.

The resolvent equation,

$$(-\mathrm{i}v\lambda\mathcal{J} - \mathcal{L})\mathbf{U}(\lambda, p) = \mathbf{F}(p), \tag{3.15}$$

has to be solved for any given $\mathbf{F} = (F_1, F_2(p), F_3, F_4(p))^{\mathrm{T}} \in \mathcal{H}$. When λ is not in the spectrum of the operator $\mathcal{L}$, the inhomogeneous problem (3.15) can be solved with the exact solution:

$$U_1(\lambda) = \frac{1}{N(\lambda)}\left[F_1 + \frac{1}{\mathrm{i}v}\int_0^1 F_2(p)\,\mathrm{e}^{\lambda(1-p)-\mathrm{i}\beta}\,\mathrm{d}p + \frac{1}{\mathrm{i}v}\int_0^{-1} F_2(p)\,\mathrm{e}^{-\lambda(1+p)+\mathrm{i}\beta}\,\mathrm{d}p\right], \tag{3.16}$$

$$U_2(\lambda, p) = U_1(\lambda)\,\mathrm{e}^{\lambda p} + \frac{1}{\mathrm{i}v}\int_0^p F_2(p')\,\mathrm{e}^{\lambda(p-p')}\,\mathrm{d}p', \tag{3.17}$$

$$U_3(\lambda) = \frac{1}{N(\lambda)}\left[F_3 - \frac{1}{\mathrm{i}v}\int_0^1 F_4(p)\,\mathrm{e}^{\lambda(1-p)+\mathrm{i}\beta}\,\mathrm{d}p - \frac{1}{\mathrm{i}v}\int_0^{-1} F_4(p)\,\mathrm{e}^{-\lambda(1+p)-\mathrm{i}\beta}\,\mathrm{d}p\right], \tag{3.18}$$

$$U_4(\lambda, p) = U_3(\lambda)\,\mathrm{e}^{\lambda p} - \frac{1}{\mathrm{i}v}\int_0^p F_4(p')\,\mathrm{e}^{\lambda(p-p')}\,\mathrm{d}p'. \tag{3.19}$$

The eigenvalues λ, defined by roots of the characteristic equations (3.11) and (3.12), appear as poles in the solution of the resolvent equation (3.15). The center manifold reductions follow from the Laurent expansions of the solutions (3.16)–(3.19) near the eigenvalues λ on the line $\mathrm{Re}(\lambda) = 0$ [17,19]. We shall study the resolvent of the operator $\mathcal{L}$ and the center manifold reductions near the zero-dispersion point (1.8).

4. Center manifold reductions near the zero-dispersion point

The zero-dispersion point (1.8) corresponds to the value $\beta = \pi/2$ on the bifurcation curve (2.4). At this point, the center manifold of the system (3.6) includes the only zero eigenvalue $\lambda = 0$, which has the algebraic multiplicity *six* and the geometric multiplicity *two*. The two eigenvectors of the kernel of $\mathcal{L}$ are

$$\mathbf{u}_0 = (1, 1, 0, 0)^{\mathrm{T}}, \qquad \mathbf{w}_0 = (0, 0, 1, 1)^{\mathrm{T}}. \tag{4.1}$$

The four eigenvectors of the generalized kernel of $\mathcal{L}$ are

$$\mathbf{u}_1 = (0, p, 0, 0)^{\mathrm{T}}, \qquad \mathbf{w}_1 = (0, 0, 0, p)^{\mathrm{T}} \tag{4.2}$$

and

$$\mathbf{u}_2 = \frac{1}{2}(0, p^2, 0, 0)^{\mathrm{T}}, \qquad \mathbf{w}_2 = \frac{1}{2}(0, 0, 0, p^2)^{\mathrm{T}}, \tag{4.3}$$

where the generalized eigenvectors are normalized by the non-homogeneous problems:

$$\mathcal{L}\mathbf{u}_k = -2\mathrm{i}\mathbf{u}_{k-1}, \qquad \mathcal{L}\mathbf{w}_k = 2\mathrm{i}\mathbf{w}_{k-1}, \quad k = 1, 2. \tag{4.4}$$

At the zero-dispersion point (1.8) with $\beta = \pi/2$, the characteristic equation $N(\lambda)$ has the following Taylor expansion near $\lambda = 0$:

$$N(\lambda) = \frac{\mathrm{i}}{3}\lambda^3\left(1 + \frac{\lambda^2}{20} + \mathrm{O}(\lambda^4)\right), \tag{4.5}$$

while the solution (3.16)–(3.19) of the resolvent equation (3.15) has the following Laurent expansion at $\lambda = 0$:

$$\begin{aligned}\mathbf{U}(\lambda, p) = {}& \frac{a_{-3}\mathbf{u}_0 + b_{-3}\mathbf{w}_0}{\lambda^3} + \frac{a_{-2}\mathbf{u}_0 + a_{-3}\mathbf{u}_1 + b_{-2}\mathbf{w}_0 + b_{-3}\mathbf{w}_1}{\lambda^2} \\ & \times\frac{\left(a_{-1} - \frac{1}{20}a_{-3}\right)\mathbf{u}_0 + a_{-2}\mathbf{u}_1 + a_{-3}\mathbf{u}_2}{\lambda} \\ & + \frac{\left(b_{-1} - \frac{1}{20}b_{-3}\right)\mathbf{w}_0 + b_{-2}\mathbf{w}_1 + b_{-3}\mathbf{w}_2}{\lambda} + \mathbf{U}_0(\lambda, p),\end{aligned} \tag{4.6}$$

where $\mathbf{U}_0(\lambda, p)$ is analytic in the neighborhood of $\lambda = 0$, the projection operators a_{-3}, a_{-2}, and a_{-1} are given explicitly as

$$a_{-3}[F_1, F_2] = -3\mathrm{i}\left(F_1 - \frac{1}{2}\int_0^1 (F_2(p) + F_2(-p))\,\mathrm{d}p\right), \tag{4.7}$$

$$a_{-2}[F_1, F_2] = \frac{3\mathrm{i}}{2}\int_0^1 (1-p)(F_2(p) - F_2(-p))\,\mathrm{d}p, \tag{4.8}$$

$$a_{-1}[F_1, F_2] = \frac{3\mathrm{i}}{4}\int_0^1 (1-p)^2(F_2(p) + F_2(-p))\,\mathrm{d}p, \tag{4.9}$$

and the projection operators b_{-3}, b_{-2}, and b_{-1} are given by the transformation:

$$b_{-k} = -a_{-k}[F_3, F_4], \quad k = 1, 2, 3. \tag{4.10}$$

When parameters (ω, v) are defined near the zero-dispersion point (1.8) according to the representation (1.9), the nonlinear problem (3.6) takes the explicit form:

$$-2\mathrm{i}\mathcal{J}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}z} = \mathcal{L}_0\mathbf{u} + \epsilon^2\mathcal{N}(\mathbf{u}), \tag{4.11}$$

where the operator $\mathcal{L}_0$ and the perturbation vector $\mathcal{N}(\mathbf{u})$ are given explicitly as:

$$\mathcal{L}_0 = \begin{pmatrix} 0 & -\mathrm{i}\delta^{+1} + \mathrm{i}\delta^{-1} & 0 & 0 \\ 0 & -2\mathrm{i}\frac{\partial}{\partial p} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i}\delta^{+1} - \mathrm{i}\delta^{-1} \\ 0 & 0 & 0 & 2\mathrm{i}\frac{\partial}{\partial p} \end{pmatrix} \tag{4.12}$$

and

$$\mathcal{N} = \begin{pmatrix} f(u_1, -\mathrm{i}\delta^{+1}u_2, \mathrm{i}\delta^{-1}u_2) - \Omega u_1 + \mathrm{i}Vu_1'(z) \\ \mathrm{i}V\left(\frac{\partial u_2}{\partial z} - \frac{\partial u_2}{\partial p}\right) \\ \bar{f}(u_3, \mathrm{i}\delta^{+1}u_4, -\mathrm{i}\delta^{-1}u_4) - \Omega u_3 - \mathrm{i}Vu_3'(z) \\ -\mathrm{i}V\left(\frac{\partial u_4}{\partial z} - \frac{\partial u_4}{\partial p}\right) \end{pmatrix}. \tag{4.13}$$

We use the Center Manifold Theorem [18] [Theorem I.4] and apply the decomposition:

$$\mathbf{u}(z) = \mathbf{u}_c(z) + \epsilon^2\mathbf{u}_h(z), \tag{4.14}$$

where $\mathbf{u}_c(z)$ is the projection to the center manifold:

$$\mathbf{u}_c(z) = A(z)\mathbf{u}_0 + B(z)\mathbf{u}_1 + C(z)\mathbf{u}_2 + \bar{A}(z)\mathbf{w}_0 + \bar{B}(z)\mathbf{w}_1 + \bar{C}(z)\mathbf{w}_2, \tag{4.15}$$

and $\mathbf{u}_h(z)$ is the projection to the rest of the spectrum of operator $\mathcal{L}_0$. The problem (4.11) is then rewritten in the equivalent form,

$$-2\mathrm{i}\mathcal{J}\frac{\mathrm{d}\mathbf{u}_h}{\mathrm{d}z} - \mathcal{L}_0\mathbf{u}_h = \mathbf{F}_h \equiv \frac{1}{\epsilon^2}\left[2\mathrm{i}\mathcal{J}\frac{\mathrm{d}\mathbf{u}_c}{\mathrm{d}z} + \mathcal{L}_0\mathbf{u}_c\right] + \mathcal{N}(\mathbf{u}_c + \epsilon^2\mathbf{u}_h), \tag{4.16}$$

where the right-hand side function $\mathbf{F}_h$ maps $\mathcal{W}_0^a(\mathcal{H})$ to itself, where $\mathcal{W}_0^a(\mathcal{H})$ is the space of functions which are exponentially decaying ($a > 0$) and growing ($a < 0$) at infinity:

$$\mathcal{W}_j^a(\mathcal{H}) = \left\{ \mathbf{F} \in C^j(\mathbb{R}, \mathcal{H}) : \|\mathbf{F}\|_j = \max_{0 \le k \le j} \sup_{z \in \mathbb{R}} \mathrm{e}^{-a|z|} \, |D^k \mathbf{F}(z)| \right\}. \tag{4.17}$$

When the pole singularities at $\lambda = 0$ are removed from the Laurent expansion (4.6) by the conditions:

$$a_{-3}[F_{h1}, F_{h2}] = a_{-2}[F_{h1}, F_{h2}] = a_{-1}[F_{h1}, F_{h2}] = 0, \tag{4.18}$$

the operator $\left(\mathcal{L}_0 + 2\mathrm{i}\mathcal{J}\frac{\mathrm{d}}{\mathrm{d}z}\right)$ can be inverted and, by the Center Manifold Theorem [5,18], there exists a continuous map $\mathcal{A}_\epsilon$ from $\mathbf{u}_c \in W_1^a(\mathcal{H})$ to $\mathbf{u}_h \in W_1^a(\mathcal{D}) \cap W_1^a(\mathcal{H})$, such that $\mathbf{u}_h = \mathcal{A}_\epsilon \mathbf{u}_c$. By explicit computations, we have

$$u_{c1} = A(z), \qquad u_{c2} = A(z) + pB(z) + \frac{1}{2} p^2 C(z), \tag{4.19}$$

and, therefore, the projection equations (4.18) are expanded as follows:

$$a_{-3} = -\frac{1}{\epsilon^2} \frac{\mathrm{d}C}{\mathrm{d}z} - 3\mathrm{i}g(A, B, C) + \mathrm{O}(\epsilon^2 \|g\| + \epsilon^2 \|\mathbf{u}_h\|), \tag{4.20}$$

$$a_{-2} = -\frac{1}{\epsilon^2} \left(\frac{\mathrm{d}B}{\mathrm{d}z} - C \right) + \mathrm{O}(\epsilon^2 \|g\| + \epsilon^2 \|\mathbf{u}_h\|), \tag{4.21}$$

$$a_{-1} = -\frac{1}{\epsilon^2} \left(\frac{\mathrm{d}A}{\mathrm{d}z} - B + \frac{1}{20} \frac{\mathrm{d}C}{\mathrm{d}z} \right) + \mathrm{O}(\epsilon^2 \|g\| + \epsilon^2 \|\mathbf{u}_h\|), \tag{4.22}$$

where

$$g(A, B, C) = f\left(A, -\mathrm{i}\left(A + B + \frac{1}{2}C \right), \mathrm{i}\left(A - B + \frac{1}{2}C \right) \right) - \Omega A + \mathrm{i}VB. \tag{4.23}$$

The function $g(A, B, C)$ can be computed explicitly for the dNLS lattice (1.3):

$$g(A, B, C) = |A|^2 A - \Omega A + \mathrm{i}VB, \tag{4.24}$$

and for the integrable AL lattice (1.4):

$$g(A, B, C) = -2\mathrm{i}|A|^2 B - \Omega A + \mathrm{i}VB. \tag{4.25}$$

When the truncation error of order $\mathrm{O}(\epsilon^2 \|g\| + \epsilon^2 \|\mathbf{u}_h\|)$ is neglected, the center manifold reductions (4.20)–(4.22) can be rewritten in the vector form:

$$\frac{\mathrm{d}}{\mathrm{d}z} \begin{pmatrix} \mathbf{x} \\ \bar{\mathbf{x}} \end{pmatrix} = \mathcal{L}_c \begin{pmatrix} \mathbf{x} \\ \bar{\mathbf{x}} \end{pmatrix} + \epsilon^2 \begin{pmatrix} \mathbf{R}(\mathbf{x}) \\ \bar{\mathbf{R}}(\mathbf{x}) \end{pmatrix}, \tag{4.26}$$

where $\mathbf{x} = (A, B, C)^{\mathrm{T}} \in \mathbb{C}^3$, the vector function $\mathbf{R} : \mathbb{C}^3 \mapsto \mathbb{C}^3$ is given by

$$\mathbf{R}(\mathbf{x}) = 3\mathrm{i}\left(\frac{1}{20}, 0, -1 \right)^{\mathrm{T}} (g_{\mathrm{nl}}(x_1, x_2, x_3) - \Omega x_1 + \mathrm{i}Vx_2) \tag{4.27}$$

and

$$g_{\mathrm{nl}}(x_1, x_2, x_3) = f\left(x_1, -\mathrm{i}\left(x_1 + x_2 + \frac{1}{2}x_3 \right), \mathrm{i}\left(x_1 - x_2 + \frac{1}{2}x_3 \right) \right), \tag{4.28}$$

and the linear operator $\mathcal{L}_c$ corresponds to the center manifold, defined by the zero eigenvalue of geometric multiplicity two and algebraic multiplicity six:

$$\mathcal{L}_c = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.29}$$

The normal form for bifurcations of travelling wave solutions near the zero-dispersion point (1.8) is derived from the center manifold reductions (4.26).

5. Normal form equations near the zero-dispersion point

Equations (4.26) for center manifold reductions can be reduced to the normal form by using nearly identical transformations:

$$\mathbf{x} = \boldsymbol{\xi} + \epsilon^2 \boldsymbol{\Phi}(\boldsymbol{\xi}), \tag{5.1}$$

where $\boldsymbol{\xi} \in \mathbb{C}^3$ is a new vector, $\boldsymbol{\Phi} : \mathbb{C}^3 \mapsto \mathbb{C}^3$ is a nearly identical transformation, and $\boldsymbol{\xi}(z)$ satisfy the normal form equations:

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} \boldsymbol{\xi} \\ \bar{\boldsymbol{\xi}} \end{pmatrix} = \mathcal{L}_c \begin{pmatrix} \boldsymbol{\xi} \\ \bar{\boldsymbol{\xi}} \end{pmatrix} + \epsilon^2 \begin{pmatrix} \mathbf{P}(\boldsymbol{\xi}) \\ \bar{\mathbf{P}}(\boldsymbol{\xi}) \end{pmatrix}. \tag{5.2}$$

The nonlinear vector function $\mathbf{P} : \mathbb{C}^3 \mapsto \mathbb{C}^3$ is referred to as the normal form. When $\mathbf{R}(\mathbf{x})$ is a polynomial function of $\mathbf{x}$, then $\mathbf{P}(\boldsymbol{\xi})$ is also a polynomial function of $\boldsymbol{\xi}$. By the Normal Form Theorem [18] [Theorem I.7], the normal form $\mathbf{P}(\boldsymbol{\xi})$ must satisfy the system of partial differential equations,

$$\mathcal{D}\mathbf{P}(\boldsymbol{\xi})\mathcal{L}_c^* \boldsymbol{\xi} = \mathcal{L}_c^* \mathbf{P}(\boldsymbol{\xi}), \tag{5.3}$$

where $\mathcal{L}_c^*$ is the adjoint operator and $\mathcal{D}$ is the Jacobian. The case of zero eigenvalue of algebraic multiplicity six and geometric multiplicity two is not included in the list of previously known examples, described in the book [18]. Therefore, we develop a general characterization of the normal form $\mathbf{P}(\boldsymbol{\xi})$ in this case. Let

$$D^* = \xi_1 \frac{\partial}{\partial \xi_2} + \xi_2 \frac{\partial}{\partial \xi_3} + \bar{\xi}_1 \frac{\partial}{\partial \bar{\xi}_2} + \bar{\xi}_2 \frac{\partial}{\partial \bar{\xi}_3}. \tag{5.4}$$

Then, equations (5.3) are rewritten explicitly as

$$D^* P_1 = 0, \qquad D^* P_2 = P_1, \qquad D^* P_3 = P_2. \tag{5.5}$$

The homogeneous equation $D^* u = 0$ is satisfied for the following five quadratic variables:

$$\begin{aligned} &u_1 = \xi_1^2, \qquad u_2 = |\xi_1|^2, \qquad u_3 = \mathrm{i}(\xi_1\bar{\xi}_2 - \bar{\xi}_1\xi_2), \qquad u_4 = \xi_2^2 - 2\xi_1\xi_3, \\ &u_5 = |\xi_2|^2 - \xi_1\bar{\xi}_3 - \bar{\xi}_1\xi_3. \end{aligned} \tag{5.6}$$

Let $\mathbf{u} = (u_1, u_2, u_3, u_4, u_5)$. In new variables, the polynomial normal forms $\mathbf{P}(\boldsymbol{\xi})$, which satisfy equations (5.5), take the following general form:

$$P_1 = \mathrm{i}\xi_1 \varphi_1(\mathbf{u}) + \mathrm{i}\bar{\xi}_1 \psi_1(\mathbf{u}), \tag{5.7}$$

$$P_2 = \xi_1\varphi_2(\mathbf{u}) + \bar{\xi}_1\psi_2(\mathbf{u}) + \mathrm{i}\xi_2\varphi_1(\mathbf{u}) + \mathrm{i}\bar{\xi}_2\psi_2(\mathbf{u}), \tag{5.8}$$

$$P_3 = \mathrm{i}\xi_1\varphi_3(\mathbf{u}) + \mathrm{i}\bar{\xi}_1\psi_3(\mathbf{u}) + \xi_2\varphi_2(\mathbf{u}) + \bar{\xi}_2\psi_2(\mathbf{u}) + \mathrm{i}\xi_3\varphi_1(\mathbf{u}) + \mathrm{i}\bar{\xi}_3\psi_1(\mathbf{u}), \tag{5.9}$$

where $\varphi_{1,2,3}(\mathbf{u})$ and $\psi_{1,2,3}(\mathbf{u})$ are polynomial functions in their variables. By the reversibility invariance (3.2), the normal form equations (5.2) must be invariant under the reversibility transformation:

$$z \mapsto -z, \qquad \xi_1 \mapsto \bar{\xi}_1, \qquad \xi_2 \mapsto -\bar{\xi}_2, \qquad \xi_3 \mapsto \bar{\xi}_3. \tag{5.10}$$

This symmetry is preserved when the polynomial functions $\varphi_{1,2,3}(\mathbf{u})$ and $\psi_{1,2,3}(\mathbf{u})$ are real-valued. By the gauge invariance (3.2), the normal form equations (5.2) must be also invariant under the gauge transformation:

$$\forall\alpha \in \mathbb{R}: \quad \xi_1 \mapsto \xi_1\,\mathrm{e}^{\mathrm{i}\alpha}, \qquad \xi_2 \mapsto \xi_2\,\mathrm{e}^{\mathrm{i}\alpha}, \qquad \xi_3 \mapsto \xi_3\,\mathrm{e}^{\mathrm{i}\alpha}. \tag{5.11}$$

This symmetry is preserved when the polynomial functions $\varphi_{1,2,3}(\mathbf{u})$ and $\psi_{1,2,3}(\mathbf{u})$ include the following zero and quadratic terms:

$$\varphi_j(\mathbf{u}) = \alpha_j + \beta_j|\xi_1|^2 + \mathrm{i}\gamma_j(\xi_1\bar{\xi}_2 - \bar{\xi}_1\xi_2) + \delta_j(|\xi_2|^2 - \xi_1\bar{\xi}_3 - \bar{\xi}_1\xi_3), \tag{5.12}$$

$$\psi_j(\mathbf{u}) = \tilde{\beta}_j\xi_1^2 + \varepsilon_j(\xi_2^2 - 2\xi_1\xi_3), \tag{5.13}$$

where α_j, β_j, γ_j, δ_j, and ε_j are independent parameters, while the terms with $\tilde{\beta}_j$ are the same as the terms with β_j. Therefore, we set $\tilde{\beta}_j = 0$ without loss of generality.

At the linear and cubic nonlinear terms, the transformation function $\mathbf{\Phi}(\boldsymbol{\xi})$ satisfies the system of equations:

$$\mathcal{D}\mathbf{\Phi}(\boldsymbol{\xi})\mathcal{L}_c\boldsymbol{\xi} - \mathcal{L}_c\mathbf{\Phi}(\boldsymbol{\xi}) = \mathbf{R}(\boldsymbol{\xi}) - \mathbf{P}(\boldsymbol{\xi}). \tag{5.14}$$

Let

$$D = -\xi_2\frac{\partial}{\partial\xi_1} - \xi_3\frac{\partial}{\partial\xi_2} - \bar{\xi}_2\frac{\partial}{\partial\bar{\xi}_1} - \bar{\xi}_3\frac{\partial}{\partial\bar{\xi}_2}. \tag{5.15}$$

Then, the system (5.14) can be rewritten explicitly:

$$\Phi_2 + D\Phi_1 = P_1 - \frac{3\mathrm{i}}{20}(g_{\mathrm{nl}}(\xi_1, \xi_2, \xi_3) - \Omega\xi_1 + \mathrm{i}V\xi_2), \tag{5.16}$$

$$\Phi_3 + D\Phi_2 = P_2, \tag{5.17}$$

$$D\Phi_3 = P_3 + 3i(g_{\mathrm{nl}}(\xi_1, \xi_2, \xi_3) - \Omega\xi_1 + \mathrm{i}V\xi_2). \tag{5.18}$$

Removing the linear terms from the system of equations (5.16)–(5.18), we recursively define the transformation $\mathbf{\Phi}(\boldsymbol{\xi})$ as

$$\Phi_1 = -\frac{3V}{20}\xi_1, \tag{5.19}$$

$$\Phi_2 = -2\mathrm{i}\alpha_1\xi_1, \tag{5.20}$$

$$\Phi_3 = (3V - \alpha_2)\xi_1 - \mathrm{i}\alpha_1\xi_2, \tag{5.21}$$

where numerical parameters for the zero-order terms in (5.12) are found to be:

$$\alpha_1 = -\frac{\Omega}{20}, \qquad \alpha_2 = \frac{3V}{2}, \qquad \alpha_3 = 3\Omega. \tag{5.22}$$

In the case of the cubic dNLS lattice (4.24), we have $g_{\rm nl}(\xi_1,\xi_2,\xi_3)=|\xi_1|^2\xi_1$. Removing the cubic terms from the system (5.16)–(5.18), we find that some parameters for quadratic terms in (5.12) and (5.13) are zero:

$$\gamma_1=\delta_1=\varepsilon_1=\beta_2=\delta_2=\varepsilon_2=\gamma_3=0, \tag{5.23}$$

while the transformation $\mathbf{\Phi}(\boldsymbol{\xi})$ is recursively defined as follows:

$$\Phi_1=0, \tag{5.24}$$

$$\Phi_2=\mathrm{i}(\delta_3-\gamma_2)|\xi_1|^2\xi_1, \tag{5.25}$$

$$\Phi_3=\mathrm{i}(\delta_3-\beta_1+2\varepsilon_3)|\xi_1|^2\xi_2+\mathrm{i}\delta_3\xi_1^2\bar{\xi}_2. \tag{5.26}$$

Once the representation (5.24)–(5.26) is substituted into the system (5.16)–(5.18), we find that $\beta_3=-3$, while β_1, γ_2, δ_3, and ε_3 satisfy the linear system:

$$-\beta_1+\gamma_2+4\delta_3+2\varepsilon_3=0, \tag{5.27}$$

$$-\beta_1-\gamma_2+\delta_3+3\varepsilon_3=0, \tag{5.28}$$

$$2\beta_1-3\gamma_2+\delta_3-2\varepsilon_3=0, \tag{5.29}$$

$$\beta_1+\gamma_2-\delta_3=\frac{3}{20}. \tag{5.30}$$

The linear system (5.27)–(5.30) has the unique solution:

$$\beta_1=\frac{3}{28},\qquad \gamma_2=\frac{1}{28},\qquad \delta_3=-\frac{1}{140},\qquad \varepsilon_3=\frac{1}{20}. \tag{5.31}$$

In the case of the integrable AL lattice (4.25), we have $g_{\rm nl}(\xi_1,\xi_2,\xi_3)=-2\mathrm{i}|\xi_1|^2\xi_2$. Removing the cubic terms from the system (5.16)–(5.18), we find that some parameters for quadratic terms in (5.12) and (5.13) are zero:

$$\beta_1=\delta_1=\varepsilon_1=\gamma_2=\beta_3=\delta_3=\varepsilon_3=0, \tag{5.32}$$

while the transformation $\mathbf{\Phi}(\boldsymbol{\xi})$ is recursively defined as follows:

$$\Phi_1=\left(\frac{2\gamma_1}{3}+2\delta_2\right)|\xi_1|^2\xi_1, \tag{5.33}$$

$$\Phi_2=\left(-\frac{\gamma_1}{3}+2\delta_2\right)|\xi_1|^2\xi_2+\left(\frac{\gamma_1}{3}+2\delta_2+4\varepsilon_2\right)\xi_1^2\bar{\xi}_2, \tag{5.34}$$

$$\begin{aligned}\Phi_3&=\gamma_3|\xi_1|^2\xi_1+\frac{\gamma_1}{3}\left(|\xi_1|^2\xi_3-\xi_1^2\bar{\xi}_3+2|\xi_2|^2\xi_1-2\xi_2^2\bar{\xi}_1\right)\\&\quad+\delta_2(\xi_1\bar{\xi}_3+\bar{\xi}_1\xi_3-|\xi_2|^2)\xi_1+\varepsilon_2(2\xi_1\xi_3-\xi_2^2)\bar{\xi}_1.\end{aligned} \tag{5.35}$$

Once the representation (5.33)–(5.35) is substituted into the system (5.16)–(5.18), we find two linear systems for β_2,γ_3:

$$\beta_2+3\gamma_3=-6, \tag{5.36}$$

$$\beta_2-\gamma_3=0, \tag{5.37}$$

and for $\gamma_1,\delta_2,\varepsilon_2$:

$$-2\gamma_1+8\delta_2+4\varepsilon_2=0, \tag{5.38}$$

$$2\gamma_1 + 2\delta_2 + 6\varepsilon_2 = 0, \tag{5.39}$$

$$2\gamma_1 + 2\delta_2 - 4\varepsilon_2 = \frac{3}{10}. \tag{5.40}$$

The two linear systems have unique solutions:

$$\beta_2 = -\frac{3}{2}, \qquad \gamma_3 = -\frac{3}{2} \tag{5.41}$$

and

$$\gamma_1 = \frac{3}{50}, \qquad \delta_2 = \frac{3}{100}, \qquad \varepsilon_2 = -\frac{3}{100}. \tag{5.42}$$

The normal-form equations for the cubic dNLS and integrable AL lattices are studied separately in Sections 6 and 7.

6. Bifurcations of travelling wave solutions in the dNLS lattice

Combining the linear and cubic nonlinear terms in the normal forms (5.7)–(5.9), we rewrite explicitly the normal form equations for the cubic dNLS lattice (1.3):

$$\frac{da}{dz} = b + i\epsilon^2(\alpha_1 a + \beta_1 |a|^2 a), \tag{6.1}$$

$$\frac{db}{dz} = c + \epsilon^2(i\alpha_1 b + \alpha_2 a + i\beta_1 |a|^2 b + i\gamma_2(a\bar{b} - \bar{a}b)a) \tag{6.2}$$

$$\frac{dc}{dz} = \epsilon^2(i\alpha_1 c + \alpha_2 b + i\alpha_3 a + i\beta_1 |a|^2 c + i\gamma_2(a\bar{b} - \bar{a}b)b - 3i|a|^2 a) + \epsilon^2(i\delta_3(|b|^2 - a\bar{c} - \bar{a}c)a + i\varepsilon_3(b^2 - 2ac)\bar{a}), \tag{6.3}$$

where numerical values for α_1, α_2, α_3, β_1, γ_2, δ_3, and ε_3 are given by (5.22) and (5.31), and we have used the representation $\boldsymbol{\xi} = (a, b, c)^{\mathrm{T}} \in \mathbb{C}^3$. The normal form equations (6.1)–(6.3) can be simplified, by using the transformation:

$$(a, b, c) = (\tilde{a}, \tilde{b}, \tilde{c}) \exp\left(i\epsilon^2\alpha_1 z + i\epsilon^2\beta_1 \int_0^z |\tilde{a}|^2(z')\,dz'\right), \tag{6.4}$$

and the representation:

$$\tilde{a} = \Phi(z), \qquad \tilde{b} = \Phi'(z), \qquad \tilde{c} = \Phi''(z) - \epsilon^2\alpha_2\Phi - i\epsilon^2\gamma_2\left(\Phi\bar{\Phi}' - \bar{\Phi}\Phi'\right)\Phi. \tag{6.5}$$

Neglecting the terms of order $O(\epsilon^4)$ beyond the normal form, we derive the scalar third-order derivative equation (1.10) from the vector normal form (6.1)–(6.3), where $h(\Phi, \Phi', \Phi'', \Phi''')$ is given by (1.12).

There exists three roots in the linear part of the scalar normal form (1.10):

$$\Phi(z) = \Phi_\infty e^{-\kappa|z|} : \frac{i\kappa^3}{3\epsilon^2} - iV\kappa - \Omega = 0. \tag{6.6}$$

One root is always purely imaginary and it corresponds to the resonant Fourier mode (2.2). Two roots are complex-valued, with positive and negative real parts, when (Ω, V) gives the perturbed point (ω, v) in the existence domain

(1.7). For example, the two roots are real-valued for $\Omega = 0$ and $V > 0$. These two roots correspond to exponentially decaying tails (2.7) of the travelling wave solution $\phi(z)$, if it exists. Since all three roots are small, of order $\mathrm{O}(\epsilon)$, when $\Omega = \mathrm{O}(\epsilon)$ and $V = \mathrm{O}(1)$, we can rescale the third-order derivative NLS equation (1.10). Assuming that $\Phi \in C^3(\mathbb{R})$ for a smooth homoclinic orbit, if it exists, we apply the scaling transformation:

$$\Phi(z) = \epsilon^{1/2} S(\zeta), \qquad \zeta = \epsilon z, \qquad V = \nu, \qquad \Omega = \epsilon\mu. \tag{6.7}$$

The function $S(\zeta)$ satisfies the re-scaled third-order equation:

$$\frac{\mathrm{i}}{3} S''' - \mathrm{i}\nu S' + \mu S = |S|^2 S + \frac{\epsilon^2}{140}(6|S|^2 S'' - 2S^2 \bar{S}'' + (S')^2 \bar{S} - 3|S'|^2 S). \tag{6.8}$$

Let us consider the truncated equation (6.8):

$$\frac{\mathrm{i}}{3} S''' - \mathrm{i}\nu S' + \mu S = |S|^2 S. \tag{6.9}$$

This equation can be reduced to the standard form, by using the scaling transformation:

$$S = 3k\sqrt{|k|} R(\eta)\, \mathrm{e}^{\frac{\mathrm{i}}{3}\eta}, \quad \eta = 3k\zeta, \quad k \in \mathbb{R}, \tag{6.10}$$

where $R = R(\eta)$ satisfies the equation:

$$R'' - \lambda R + \sigma |R|^2 R = \mathrm{i}(R''' - \lambda R'), \quad \sigma = \mathrm{sign}(k), \tag{6.11}$$

and the two parameters (μ, ν) are related to new parameters (λ, k) as follows:

$$\mu = 6k^3\lambda + \frac{2}{3}k^3, \qquad \nu = -k^2 + 3\lambda k^2. \tag{6.12}$$

In the standard form (6.11), exponentially decaying and oscillatory tails are scaled as follows: $R(\eta) = R_\infty\, \mathrm{e}^{-\sqrt{\lambda}|\eta|}$ and $R(\eta) = R_0\, \mathrm{e}^{-\mathrm{i}\eta}$. Since exponentially decaying solutions for $R(\eta)$ have real-valued decay rate, we match scaling transformations (6.7) and (6.10) with the travelling wave solution anzats (1.5) and (1.6), such that the parameters (β, κ) are given at the leading order:

$$\kappa = 3\epsilon k\sqrt{\lambda}, \qquad \beta = \frac{\pi}{2} - \epsilon k, \quad \lambda > 0. \tag{6.13}$$

Additionally, the transformations (1.9), (6.7), and (6.12) define the leading-order expansions of parameters (ω, v) in the travelling wave solution (1.5):

$$\omega + 2 = \epsilon k v + \epsilon^3 \mu = 2\epsilon k + \epsilon^3(k\nu + \mu) = 2\epsilon k + \epsilon^3 k^3 \left(9\lambda - \frac{1}{3}\right), \tag{6.14}$$

$$v - 2 = \epsilon^2 \nu = \epsilon^2 k^2 (3\lambda - 1), \tag{6.15}$$

where $k \in \mathbb{R}$ and $\lambda > 0$. The leading-order expansion (6.14) and (6.15) agree with the relations (2.8) and (6.13).

The truncated equation (6.11) was studied in the focusing case $\sigma = +1$ ($k > 0$). It was proved analytically in [16] that no single-humped homoclinic orbits exists in the truncated equation (6.11) with $\sigma = +1$ for any value of λ. Therefore, no single-humped homoclinic orbits exists in the full normal form (6.8) for small $\epsilon^2 \neq 0$.

On the other hand, it was proved with the same method in [6] that double-humped and multi-humped homoclinic orbits exist in the truncated equation (6.11) with $\sigma = +1$ for special values of parameter λ. These multi-humped homoclinic orbits are associated with a positive Birkhoff signature in the normal form computations [25]. It was proved in [28,7] that such multi-humped homoclinic orbits persist in the full normal form (6.8), even beyond all orders in ϵ.

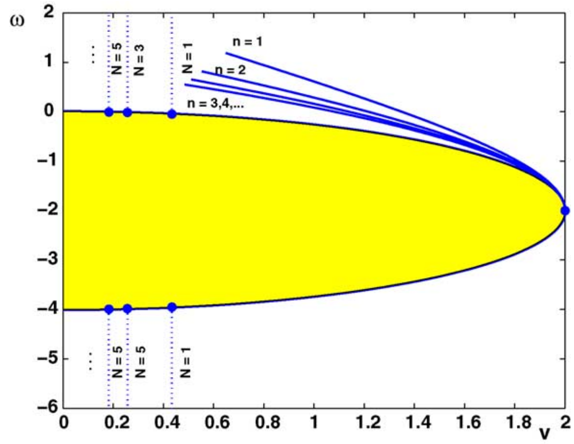

Fig. 3. Asymptotic approximations of the bifurcation curves for two-humped travelling wave solutions of the dNLS lattice (1.3) on the parameter plane (ω, v). The four curves are shown for four values of λ_n, according to the expansions (6.14) and (6.15). See Fig. 2 for other notations.

Numerical approximations of the double-humped homoclinic orbits were reported in [24,30]. The discrete infinite set of values of $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ starts with the first values: $\lambda_1 = 0.0668$, $\lambda_2 = 0.0480$, $\lambda_3 = 0.0397$, $\lambda_4 = 0.0345$ and extends to $\lim_{n\to\infty} \lambda_n = 0$. Different solutions from the set $R = \{R_n(\eta)\}_{n=1}^{\infty}$ are characterized by the different distances between the two humps, such that the solutions with larger *n* have larger distance between the two humps. Fig. 3 plots families of the doubled-humped solutions on the parameter plane (ω, v), predicted from the leading-order expansions (6.14) and (6.15). It is expected that double-humped travelling wave solutions in the discrete NLS lattice (1.3) exist in a neighborhood of these curves. Numerical search for double-humped travelling wave solutions in the dNLS lattice (1.3) is beyond the scope of this paper. We also mention that no results exist in the literature on the double-humped homoclinic orbit in the defocusing case $\sigma = -1$ $(k < 0)$, which corresponds to the lower semi-circle on Fig. 3. Additional studies of the truncated equation (6.11) with $\sigma = -1$ need to be developed in the context of bifurcations of travelling wave solutions in the dNLS lattice (1.3).

7. Bifurcations of travelling wave solutions in the AL lattice

The normal form equations for the integrable AL lattice (1.4) are rewritten explicitly as

$$\frac{da}{dz} = b + \epsilon^2(i\alpha_1 a - \gamma_1(a\bar{b} - \bar{a}b)a), \tag{7.1}$$

$$\frac{db}{dz} = c + \epsilon^2(i\alpha_1 b + \alpha_2 a - \gamma_1(a\bar{b} - \bar{a}b)b + \beta_2|a|^2 a) + \epsilon^2(\delta_2(|b|^2 - a\bar{c} - \bar{a}c)a + \varepsilon_2(b^2 - 2ac)\bar{a}) \tag{7.2}$$

$$\frac{dc}{dz} = \epsilon^2(i\alpha_1 c + \alpha_2 b + i\alpha_3 a - \gamma_1(a\bar{b} - \bar{a}b)c + \beta_2|a|^2 b) + \epsilon^2(\delta_2(|b|^2 - a\bar{c} - \bar{a}c)b + \varepsilon_2(b^2 - 2ac)\bar{b} - \gamma_3(a\bar{b} - \bar{a}b)a), \tag{7.3}$$

where numerical values for α_1, α_2, α_3, γ_1, β_2, δ_2, ε_2, and γ_3 are given by (5.22), (5.41), and (5.42). The normal form equations (7.1)–(7.3) can be simplified, by using the transformation:

$$(a, b, c) = (\tilde{a}, \tilde{b}, \tilde{c}) \exp\left(\mathrm{i}\epsilon^2\alpha_1 z - \epsilon^2\gamma_1 \int_0^z (\tilde{a}\bar{\tilde{b}} - \bar{\tilde{a}}\tilde{b})(z')\,\mathrm{d}z'\right), \tag{7.4}$$

and the representation $\tilde{a} = \Phi(z)$, $\tilde{b} = \Phi'(z)$, and

$$\tilde{c} = \Phi''(z) - \epsilon^2(\alpha_2\Phi + \beta_2|\Phi|^2\Phi + \delta_2(|\Phi'|^2 - \Phi\bar{\Phi}'' - \bar{\Phi}\Phi'')\Phi + \varepsilon_2((\Phi')^2 - 2\Phi\Phi'')\bar{\Phi}). \tag{7.5}$$

Neglecting the terms of order $\mathrm{O}(\epsilon^4)$ beyond the normal form, we derive the scalar third-order derivative equation (1.10) from the vector normal form (7.1)–(7.3), where $h(\Phi, \Phi', \Phi'', \Phi''')$ is given by (1.13).

Using the same assumption that $\Phi \in C^3(\mathbb{R})$ for a smooth homoclinic orbit, if it exists, we apply the scaling transformation:

$$\Phi(z) = S(\zeta), \qquad \zeta = \epsilon z, \qquad V = \nu, \qquad \Omega = \epsilon\mu, \tag{7.6}$$

where the function $S(\zeta)$ satisfies the re-scaled third-order equation:

$$\frac{\mathrm{i}}{3}S''' - \mathrm{i}\nu S' + \mu S = -2\mathrm{i}|S|^2S' + \frac{\mathrm{i}\epsilon^2}{100}(4SS''\bar{S}' - 2SS'\bar{S}'' - 2S'S''\bar{S} + |S|^2S''' - S^2\bar{S}'''). \tag{7.7}$$

Let us consider the truncated equation (7.7):

$$\frac{\mathrm{i}}{3}S''' - \mathrm{i}\nu S' + \mu S = -2\mathrm{i}|S|^2S'. \tag{7.8}$$

This equation can be reduced to the standard form, by using the scaling transformation:

$$S = R(\zeta)\,\mathrm{e}^{\mathrm{i}k\zeta}, \quad k \in \mathbb{R}, \tag{7.9}$$

where $R(\eta)$ satisfies the equation:

$$3k(R'' - \lambda R + 2|R|^2R) = \mathrm{i}(R''' - \lambda R' + 6|R|^2R'), \tag{7.10}$$

and the two parameters (μ, ν) are related to new parameters (λ, k) as follows:

$$\mu = \frac{2}{3}k(\lambda + k^2), \qquad \nu = -k^2 + \frac{1}{3}\lambda. \tag{7.11}$$

The standard form (7.10) has the exact solution for single-humped homoclinic orbits:

$$R = \sqrt{\lambda}\,\mathrm{sech}(\sqrt{\lambda}\eta), \quad \lambda > 0. \tag{7.12}$$

The exact solution (7.12) corresponds to the exact single-humped travelling wave solution of the integrable AL lattice (1.4):

$$\phi(z) = \epsilon^{-1}\sinh(\kappa)\,\mathrm{sech}(\kappa z), \tag{7.13}$$

where κ defines the asymptotic tail at infinity (1.6), such that parameters (β, κ) are related to parameters (ω, υ) by virtue of the transformation (2.8). The single-humped solutions (7.13) exist everywhere in the domain (1.7), outside the shaded area on Fig. 2. The exact solutions (7.12) and (7.13) agree asymptotically as $\epsilon \to 0$, with the correspondence:

$$\kappa = \epsilon\sqrt{\lambda}, \qquad \beta = \frac{\pi}{2} - \epsilon k \tag{7.14}$$

and

$$\omega + 2 = 2\epsilon k + \epsilon^3 k\left(\lambda - \frac{1}{3}k^2\right), \tag{7.15}$$

$$v - 2 = \epsilon^2 k^2\left(\frac{1}{3}\lambda - k^2\right), \tag{7.16}$$

where $k \in \mathbb{R}$ and $\lambda > 0$. Since the exact solution for single-humped travelling wave (7.13) is well-known for the integrable AL lattice (1.4), we do not consider the problem of persistence of the single-humped homoclinic orbit (7.9) and (7.12) within the full normal form (7.7). When $k = 0$ ($\mu = 0$), the exact solution (7.12) satisfies (7.7) identically. When $k \neq 0$ ($\mu \neq 0$), corrections to (7.12) appear due to the perturbation terms, which are of the same order as the order of truncation of the vector normal form (7.1)–(7.3). Therefore, the persistence problem would involve higher-order and beyond-all-orders asymptotic expansions, where all resonances (Stokes constants) are supposed to vanish due to the existence of the exact solution (7.13).

8. Discussions

We have applied central manifold and normal form reductions for derivation of the third-order differential equation (1.10) from the differential advance–delay equation (2.1). These reductions allowed us to analyze bifurcations of travelling wave solutions in the dNLS and AL lattices (1.3) and (1.4). In particular, we showed that no single-humped travelling wave solutions exist in the dNLS lattice (1.3) near the zero-dispersion limit, while two-parameter single-humped travelling wave solutions exist in the AL lattice (1.4). On the other hand, an infinite discrete set of one-parameter double-humped travelling wave solutions may exist in the dNLS lattice near the zero-dispersion limit.

Similar but numerical results show existence of an infinite discrete set of one-parameter families of multi-humped travelling kinks in the Frenkel–Kontorova lattices [8]. Numerical computations of the full discrete problem show that the families extend in the existence domain with $N = 1$ but terminate in the domains with $N > 1$ [4]. Following to these connections, one can study two open problems: (i) derivation and analysis of a normal form for the special point ($c = 1, \gamma = 0$) of the travelling kink problem in the Frenkel–Kontorova lattices and (ii) numerical approximations of one-parameter families of double-humped travelling wave solutions in the discrete NLS equation (1.1).

In addition, we mention that Kevrekidis [22] constructed recently various discretizations of the NLS and Klein–Gordon equations that possess an additional momentum conserved quantity in spite of the broken translational invariance. He also conjectured that the single-humped travelling wave solutions should exist in the systems with the momentum conservation. The integrable AL lattice (1.4) satisfies the latter class of equations and it has the travelling wave solutions (7.13). The proof of this conjecture is left open for further studies.

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Appendix A. Formal derivation of the third-order ODEs (1.10)

Here we show that the truncated third-order ODEs (6.9) and (7.8) can be derived from the differential advance–delay equation (2.1) with the formal asymptotic multi-scale expansion methods.

Let $\beta = \pi/2$ and (ω, υ) be defined by (1.9). Then, equation (2.1) takes the form:

$$\begin{aligned} &\mathrm{i}(\phi(z+1) - \phi(z-1) - 2\phi'(z)) \\ &\quad = \epsilon^2 (f(\phi(z), -\mathrm{i}\phi(z+1), \mathrm{i}\phi(z-1)) - \Omega\phi(z) + \mathrm{i}V\phi'(z)). \end{aligned} \tag{A.1}$$

Assuming the slow variations of $\phi(z)$, we can reduce (A.1) to a differential equation in the Taylor series approximation. The order of the asymptotic truncation of the Taylor series depends on the form of the nonlinearity function $f = f(\phi(z), -\mathrm{i}\phi(z+1), \mathrm{i}\phi(z-1))$.

Let us consider the dNLS lattice (1.3), such that $f = |\phi(z)|^2\phi(z)$. Then, we apply the scaling transformation,

$$\phi(z) = \sqrt{\epsilon} S(\zeta), \qquad \zeta = \epsilon z, \qquad V = \nu, \qquad \Omega = \epsilon\mu, \tag{A.2}$$

and reduce the problem (A.1) to the form:

$$\frac{\mathrm{i}}{3} S''' - \mathrm{i}\nu S' + \mu S = |S|^2 S + \mathrm{O}(\epsilon^2), \tag{A.3}$$

which is nothing but the normal form (6.8).

Let us now consider the AL lattice (1.3), such that $f = -\mathrm{i}|\phi(z)|^2(\phi(z+1) - \phi(z-1))$. Then, we apply the scaling transformation,

$$\phi(z) = S(\zeta), \qquad \zeta = \epsilon z, \qquad V = \nu, \qquad \Omega = \epsilon\mu, \tag{A.4}$$

and reduce the problem (A.1) to the form:

$$\frac{\mathrm{i}}{3} S''' - \mathrm{i}\nu S' + \mu S = -2i|S|^2 S' + \mathrm{O}(\epsilon^2), \tag{A.5}$$

which is nothing but the normal form (7.7).

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