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Eigenfunctions and Eigenvalues for a Scalar Riemann–Hilbert Problem Associated to Inverse Scattering

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Abstract: A complete set of eigenfunctions is introduced within the Riemann–Hilbert formalism for spectral problems associated to some solvable nonlinear evolution equations. In particular, we consider the time-independent and time-dependent Schrödinger problems which are related to the KdV and KPI equations possessing solitons and lumps, respectively. Non-standard scalar products, orthogonality and completeness relations are derived for these problems. The complete set of eigenfunctions is used for perturbation theory and bifurcation analysis of eigenvalues supported by the potentials under perturbations. We classify two different types of bifurcations of new eigenvalues and analyze their characteristic features. One type corresponds to thresholdless generation of solitons in the KdV equation, while the other predicts a threshold for generation of lumps in the KPI equation.

1. Introduction

1.1. Motivations. Some nonlinear evolution equations have attracted intense studies in past years for their universal appearance in the mathematical description of wave processes in dispersive systems and their remarkable analytical properties. In particular, they are related to linear scattering problems in such a way that the nonlinear analysis of wave systems is possible through the Fourier-type analysis of the direct and inverse scattering transform of their linear counterparts [1]. The spectral data in inverse scattering consist typically of the *continuous spectrum* eigenfunctions and a discrete number of *bound states*. The bound states correspond to localized steady-state disturbances such as solitons, lumps, dromions and instantons.

Among many universal properties in inverse scattering, Ablowitz, Kaup, Newell and Segur noticed in their pioneer paper [1] that the set of eigenfunctions for the continuous and discrete spectrum for the AKNS spectral problem is complete, i.e. an arbitrary vector-function with appropriate boundary conditions at infinity can be decomposed through this set of eigenfunctions. This property generalizes the Fourier decomposition [2] and is well-known in spectral theory of linear self-adjoint operators [3,4]. The completeness relation was proved in Ref. [1] by means of the Gelfand–Levitan–Marchenko (GLM) integral equations which appear in the formalism of the inverse scattering transform.

In the AKNS spectral problem, the isolated eigenvalues appear as poles of transmission coefficients and correspond to *exponentially localized* bound states associated to *solitons* in nonlinear evolution equations. Further development in inverse scattering led to the construction of new linear scattering problems associated to the nonlinear evolution equation in one and two dimensions (see review in [5,6]). In the latter problems, Fokas and Ablowitz [7–9] showed that the isolated eigenvalues appear in homogeneous integral Fredholm equations and the corresponding bound states have *algebraic decay* at infinity. These bound states are associated to *lumps* or *algebraic solitons* in nonlinear evolution equations.

The most general formulation of the inverse scattering transform relies on the Riemann-Hilbert (RH) boundary value problem or its generalization, the $\bar{\partial}$ problem. This setting requires new methods for constructing and studying complete sets of eigenfunctions. A particular spectral system associated to the Benjamin-Ono (BO) equation was studied recently by Kaup, Lakoba, and Matsuno [10–12]. Their results serve as a pivot for our approach to integrable problems associated to the RH formalism.

Studies of complete sets of eigenfunctions have many different prospects. First, they provide a basis for the spectral decomposition associated to the given linear problem. Second, they enable us to develop a perturbation theory and study variations of spectral data and eigenfunctions induced by perturbations of the potential. Third, bifurcations of eigenvalues can be analyzed through the expansions over a complete basis, while a standard perturbative analysis usually misses the possibility of such bifurcations. We recently obtained [13,14] that, for the spectral problem associated to the BO equation, this bifurcation may happen from the edge of the continuous spectrum when the potential of the scattering problem satisfies a condition of non-genericity. Finally, the orthogonality and completeness relations are used in Hamiltonian formalism of nonlinear evolution equations and construction of Poisson brackets and canonical variables [15].

In this paper we construct a complete set of eigenfunctions associated to the scalar RH formalism. Although our analysis is based on two canonical and physically important problems (Sect. 1.2), it can also be formulated in an abstract form (Sect. 1.3). The main analysis concentrates on the time-independent Schrödinger problem which is associated to solitons of the Korteweg–de Vries (KdV) equation and the time-dependent Schrödinger problem which is associated to lumps of the Kadomtsev–Petviashvili (KPI) equation. We derive non-standard scalar products and orthogonality relations and prove the completeness formula by means of the RH formalism. We then develop a regular perturbation theory from the integral representation of the linear eigenvalue problem and calculate variational derivatives of spectral data in the absence of bifurcations of new eigenvalues. When the integral representation becomes singular, we find the conditions for a new eigenvalue to emerge from the continuous spectrum. These bifurcations are classified into two general types.

The *type I bifurcation* occurs when the marginal eigenfunction at the edge of the continuous spectrum becomes bounded (nonsecular) in space and belongs to the spectrum in contrast to a generic secular eigenfunction which is excluded from the spectrum. The multisoliton solutions are examples of nongeneric potentials and the type I bifurcation occurs under a certain *thresholdless* perturbation of multisoliton potentials. The *type II bifurcation* occurs when a new bound state is embedded into the continuous spectrum at the bifurcation point and splits apart from the continuous spectrum or disappears upon a perturbation. This type is not supported by multisoliton potentials. A new eigenvalue appears above a certain *threshold* on the amplitude of a perturbation to the multisoliton potential.

The type I bifurcation is illustrated in Sect. 2 for the time-independent Schrödinger equation. Although our results recover the standard inverse scattering formalism associated to this equation (see Appendix A.2 of Ref. [2]), we introduce and study a new non-standard basis of eigenfunctions within the RH formalism. The type II bifurcation is illustrated in Sect. 3 for the time-dependent Schrödinger equation. We find for the first time to our knowledge a complete set of eigenfunctions associated to this equation. The methods and results derived for these two basic problems can be generalized for other examples in inverse scattering which include differential-difference linear systems associated to the Intermediate Long-Wave (ILW) equation and the BO equation as well as vector eigenvalue problems such as the AKNS spectral system in one and two dimensions. A brief review of these spectral problems is discussed in Sect. 4.

1.2. Linear eigenvalue problems. The inverse scattering theory has been developed for several prototypical examples which include the Kadomtsev–Petviashvili equation referred to as the KPI equation,

$$(u_t + 6uu_x + u_{xxx})_x = 3u_{yy}. \tag{1.1}$$

It is associated with the time-dependent Schrödinger equation,

$$i\varphi_y + \varphi_{xx} + u\varphi = 0, \tag{1.2}$$

where $u = u(x, y, t)$ satisfies Eq. (1.1). Inverse scattering for the KPI equation was initiated by Manakov [16] and developed by Fokas and Ablowitz [8] by means of a (nonlocal) RH boundary value problem. In particular, the authors of [8] defined proper eigenfunctions $M_\pm$ and $N_\pm$ of the time-dependent Schrödinger equation (1.2) and incorporated the lump solutions in the inverse scattering scheme. Rigorous results on the solvability of direct and inverse scattering transforms were reported by Beals and Coifman [17], Zhou [18], and Fokas and Sung [19]. More complete results on existence and classification of multiple bound states in the discrete spectrum of the time-dependent Schrödinger equations were recently found by Ablowitz and Villaroel [20–22].

A complete version of the spectral transform for the KPI equation was derived by Boiti et al. [23–25] by means of a formal resolvent approach based on some orthogonality relations for the eigenfunctions of Eq. (1.2). However, their approach does not provide a complete basis of eigenfunctions for the perturbation theory and bifurcation analysis of weakly localized potentials such as multilump potentials. This problem was discussed by Kaup [26] who pointed out that the eigenfunctions of (1.2) are unbounded and incomplete in the Hilbert space if the potential $u(x, y)$ is not absolutely integrable.

Recently the inverse scattering transform theory was applied to solve rigorously the initial-value problem for the KPI equation (1.1) with and without the zero mass constaint [27–29]. Uniqueness and existence of the solution was also proved by Fokas and Sung [30,31] under the assumption that the initial data is a “small” function in the Schwartz space. The latter assumption was used to exclude generation of lumps (two-dimensional solitons) in the KPI equation (1.1) by localized initial data.

The problem of lump generation in the KPI equation remains open in spite of its applications in water wave theory [6]. Kuznetsov and Turitsyn [32] showed that a single KPI lump is stable against small perturbations. Recent numerical simulations of the

KPI equation (1.1) by He [33] showed that a localized initial condition may lead to the formation of KPI lumps if the amplitude of the initial pulse exceeds a certain threshold value.

In the case of y-independent solutions, the KPI equation reduces to the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.3}$$

and the linear system (1.2) to the time-independent Schrödinger equation,

$$\varphi_{xx} + (u + k^2)\varphi = 0, \tag{1.4}$$

where k is a spectral parameter and $u = u(x, t)$ satisfies Eq. (1.3). The standard complete set of eigenfunctions for this problem is described in Ref. [2]. The spectral properties of the eigenfunctions at the edge of the continuous spectrum were also studied in relation to bifurcation of new eigenvalues in the problem (1.4) [34]. The appearance of a single eigenvalue supported by a small potential for the problem (1.4) was analyzed by direct methods in [35–37]. It was found that the linear problem (1.4) exhibits a single small eigenvalue for infinitely small potentials under the constraint that the area integral of the potential is positive. The same conclusion was also formulated for a perturbation of a single soliton potential [38].

In this paper, we present a systematic method based on the RH problem to derive the spectral decomposition associated to the linear eigenvalue problems in inverse scattering. For the sake of clarity, it is first presented in the context of the time-independent Schrödinger equation (1.4) and then extended to the time-dependent Schrödinger equation (1.2) which is more difficult. In both cases, we use the completeness properties to study bifurcation of new eigenvalues. We recover and generalize some of the results discussed above. In particular, we show that for the spectral problem (1.4), an eigenvalue and its associated bound state exist for an arbitrary small potential $u = u(x)$, while the spectral problem (1.2) does not have eigenvalues and bound states for small potentials $u = u(x, y)$. This feature illustrates the different types of bifurcation of eigenvalues for Eqs. (1.2) and (1.4) (types I and II).

1.3. RH formalism and eigenfunctions. A Riemann–Hilbert boundary value problem in a complex plane ($z \in \mathcal{C}$) consists in reconstructing meromorphic functions $\mu^{\pm}(z)$ outside of a contour $\Gamma \in \mathcal{C}$ according to a given jump at the contour,

$$\mu^+(z) - \mu^-(z) = \mathbf{T}\left[\mu^-(z)\right], \tag{1.5}$$

where $\mathbf{T}$ is an operator and the functions $\mu^{\pm}(z)$ satisfy the boundary conditions,

$$\lim_{|z|\to\infty} \mu^{\pm}(z) = 1,$$

in the corresponding domains of $\mathcal{C}$.

In inverse scattering [6,5], the RH problem appears typically if the scattering problem has a single spectral parameter (say k) and the continuous spectrum is located for real values of k, i.e. $z = k$ and $\Gamma = \mathrm{Re}(k)$. This problem relates two Jost functions $M_{\pm}(x, k)$ which are generally $(n \times n)$ matrices and depend on m variables $x_1, x_2, ..., x_m$. In what follows, we restrict ourselves to scalar RH problems ($n = 1$) in one dimension ($x_1 = x$)

or two dimensions ($x_1 = x$ and $x_2 = y$). The Jost functions $M^{\pm}(x, k)$ are introduced as particular solutions of Fredholm's integral equations in Green's function representation,

$$M_{\pm}(x, k) = 1 + \int_{-\infty}^{\infty} G_{\pm}(x - x', k)u(x')M_{\pm}(x', k)dx'. \tag{1.6}$$

Here $u(x)$ is a real-valued potential, $G_+(x, k)$ and $G_-(x, k)$ are Green's functions which are supposed to be analytic in $\mathrm{Im}(k) \geq 0$ and $\mathrm{Im}(k) \leq 0$ respectively, and satisfy $\lim_{|k|\to\infty} G_{\pm}(x, k) = 0$.

Taking the derivative $\partial/\partial\bar{k}$ in Eq. (1.6), where $\bar{k}$ is the complex conjugate of k, we find that the eigenfunctions $M_{\pm}(x, k)$ are analytic functions of k in the domains of analyticity of $G_{\pm}(x, k)$ if there are no homogeneous solutions of Fredholm's integral equations (1.6). On the other hand, if Fredholm's integral equations (1.6) do possess homogeneous solutions in a number of isolated points k of the complex domain, then the eigenfunctions $M_{\pm}(x, k)$ are meromorphic functions of k. We refer to the bound states in the former case as *solitons* and in the latter case as *lumps*.

At real k, the limiting values of the eigenfunctions $M_{\pm}(x, k)$ are related by the following scattering problems,

$$\frac{M_+(x, k)}{a_+(k)} - M_-(x, k) = \rho_-(k)N_-(x, k), \tag{1.7}$$

$$M_+(x, k) - \frac{M_-(x, k)}{a_-(k)} = \rho_+(k)N_+(x, k). \tag{1.8}$$

Here $a_{\pm}(k)$ are the inverse transmission coefficients. The coefficients $\rho_{\pm}(k)$ represent scattering data and the eigenfunctions $N_{\pm}(x, k)$ are linearly independent solutions of the spectral system with *non-constant* boundary conditions at infinity,

$$N_{\pm}(x, k) = e^{i\beta(x,k)} + \int_{-\infty}^{\infty} G_{\pm}(x - x', k)u(x')N_{\pm}(x', k)dx'. \tag{1.9}$$

The coefficients $a_{\pm}(k)$ are identically equal to unity for problems associated to lumps and are not constant for problems associated to solitons. In the latter problems, the coefficients $a_{\pm}(k)$ have the same analyticity properties as the eigenfunctions $M_{\pm}(x, k)$ subject to the following boundary conditions,

$$\lim_{|k|\to\infty} M_{\pm}(x, k) = \lim_{|k|\to\infty} a_{\pm}(k) = 1. \tag{1.10}$$

Combining all these facts, the scattering problem (1.7) (or, equivalently, Eq. (1.8)) defines a RH boundary-value problem, if the eigenfunctions $N_{\pm}(x, k)$ can be expressed through $M_{\pm}(x, k)$ by additional symmetry formulas,

$$N_{\pm}(x, k) = \mathbf{F}M_{\pm}(x, k)e^{i\beta(x,k)}, \tag{1.11}$$

where $\mathbf{F}$ is an operator. Bound states are to be added to the problem (1.7) and (1.11) as pole contributions in the meromorphic functions $[a_{\pm}(k)]^{-1}M_{\pm}(x, k)$. Then, a closed solution of the RH problem can be found (see Appendix A1 in [6]), from which the potential is recovered.

In a simplified version, the inverse scattering scheme is a sequence of transformations from the given potential $u = u(x, 0)$ to the set of eigenfunctions $S(0)$ for the associated linear problem, then to the spectral data $R(0)$ with simple evolution in time $R = R(t)$, then back to the set of eigenfunctions $S = S(t)$ via self-consistent integral equations and finally back to the potential $u = u(x, t)$. This sequence of transformations generalizes the Fourier transform which is based on orthogonality and completeness relations of the trigonometric functions. Similarly, a closure of the general scheme at $t = 0$ implies the existence of a complete basis of eigenfunctions for the direct and inverse spectral transforms. However, the orthogonality and completeness relations for the eigenfunctions used in inverse scattering are not usually under consideration because their derivation may be labourous. Moreover, it is not always clear how to choose a proper basis for these transformations. For example, it is natural to use the eigenfunctions $M_\pm(x, k)$ for a characterization of the inverse scattering problem whereas these functions do not form a complete basis.

Our main idea is that each linear problem associated to a nonlinear evolution equation provides a natural set of orthogonal and complete eigenfunctions which forms a basis of the inverse scattering transform. The complete set of eigenfunctions consists of the eigenfunctions $N_\pm(x, k)$ and associated bound states and characterize all other data of the spectral transform, including the associated eigenfunctions $M_\pm(x, k)$, the spectral data $a_\pm(k)$ and $\rho_\pm(k)$ and the potential $u(x)$. We prove this statement in Sects. 2 and 3 for the particular scattering problems (1.2) and (1.4).

2. Time-Independent Schrödinger Equation

The local RH problem (1.7) appears for the spectral problem (1.4) after the transformation, $\varphi = me^{-ikx}$, where the function $m = m(x, k)$ satisfies the problem,

$$m_{xx} - 2ikm_x + u(x)m = 0. \tag{2.1}$$

We suppose that the function $u(x)$ is real, smooth and belongs to L^p for any $p \geq 1$. These requirements are satisfied for multisoliton potentials of the KdV equation (3.8) since such potentials have an exponential decay at infinity. The dependence of the potential and the eigenfunctions on evolution time t will be omitted henceforth. The standard complete set of eigenfunctions is described in Appendix A.2 of Ref. [2]. Here we view the problem by means of the RH formalism and introduce a new non-standard complete set of eigenfunctions.

2.1. Spectrum and scattering data. Two fundamental solutions $M_\pm(x, k)$ of Eq. (2.1) can be extended analytically for $\mathrm{Im}(k) \geq 0$ and $\mathrm{Im}(k) \leq 0$ according to the integral representation (1.6). The corresponding Green's functions have the form [6]:

$$G_\pm(x, k) = \pm\frac{1}{2ik}(1 - e^{2ikx})\Theta(\pm x), \tag{2.2}$$

where $\Theta(x) = 1$ if $x > 0$ and $\Theta(x) = 0$ if $x < 0$. The other two fundamental solutions $N_\pm(x, k)$ can be found from Eqs. (1.9) with $\beta(x, k) = 2kx$. The eigenfunctions $M_\pm(x, k)$ and $N_\pm(x, k)$ satisfy the following boundary conditions in the limit $x \to \mp\infty$,

$$M_\pm(x, k) \to 1, \quad N_\pm(x, k) \to e^{2ikx}. \tag{2.3}$$

Taking the limits $x \to \pm\infty$ in the Green's function representation (1.6) and using Eqs. (2.2) and (2.3), we find the scattering relations (1.7) and (1.8) with the spectral data $\rho_\pm = b_\mp(k)/a_\mp(k)$. The coefficients $a_\pm(k)$ and $b_\pm(k)$ can be expressed through $M_\pm(x,k)$ as

$$a_\pm(k) = 1 \pm \frac{1}{2ik}\int_{-\infty}^{\infty} u(x)M_\pm(x,k)dx, \tag{2.4}$$

$$b_\pm(k) = -\frac{1}{2ik}\int_{-\infty}^{\infty} u(x)M_\pm(x,k)e^{-2ikx}dx. \tag{2.5}$$

The scattering coefficients satisfy the constraints [6]

$$a_-(k) = a_+^*(k), \qquad b_-(k) = b_+(k), \tag{2.6}$$

$$a_\pm^*(k) = a_\pm(-k), \qquad b_\pm^*(k) = b_\pm(-k), \tag{2.7}$$

and

$$|a_+(k)|^2 = 1 + |b_+(k)|^2. \tag{2.8}$$

Using these relations, we deduce from Eqs. (1.7) and (1.8) the boundary conditions for the eigenfunctions $M_\pm(x,k)$ and $N_\pm(x,k)$ in the limits $x \to \pm\infty$,

$$M_\pm(x,k) \to a_\pm(k) \pm b_\pm(k)e^{2ikx}, \tag{2.9}$$

$$N_\pm(x,k) \to a_\mp(k)e^{2ikx} \pm b_\pm^*(k). \tag{2.10}$$

When $k \to \infty \pm i0$, the eigenfunctions $M_\pm(x,k)$ have the asymptotic representation,

$$M_\pm(x,k) = 1 + \frac{1}{2ik}\int_{\mp\infty}^{x} u(x')dx' + \mathrm{O}(k^{-2}). \tag{2.11}$$

This formula follows from Eqs. (1.6) and (2.2).

The scattering relation (1.7) defines the (local) RH boundary-value problem for $M_\pm(x,k)$. The closure relations (1.11) follow from the symmetry of the Green's functions, $G_\pm(x,k) = G_\pm^*(x,-k) = G_\mp^*(x,k)e^{2ikx}$ and have the form,

$$N_\pm(x,k) = N_\pm^*(x,-k) = M_\pm^*(x,k)e^{2ikx}. \tag{2.12}$$

Bound states for Eq. (2.1) exist for eigenvalues given by the zeros of $a_+(k)$ in the upper half-plane of k and the zeros of $a_-(k)$ in the lower half-plane. Zeros of $a_\pm(k)$ are simple [6] and located symmetrically on the imaginary axis of k due to the constraints imposed on $a_\pm(k)$. These bound states correspond to exponentially localized solitons of the KdV equation (1.3).

The two RH problems (1.7) and (1.8) supplemented by the boundary conditions (1.10) and the closure relation (2.12) can be solved in the form

$$M_\pm(x,k) = 1 + \sum_{j=1}^{n} \frac{c_j^\mp \Phi_j^\mp(x)}{k - k_j^\mp} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_\pm(k')N_\pm(x,k')dk'}{k' - (k \pm i0)}, \tag{2.13}$$

or, equivalently,

$$\frac{M_\pm(x,k)}{a_\pm(k)} = 1 + \sum_{j=1}^{n} \frac{c_j^\pm \Phi_j^\pm(x)}{k - k_j^\pm} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_\mp(k')N_\mp(x,k')dk'}{k' - (k \pm i0)}, \tag{2.14}$$

where $\Phi_j^\pm(x)$ are the bound states, the eigenvalues $k_j^\pm$ satisfy the constraints $k_j^\pm = \pm i\kappa_j$ due to the symmetry ($\kappa_j > 0$), n is the number of bound states, and $c_j^\pm$ are renormalization constants. The limiting relations for the eigenfunctions $M_\pm(x,k)$ approaching bound states are

$$\lim_{k\to k_j^\pm} M_\pm(x,k) = \gamma_j^\pm \Phi_j^\pm(x), \tag{2.15}$$

where $\gamma_j^\pm = c_j^\pm (a_j')^\pm$ are real coefficients. Using the symmetry (2.6) and (2.7), we write the coefficients as $c_j^\pm = \pm i C_j^\pm$ and

$$\left(a_j'\right)^\pm = \left.\frac{da_\pm(k)}{dk}\right|_{k=k_j^\pm} = \pm i a_j',$$

where $C_j^\pm$ and a_j' are real. The bound states $\Phi_j^\pm(x)$ are real functions satisfying the inhomogeneous integral equations,

$$\Phi_j^\pm(x) = (\gamma_j^\pm)^{-1} + \int_{-\infty}^{\infty} G_\pm(x-x',k_j^\pm)u(x')\Phi_j^\pm(x')dx', \tag{2.16}$$

with the boundary conditions

$$\Phi_j^\pm(x) \to \begin{cases} (\gamma_j^\pm)^{-1} & \text{as } x\to\mp\infty \\ \mathrm{O}(e^{\mp 2\kappa_j x}) & \text{as } x\to\pm\infty \end{cases} \tag{2.17}$$

We notice that the bound states $\Phi_j^\pm(x)$ are not localized in the limit $x\to\mp\infty$. Using the boundary conditions for $G_\pm(x,k)$ we find the following integral representation,

$$\kappa_j = \frac{\gamma_j^\pm}{2}\int_{-\infty}^{\infty} u(x)\Phi_j^\pm(x)dx, \tag{2.18}$$

which is also a condition for $a_\pm(k)$ to have a zero at $k = k_j^\pm = \pm i\kappa_j$ (see Eq. (2.4)). In addition, comparing the boundary values (2.3) and (2.17), we normalize the bound states according to the limiting relations,

$$\lim_{k\to k_j^\mp} N_\pm(x,k) = \Phi_j^\mp(x), \tag{2.19}$$

or, equivalently, according to the boundary conditions $\Phi_j^\pm(x) \to e^{\mp 2\kappa_j x}$ as $x\to\pm\infty$. This renormalization leads by virtue of Eqs. (2.12) to the relations

$$\Phi_j^\mp(x) = \gamma_j^\pm \Phi_j^\pm(x) e^{\pm 2\kappa_j x}. \tag{2.20}$$

It follows from Eqs. (2.20) that the coefficients $C_j^\pm$ and $\gamma_j^\pm$ satisfy the constraints,

$$C_j^+ C_j^- \left(a_j'\right)^2 = 1, \quad \gamma_j^+\gamma_j^- = 1. \tag{2.21}$$

The set of coefficients $\{a_\pm(k), b_\pm(k)\}$ represents the spectral data for the continuous spectrum of the linear problem (2.1) while the set $\{k_j^\pm, \gamma_j^\pm\}_{j=1}^m$ corresponds to the data for the discrete spectrum. The separation of the discrete and continuous spectra follows from the analysis of the asymptotic behavior of the spectral data in the limit $k\to 0$.

Definition 2.1. *The potential $u(x)$ is called* **generic** *potential of type I if the limiting point $k = 0$ is excluded from the continuous spectrum, i.e. the limiting eigenfunctions $M_{\pm}(x, 0)$ are not bounded in x as $x \to \infty$ and the spectral coefficients $a_{\pm}(k)$ are not bounded in k as $k \to 0$, so that* $\lim_{k\to 0}[a_{\pm}(k)]^{-1}M_{\pm}(x, k) = 0$*. Otherwise, the potential is called* **nongeneric** *potential of type I.*

The asymptotic behavior of the scattering data as $k \to 0$ follows from Eqs. (2.4),

$$a_{\pm}(k) \to \pm\frac{m_{-1}}{2ik} + \mathrm{O}(1), \quad b_{\pm}(k) \to -\frac{m_{-1}}{2ik} + \mathrm{O}(1), \tag{2.22}$$

where

$$m_{-1} = \int_{-\infty}^{\infty} u(x)M_{+}(x, 0)dx = \int_{-\infty}^{\infty} u(x)M_{-}(x, 0)dx, \tag{2.23}$$

and the eigenfunctions $M_{\pm}(x, 0)$ are real and satisfy the integral equations,

$$M_{\pm}(x, 0) = 1 - \int_{\mp\infty}^{x} (x - x')u(x')M_{\pm}(x', 0)dx'. \tag{2.24}$$

These eigenfunctions have a secular growth in x at infinity according to the boundary conditions,

$$M_{\pm}(x, 0) \to \begin{cases} 1 & \text{as } x \to \mp\infty \\ 1 \pm m_0^{\pm} \mp m_{-1}x & \text{as } x \to \pm\infty, \end{cases} \tag{2.25}$$

where

$$m_0^{\pm} = \int_{-\infty}^{\infty} xu(x)M_{\pm}(x, 0)dx. \tag{2.26}$$

Thus, if $m_{-1} \neq 0$, the limiting point $k = 0$ is excluded from the continuous spectrum and the potential $u(x)$ is a generic potential of type I. On the other hand, if $m_{-1} = 0$, the secularities of the spectral data as $k \to 0$ disappear and the limiting eigenfunctions $M_{\pm}(x, 0)$ become bounded and related as

$$M_{-}(x, 0) = (1 - m_0^{-})M_{+}(x, 0), \quad M_{+}(x, 0) = (1 + m_0^{+})M_{-}(x, 0). \tag{2.27}$$

In this case, the potential $u(x)$ is a nongeneric potential of type I and the limiting point $k \to 0$ belongs to the continuous spectrum as

$$a_{\pm}(k) \to a_0 + \mathrm{O}(k), \quad b_{\pm}(k) \to b_0 + \mathrm{O}(k), \tag{2.28}$$

where real coefficients a_0 and b_0 are expressed through m_0^{+},

$$a_0 = 1 + \frac{m_0^{+2}}{2(1 + m_0^{+})}, \quad b_0 = m_0^{+} - \frac{m_0^{+2}}{2(1 + m_0^{+})}, \tag{2.29}$$

or, equivalently, through m_0^{-} according to the relation,

$$m_0^{-} = \frac{m_0^{+}}{1 + m_0^{+}}.$$

Multisoliton potentials are particular examples of nongeneric potentials of type I since they display a non-secular behavior of $b_\pm(k)$, i.e. $b_\pm(k) \equiv 0$ [2]. The asymptotic expressions (2.22) for spectral data were analyzed in Ref. [34], where the number n of bound states was related to a finite value of $\arg a_\pm(0)$,

$$\arg a_\pm(0) = \mp \begin{cases} \pi\left(n - \frac{1}{2}\right) & \text{if } m_{-1} \neq 0 \\ \pi n & \text{if } m_{-1} = 0. \end{cases} \tag{2.30}$$

Thus, the constraint $m_{-1} = 0$ changes the spectral data and may result in a change of the number of bound states (i.e. in bifurcation of a new eigenvalue of the linear system). This is the type I bifurcation analyzed in Sect. 2.4.

2.2. Scalar products, orthogonality and completeness relations. According to Eqs. (2.13) and (2.14), the eigenfunctions $M_\pm(x,k)$ can be characterized by either of the two sets $S^\pm = \left[N_\pm(x,k), \{\Phi_j^\mp(x)\}_{j=1}^n\right]$. Furthermore, the spectral data $\{a_\pm(k), b_\pm(k)\}$ and $\{k_j^\pm, \gamma_j^\pm\}_{j=1}^n$ can be expressed through the functions of the sets $S^\pm$ according to formulas (2.4), (2.5), (2.12), (2.17), and (2.18). Also the potential $u(x)$ is related to the sets $S^\pm$ by

$$\int_{\mp\infty}^x u(x)dx = -\frac{1}{\pi}\int_{-\infty}^{\infty} \rho_\pm(k) N_\pm(x,k)dk \pm 2\sum_{j=1}^n C_j^\mp \Phi_j^\mp(x). \tag{2.31}$$

This formula results from Eqs. (2.11) and (2.13) in the limit $k \to \infty$. Thus, the scheme for closure of the spectral transform holds for the sets $S^\pm$. We now prove the following main result.

Proposition 2.2. *An arbitrary scalar function $f(x)$ with the boundary conditions*

$$\lim_{x\to\pm\infty} f(x) = f_\pm,$$

where $f_\pm$ are constants, can be decomposed through the orthogonal and complete set of eigenfunctions S^+ if $f_- = 0$ or through its dual set S^- if $f_+ = 0$.

The proof of this proposition is based on two lemmas.

Lemma 2.3. *The eigenfunctions $N_\pm(x,k)$ and $\{\Phi_j^\mp(x)\}_{j=1}^n$ introduced in Sect. 2.1 satisfy the orthogonality relations,*

$$\langle N_\mp(k')|N_\pm(k)\rangle = 2\pi i k a_\mp(k)\delta(k-k'), \tag{2.32}$$

$$\langle \Phi_j^\pm|N_\pm(k)\rangle = \langle N_\pm(k)|\Phi_j^\pm\rangle = 0, \tag{2.33}$$

$$\langle \Phi_l^\pm|\Phi_j^\mp\rangle = \mp\kappa_j a_j' \delta_{jl}, \tag{2.34}$$

where the scalar product is defined by

$$\langle g(k')|h(k)\rangle = \int_{-\infty}^{\infty} g^*(x,k')\partial_x h(x,k)dx. \tag{2.35}$$

Proof. First, we derive the Wronskian relation for two solutions $h(k)$ and $g(k')$ of Eq. (2.1) with a real potential $u(x)$,

$$\frac{d}{dx}\left[g^*(k')h_x(k) - g_x^*(k')h(k) - 2ik'g^*(k')h(k)\right] = 2i(k-k')g^*(k')h_x(k). \quad (2.36)$$

Then, we integrate Eq. (2.36) for $h(k) = N_\pm(x,k)$ and $g^*(k') = N_\mp^*(x,k')$ over x and use the boundary conditions (2.10) and the formula for generalized functions,

$$\lim_{L\to\pm\infty} e^{ikL} = \pm\pi ik\delta(k). \quad (2.37)$$

As a result, we find Eq. (2.32). The zero scalar products in Eqs. (2.33) and (2.34) follow also from Eq. (2.36) for different bound states. In order to find the nonzero scalar products (2.34), we integrate Eq. (2.36) for $h(k) = M_\pm(x,k)$ and $g^*(k') = \Phi^\pm(x)$ over x and use the boundary conditions (2.3) and (2.17). As a result, we find the integral relation,

$$2i(k - k_j^\mp)\int_{-\infty}^{\infty}\Phi_j^\pm(x)\partial_x M_\pm(x,k)dx = 2\kappa_j\left(\gamma_j^\pm\right)^{-1}.$$

This equation reduces to Eq. (2.34) after computing the integral on the left-hand side with the help of Eq. (2.13) and the zero scalar products (2.33) and (2.34). □

The proof of the orthogonality relations uses only the direct analysis of the spectral problem (2.1). The next lemma formulates the completeness relation. It will be proved by using equations of the inverse scattering transform.

Lemma 2.4. *The eigenfunctions $N_\pm(x,k)$ and $\{\Phi_j^\mp(x)\}_{j=1}^n$ satisfy the completeness relations,*

$$\pm\Theta[\pm(x-y)] = \int_{-\infty}^{\infty}\frac{N_\mp^*(y,k)N_\pm(x,k)dk}{2\pi i(k\mp i0)a_\mp(k)} \mp \sum_{j=1}^{n}\frac{\Phi_j^\pm(y)\Phi_j^\mp(x)}{\kappa_j a_j'}. \quad (2.38)$$

Proof. First, we close Eq. (2.13) with the help of Eqs. (2.12) and (2.15). As a result, we find a system of integral and algebraic relations for the eigenfunctions $N_\pm(x,k)$ and $\Phi_j^\mp(x)$,

$$N_\pm(x,k) = e^{2ikx}\left[1 \pm \sum_{j=1}^{n}\frac{iC_j^\mp\Phi_j^\mp(x)}{k\mp i\kappa_j} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho_\pm(k')N_\pm(x,k')dk'}{k'+k\mp i0}\right], \quad (2.39)$$

$$\Phi_j^\mp(x) = e^{\pm 2\kappa_j x}\left[1 - \sum_{j=1}^{n}\frac{C_l^\mp\Phi_l^\mp(x)}{\kappa_j+\kappa_l} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho_\pm(k)N_\pm(x,k)dk}{k\mp i\kappa_j}\right]. \quad (2.40)$$

We express $N_\mp^*(y,k)$ by using Eqs. (1.7) and (2.12),

$$N_\mp^*(y,k) = a_\mp(k)N_\pm^*(y,k) \mp b_\mp(k)N_\pm(y,k)e^{-2iky}. \quad (2.41)$$

The product $N_\pm^*(y,k)N_\pm(x,k)$ can be found from Eqs. (2.39) and (2.40) using the pole decomposition,

$$N_\pm^*(y,k)N_\pm(x,k) = e^{2ik(x-y)}\left[1 \pm \sum_{j=1}^n \frac{\Phi_j^\pm(y)\Phi_j^\mp(x)}{i(k\mp i\kappa_j)a_j'} \pm \sum_{j=1}^n \frac{\Phi_j^\mp(y)\Phi_j^\pm(x)}{i(k\pm i\kappa_j)a_j'} \right.$$
$$+ \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho_\pm(k')N_\pm(y,k')N_\pm(x,k')e^{-2ik'y}dk'}{k'+k\mp i0}$$
$$\left. + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho_\pm(k')N_\pm(y,k')N_\pm(x,k')e^{-2ik'x}dk'}{k'-k\mp i0}\right].$$

Then, the following integral can be evaluated using the residue theorem,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{N_\pm^*(y,k)N_\pm(x,k)dk}{k\mp i0} =$$
$$\pm\,\Theta[\pm(x-y)]\left(1+\sum_{j=1}^n\frac{\Phi_j^\pm(y)\Phi_j^\mp(x)}{\kappa_j a_j'}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho_\pm(k)N_\pm(y,k)N_\pm(x,k)e^{-2iky}dk}{k\mp i0}\right)$$
$$\pm\,\Theta[\mp(x-y)]\left(\sum_{j=1}^n\frac{\Phi_j^\pm(y)\Phi_j^\mp(x)}{\kappa_j a_j'}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho_\pm(k)N_\pm(y,k)N_\pm(x,k)e^{-2iky}dk}{k\mp i0}\right). \tag{2.42}$$

Substituting Eqs. (2.41) and (2.42) into the integral on the right-hand side we derive the completeness relation (2.38). □

Proof of Proposition 2.2. Using Lemmas 2.3 and 2.4, we decompose the function $f(x)$ into two equivalent integral representations,

$$f(x) = f_\mp + \int_{-\infty}^{\infty}\alpha_\pm(k)N_\pm(x,k)dk + \sum_{j=1}^n \alpha_j^\mp\Phi_j^\mp(x), \tag{2.43}$$

where $\alpha_\pm(k)$ and $\alpha_j^\pm$ are coefficients of the expansion and $f_\pm$ are constants defined by boundary conditions for $f(x)$. The coefficients of the expansion can be expressed through the function $f(x)$ by means of Eqs. (2.32)–(2.34),

$$\alpha_\pm(k) = \frac{\langle N_\mp(k)|f\rangle}{2\pi i(k\mp i0)a_\mp(k)}, \quad \alpha_j^\mp = \mp\frac{\langle\Phi_j^\pm|f\rangle}{\kappa_j a_j'}.$$

Then, Eq. (2.43) reduces to an identity by means of Eq. (2.38). □

The completeness relations (2.38) and scalar products (2.35) for a new complete set of eigenfunctions differ from the standard relations for Jost eigenfunctions of the time-independent Schrödinger problem (see Appendix A.2 in Ref. [2]). This is due to the derivative term ∂_x appearing in the problem (2.1) in front of the spectral parameter k. Since only the derivatives of $f(x)$ determine the coefficients in Eq. (2.43), an arbitrary function $f(x)$ may not be localized at infinity. Another related feature is that we have to pass by the singular point $k = 0$ in the completeness relations (2.38) into the corresponding complex extensions of k, where the functions $a_{\pm}(k)$ are analytic. We notice that the spectral problem (2.1) is not self-adjoint in contrast with the original problem (1.4). Furthermore, the scalar product (2.35) is not a proper inner product since it is not sign-definite [4]. However, the problem (2.1) inherits some features of the self-adjoint problem. In particular, the orthogonal sets S^{+} and S^{-} are self-dual, i.e. complex-conjugate eigenfunctions of S^{-} are adjoint to eigenfunctions of S^{+} and vice versa.

It is important to point out that the relation (2.31) for the inverse scattering transform is a particular application of Eq. (2.43). The coefficients $\rho_{\pm}(k)$ and $C_j^{\mp}$ play the role of Fourier coefficients. Indeed, using the orthogonality relations (2.32) and (2.34), one can express these coefficients through the potential $u(x)$ according to Eqs. (2.5), (2.12), and (2.18). Thus, the sets $S^{\pm}$ represent the only basis for closure of direct and inverse scattering transforms.

Since two alternative (self-dual) orthogonal and complete sets of eigenfunctions have been constructed, we can now study perturbations of the potential and the associated transformation of the spectrum of Eq. (2.1).

2.3. Perturbation theory for spectral data. The spectral data can be evaluated explicitly only in some special cases such as multisoliton potentials. Therefore, perturbation theory for the scattering data under a perturbation of the potential is an effective tool to study characteristic features of a given scattering problem. Furthermore, the dynamics of solitons in nearly integrable systems can be investigated with the help of the same perturbation theory (see reviews in Refs. [39,40]). The results of the perturbation theory for the time-independent Schrödinger equation are now well-known and have been used many times. Here we reproduce these results within the self-consistent scheme given in Sect. 2.2.

Suppose that the potential can be decomposed as $u^{\epsilon} = u(x) + \epsilon\Delta u(x)$, where $\epsilon \ll 1$ and the complete sets of eigenfunctions $S^{\pm} = \left[N_{\pm}(x,k), \{\Phi_j^{\mp}(x)\}_{j=1}^{n}\right]$ are associated to the potential $u(x)$. Here we evaluate variations of the spectral data due to the perturbation $\Delta u(x)$.

2.3.1. Variations of data of discrete spectrum. Suppose that $\Phi_j^{\mp\epsilon}(x)$ solves Eq. (2.1) for $u^{\epsilon} = u(x) + \epsilon\Delta u(x)$ with the eigenvalue $k = k_j^{\mp\epsilon} = \mp i\kappa_j^{\epsilon}$. We expand $\Phi_j^{\mp\epsilon}(x)$ through the sets $S^{\pm}$ according to Eq. (2.43) rewritten as

$$\Phi_j^{\mp\epsilon}(x) = \int_{-\infty}^{\infty} \frac{\alpha_{\pm}(k)N_{\pm}(x,k)dk}{4\pi(k \mp i0)a_{\mp}(k)(k \pm i\kappa_j^{\epsilon})} + \sum_{l=1}^{n} \frac{\alpha_l^{\mp}\Phi_l^{\mp}(x)}{2\kappa_l a_l'(\kappa_l - \kappa_j^{\epsilon})}. \tag{2.44}$$

The eigenvalue problem (2.1) reduces with the help of Eqs. (2.32)–(2.34) and (2.44) to an equivalent set of homogeneous integral equations for the coefficients $\alpha_{\pm}(k)$ and $\alpha_l^{\mp}$,

$$\alpha_{\pm}(k) = \epsilon\left[\int_{-\infty}^{\infty} \frac{K_{\pm}(k,k')\alpha_{\pm}(k')dk'}{4\pi(k' \mp i0)a_{\mp}(k')(k' \pm i\kappa_j^{\epsilon})} + \sum_{l=1}^{n} \frac{K_{\mp l}(k)\alpha_l^{\mp}}{2\kappa_l a_l'(\kappa_l - \kappa_j^{\epsilon})}\right], \tag{2.45}$$

$$\alpha_l^{\mp} = \epsilon\left[\int_{-\infty}^{\infty} \frac{K_{\pm l}^{*}(k)\alpha_{\pm}(k)dk}{4\pi(k \mp i0)a_{\mp}(k)(k \pm i\kappa_j^{\epsilon})} + \sum_{m=1}^{n} \frac{K_{\mp lm}\alpha_m^{\mp}}{2\kappa_m a_m'(\kappa_m - \kappa_j^{\epsilon})}\right], \tag{2.46}$$

where the integral elements are

$$K_\pm(k,k') = \int_{-\infty}^{\infty} \Delta u(x) N^*_\mp(x,k) N_\pm(x,k') dx,$$
$$K_{\pm j}(k) = \int_{-\infty}^{\infty} \Delta u(x) N^*_\pm(x,k) \Phi_j^\pm(x) dx,$$

and

$$K_{\pm jl} = \int_{-\infty}^{\infty} \Delta u(x) \Phi_j^\mp(x) \Phi_l^\pm(x) dx.$$

We look for solutions of Eqs. (2.45) and (2.46) in the asymptotic limit $\epsilon \to 0$. The results are summarized in the following proposition.

Proposition 2.5. *Variational derivatives of data $\{\kappa_j, \gamma_j^\pm\}_{j=1}^n$ of the discrete spectrum of Eq. (2.1) with respect to the potential $u(x)$ are given by*

$$\frac{\delta \kappa_j}{\delta u(x)} = -\frac{\Phi_j^-(x)\Phi_j^+(x)}{2\kappa_j a'_j}, \tag{2.47}$$

$$\frac{\delta \ln \gamma_j^\pm}{\delta u(x)} = \mp \frac{x\Phi_j^-(x)\Phi_j^+(x)}{\kappa_j a'_j} + \frac{1}{2\kappa_j}\left[\gamma_j^\mp \Phi_j^\mp(x)\mu_j^\pm(x) - \gamma_j^\pm \Phi_j^\pm(x)\mu_j^\mp(x)\right], \tag{2.48}$$

where the real functions $\mu_j^\pm(x)$ are introduced as the limits,

$$\lim_{k\to k_j^\pm}\left[\frac{M_\pm(x,k)}{a_\pm(k)} - \frac{c_j^\pm \Phi_j^\pm(x)}{k-k_j^\pm}\right] = \mu_j^\pm(x). \tag{2.49}$$

Proof. It follows from the self-consistency condition for Eq. (2.46) at $l = j$ that κ_j^ϵ can be expanded into the asymptotic series,

$$\kappa_j^\epsilon = \kappa_j + \epsilon\Delta\kappa_j + \epsilon^2 \Delta_2\kappa_j + \mathrm{O}(\epsilon^3),$$

where

$$\Delta\kappa_j = -\frac{K_{\mp jj}}{2\kappa_j a'_j}. \tag{2.50}$$

This formula is equivalent to Eq. (2.47). Using Eq. (2.50), we construct an asymptotic solution of Eq. (2.45) and (2.46) to the first order of the perturbation theory,

$$\alpha_\pm(k) = \epsilon K_\mp(k) + \mathrm{O}(\epsilon^2),$$
$$\alpha_l^\mp = \epsilon K_{\mp lj} + \mathrm{O}(\epsilon^2).$$

As a result, we find a perturbation to the bound state, $\Phi_j^{\mp\epsilon}(x) = \Phi_j^\mp(x) + \epsilon\Delta\Phi_j^\mp(x) + \mathrm{O}(\epsilon^2)$, in the form,

$$\Delta\Phi_j^\mp(x) = \Delta\alpha_j^\mp \Phi_j^\mp(x) + \int_{-\infty}^{\infty} \frac{K_{\mp j}(k) N_\pm(x,k) dk}{4\pi(k \mp i0) a_\mp(k)(k \pm i\kappa_j)} + \sum_{l\neq j} \frac{K_{\mp lj}\Phi_l^\mp(x)}{2\kappa_l a'_l(\kappa_l - \kappa_j)}, \tag{2.51}$$

where $\Delta\alpha_j^{\mp}$ is defined through the corrections to $\alpha_j^{\mp}$ and κ_j. We use the boundary conditions (2.3) and (2.17) as $x \to \mp\infty$ and evaluate the contribution from a double pole at $k = \mp i\kappa_j$ in Eq. (2.51). Then, the term $\mathrm{O}(e^{\pm 2\kappa_j x})$ should be removed from the asymptotic representation for $\Delta\Phi_j^{\mp}(x)$ as $x \to \mp\infty$ by specifying the correction $\Delta\alpha_j^{\mp}$ in the form,

$$\Delta\alpha_j^{\mp} = \frac{\Delta\kappa_j}{\kappa_j} \mp \frac{1}{\kappa_j a_j'} \int_{-\infty}^{\infty} x\,\Delta u(x)\Phi_j^{\mp}(x)\Phi_j^{\pm}(x)dx - \frac{\gamma_j^{\pm}}{2\kappa_j} \int_{-\infty}^{\infty} \Delta u(x)\Phi_j^{\pm}(x)\mu_j^{\mp}(x)dx, \tag{2.52}$$

where $\mu_j^{\pm}(x)$ is defined by Eq. (2.49). On the other hand, we assume an expansion $\gamma_j^{\pm\epsilon} = \gamma_j^{\pm} + \epsilon\Delta\gamma_j^{\pm} + \mathrm{O}(\epsilon^2)$ and find the correction $\Delta\gamma_j^{\pm}$,

$$\Delta\gamma_j^{\pm} = \gamma_j^{\pm}\Delta\alpha_j^{\mp} + \frac{K_{\mp j}(0)}{2\kappa_j} \pm \int_{-\infty}^{\infty} \frac{K_{\mp j}(k)\rho_{\mp}^*(k)dk}{4\pi(k \mp i0)(k \pm i\kappa_j)} + \sum_{l\neq j} \frac{K_{\pm jl}\gamma_l^{\pm}}{2\kappa_l a_l'(\kappa_l - \kappa_j)}. \tag{2.53}$$

This formula follows from Eq. (2.51) in the limit $x \to \pm\infty$ with the help of Eqs. (2.10) and (2.17). In order to simplify this formula, we rewrite Eqs. (2.14) and (2.49) in the form,

$$\mu_j^{\pm}(x) - M_{\mp}(x,0) = \frac{C_j^{\pm}\Phi_j^{\pm}(x)}{\kappa_j} - \kappa_j \sum_{l\neq j} \frac{C_l^{\pm}\Phi_l^{\pm}(x)}{\kappa_l(\kappa_l - \kappa_j)} \pm \kappa_j \int_{-\infty}^{\infty} \frac{\rho_{\mp}^*(k)N_{\mp}^*(x,k)dk}{2\pi(k \mp i0)(k \pm i\kappa_j)}, \tag{2.54}$$

where we have used Eqs. (2.7) and (2.12). Substitution of Eq. (2.54) into Eq. (2.53) gives

$$\Delta\gamma_j^{\pm} = \gamma_j^{\pm}\left(\Delta\alpha_j^{\mp} - \frac{\Delta\kappa_j}{\kappa_j}\right) + \frac{1}{2\kappa_j}\int_{-\infty}^{\infty} \Delta u(x)\Phi_j^{\mp}(x)\mu_j^{\pm}(x)dx.$$

This expression reduces to Eq. (2.48) with the help of Eq. (2.52) □

Formulas (2.47) and (2.48) for the variations of data of discrete spectrum coincide with those derived from the standard perturbation theory of Eq. (1.4) (see Ref. [39]). Here we have derived these formulas by using the non-standard complete sets $S^{\pm}$ of Eq. (2.1). In addition, the solution of the first order of the perturbation theory enables us to evaluate from Eq. (2.46) at $l = j$ the next-order correction $\Delta_2\kappa_j$,

$$\Delta_2\kappa_j = -\int_{-\infty}^{\infty} \frac{K_{\pm j}^*(k)K_{\mp j}(k)dk}{8\pi\kappa_j a_j'(k \mp i0)a_{\mp}(k)(k \pm i\kappa_j)} - \sum_{l\neq j} \frac{K_{\mp jl}K_{\mp lj}}{4\kappa_j\kappa_l a_j' a_j'(\kappa_l - \kappa_j)}. \tag{2.55}$$

2.3.2. Variations of data of continuous spectrum. The eigenfunctions of the continuous spectrum can also be decomposed through the sets $S^{\pm}$ as in Sect. 2.3.1. Suppose that $N^{\epsilon}_{\pm}(x,k)$ solves Eq. (2.1) for $u^{\epsilon} = u(x) + \epsilon \Delta u(x)$. We expand them to the first order of the perturbation theory, $N^{\epsilon}_{\pm}(x,k) = N_{\pm}(x,k) + \epsilon \Delta N_{\pm}(x,k) + \mathrm{O}(\epsilon^2)$, and find the correction $\Delta N_{\pm}(x,k)$ in the form,

$$\Delta N_{\pm}(x,k) = \int_{-\infty}^{\infty} \frac{K_{\pm}(k',k) N_{\pm}(x,k') dk'}{4\pi(k' \mp i0) a_{\mp}(k')(k' - (k \pm i0))} \mp \sum_{j=1}^{n} \frac{K^*_{\pm j}(k)\Phi^{\mp}_j(x)}{2i\kappa_j a'_j(k \pm i\kappa_j)}. \tag{2.56}$$

Proposition 2.6. *Variational derivatives of data $\{a_{\pm}(k), b_{\pm}(k)\}$ of the continuous spectrum of Eq. (2.1) with respect to the potential $u(x)$ are given by*

$$\frac{\delta a_{\pm}(k)}{\delta u(x)} = \pm \frac{N^*_{\mp}(x,k) N_{\pm}(x,k)}{2ik}, \tag{2.57}$$

$$\frac{\delta b_{\pm}(k)}{\delta u(x)} = -\frac{N^*_{\pm}(x,k) M_{\mp}(x,k)}{2ik}. \tag{2.58}$$

Proof. We expand the scattering data in the form

$$a^{\epsilon}_{\pm}(k) = a_{\pm}(k) + \epsilon \Delta a_{\pm}(k) + \mathrm{O}(\epsilon^2),$$
$$b^{\epsilon}_{\pm}(k) = b_{\pm}(k) + \epsilon \Delta b_{\pm}(k) + \mathrm{O}(\epsilon^2),$$

and analyze Eq. (2.56) in the limits $x \to \pm\infty$ by comparin0g with the boundary conditions (2.10) and (2.17). We find explicit solutions

$$\Delta a_{\pm}(k) = \pm \frac{K_{\pm}(k,k)}{2ik} \tag{2.59}$$

and

$$\Delta b_{\pm}(k) = -\frac{K_{\mp}(k,0)}{2ik} + \int_{-\infty}^{\infty} \frac{K_{\mp}(k,k') \rho_{\mp}(k') dk'}{4\pi(k' \pm i0)(k' - (k \mp i0))} - \sum_{j=1}^{n} \frac{K_{\pm j}(k) C^{\pm}_j}{2i\kappa_j(k \mp i\kappa_j)}. \tag{2.60}$$

Then, Eq. (2.57) follows directly from Eq. (2.59). Furthermore, using Eq. (2.13), we derive the relation,

$$M_{\pm}(x,k) - M_{\pm}(x,0) = \frac{k}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_{\pm}(k') N_{\pm}(x,k') dk'}{(k' \mp i0)(k' - (k \pm i0))} + k \sum_{j=1}^{n} \frac{C^{\mp}_j \Phi^{\mp}_j(x)}{\kappa_j(k \pm i\kappa_j)}.$$

Substituting this formula to Eq. (2.60), we recover Eq. (2.58). □

Formulas (2.57) and (2.58) for the variations of data of continuous spectrum coincide with those obtained from the standard perturbation theory of Eq. (1.4) (see Ref. [39]). We notice that in the limit $k \to 0$, the variations $\Delta a_{\pm}(k)$ and $\Delta b_{\pm}(k)$ are divergent, i.e.

$$\Delta a_{\pm}(k) \to \pm \frac{K_{\pm}(0,0)}{2ik}, \quad \Delta b_{\pm}(k) \to -\frac{K_{\mp}(0,0)}{2ik}. \tag{2.61}$$

Comparing with Eq. (2.22) we identify the expansion of the parameter $m^{\epsilon}_{-1} = m_{-1} + \epsilon\Delta m_{-1} + \mathrm{O}(\epsilon^2)$, where

$$\Delta m_{-1} = K_{\pm}(0,0) = \int_{-\infty}^{\infty} \Delta u(x) M_{-}(x,0) M_{+}(x,0) dx. \tag{2.62}$$

2.3.3. Example: A single soliton potential. We solve Eq. (2.40) for $\rho_{\pm}(k) = 0$ and $n = 1$ in the form,

$$\Phi_1^{\mp}(x) = \frac{e^{\pm 2\kappa_1 x}}{1 + e^{\pm 2\kappa_1 (x - x_0)}}, \tag{2.63}$$

where we have used the parametrization,

$$C_1^{\mp} = 2\kappa_1 e^{\mp 2\kappa_1 x_0}.$$

Then, the soliton of the KdV equation is

$$u_s(x) = 2\kappa_1^2 \mathrm{sech}^2 \kappa_1 (x - x_0). \tag{2.64}$$

Using the following data for the single soliton potential,

$$a_{+}(k) = a_{-}^{*}(k) = \frac{k - i\kappa_1}{k + i\kappa_1}, \quad a_1' = -\frac{1}{2\kappa_1}, \quad \gamma_1^{\pm} = e^{\pm 2\kappa_1 x_0},$$

we evaluate the corrections of the first order of the perturbation theory from Eqs. (2.47), (2.48), (2.57), and (2.58),

$$\Delta\kappa_1 = \frac{1}{8\kappa_1^2} \int_{-\infty}^{\infty} \Delta u(x) u_s(x) dx, \tag{2.65}$$

$$\frac{\Delta\gamma_1^{\pm}}{\gamma_1^{\pm}} = \pm \frac{1}{4\kappa_1^2} \int_{-\infty}^{\infty} x\Delta u(x) u_s(x) dx \pm \frac{1}{2\kappa_1} \int_{-\infty}^{\infty} \Delta u(x) \tanh[\kappa_1 (x - x_0)] dx, \tag{2.66}$$

$$\Delta a_{\pm}(k) = \pm \frac{1}{2ik(k \mp i\kappa_1)^2} \int_{-\infty}^{\infty} \Delta u(x) \left(k^2 + \kappa_1^2 \tanh^2[\kappa_1 (x - x_0)]\right) dx, \tag{2.67}$$

$$\Delta b_{\pm}(k) = -\frac{1}{2ik(k^2 + \kappa_1^2)} \int_{-\infty}^{\infty} \Delta u(x) \left(k - i\kappa_1 \tanh[\kappa_1 (x - x_0)]\right)^2 e^{-2ikx} dx. \tag{2.68}$$

The results for a single KdV soliton can be found in Refs. [39,40]. We notice that the integral in Eq. (2.65) identifies with ΔP_s, the correction to the momentum $P_s = \frac{1}{2} \int_{-\infty}^{\infty} u_s^2 dx$ of the KdV soliton. Computing P_s from Eq. (2.64) as $P_s = \frac{8}{3}\kappa_1^3$, we conclude from Eq. (2.65) that the correction ΔP_s defines completely the renormalization of the parameter κ_1 of the KdV soliton. The momentum of the continuous spectrum is therefore affected at the order of $\mathrm{O}(\epsilon^2)$. This result is associated to the stability of a KdV soliton against small perturbations (see Ref. [41] for other examples).

2.4. Type I bifurcation of new eigenvalues. The number of bound states may change if the potential $u(x)$ is a nongeneric potential of type I, i.e. the criterion $m_{-1} = 0$ is met (see Definition 2.1). Using the asymptotic formulas (2.61) and (2.62), we find a necessary condition for this bifurcation. Suppose that the nongeneric potential $u(x)$ has n bound states. Then, the coefficient $a_{\pm}^{\epsilon}(k)$ for the perturbed potential $u^{\epsilon} = u(x) + \epsilon \Delta u(x)$ has the following behavior as $k \to 0$,

$$a_{\pm}^{\epsilon}(k) = a_0 \pm \frac{\epsilon \Delta m_{-1}}{2ik} + \mathrm{O}(\epsilon, k),$$

where a_0 is given by Eq. (2.29) and Δm_{-1} is defined by Eq. (2.62). Then, we take into account Eq. (2.30) for the potential $u(x)$ and derive the extension of this formula for the perturbed potential $u^{\epsilon}(x)$,

$$\arg a_{\pm}^{\epsilon}(0) = \mp \pi \left(n + \frac{\sigma}{2} \right), \tag{2.69}$$

where $\sigma = \mathrm{sign}(\epsilon \Delta m_{-1}/a_0)$. Comparing Eqs. (2.30) and (2.69) we conclude that a new $(n+1)^{\mathrm{th}}$ eigenvalue detaches from the edge of the continuous spectrum if $\sigma = +1$. Here we derive asymptotic expansions for the data of discrete spectrum corresponding to the new bound state. Also we discuss applications of these results to some physical problems such as soliton generation in the KdV equation and bifurcation of oscillatory modes in nonlinear Klein–Gordon equations.

2.4.1. Asymptotic expressions for a new eigenvalue and bound state. Suppose that the potential $u(x)$ is nongeneric and the perturbation $\Delta u(x)$ supports the type I bifurcation. A new bound state $\Phi_{n+1}^{\mp\epsilon}(x)$ can be decomposed through the complete sets $S^{\pm}$ according to the same integral representation (2.44) but with the eigenvalue $k = \mp i \kappa_{n+1}^{\epsilon}$ such that $\lim_{\epsilon \to 0} \kappa_{n+1}^{\epsilon} = 0$. A self-consistent solution to the homogeneous integral equation (2.45) may appear only if the kernel of the integral transform becomes singular as $k' \to 0$ for $\kappa_{n+1}^{\epsilon} \to 0$. Indeed, this is the case for a nongeneric potential, when the coefficients $a_{\pm}(k)$ satisfy the asymptotic representation (2.28). Solving Eqs. (2.45) and (2.46) into the limit $\epsilon \to 0$ we derive the following result.

Proposition 2.7. *Under the conditions that the potential $u(x)$ satisfies $m_{-1} = 0$ and the perturbation $\Delta u(x)$ satisfies $a_0^{-1} \Delta m_{-1} > 0$, the potential $u^{\epsilon} = u(x) + \epsilon \Delta u(x)$ supports a bound state in a neighbourhood of $k = 0$ for $\epsilon > 0$. The spectral data $(\kappa_{n+1}^{\epsilon}, \gamma_{n+1}^{\pm\epsilon})$ for the new bound state are defined by*

$$\kappa_{n+1}^{\epsilon} = \epsilon \Delta \kappa + \epsilon^2 \Delta_2 \kappa + \mathrm{O}(\epsilon^3),$$
$$\gamma_{n+1}^{\pm\epsilon} = \Delta \gamma^{\pm} + \mathrm{O}(\epsilon),$$

where

$$\Delta \kappa = \frac{\Delta m_{-1}}{2a_0} > 0, \tag{2.70}$$

$$\Delta_2 \kappa = \frac{1}{2a_0} \left[\mathrm{p.v.} \int_{-\infty}^{\infty} \frac{a_-^{-1}(k) K_+(0,k) K_+(k,0) - a_0^{-1} (\Delta m_{-1})^2}{4\pi k^2} dk + \sum_{j=1}^{n} \frac{K_{-j}(0) K_{+j}(0)}{2\kappa_j^2 a_j'} \right], \tag{2.71}$$

and

$$\Delta\gamma^{\pm} = 1 \pm m_0^{\pm}, \tag{2.72}$$

where $m_0^{\pm}$ is defined by Eq. (2.26).

Proof. Evaluating the singular contribution from the pole $k' = 0$ in Eq. (2.45), we find the leading order term in the form,

$$\alpha_{\pm}(k) \to \frac{\epsilon K_{\pm}(k,0)\alpha_{\pm}(0)}{4\pi a_0} Q_{\pm}, \tag{2.73}$$

where

$$Q_{\pm} = \int_{-\infty}^{\infty} \frac{dk'}{(k' \mp i0)(k' \pm i\epsilon\Delta\kappa)} = \frac{2\pi}{\epsilon\Delta\kappa},$$

if $\epsilon\Delta\kappa > 0$, and $Q_{\pm} = 0$, if $\epsilon\Delta\kappa < 0$. Therefore, the new eigenvalue exists under the condition $\Delta\kappa > 0$ (assuming $\epsilon > 0$). Writing Eq. (2.73) at $k = 0$ gives the asymptotic expression (2.70). The new bound state $\Phi_{n+1}^{\mp\epsilon}(x)$ is defined by Eq. (2.44) and (2.73). Using the boundary condition (2.3) for $N_{\pm}(x,k)$, we take the limit $x \to \mp\infty$ for $\Phi_{n+1}^{\mp\epsilon}(x)$ and find

$$\Phi_{n+1}^{\mp\epsilon}(x) \to \frac{\alpha_{\pm}(0)}{\epsilon K_{\pm}(0,0)} e^{\mp 2\epsilon\Delta\kappa x}.$$

The boundary condition (2.17) is met if $\alpha_{\pm}(0) = \epsilon K_{\pm}(0,0)$. Then, Eqs. (2.44) and (2.73) reduce to an asymptotic expression for the new bound state,

$$\Phi_{n+1}^{\mp\epsilon}(x) = \epsilon \int_{-\infty}^{\infty} \frac{K_{\pm}(k,0)N_{\pm}(x,k)dk}{4\pi(k \mp i0)a_{\mp}(k)(k \pm i\epsilon\Delta\kappa)} + \mathrm{O}(\epsilon), \tag{2.74}$$

where the integral term is an order of $\mathrm{O}(1)$. At the intermediate scale for finite x, we find from Eq. (2.74) that $\Phi_{n+1}^{\mp\epsilon}(x) = N_{\pm}(x,0) + \mathrm{O}(\epsilon)$. Therefore, the bound state approaches a delocalized limiting eigenfunction of the continuous spectrum for finite x. Then, using Eqs. (2.10) and (2.17), we take the limit $x \to \pm\infty$ in Eq. (2.74) and find $\Delta\gamma_j^{\pm}$ in the form,

$$\Delta\gamma^{\pm} = a_0 \pm b_0.$$

This expression reduces to Eq. (2.72) with the help of Eqs. (2.29). Finally, Eq. (2.71) follows from Eq. (2.45) at $k = 0$ by substituting the results of the first order of the perturbation theory. □

We notice that the asymptotic approximation for $\Delta\kappa$ can be equivalently written from Eqs. (2.27), (2.62), and (2.70) as

$$\Delta\kappa = \frac{1}{2}\left(1 - \frac{b_0}{a_0}\right) \int_{-\infty}^{\infty} dx\, \Delta u(x) \left[M_{+}(x,0)\right]^2 > 0. \tag{2.75}$$

We have thus obtained that, for the type I bifurcation, a new eigenvalue is located in a neighbourhood of the edge of the continuous spectrum (e.g. $k = 0$) and a new (localized) bound state arises from a delocalized critical eigenfunction that exists in a nongeneric case.

2.4.2. Example: A new eigenvalue supported by a small potential. Suppose that the initial potential is small, i.e. $u(x) = 0$ and $u^\epsilon(x) = \epsilon \Delta u(x)$. Then, the spectrum of the unperturbed problem is obvious:

$$n = 0, \quad a_\pm(k) = 1, \quad b_\pm(k) = 0, \quad M_\pm(x,k) = 0, \quad N_\pm(x,k) = e^{2ikx}.$$

Since $m_{-1} = 0$, the zero background belongs to the class of nongeneric potentials of type I. Therefore, the type I bifurcation is possible, i.e. an infinitesimal initial disturbance can support a single eigenvalue in the problem (2.1). The criterion for this bifurcation follows from Eq. (2.75) as

$$\Delta\kappa = \frac{1}{2}\Delta M = \frac{1}{2}\int_{-\infty}^{\infty} \Delta u(x) dx > 0, \tag{2.76}$$

where ΔM is the area integral which is the mass invariant for the KdV equation (1.3). This result is well-known as a Peierls problem in quantum mechanics [36]. In application to the KdV equation, we illustrate this phenomenon in Fig.1, where numerical simulations of Eq. (1.3) are presented.

Figure 1(a) shows the evolution of the initial data $u(x,0) = 2a\mathrm{sech}^2 x$ with $a = 0.5$. This corresponds to a disturbance of u with $\Delta M > 0$. We observe that the initial pulse evolves into a soliton propagating to the right and a radiative wave packet propagating to the left. The soliton has the mass $M_{sol} = 2\epsilon\Delta M$, while the radiation has the mass $M_{rad} = -\epsilon\Delta M$. On the other hand, the same initial pulse but with $a = -0.4$, which corresponds to the case $\Delta M < 0$, transforms solely into a linear radiative wave packet as seen in Fig. 1(b). No soliton is generated for this case.

In the critical case $\Delta M = 0$ (e.g. for asymmetric pulses $u(-x) = -u(x)$), the type I bifurcation may still take place if $\Delta_2\kappa > 0$. Inspecting the expression (2.71), we transform it to the form,

$$\Delta_2\kappa = -\frac{1}{4}\iint_{-\infty}^{\infty} dx dy \Delta u(x)\Delta u(y)|x-y|,$$

or, equivalently,

$$\Delta_2\kappa = -\frac{1}{2}\int_{-\infty}^{\infty} dx \left(\int_{-\infty}^{x} \Delta u(y) dy\right)\left(\int_{x}^{\infty} \Delta u(y) dy\right). \tag{2.77}$$

It is clear from Eq. (2.77) that $\Delta_2\kappa > 0$ if $\Delta\kappa = 0$. Therefore, the soliton generation always occurs even for critical initial disturbance with $\Delta M = 0$ (see also Ref. [37] for the same conclusion). Moreover, for small negative ΔM the soliton generation still occurs if $\kappa^\epsilon = \epsilon\Delta\kappa + \epsilon^2\Delta_2\kappa + \mathrm{O}(\epsilon^3) > 0$.

Preliminary results on soliton generation in the critical case $\Delta M = 0$ were reported by Karpman (see Chap. 21 in [35]). Using physical motivations and analysis of quasi-linear self-similar solutions, he found that the quasi-linear solutions of the KdV equation (1.3) exist for $\Delta M = 0$ and

$$p_1 = \int_{\infty}^{\infty} x u(x,0) dx < p_{cr},$$

where $p_{cr} \approx 7$. As a result, he concluded that no soliton can be generated by a small initial perturbation with $p_1 < p_{cr}$. This conclusion together with early numerical simulations (see Fig. 21.1 in Ref. [35]) are not confirmed by the analysis developed here.

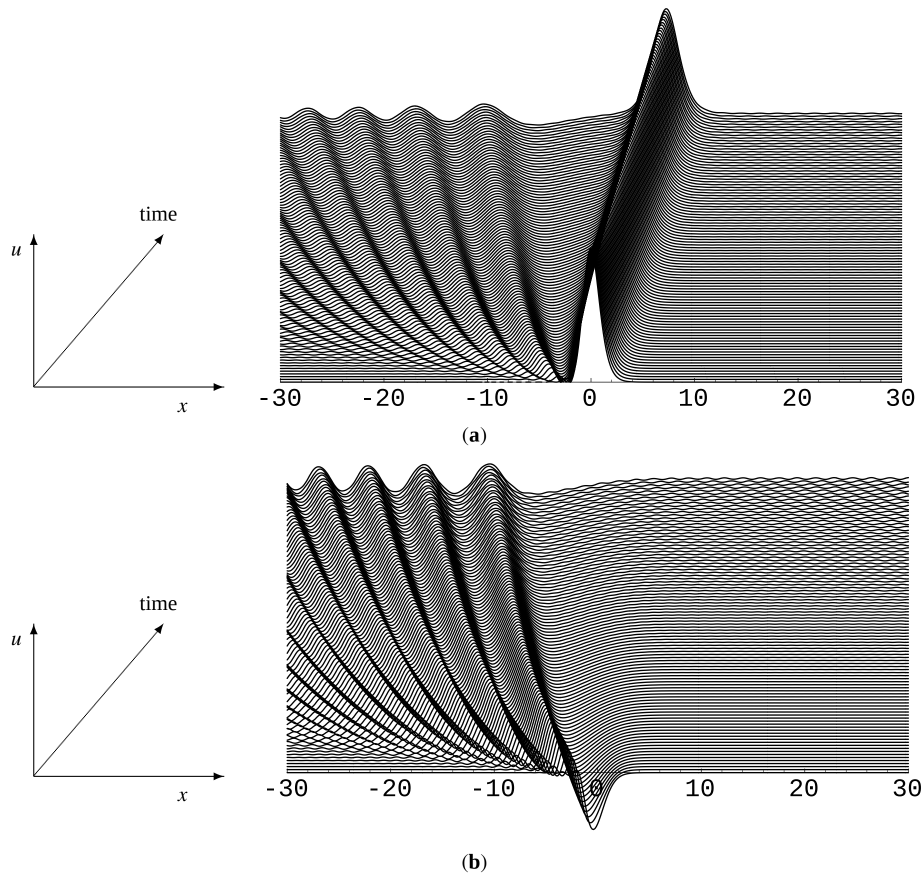

Fig. 2.1. Time evolution of the solution of the KdV equation (1.3) for the initial condition, $u(x, 0) = 2a\mathrm{sech}^2(x)$. **(a)** Formation of a new soliton for $a = 0.5$. **(b)** Transformation of an initial pulse to a linear wave packet for $a = -0.4$

2.4.3. Example: A new eigenvalue supported by a perturbed single soliton potential. Multisoliton potentials also belong to the class of nongeneric potentials of type I for the problem (2.1). Therefore, perturbation of multisoliton potentials may generate a new eigenvalue and a bound state provided the condition (2.75) is met. In particular, a perturbation to a single soliton generates a new bound state if

$$\Delta\kappa = \frac{1}{2}\int_{-\infty}^{\infty} \Delta u(x) \tanh^2[\kappa(x - x_0)]dx > 0, \tag{2.78}$$

where x_0 is defined in Eq. (2.63).

This bifurcation was analyzed in Ref. [38] for the problem of soliton production from a shelf emitted by a moving soliton. The account of a secondary soliton allowed one to satisfy the mass conservation in the KdV equation perturbed by an external (dissipative) term.

Recently, the same bifurcation was analyzed for the problem of existence of internal (oscillation) modes of kinks in nonlinear Klein-Gordon equations [42]. The criterion (2.78) was compared with numerical data for the oscillation mode in the spectrum of a double sine–Gordon equation.

3. Time-Dependent Schrödinger Problem

The (nonlocal) RH formalism for the linear equation (1.2) can be developed after the transformation, $\varphi = me^{-ikx-ik^2y}$, where $m = m(x, y, k)$ satisfies the problem,

$$im_y + m_{xx} - 2ikm_x + u(x, y)m = 0. \tag{3.1}$$

We assume that the function $u(x, y)$ is real, smooth and belongs to L^p for any $p \geq 2$. Also we assume the boundary condition for $u(x, y)$ in the form, $u(x, y) \sim \mathrm{O}(R^{-2})$ as $R = \sqrt{x^2 + y^2} \to \infty$, which includes the class of multilump potentials. Note that the function $u(x, y)$ for the multilump potentials is not in L^1. A solution $u = u(x, y, t)$ of the KPI equation (1.1) satisfies the constraint for $t > 0$ [27,28],

$$\int_{-\infty}^{\infty} u(x, y, t)dx = 0. \tag{3.2}$$

If the initial data $u = u(x, y, 0)$ does not satisfy this constraint, the instant transformation of a solution occurs in an initial time layer so that the solution has the jump discontinuity at $t \to 0^{\pm}$ [28–30].

Since the potential $u(x, y)$ of the linear system (3.1) corresponds to any solution of the KPI equation (1.1) including the initial data $u = u(x, y, 0)$, we do not impose the constraint (3.2) in our analysis and omit again the dependence on time. However, we assume the convergence of the following integral,

$$\left| \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx u(x, y) \right| < \infty. \tag{3.3}$$

Under this assumption, the integrals involving the eigenfunctions of Eq. (3.1), the spectral data and the potential $u(x, y)$ are bounded in the scheme developed below (see e.g. Eq. (3.65)).

3.1. Spectrum and scattering data. Here we construct the continuous and discrete spectrum for Eq. (3.1) according to previous approaches [8,23] and also derive additional relations between the spectral data.

3.1.1. Green's functions. The Green's functions $G_{\pm}(x, y, k)$ associated to the problem (3.1) have the form [6],

$$G_{\pm}(x, y, k) = \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{i(\xi x + 2\xi k y - \xi^2 y)} \left[\Theta(y)\Theta(\pm\xi) - \Theta(-y)\Theta(\mp\xi) \right] d\xi. \tag{3.4}$$

The Green's functions $G_+(x, y, k)$ and $G_-(x, y, k)$ are analytic in the domains $\mathrm{Im}(k) \geq 0$ and $\mathrm{Im}(k) \leq 0$ respectively and have a jump at $\mathrm{Im}(k) = 0$,

$$G_+(x, y, k) - G_-(x, y, k) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \mathrm{sign}(\xi) e^{i(\xi x + 2\xi k y - \xi^2 y)} d\xi. \tag{3.5}$$

In addition, the Green's functions have two symmetry properties:

$$\frac{\partial G_{\pm}(x,y,k)}{\partial k} = i(x+2ky)G_{\pm}(x,y,k) \pm \frac{i}{2\pi} \tag{3.6}$$

and

$$G_{\pm}(x,y,k) = G^*_{\mp}(-x,-y,k). \tag{3.7}$$

It follows from Eq. (3.4), or, equivalently, from Eq. (3.6) that the Green's functions are localized in the limit $R = \sqrt{x^2+y^2} \to \infty$,

$$G_{\pm}(x,y,k) \to \mp\frac{1}{2\pi(x+2(k\pm i0)y)} + \mathrm{O}(R^{-2}), \tag{3.8}$$

subject to $x+2ky \neq 0$. This expression is exact for $y=0$. Furthermore, the Green's functions are weakly localized along the singular line $x+2ky=0$, where $G_{\pm}(x,y,k) \to \mathrm{O}(R^{-1/2})$ as $R\to\infty$.

The boundary value (3.8) implies the following asymptotic expansion in the limit $k \to \infty \pm i0$,

$$G_{\pm}(x,y,k) \to \mp\frac{1}{4\pi k(y \mp i0\,\mathrm{sign}(x))} + \mathrm{O}(k^{-2}). \tag{3.9}$$

Using the relation,

$$\frac{1}{z\mp i0} = \pm\pi i\delta(z) + \text{p.v.}\left(\frac{1}{z}\right), \tag{3.10}$$

we express Eq. (3.9) in the form,

$$G_{\pm}(x,y,k) \to \frac{1}{4ik}\mathrm{sign}(x)\delta(y) \mp \frac{1}{4\pi ky} + \mathrm{O}(k^{-2}). \tag{3.11}$$

This result agrees with the analysis of Ref. [28].

Remark. The order of integration becomes important for computing spectral data for the problem (3.1) when the potential $u(x,y)$ is not absolutely integrable. Moreover, the result of integration of the Green's functions (3.4) depends on the order in the double integrals,

$$\int_{-\infty}^{\infty} dy \int_{\infty}^{\infty} dx G_{\pm}(x,y,k) = 0,$$

while

$$\int_{-\infty}^{\infty} dx \int_{\infty}^{\infty} dy G_{\pm}(x,y,k) = -\frac{1}{4(k\pm i0)^2}.$$

According to this result, we define all data for the former order of integration and use the following notation,

$$\iint_R dydx = \text{p.v.}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx. \tag{3.12}$$

3.1.2. Continuous spectrum. The eigenfunctions $M_\pm(x, y, k)$ and $N_\pm(x, y, k, l)$ of Eq. (3.1) satisfy Fredholm's integral equations,

$$M_\pm(x, y, k) = 1 + \iint_R dy' dx' G_\pm(x - x', y - y', k) u(x', y') M_\pm(x', y', k) \quad (3.13)$$

and

$$\begin{aligned} N_\pm(x, y, k, l) = & e^{i\beta(x,y,k,l)} \\ & + \int_R dy' dx' G_\pm(x - x', y - y', k) u(x', y') N_\pm(x', y', k, l), \end{aligned} \quad (3.14)$$

where $\beta(x, y, k, l) = (k - l)x + (k^2 - l^2)y$. The additional implicit parameter l appears for the eigenfunctions $N_\pm(x, y, k, l)$ according to the most general Fourier solution of Eq. (3.1) for the case $u(x, y) = 0$.

Applying the boundary conditions (3.8) to Eqs. (3.13) and (3.14), we find that the eigenfunctions $M_\pm(x, y, k)$ and $N_\pm(x, y, k, l)$ are not secular in x and y for any k and l. As $R = \sqrt{x^2 + y^2} \to \infty$ and $x + 2ky \neq 0$, they approach the boundary conditions,

$$M_\pm(x, y, k) \to 1 + \mathrm{O}(R^{-1}), \quad N_\pm(x, y, k, l) \to e^{i\beta(x,y,k,l)} + \mathrm{O}(R^{-1}). \quad (3.15)$$

The asymptotic representation of the eigenfunctions $M_\pm(x, y, k)$ in the limit $k \to \infty \pm i0$ follows from Eqs. (3.11) and (3.13),

$$\begin{aligned} M_\pm(x, y, k) = & 1 + \frac{1}{4ik} \left(\int_{-\infty}^{x} - \int_{x}^{\infty} \right) dx' u(x', y) \\ & \pm \frac{1}{4\pi k} \int_{-\infty}^{\infty} \frac{dy'}{y' - y} \int_{-\infty}^{\infty} dx' u(x', y') + \mathrm{O}(k^{-2}). \end{aligned} \quad (3.16)$$

Using Eq.(3.5), we find the RH boundary value problem for the eigenfunctions $M_\pm(x, y, k)$ at $\mathrm{Im}(k) = 0$,

$$M_+(x, y, k) - M_-(x, y, k) = -\left(\int_{-\infty}^{k} - \int_{k}^{\infty} \right) dl r_\mp(k, l) N_\pm(x, y, k, l), \quad (3.17)$$

where $r_\pm = r_\pm(k, l)$ is the spectral transform [8,23],

$$r_\pm(k, l) = \frac{1}{2\pi i} \iint_R dy dx u(x, y) M_\pm(x, y, k) e^{-i\beta(x,y,k,l)}. \quad (3.18)$$

The RH problem (3.17) is equivalent to the nonlocal form of Eq. (1.7) with $a_\pm(k) \equiv 1$. The closure relations (1.11) between the eigenfunctions $N_\pm(x, y, k, l)$ and $M_\pm(x, y, k)$ follow from Eq. (3.6),

$$\frac{\partial N_\pm(x, y, k, l)}{\partial k} = i(x + 2ky) N_\pm(x, y, k, l) \pm F_\pm(k, l) M_\pm(x, y, k), \quad (3.19)$$

where

$$F_\pm(k, l) = -\frac{1}{2\pi i} \iint_R dy dx u(x, y) N_\pm(x, y, k, l). \quad (3.20)$$

This relation is to be complemented by the boundary conditions following from the uniqueness of solutions of Eqs. (3.13) and (3.14),

$$N_{\pm}(x, y, k, k) = M_{\pm}(x, y, k). \tag{3.21}$$

Using these relations, we integrate Eq. (3.17) and obtain,

$$N_{\pm}(x, y, k, l) = M_{\pm}(x, y, l)e^{i\beta(x,y,k,l)} \pm \int_l^k F_{\pm}(p, l)M_{\pm}(x, y, p)e^{i\beta(x,y,k,p)}dp. \tag{3.22}$$

In addition to the spectral data $r_{\pm}(k, l)$ and $F_{\pm}(k, l)$, we consider also the spectral data $T_{\pm}(k, l, p)$ which appear in the relationship between the eigenfunctions $N_{\pm}(x, y, k, l)$ following from Eq. (3.14),

$$N_{+}(x, y, k, l) - N_{-}(x, y, k, l) = -\left(\int_{-\infty}^{k} - \int_{k}^{\infty}\right) dpT_{\mp}(k, l, p)N_{\pm}(x, y, k, p), \tag{3.23}$$

where

$$T_{\pm}(k, l, p) = \frac{1}{2\pi i}\iint_R dydxu(x, y)N_{\pm}(x, y, k, l)e^{-i\beta(x,y,k,p)}. \tag{3.24}$$

We point out that the relation (3.23) is not a RH boundary value problem since the eigenfunctions $N_{\pm}(x, y, k, l)$ have no meromorphic continuation in a complex domain of k. Still, the relation (3.23) is formally valid for real k. Furthermore, the integrals (3.18), (3.20), and (3.24) for the spectral data are not absolutely integrable and, therefore, the order of integration specified by Eq. (3.12) cannot be interchanged. On the other hand, the integrals in Eqs. (3.13) and (3.14) converge absolutely and the order of integration can be interchanged in these integrals and also in further integration with respect to x, y and k.

The spectral data $r_{\pm}(k, l)$ define the continuous spectrum of the problem (3.1) and satisfy the integral relations [23],

$$r_{\pm}(k, l) + r_{\pm}^{*}(l, k) \mp \left(\int_{-\infty}^{l} - \int_{k}^{\infty}\right) dpr_{\pm}(k, p)r_{\pm}^{*}(l, p) = 0,$$

$$r_{\pm}(k, l) + r_{\mp}^{*}(l, k) \pm \int_{k}^{l} dpr_{\pm}(k, p)r_{\mp}^{*}(l, p) = 0.$$

These equations were used in Ref. [23] to factorize the RH boundary-value nonlocal problem (3.17) and eliminate the set of eigenfunctions $N_{\pm}(x, y, k, l)$ from the problem. We intend to solve here a different problem: we express all eigenfuctions and scattering data in terms of the sets involving the eigenfunctions $N_{\pm}(x, y, k, l)$. In this respect, the following result completes the construction of the continuous spectrum for the problem (3.1).

Proposition 3.1. *The spectral data $r_{\pm}(k, l)$, $F_{\pm}(k, l)$ and $T_{\pm}(k, l, p)$ defined by Eqs. (3.18), (3.20) and (3.24) are related algebraically by*

$$r_{\pm}(k, l) = F_{\mp}^{*}(k, l), \tag{3.25}$$

$$T_{\pm}(k, l, p) = -T_{\mp}^{*}(k, p, l). \tag{3.26}$$

Proof. We multiply Eq. (3.14) by $u(x, y)M^*_\mp(x, y, k)$ and integrate over y and x. Using the symmetry relation (3.7) and integral equations (3.13) for $M^*_\mp(x, y, k)$ we find a simple formula,

$$0 = -2\pi i r^*_\mp(k, l) + 2\pi i F_\pm(k, l),$$

where we have used definitions (3.18) and (3.20). This formula is nothing but Eq. (3.25). The proof of Eq. (3.26) can be done by the same method starting with Eq. (3.14) and multiplying it by $u(x, y)N^*_\mp(x, y, k, l)$. □

3.1.3. Discrete spectrum. Bound states for Eq. (3.1) exist as homogeneous solutions of Fredholm's integral equations (3.13) for isolated complex values of k (eigenvalues). The eigenvalues are located symmetrically in upper and lower half-planes [8]. The bound states correspond to algebraically decaying lumps of the KPI equation (1.1). It was proved [20,21] that the bound states may appear as multiple poles in the complex plane of k. Here we restrict ourselves to the case when the bound states are not multiple.

The RH problem (3.17) coupled by the boundary conditions (1.10) and the closure relations (3.22) can be solved in the form,

$$M_\pm(x, y, k) = 1 + \sum_{j=1}^{n} \left[\frac{c_j^+ \Phi_j^+(x, y)}{k - k_j^+} + \frac{c_j^- \Phi_j^-(x, y)}{k - k_j^-} \right] - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dk'}{k' - (k \pm i0)} \left(\int_{-\infty}^{k'} - \int_{k'}^{\infty} \right) dl r_{-\sigma}(k', l) N_{+\sigma}(x, y, k', l), \tag{3.27}$$

where $\sigma = +1$ or $\sigma = -1$, $\Phi_j^\pm(x, y)$ are the bound states, n is the number of bound states, and $c_j^\pm$ are renormalization constants. The bound states $\Phi_j^\pm(x, y)$ are complex functions satisfying the homogeneous integral equations,

$$\Phi_j^\pm(x, y) = \iint_R dy' dx' G_\pm(x - x', y - y', k_j^\pm) u(x', y') \Phi_j^\pm(x', y'). \tag{3.28}$$

It follows from Eq. (3.8) that they can be renormalized according to the boundary conditions as $R = \sqrt{x^2 + y^2} \to \infty$,

$$\Phi_j^\pm(x, y) \to \frac{1}{x + 2k_j^\pm y} + \mathrm{O}(R^{-2}), \tag{3.29}$$

subject to the normalization constraints,

$$Q^\pm = \mp \frac{1}{2\pi} \iint_R dy dx u(x, y) \Phi_j^\pm(x, y) = 1. \tag{3.30}$$

Multiple bound states also occur for the KPI equation when the quantities $Q^\pm$ vanish. In this case, the expression (3.27) should be modified by multiple pole contributions [20, 21]. We consider only potentials $u(x, y)$ for which the renormalization (3.30) holds.

The limiting relations for the eigenfunctions $M_\pm(x, y, k)$ approaching bound states can be derived from Eq. (3.1) in the form,

$$\lim_{k \to k_j^\pm} \left[M_\pm(x, y, k) - \frac{c_j^\pm \Phi_j^\pm(x, y)}{k - k_j^\pm} \right] = \mu_j^\pm(x, y) = i c_j^\pm (x + 2k_j^\pm y + \gamma_j^\pm) \Phi_j^\pm(x, y). \tag{3.31}$$

Taking the limit $R = \sqrt{x^2 + y^2} \to \infty$ in Eq. (3.31) with the help of Eqs. (3.15) and (3.29) we find that

$$c_j^\pm = -i.$$

We notice that this constraint does not hold for the problem (2.1) in one dimension, where $c_j^\pm$ is related to $\gamma_j^\pm$. The data $\{k_j^\pm, \gamma_j^\pm\}_{j=1}^n$ defines the discrete spectrum of the problem (3.1) subject to the symmetry constraints,

$$(k_j^-)^* = k_j^+, \quad (\gamma_j^-)^* = \gamma_j^+. \tag{3.32}$$

The first symmetry constraint can be proved by means of the relation,

$$R_\pm(k) \equiv r_\pm(k, k) = -r_\mp^*(k, k), \tag{3.33}$$

which follows from Eqs. (3.21) and (3.25). The coefficients $R_+(k)$ and $R_-(k)$ are meromorphic functions in $\mathrm{Im}(k) \geq 0$ and $\mathrm{Im}(k) \leq 0$ respectively (see Eqs. (3.18) and (3.27)). Therefore, the symmetry constraint (3.33) implies that the location of the poles $R_+(k)$ and $R_-^*(k)$ coincides, i.e. the first relation in Eq. (3.32). The second symmetry constraint in Eq. (3.32) follows from Eqs. (3.70) and (3.71) below. Notice that the coefficients $R_\pm(k)$ play now the same role as the coefficients $\left[a_\pm(k)\right]^{-1}$ in the problem (2.1) despite the fact that $a_\pm(k) = 1$ for the RH problem (3.17).

3.1.4. Embedded eigenvalues. The continuous spectrum in the problem (3.1) has no edge points which separate it from the discrete spectrum. Recall that the problem (2.1) has the edge point at $k = 0$. Indeed, the spectral data $r_\pm(k, l)$ are not singular for real k and l in the general case (see Eqs. (3.18) and (3.27)), and the eigenfunctions $M_\pm(x, y, k)$ are not growing in x and y as $R = \sqrt{x^2 + y^2} \to \infty$ (see Eq. (3.15)). Still there are special (nongeneric) potentials $u(x, y)$ for which the spectral data become singular at a certain point $k = k_0$ at the real axis.

Definition 3.2. *The potential $u(x, y)$ is called* **nongeneric** *of type II if there is at least one eigenvalue embedded into the continuous spectrum, i.e. the homogeneous Fredholm's equations (3.28) exhibit bounded solutions at real $k = k_0$. Otherwise, the potential is called* **generic** *of type II.*

If the eigenvalue $k = k_0$ is embedded into the continuous spectrum, the eigenfunctions $M_\pm(x, y, k)$, $N_\pm(x, y, k, l)$, and the spectral data $r_\pm(k, l)$ have a resonant pole at $k = k_0$. This pole is produced by the integral part in the solution of the RH problem (3.27). We introduce the singular behavior of $M_\pm(x, y, k)$ as $k \to k_0$ according to a limiting relation as $k \to k_0$,

$$M_\pm(x, y, k) \to \frac{-i \Phi_0^\pm(x, y)}{k - (k_0 \mp i0)}, \tag{3.34}$$

where $\Phi_0^{\pm}(x, y)$ are solutions of Eq. (3.28) for $k = k_0$. It follows from Eq. (3.18) that

$$r_{\pm}(k,l) \to \frac{r_0^{\pm}(l)}{k - (k_0 \mp i0)} \quad \text{as } k \to k_0, \tag{3.35}$$

where

$$r_0^{\pm}(l) = -\frac{1}{2\pi} \iint_R dydxu(x, y)\Phi_0^{\pm}(x, y)e^{-i\beta(x,y,k_0,l)}.$$

We normalize the bound state $\Phi_0^{\pm}(x, y)$ according to the same constraint (3.30) so that

$$r_0^{\pm}(k_0) = \pm 1.$$

However, the boundary conditions (3.29) are no longer valid due to the singular line at $x + 2k_0 y = 0$ and the bound states $\Phi_0^{\pm}(x, y)$ are weakly localized as $R = \sqrt{x^2 + y^2} \to \infty$,

$$\Phi_0^{\pm}(x, y) \to \mathrm{O}(R^{-1}) \quad \text{as } x + 2k_0 y \neq 0 \tag{3.36}$$

and

$$\Phi_0^{\pm}(x, y) \to \mathrm{O}(R^{-1/2}) \quad \text{as } x + 2k_0 y = 0. \tag{3.37}$$

According to Eqs. (3.22) and (3.25), the eigenfunctions $N_{\pm}(x, y, k, l)$ are not singular as $l \to k_0$ and have the dominant behavior as $k \to k_0$,

$$N_{\pm}(x, y, k, l) \to \pm \frac{i r_0^{\mp *}(l)\Phi_0^{\pm}(x, y)}{k - (k_0 \mp i0)}. \tag{3.38}$$

Using Eqs. (3.27), (3.34), (3.35) and (3.38), we find that the asymptotic expressions are self-consistent provided the following constraints are satisfied,

$$\Phi_0^{+}(x, y) = -\Phi_0^{-}(x, y) \equiv \Phi_0(x, y) \tag{3.39}$$

and

$$\int_{-\infty}^{\infty} dl \; \mathrm{sign}(k_0 - l) \; |r^{\pm}(l)|^2 = 0. \tag{3.40}$$

These constraints can be derived by evaluating the residue contributions at $k = k_0$ in Eq. (3.27) with the help of the formal expansion,

$$\mathrm{sign}(k - l) = \mathrm{sign}(k_0 - l) + 2(k - k_0)\delta(k_0 - l) + \mathrm{O}(k - k_0)^2. \tag{3.41}$$

These eigenstates $\Phi_0^{\pm}(x, y)$ are called *half-bound states* since they are weakly localized as $R \to \infty$ and their spectral data consist only of the embedded eigenvalue k_0.

Embedded eigenvalues and half-bound states are structurally unstable under a perturbation of the potential according to the theory of quantum resonances [43,44]. Therefore, we expect that the perturbation leads either to disappearance of the embedded eigenvalues at $k = k_0$ or to their emergency into the complex domain as true eigenvalues. This is the type II bifurcation analyzed in Sect. 3.4.

3.2. Spectral decompositions. Here we study the spectral decomposition based on the eigenfunctions of the problem (3.1) for a potential $u(x, y)$. Our analysis is not affected by the presence of embedded eigenvalues. The only assumption required for the potential $u(x, y)$ is that it does not support multiple poles in the expansion (3.27). Non-standard orthogonality and completeness relations for the eigenfunctions of Eq. (3.1) are obtained in Sect. 3.2.1. Additional integral relations for the data $\gamma_j^{\pm}$ of the discrete spectrum are derived in Sect. 3.2.2.

3.2.1. Scalar products, orthogonality and completeness relations. The eigenfunctions $M_{\pm}(x, y, k)$ are characterized through the sets of eigenfunctions $S^{\pm} = [N_{\pm}(x, y, k, l), \{\Phi_j^{+}(x, y), \Phi_j^{-}(x, y)\}_{j=1}^{n}]$ by means of Eq. (3.27). The spectral data $r_{\pm}(k, l)$ and $\{k_j^{\pm}, \gamma_j^{\pm}\}_{j=1}^{n}$ are defined by the sets $S^{\pm}$ through Eqs. (3.20), (3.25), (3.30) and (3.31) (see also the additional Eq. (3.70) below). The potential $u(x, y)$ is related to the sets $S^{\pm}$ as follows [6]

$$\frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right) u(x', y)dx' = \frac{1}{\pi}\int_{-\infty}^{\infty} dk \left(\int_{-\infty}^{k} - \int_{k}^{\infty}\right) dl r_{\mp}(k, l) N_{\pm}(x, y, k, l) + 2\sum_{j=1}^{n}\left[\Phi_j^{+}(x, y) + \Phi_j^{-}(x, y)\right]. \tag{3.42}$$

This formula results from Eqs. (3.16) and (3.27) in the limit $k \to \infty$. Thus, the scheme for closure of the integral transform holds for the sets $S^{\pm}$ and we state the following main result.

Proposition 3.3. *An arbitrary scalar function $f(x, y)$ with the boundary conditions $\lim_{x\to\pm\infty} f(x, y) = f_{\pm}(y)$ can be decomposed through any of the orthogonal and complete sets of eigenfunctions $S^{\pm}$ if $f_{+}(y) + f_{-}(y) = 0$.*

The proof of this proposition is based on two lemmas.

Lemma 3.4. *The eigenfunctions $N_{\pm}(x, y, k, l)$ and $\{\Phi_j^{+}(x, y), \Phi_j^{-}(x, y)\}_{j=1}^{n}$ introduced in Sects. 3.1.2 and 3.1.3 satisfy the orthogonality relations,*

$$\langle N_{\pm}(k', l')|N_{\pm}(k, l)\rangle = 2\pi^2 i \,\mathrm{sign}(k - l)\, \delta(k - k')\, \delta(l - l'), \tag{3.43}$$

$$\langle \Phi_j^{\pm}|N_{\pm}(k, l)\rangle = \langle N_{\pm}(k, l)|\Phi_j^{\pm}\rangle = \langle \Phi_j^{\mp}|N_{\pm}(k, l)\rangle = \langle N_{\pm}(k, l)|\Phi_j^{\mp}\rangle = 0, \tag{3.44}$$

$$\langle \Phi_l^{\pm}|\Phi_j^{\pm}\rangle = 0, \quad \langle \Phi_l^{\mp}|\Phi_j^{\pm}\rangle = \pm\pi\delta_{jl}, \tag{3.45}$$

where the scalar product is given by

$$\langle g(k', l')|h(k, l)\rangle = \iint_{R} dydx g^{*}(x, y, k', l')\partial_x h(x, y, k, l). \tag{3.46}$$

Proof. First, we derive a balance equation for two solutions $h(k,l)$ and $g(k',l')$ of Eq. (3.1) with a real potential $u(x,y)$,

$$i\frac{\partial}{\partial y}\left(g^*(k',l')h(k,l)\right)+\frac{\partial}{\partial x}\left(g^*(k',l')h_x(k,l)-g_x^*(k',l')h(k,l)\right. \\ \left.-2ik'g^*(k',l')h(k,l)\right)=2i(k-k')g^*(k',l')h_x(k,l). \tag{3.47}$$

We integrate this equation for $h=N_\pm(x,y,k,l)$ and $g^*=N_\pm^*(x,y,k',l')$ over x and then over y. Using Eqs. (2.37) and (3.15), we derive the relation,

$$\langle N_\pm(k',l')|N_\pm(k,l)\rangle=\frac{1}{2(k-k')}\left[\lim_{y\to\infty}-\lim_{y\to-\infty}\right] \\ \cdot\int_{-\infty}^{\infty}dxN_\pm^*(x,y,k',l')N_\pm(x,y,k,l) \\ +2\pi^2 i\frac{(k-k'-l)^2-l'^2}{k-k'}\delta(k-k'-l+l')\delta(k^2-l^2-k'^2+l'^2). \tag{3.48}$$

We substitute the integral representation (3.14) to evaluate the first term in Eq. (3.48) and integrate the Green's functions according to Eq. (3.4). Then, the relation (3.48) reduces to the formula,

$$\langle N_\pm(k',l')|N_\pm(k,l)\rangle=4\pi^2 i(k-l)\delta(k-l-k'+l')\delta(k^2-l^2-k'^2+l'^2) \\ \mp\pi^2 i\delta(k-k')\mathcal{R}_\pm, \tag{3.49}$$

where

$$\mathcal{R}_\pm=T_\pm(k,l,l')+T_\pm^*(k,l',l)\mp\left[\int_{-\infty}^{k}-\int_{k}^{\infty}\right]dpT_\pm(k,l,p)T_\pm^*(k,l',p)$$

and $T_\pm(k,l,p)$ is given by Eq. (3.24). In the derivation of Eq. (3.49) we have supposed that $k\neq l$ and $k'\neq l'$, i.e. the eigenfunction $N_\pm(x,y,k,l)$ is not degenerate [cf. Eq. (3.21)]. Under these conditions, zeros of both δ-functions in Eq. (3.49) occur only for $k=k'$ and $l=l'$. Therefore, we simplify Eq. (3.49) by using the following formulas,

$$\alpha\delta(\alpha x)=\mathrm{sign}(\alpha)\delta(x),\quad 2\delta(x+y)\delta(x-y)=\delta(x)\delta(y). \tag{3.50}$$

Then, Eq. (3.49) reduces to Eq. (3.43) provided $\mathcal{R}_\pm=0$. The latter identity follows from the relation (3.26) and the explicit expressions (3.23) and (3.24).

The zero scalar products (3.44) and (3.45) can also be found from Eq. (3.47) for bound states. In order to find the nonzero inner products in Eqs. (3.45), we integrate Eq. (3.47) for $h=M_\pm(x,y,k)$ and $g^*=\Phi_j^{\mp*}(x,y)$ over x and then over y and use the boundary conditions (3.15) and (3.29). As a result, we derive the integral relation,

$$2i(k-k_j^\pm)\iint_R dydx\Phi_j^{\mp*}(x,y)\partial_x M_\pm(x,y,k)=\pm 2\pi. \tag{3.51}$$

This relation reduces to Eq. (3.45) after substitution of Eq. (3.27) for $M_\pm(x,y,k)$ and use of the zero scalar products (3.44) and (3.45). □

We notice that Boiti et al. [23] used different scalar products for the orthogonality relations,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx M_{\mp}^{*}(x,y,l)M_{\pm}(x,y,k)e^{i\beta(x,y,l,k)} = \delta(l-k). \tag{3.52}$$

These products generalize the results of the time-independent Schrödinger equation and are independent on y for solutions of Eq. (3.1). However, an arbitrary scalar function in two dimensions cannot be decomposed through the eigenfunctions

$$M_{\pm}(x,y,k)e^{-i(kx+k^2y)}$$

which depend on one spectral parameter. Note that, in the time-dependent problem, the eigenfunctions $N_{\pm}(x,y,k,l)$ and $N_{\pm}(x,y,k',l')$ are orthogonal while $N_{\mp}(x,y,k,l)$ and $N_{\pm}(x,y,k',l')$ are not (the time-independent problem has the opposite property, see Eq. (2.34)).

Lemma 3.5. *The eigenfunctions $N_{\pm}(x,y,k,l)$ and $\{\Phi_j^{+}(x,y),\Phi_j^{-}(x,y)\}_{j=1}^{n}$ satisfy the completeness relation,*

$$\frac{1}{2}\mathrm{sign}(x-x')\delta(y-y') = \frac{1}{2\pi^2 i}\iint_D dkdl N_{\pm}^{*}(x',y',k,l)N_{\pm}(x,y,k,l)$$

$$+\frac{1}{\pi}\sum_{j=1}^{n}\left[\Phi_j^{-*}(x',y')\Phi_j^{+}(x,y) - \Phi_j^{+*}(x',y')\Phi_j^{-}(x,y)\right], \tag{3.53}$$

where we have used the notation,

$$\iint_D dkdl \equiv \int_{-\infty}^{\infty} dk\left(\int_{-\infty}^{k} - \int_{k}^{\infty}\right)dl.$$

Proof. We start with transforming Eq. (3.19) to the form,

$$\begin{aligned}
&\frac{\partial}{\partial k}\left[N_{\pm}^{*}(x',y',k,l)N_{\pm}(x,y,k,l)\right]\\
&= i\left[x-x'+2k(y-y')\right]N_{\pm}^{*}(x',y',k,l)N_{\pm}(x,y,k,l)\\
&\pm\left[M_{\pm}(x,y,k)r_{\mp}^{*}(k,l)N_{\pm}^{*}(x',y',k,l) + M_{\pm}^{*}(x',y',k)r_{\mp}(k,l)N_{\pm}(x,y,k,l)\right],
\end{aligned}$$

where we have used Eq. (3.25). Multiplying this equation by $\mathrm{sign}(k-l)$ and integrating over l, we derive the expression,

$$\begin{aligned}
\frac{\partial W(k)}{\partial k} &= i\left[x-x'+2k(y-y')\right]W(k) + M_{-}^{*}(x',y',k)M_{+}(x,y,k)\\
&\quad + M_{+}^{*}(x',y',k)M_{-}(x,y,k),
\end{aligned} \tag{3.54}$$

where

$$W(k) = \left[\int_{-\infty}^{k} - \int_{k}^{\infty}\right] dl N_{\pm}^{*}(x',y',k,l)N_{\pm}(x,y,k,l).$$

The functions $M_{-}^{*}(x',y',k)M_{+}(x,y,k)$ and $M_{+}^{*}(x',y',k)M_{-}(x,y,k)$ are meromorphic in $\mathrm{Im}(k) \geq 0$ and $\mathrm{Im}(k) \leq 0$ respectively. We apply the Plemelj formula (see

Appendix A1 in [6]) to reconstruct their sum from a given jump at $\mathrm{Im}(k) = 0$. Evaluating the pole contribution according to Eqs. (3.31) and (3.32), we derive the following representation,

$$\begin{aligned} &\frac{1}{2}\left[M_-^*(x', y', k)M_+(x, y, k) + M_+^*(x', y', k)M_-(x, y, k)\right] \\ &= R(k) + \frac{1}{2}\left[\Delta^+(k) - \Delta^-(k)\right], \end{aligned} \tag{3.55}$$

where

$$\begin{aligned} R(k) = 1 &+ \sum_{j=1}^{n} \Phi_j^{+*}(x', y')\Phi_j^-(x, y)\left[\frac{i(x - x' + 2k_j^-(y - y'))}{k - k_j^-} + \frac{1}{(k - k_j^-)^2}\right] \\ &+ \sum_{j=1}^{n} \Phi_j^{-*}(x', y')\Phi_j^+(x, y)\left[\frac{i(x - x' + 2k_j^+(y - y'))}{k - k_j^+} + \frac{1}{(k - k_j^+)^2}\right] \end{aligned} \tag{3.56}$$

and

$$\begin{aligned} \Delta^{\pm}(k) = &\pm \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dk'}{k' - (k \pm i0)} \\ &\cdot \left[M_-^*(x', y', k')M_+(x, y, k') - M_+^*(x', y', k')M_-(x, y, k')\right]. \end{aligned} \tag{3.57}$$

The functions $\Delta^+(k)$ and $\Delta^-(k)$ represent the boundary values at real k of analytical functions in the upper and lower half-plane of k, respectively, subject to the boundary conditions in the limit $k \to \infty \pm i0$,

$$\Delta^{\pm}(k) \to \pm \frac{\Delta_{\infty}^{\pm}}{k} + \mathrm{O}(k^{-2}), \tag{3.58}$$

where $\Delta_{\infty}^{\pm}$ are given by

$$\begin{aligned} \Delta_{\infty}^{\pm} = &-\frac{1}{2\pi i}\int_{-\infty}^{\infty} dk\left[M_-^*(x', y', k)M_+(x, y, k) - M_+^*(x', y', k)M_-(x, y, k)\right] \\ &\pm \frac{(y - y')}{4\pi}\int_{-\infty}^{\infty} \frac{dy''}{(y'' - y)(y'' - y')}\int_{-\infty}^{\infty} dx'' u(x'', y''). \end{aligned} \tag{3.59}$$

Solving Eq. (3.54) as a differential equation in k, we derive explicitly,

$$\int_{-\infty}^{\infty} W(k)dk = \int_{-\infty}^{\infty} W^+(k)dk + \int_{-\infty}^{\infty} W^-(k)dk + \iint_D dk dl R(l) e^{i\beta(x - x', y - y', k, l)}, \tag{3.60}$$

where the functions $W^{\pm}(k)$ solve the differential equations,

$$\frac{\partial W^{\pm}(k)}{\partial k} = i\left[x - x' + 2k(y - y')\right]W^{\pm}(k) \pm \Delta^{\pm}(k). \tag{3.61}$$

In order to evaluate the last integral in Eq. (3.60) from Eq. (3.56), we transform the variables,

$$k = \frac{1}{2}(p + \kappa), \quad l = \frac{1}{2}(p - \kappa),$$

and integrate first over p and then over κ with the use of the residue theorem. Then, Eq. (3.60) reproduces exactly the completeness relation (3.53) subject to the following constraint,

$$\int_{-\infty}^{\infty} W^+(k)dk + \int_{-\infty}^{\infty} W^-(k)dk = 0. \tag{3.62}$$

Now we show that this constraint is satisfied for the functions $W^{\pm}(k)$ defined by Eqs. (3.61). Since the right-hand-sides $\Delta^{\pm}(k)$ are analytical functions in the upper/lower half-planes of k, the functions $W^{\pm}(k)$ solving Eq. (3.61) can be analytically continued in the corresponding domains of k subject to the constraint,

$$(y - y') \int_{-\infty}^{\infty} W^{\pm}(k)dk = 0. \tag{3.63}$$

In the case $y \neq y'$, the boundary conditions for $W^{\pm}(k)$ follow from Eqs. (3.58) and (3.61) as $W^{\pm}(k) \sim \mathrm{O}(k^{-2})$. Therefore, the constraints (3.62) and (3.63) are satisfied. In the case $y = y'$, the constraint (3.63) is still met and the functions $W^{\pm}(k)$ have the boundary conditions as $k \to \infty \pm i0$,

$$W^{\pm}(k) \to \frac{i\Delta_{\infty}^{\pm}}{k(x - x')} + \mathrm{O}(k^{-2}).$$

As a result, we find explicitly that

$$\int_{-\infty}^{\infty} W^{\pm}(k)dk = \pm\frac{\pi\Delta_{\infty}^{\pm}}{x - x'}.$$

However, it follows from Eq. (3.59) that $\Delta_{\infty}^{+} = \Delta_{\infty}^{-}$ when $y = y'$ and, therefore, the constraint (3.62) is satisfied. □

Proof of Proposition 3.3. We decompose a scalar function $f(x, y)$ in the form,

$$\begin{aligned} f(x, y) = \frac{1}{2}\left(f_+(y) + f_-(y)\right) + \iint_D dkdl\alpha_{\pm}(k, l)N_{\pm}(x, y, k, l) \\ + \sum_{j=1}^{n}\left(\alpha_j^+\Phi_j^+(x, y) + \alpha_j^-\Phi_j^-(x, y)\right). \end{aligned} \tag{3.64}$$

The coefficients of the expansion can be expressed through the derivative $f_x(x, y)$ according to Eqs. (3.43)–(3.45),

$$\alpha_{\pm}(k, l) = \frac{\langle N_{\pm}(k, l)|f\rangle}{2\pi^2 i}, \quad \alpha_j^{\pm} = \pm\frac{\langle\Phi_j^{\mp}|f\rangle}{\pi}.$$

Then, Eq. (3.64) reduces to an identity by means of Eq. (3.53). □

We conclude that the relation (3.42) for the inverse scattering transform is a particular application of Eq. (3.64). The coefficients $r_{\pm}(l, k)$ play the role of Fourier coefficients and they can be found from Eqs. (3.42) and (3.43) in the form (3.20) and (3.25). The coefficients $c_j^{\pm}$ for the discrete spectrum are all fixed, $c_j^{\pm} = -i$, due to the renormalization conditions (3.30). These conditions are consistent with Eqs. (3.42) and (3.45). Notice that the formula (3.42) gives a nontrivial limit at $x \to \pm\infty$ even in the case when the constraint (3.2) does not hold. Indeed, integrating Eq. (3.42) over y and then taking the limit $x \to \infty$, we derive the explicit expression,

$$\iint_R dydxu(x,y) = 2\pi i R_{\pm}(0) - \iint_D dkdl \frac{|r_{\pm}(k,l)|^2}{k \mp i0} + \pi i \sum_{j=1}^{n} \frac{k_j^- - k_j^+}{k_j^+ k_j^-}, \tag{3.65}$$

where we have used the relations following from Eqs. (3.4), (3.14) and (3.28),

$$\left[\lim_{x\to\infty} - \lim_{x\to-\infty}\right] \int_{-\infty}^{\infty} dy\ N_{\pm}(x,y,k,l) = 2\pi^2 i\ \mathrm{sign}(k-l)\ \delta(k)\ \delta(l) - \frac{\pi r_{\mp}^*(k,l)}{k \pm i0} \tag{3.66}$$

and

$$\left[\lim_{x\to\infty} - \lim_{x\to-\infty}\right] \int_{-\infty}^{\infty} dy\ \Phi_j^{\pm}(x,y) = \pm \frac{\pi i}{k_j^{\pm}}. \tag{3.67}$$

The relation (3.65) can also be derived from Eq. (3.27). We notice that the spectral decomposition gives an explicit value for the mass integral (3.65) provided the order of integration is specified according to Eq. (3.12).

3.2.2. Characterization of the data of the discrete spectrum. Here we use the orthogonality relations (3.43)–(3.45) and find an integral representation for the parameters $\gamma_j^{\pm}$ of the bound states. First, it follows from Eqs. (3.27) and (3.31) that the functions $\Phi_j^{\pm}(x,y)$ satisfy the system of algebraic equations,

$$\begin{aligned}(x + 2k_j^{\pm} y + \gamma_j^{\pm})\Phi_j^{\pm}(x,y) &= 1 - i {\sum_{l=1}^{n}}' \left[\frac{\Phi_l^{+}(x,y)}{k_j^{\pm} - k_l^{+}} + \frac{\Phi_l^{-}(x,y)}{k_j^{\pm} - k_l^{-}}\right] \\ &- \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dk}{k - k_j^{\pm}} \left(\int_{-\infty}^{k} - \int_{k}^{\infty}\right) dl r_{-\sigma}(k,l) N_{+\sigma}(x,y,k,l),\end{aligned} \tag{3.68}$$

where $\sigma = +1$ or $\sigma = -1$ and $\sum'$ stands for sum without the singular term at $k_l^{\pm} = k_j^{\pm}$. Equation (3.68) can be viewed as a spectral decomposition of the functions $\mu_j^{\pm}(x,y)$ defined by Eq. (3.31) through the complete sets $S^{\pm}$. It follows from Eq. (3.68) and Eqs. (3.43)–(3.45) that

$$\iint_R dydx \Phi_j^{\mp *}(x,y) \left((x + 2k_j^{\pm} y + \gamma_j^{\pm})\Phi_j^{\pm}(x,y)\right)_x = 0. \tag{3.69}$$

As a result, the spectral data $\gamma_j^{\pm}$ can be expressed from Eqs. (3.45) and (3.69) as

$$\gamma_j^{\pm} = \mp \frac{1}{\pi} \iint_R dydx \Phi_j^{\mp *}(x,y)(x + 2k_j^{\pm} y)\Phi_{jx}^{\pm}(x,y), \tag{3.70}$$

subject to the constraint,

$$\int_{-\infty}^{\infty} dx \Phi_j^{\mp *}(x, y) \Phi_j^{\pm}(x, y) = 0. \tag{3.71}$$

This constraint can be derived by integrating Eq. (3.47) for $h = \Phi_j^{\pm}(x, y)$ and $g^* = \Phi_j^{\mp *}(x, y)$ over x subject to the zero boundary conditions as $y \to \infty$ (see Eq. (3.29)). We notice that Eqs. (3.70) and (3.71) imply the symmetry constraint $\left(\gamma_j^-\right)^* = \gamma_j^+$, i.e. the second relation in Eq. (3.32).

We apply the orthogonal and complete sets of eigenfunctions $S^{\pm}$ to study perturbations of the potential and variation of the spectral data for Eq. (3.1).

3.3. Perturbation theory. Suppose that the potential can be decomposed as $u^{\epsilon} = u(x, y) + \epsilon \Delta u(x, y)$, where $\epsilon \ll 1$. We assume that the potential $u(x, y)$ supports the complete sets of eigenfunctions $S^{\pm} = \left[N_{\pm}(x, y, k, l), \{\Phi_j^+(x, y), \Phi_j^-(x, y)\}_{j=1}^n\right]$. Also we assume that the perturbation $\Delta u(x, y) \sim \mathrm{O}(1)$ as $\epsilon \to 0$. Here we evaluate variations of the spectral data due to the perturbation $\Delta u(x, y)$.

3.3.1. Variations of data of discrete spectrum. Suppose that $\Phi_j^{\pm\epsilon}(x, y)$ solves Eq. (3.1) for $u^{\epsilon} = u(x, y) + \epsilon \Delta u(x, y)$ with the eigenvalue $k = k_j^{\pm\epsilon}$. We expand $\Phi_j^{\pm\epsilon}(x, y)$ through the sets $S^{\pm}$ according to Eq. (3.64) rewritten as

$$\begin{aligned} \Phi_j^{\pm\epsilon}(x, y) = & \iint_D dkdl \frac{\alpha_{\pm}(k, l) N_{\pm}(x, y, k, l)}{4\pi^2 (k - k_j^{\pm\epsilon})} \\ & \pm \frac{1}{2\pi i} \sum_{l=1}^{n} \left[\frac{\alpha_l^{\pm} \Phi_l^{\pm}(x, y)}{k_j^{\pm\epsilon} - k_l^{\pm}} - \frac{\tilde{\alpha}_l^{\mp} \Phi_l^{\mp}(x, y)}{k_j^{\pm\epsilon} - k_l^{\mp}} \right]. \end{aligned} \tag{3.72}$$

The eigenvalue problem (3.1) reduces with the help of Eqs. (3.43)–(3.45) and (3.72) to a set of homogeneous integral equations,

$$\begin{aligned} \alpha_{\pm}(k, l) = \epsilon & \left[\iint_D dk'dl' \frac{K_{\pm}(k, k', l, l') \alpha_{\pm}(k', l')}{4\pi^2 (k' - k_j^{\pm\epsilon})} \right. \\ & \left. \pm \frac{1}{2\pi i} \sum_{l=1}^{n} \left(\frac{K_{\pm l}(k, l) \alpha_l^{\pm}}{k_j^{\pm\epsilon} - k_l^{\pm}} - \frac{\tilde{K}_{\mp l}(k, l) \tilde{\alpha}_l^{\mp}}{k_j^{\pm\epsilon} - k_l^{\mp}} \right) \right], \end{aligned} \tag{3.73}$$

$$\alpha_l^{\pm} = \epsilon \left[\iint_D dkdl \frac{\tilde{K}_{\mp l}^*(k, l) \alpha_{\pm}(k, l)}{4\pi^2 (k - k_j^{\pm\epsilon})} \pm \frac{1}{2\pi i} \sum_{m=1}^{n} \left(\frac{K_{\pm lm} \alpha_m^{\pm}}{k_j^{\pm\epsilon} - k_m^{\pm}} - \frac{\tilde{K}_{\mp lm} \tilde{\alpha}_m^{\mp}}{k_j^{\pm\epsilon} - k_m^{\mp}} \right) \right], \tag{3.74}$$

$$\tilde{\alpha}_l^{\mp} = \epsilon \left[\iint_D dkdl \frac{K_{\pm l}^*(k, l) \alpha_{\pm}(k, l)}{4\pi^2 (k - k_j^{\pm\epsilon})} \pm \frac{1}{2\pi i} \sum_{m=1}^{n} \left(\frac{\tilde{K}_{\pm lm} \alpha_m^{\pm}}{k_j^{\pm\epsilon} - k_m^{\pm}} - \frac{K_{\mp lm} \tilde{\alpha}_m^{\mp}}{k_j^{\pm\epsilon} - k_m^{\mp}} \right) \right], \tag{3.75}$$

where the integral elements are

$$K_{\pm}(k,k',l,l') = \iint_R dydx\Delta u(x,y) N_{\pm}^*(x,y,k,l) N_{\pm}(x,y,k',l'),$$

$$K_{\pm j}(k,l) = \iint_R dydx\Delta u(x,y) N_{\pm}^*(x,y,k,l) \Phi_j^{\pm}(x,y),$$

$$\tilde{K}_{\pm j}(k,l) = \iint_R dydx\Delta u(x,y) N_{\mp}^*(x,y,k,l) \Phi_j^{\pm}(x,y),$$

$$K_{\pm jl} = \iint_R dydx\Delta u(x,y) \Phi_j^{\mp *}(x,y) \Phi_l^{\pm}(x,y),$$

and

$$\tilde{K}_{\pm jl} = \iint_R dydx\Delta u(x,y) \Phi_j^{\pm *}(x,y) \Phi_l^{\pm}(x,y).$$

The results of the asymptotic analysis of Eqs. (3.73)–(3.75) in the limit $\epsilon \to 0$ are summarized in the following proposition.

Proposition 3.6. *Variational derivatives of the data* $\{k_j^{\pm}, \gamma_j^{\pm}\}_{j=1}^n$ *of the discrete spectrum of Eq. (3.1) with respect to the potential* $u(x,y)$ *are defined by*

$$\frac{\delta k_j^{\pm}}{\delta u(x,y)} = \pm \frac{\Phi_j^{\mp *}(x,y)\Phi_j^{\pm}(x,y)}{2\pi i}, \tag{3.76}$$

$$\begin{aligned}\frac{\delta \gamma_j^{\pm}}{\delta u(x,y)} &= \mp \frac{y\Phi_j^{\mp *}(x,y)\Phi_j^{\pm}(x,y)}{\pi i} \\ &\pm \frac{1}{2\pi}{\sum_{l=1}^{n}}' \left[\frac{\Phi_j^{\mp *}\Phi_l^{\pm} - \Phi_j^{\pm}\Phi_l^{\mp *}}{(k_j^{\pm} - k_l^{\pm})^2} + \frac{\Phi_j^{\mp *}\Phi_l^{\mp} - \Phi_j^{\pm}\Phi_l^{\pm *}}{(k_j^{\pm} - k_l^{\mp})^2} \right] \\ &\pm \iint_D dkdl \frac{r_{\pm}(k,l)N_{\mp}(x,y,k,l)\Phi_j^{\mp *}(x,y) - r_{\mp}^*(k,l)N_{\pm}^*(x,y,k,l)\Phi_j^{\pm}(x,y)}{4\pi^2(k-k_j^{\pm})^2},\end{aligned} \tag{3.77}$$

where $\sum_l'$ *stands for sum excluding the singular term at* $k_l^{\pm} = k_j^{\pm}$.

Proof. The self-consistency condition of Eq. (3.74) at $l = j$ defines the expansion of the eigenvalue $k_j^{\pm\epsilon}$ as

$$k_j^{\pm\epsilon} = k_j^{\pm} + \epsilon\Delta k_j^{\pm} + \mathrm{O}(\epsilon^2),$$

where

$$\Delta k_j^{\pm} = \pm \frac{K_{\pm jj}}{2\pi i}. \tag{3.78}$$

This formula is equivalent to Eq. (3.76). We notice that the symmetry $k_j^- = k_j^{+*}$ is preserved in the perturbation theory for real $\Delta u(x,y)$. The set of integral equations is solved at the leading order as

$$\alpha_{\pm}(k,l) = \epsilon K_{\pm j}(k,l) + \mathrm{O}(\epsilon^2),$$

$$\alpha_l^{\pm} = \epsilon K_{\pm lj} + \mathrm{O}(\epsilon^2),$$

$$\tilde{\alpha}_l^{\mp} = \epsilon \tilde{K}_{\pm lj} + \mathrm{O}(\epsilon^2).$$

This solution defines a perturbation to the bound state, $\Phi_j^{\pm\epsilon}(x, y) = \Phi_j^{\pm}(x, y) + \epsilon\Delta\Phi_j^{\pm}(x, y) + O(\epsilon^2)$, in the form,

$$\Delta\Phi_j^{\pm}(x, y) = \Delta\alpha_j^{\pm}\Phi_j^{\pm}(x, y) \pm \frac{1}{2\pi i}\sum_{l=1}^{n}{}' \left[\frac{K_{\pm lj}\Phi_l^{\pm}(x, y)}{k_j^{\pm} - k_l^{\pm}} - \frac{\tilde{K}_{\pm lj}\Phi_l^{\mp}(x, y)}{k_j^{\pm} - k_l^{\mp}} \right] + \iint_D dkdl \frac{K_{\pm j}(k, l)N_{\pm}(x, y, k, l)}{4\pi^2(k - k_j^{\pm})}, \tag{3.79}$$

where the coefficient $\Delta\alpha_j^{\pm}$ is expressed through the corrections of $\alpha_j^{\pm}$ and $k_j^{\pm\epsilon}$. This coefficient should be specified by normalizing the bound state $\Phi_j^{\pm\epsilon}(x, y)$ as $R = \sqrt{x^2 + y^2} \to \infty$ according to Eq. (3.29) or, equivalently, Eq. (3.30). The latter constraint can be expanded to the first order in ϵ as

$$\mp\frac{1}{2\pi}\iint_R dydx \left(u(x, y)\Delta\Phi_j^{\pm}(x, y) + \Delta u(x, y)\Phi_j^{\pm}(x, y) \right) = 0. \tag{3.80}$$

We prove that this integral equation defines the correction $\Delta\alpha_j^{\pm}$ in the form,

$$\Delta\alpha_j^{\pm} = -\frac{\Delta k_j^{\pm}}{k_j^{\pm}} \pm \frac{k_j^{\pm}}{2\pi i}\sum_{l=1}^{n}{}' \left[\frac{K_{\pm lj}}{k_l^{\pm}(k_l^{\pm} - k_j^{\pm})} + \frac{\tilde{K}_{\pm lj}}{k_l^{\mp}(k_l^{\mp} - k_j^{\pm})} \right] \pm \frac{1}{2\pi}K_{\pm j}(0, 0) \pm \frac{k_j^{\pm}}{4\pi^2 i}\iint_D dkdl \frac{K_{\pm j}(k, l)r_{\mp}^*(k, l)}{(k \pm i0)(k - k_j^{\pm})}. \tag{3.81}$$

We evaluate the first integral in Eq. (3.80) by substituting Eq. (3.79) and using Eqs. (3.20), (3.25), and (3.30),

$$\mp\frac{1}{2\pi}\iint_R dydxu(x, y)\Delta\Phi_j^{\pm}(x, y) = \Delta\alpha_j^{\pm} \pm \frac{1}{2\pi i}\sum_{l=1}^{n}{}' \left[\frac{K_{\pm lj}}{k_j^{\pm} - k_l^{\pm}} + \frac{\tilde{K}_{\pm lj}}{k_j^{\pm} - k_l^{\mp}} \right] \mp \int_D dkdl \frac{K_{\pm j}(k, l)r_{\mp}^*(k, l)}{4\pi^2 i(k - k_j^{\pm})}. \tag{3.82}$$

Then, we evaluate the second integral in Eq. (3.80) by using the spectral decomposition,

$$\frac{1}{2}\left(\int_{\infty}^{x_0} - \int_{x_0}^{\infty} \right) dx\Delta u(x, y)\Phi_j^{\pm}(x, y) = \frac{1}{2\pi^2 i}\iint_D dkdlK_{\pm j}(k, l)N_{\pm}(x_0, y, k, l) \pm \frac{1}{\pi}\sum_{l=1}^{n}\left(K_{\pm lj}\Phi_l^{\pm}(x_0, y) - \tilde{K}_{\pm lj}\Phi_l^{\mp}(x_0, y) \right). \tag{3.83}$$

Integrating this expression first in y and then taking the limit $x_0 \to \infty$, we find the second integral in Eq. (3.80) with the use of Eqs. (3.66) and (3.67),

$$\mp\frac{1}{2\pi}\iint_R dydx\Delta u(x,y)\Phi_j^{\pm}(x,y)=\pm\frac{1}{2\pi i}\sum_{l=1}^{n}\left[\frac{K_{\pm lj}}{k_l^{\pm}}+\frac{\tilde{K}_{\pm lj}}{k_l^{\mp}}\right]$$
$$\mp\frac{1}{2\pi}K_{\pm j}(0,0)\pm\iint_D dkdl\frac{K_{\pm j}(k,l)r_{\mp}^*(k,l)}{4\pi^2 i(k\pm i0)}. \quad (3.84)$$

Formulas (3.82) and (3.84) reduce Eq. (3.80) to the form (3.81). Furthermore, we simplify Eq. (3.81) as follows

$$\Delta\alpha_j^{\pm}=\pm\frac{1}{2\pi}\iint_R dydx\Delta u(x,y)(x+2k_j^{\pm}y+\gamma_j^{\pm})\Phi_j^{\mp *}(x,y)\Phi_j^{\pm}(x,y). \quad (3.85)$$

This transformation is based on the relation following from Eqs. (3.27) and (3.68),

$$\mu_j^{\pm}(x,y)-M_{\mp}(x,y,0)=-\frac{i\Phi_j^{\pm}(x,y)}{k_j^{\pm}}$$
$$-ik_j^{\pm}\sum_{l=1}^{n}{}'\left[\frac{\Phi_l^{+}(x,y)}{k_l^{+}(k_j^{\pm}-k_l^{+})}+\frac{\Phi_l^{-}(x,y)}{k_l^{-}(k_j^{\pm}-k_l^{-})}\right]$$
$$-\frac{k_j^{\pm}}{2\pi i}\iint_D dkdl\frac{r_{\pm}(k,l)N_{\mp}(x,y,k,l)}{(k\pm i0)(k-k_j^{\pm})},$$

where $\mu_j^{\pm}(x,y)$ is defined by Eq. (3.31).

In order to prove Eq. (3.77) we assume the asymptotic expansion, $\gamma_j^{\pm\epsilon}=\gamma_j^{\pm}+\epsilon\Delta\gamma_j^{\pm}+O(\epsilon^2)$, and express the correction $\Delta\gamma_j^{\pm}$ from Eq. (3.69) in the form,

$$\Delta\gamma_j^{\pm}=\mp\frac{1}{\pi}\iint_R dydx\Delta\Phi_j^{\mp *}(x,y)\left[(x+2k_j^{\pm}y+\gamma_j^{\pm})\Phi_j^{\pm}(x,y)\right]_x$$
$$\mp\frac{1}{\pi}\iint_R dydx\Phi_j^{\mp *}(x,y)\left[(x+2k_j^{\pm}y+\gamma_j^{\pm})\Delta\Phi_j^{\pm}(x,y)\right]_x \quad (3.86)$$
$$\mp\frac{2\Delta k_j^{\pm}}{\pi}\iint_R dydxy\Phi_j^{\mp *}(x,y)\Phi_{jx}^{\pm}(x,y).$$

The first two terms can be evaluated by means of direct substitution of Eq. (3.79) and use of Eq. (3.68) and the orthogonality relations (3.43)–(3.45). The result is given by the expression,

$$\Delta\gamma_j^{\pm}=\pm\frac{1}{2\pi}\sum_{l=1}^{n}{}'\left[\frac{K_{\mp lj}^*-K_{\pm lj}}{(k_j^{\pm}-k_l^{\pm})^2}+\frac{\tilde{K}_{\mp lj}^*-\tilde{K}_{\pm lj}}{(k_j^{\pm}-k_l^{\mp})^2}\right]$$
$$\pm\iint_D dkdl\frac{r_{\pm}(k,l)K_{\mp j}^*(k,l)-r_{\mp}^*(k,l)K_{\pm j}(k,l)}{4\pi^2(k-k_j^{\pm})^2} \quad (3.87)$$
$$\mp\frac{2\Delta k_j^{\pm}}{\pi}\iint_R dydxy\Phi_j^{\mp *}(x,y)\Phi_{jx}^{\pm}(x,y)$$
$$\mp\frac{1}{\pi}\iint_R dydx\Phi_j^{\mp *}(x,y)\Delta\Phi_j^{\pm}(x,y).$$

In order to evaluate the last two terms in this expression, we use the equation for $\Delta\Phi_j^{\pm}$,

$$i\Delta\Phi_{jy}^{\pm} + \Delta\Phi_{jxx}^{\pm} - 2ik_j^{\pm}\Delta\Phi_{jx}^{\pm} + u\Delta\Phi_j^{\pm} = 2i\Delta k_j^{\pm}\Phi_{jx}^{\pm} - \Delta u\Phi_j^{\pm} \tag{3.88}$$

with the boundary condition as $R = \sqrt{x^2+y^2} \to \infty$,

$$\Delta\Phi_j^{\pm} \to -\frac{2\Delta k_j^{\pm} y}{(x+2k_j^{\pm}y)^2} + \mathrm{O}(R^{-2}). \tag{3.89}$$

The balance equation following from Eqs. (3.1) and (3.88) can be integrated to the form,

$$\begin{aligned}&\frac{\partial}{\partial y}\left(\int_{-\infty}^{\infty} dx\,\Phi_j^{\mp *}(x,y)\Delta\Phi_j^{\pm}(x,y)\right)\\ &= 2\Delta k_j^{\pm}\int_{-\infty}^{\infty} dx\,\Phi_j^{\mp *}(x,y)\Phi_{jx}^{\pm}(x,y) + i\int_{-\infty}^{\infty} dx\,\Delta u(x,y)\Phi_j^{\mp *}(x,y)\Phi_j^{\pm}(x,y).\end{aligned} \tag{3.90}$$

Multiplying this equation by y and integrating over y with the boundary conditions (3.29) and (3.89), we find

$$\begin{aligned}&\mp\frac{2\Delta k_j^{\pm}}{\pi}\iint_R dy\,dx\,y\,\Phi_j^{\mp *}(x,y)\Phi_{jx}^{\pm}(x,y) \mp \frac{1}{\pi}\iint_R dy\,dx\,\Phi_j^{\mp *}(x,y)\Delta\Phi_j^{\pm}(x,y)\\ &\qquad = \mp\frac{1}{\pi i}\iint_R dy\,dx\,y\,\Delta u(x,y)\Phi_j^{\mp *}(x,y)\Phi_j^{\pm}(x,y).\end{aligned} \tag{3.91}$$

Formulas (3.87) and (3.91) reduce to Eq. (3.77). We notice that the symmetry constraint $(\gamma_j^-)^* = \gamma_j^+$ is preserved by the real potential $\Delta u(x,y)$. □

The results formulated in Proposition 3.6 constitute the basis for the analysis of dynamics of the KPI lumps under small perturbations, e.g. under distortions of their shapes.

3.3.2. Variations of data of continuous spectrum. Suppose that $N_{\pm}^{\epsilon}(x,y,k,l)$ solves Eq. (3.1) for $u^{\epsilon} = u(x,y) + \epsilon\Delta u(x,y)$. We expand it to the first order, $N_{\pm}^{\epsilon}(x,y,k,l) = N_{\pm}(x,y,k,l) + \epsilon\Delta N_{\pm}(x,y,k,l) + \mathrm{O}(\epsilon^2)$, and find the correction $\Delta N_{\pm}(x,y,k,l)$ in the form,

$$\begin{aligned}\Delta N_{\pm}(x,y,k,l) = &\iint_D dk'dl'\,\frac{K_{\pm}(k',k,l',l)N_{\pm}(x,y,k',l')}{4\pi^2(k'-(k\pm i0))}\\ &\pm\frac{1}{2\pi i}\sum_{l=1}^{n}\left[\frac{\tilde{K}_{\mp l}^{*}(k,l)\Phi_l^{\pm}(x,y)}{k-k_l^{\pm}} - \frac{K_{\pm l}^{*}(k,l)\Phi_l^{\mp}(x,y)}{k-k_l^{\mp}}\right].\end{aligned} \tag{3.92}$$

The main result of this subsection is formulated in the following proposition.

Proposition 3.7. *Variational derivatives of the data $r_{\pm}(k,l)$ of the continuous spectrum of Eq. (3.1) with respect to the potential $u(x,y)$ are given by*

$$\frac{\delta r_{\pm}(k,l)}{\delta u(x,y)} = \frac{N_{\mp}^{*}(x,y,k,l)M_{\pm}(x,y,k)}{2\pi i}. \tag{3.93}$$

Proof. The derivation follows the proof of Proposition 3.6. First, we expand the scattering data as $r^{\epsilon}_{\pm}(k,l) = r_{\pm}(k,l) + \epsilon \Delta r_{\pm}(k,l) + \mathrm{O}(\epsilon^2)$ and use Eqs. (3.20) and (3.25) to express $\Delta r_{\pm}(k,l)$ as

$$\Delta r_{\pm}(k,l) = \frac{1}{2\pi i} \iint_R dy dx \left(u(x,y) \Delta N^*_{\mp}(x,y,k,l) + \Delta u(x,y) N^*_{\mp}(x,y,k,l) \right). \tag{3.94}$$

The first integral can be evaluated explicitly through the substitution of Eqs. (3.92) and use of Eqs. (3.20), (3.25), and (3.30). The second integral can be found by integrating the spectral decomposition,

$$\begin{aligned} &\frac{1}{2}\left(\int_{-\infty}^{x_0} - \int_{x_0}^{\infty} \right) dx \Delta u(x,y) N^*_{\mp}(x,y,k,l) \\ &= \frac{1}{2\pi^2 i} \iint_D dk' dl' K^*_{\mp}(k',k,l',l) N^*_{\mp}(x_0,y,k',l') \\ &\pm \frac{1}{\pi} \sum_{l=1}^{n} \left(\tilde{K}_{\pm l}(k,l) \Phi^{\mp *}_l(x_0,y) - K_{\mp l}(k,l) \Phi^{\pm *}_l(x_0,y) \right), \end{aligned}$$

over y in the limit $x_0 \to \infty$. As a result, we deduce

$$\begin{aligned} \Delta r_{\pm}(k,l) = &\frac{1}{2\pi i} K^*_{\mp}(0,k,0,l) + \frac{k}{4\pi^2} \iint_D dk' dl' \frac{K^*_{\mp}(k',k,l',l) r_{\pm}(k',l')}{(k' \pm i0)(k' - (k \pm i0))} \\ &+ \frac{k}{2\pi} \sum_{l=1}^{n} \left(\frac{\tilde{K}_{\pm l}(k,l)}{k_l^{\pm}(k_l^{\pm} - k)} + \frac{K_{\mp l}(k,l)}{k_l^{\mp}(k_l^{\mp} - k)} \right). \end{aligned} \tag{3.95}$$

Using Eqs. (3.27) for $M_{\pm}(x,y,k) - M_{\mp}(x,y,0)$, we conclude that Eq. (3.95) reduces to Eq. (3.93). □

3.3.3. Example: A single lump potential. We solve Eq. (3.68) for $r_{\pm}(k,l) = 0$ at $l \neq k$ and $n = 1$ in the form,

$$\Phi^+_1(x,y) = 2\kappa_1 \frac{2\kappa_1 X - 4i\kappa_1^2 Y - 1}{4\kappa_1^2 X^2 + 16\kappa_1^4 Y^2 + 1}, \tag{3.96}$$

$$\Phi^-_1(x,y) = 2\kappa_1 \frac{2\kappa_1 X + 4i\kappa_1^2 Y + 1}{4\kappa_1^2 X^2 + 16\kappa_1^4 Y^2 + 1}, \tag{3.97}$$

where we have used the parametrization,

$$k_1^{\pm} = p_1 \pm i\kappa_1, \quad \gamma_1^{\pm} = -x_0 - 2k_1^{\pm} y_0, \quad X = x - x_0 + 2p_1(y - y_0), \quad Y = y - y_0.$$

Then, the lump of the KPI equation (1.1) is

$$u_s(x,y) = w_{sX}(X,Y), \quad w_s(X,Y) = \frac{16\kappa_1^2 X}{4\kappa_1^2 X^2 + 16\kappa_1^4 Y^2 + 1}, \tag{3.98}$$

which satisfies the constraint (3.2). We use the following relation,

$$\Phi_1^{\mp *}(x, y)\Phi_1^{\pm}(x, y) = -\frac{1}{4}u_s(X, Y) \pm \frac{i}{8\kappa_1}w_{sY}(X, Y),$$

and evaluate explicitly the perturbation corrections of the first order of the perturbation theory (3.76) and (3.77),

$$\Delta k_1^{\pm} = \frac{1}{16\pi\kappa_1}\left(\Delta P_{sy} - 2p_1\Delta P_{sx}\right) \pm \frac{i}{8\pi}\Delta P_{sx}, \tag{3.99}$$

$$\begin{aligned}\Delta\gamma_1^{\pm} = \frac{4\kappa_1}{\pi}\iint_R dydx\frac{X(16\kappa_1^4 yY - 1)}{(4\kappa_1^2X^2 + 16\kappa_1^4Y^2 + 1)^2}\Delta u(x, y)\\ \pm\frac{1}{4\pi i}\iint_R dydx\ yu_s(x, y)\Delta u(x, y),\end{aligned} \tag{3.100}$$

where ΔP_{sx} and ΔP_{sy} are corrections to the x and y-projections of the momentum for the KPI equation (1.1),

$$P_{sx} = \frac{1}{2}\iint_R dydxw_{sx}^2, \quad P_{sy} = \frac{1}{2}\iint_R dydxw_{sx}w_{sy}.$$

The perturbation of the data of the continuous spectrum $\Delta r_{\pm}(k, l)$ can be found from Eq. (3.93) by using the explicit relation (see Eqs. (3.22) and (3.27)),

$$N_{\pm}(x, y, k, l) = \left[1 - \frac{4i\kappa_1^2(2(l - p_1)X + 4\kappa_1^2Y - i)}{((l - p_1)^2 + \kappa_1^2)(4\kappa_1^2X^2 + 16\kappa_1^4Y^2 + 1)}\right]e^{i\beta(x,y,k,l)}. \tag{3.101}$$

We notice that

$$R_{\pm}(k) = r_{\pm}(k, k) = -\frac{4\pi\kappa_1}{(k - p_1)^2 + \kappa_1^2},$$

i.e. $R(k) \neq 0$. On the other hand, we confirm from Eqs. (3.20), (3.25), and (3.101) that $r_{\pm}(k, l) = 0$ for any $l \neq k$.

Since the projections of the momentum at the KPI lump (3.98) are $P_{sx} = 8\pi\kappa_1$ and $P_{sy} = 16\pi\kappa_1 p_1$, we check from Eq. (3.99) that the first-order corrections ΔP_{sx} and ΔP_{sy} define completely the renormalization of the parameters κ_1 and p_1 of the KPI lump (3.98) and affect the excitation of the momentum of the continuous spectrum in the order of $O(\epsilon^2)$. This result confirms the stability of the single KPI lump against small perturbations [32].

3.4. Type II bifurcation of new eigenvalues. The results of Sects. 3.2 and 3.3 remain valid even if the potential $u(x, y)$ is a nongeneric potential of type II, i.e. it supports an embedded eigenvalue at $k = k_0$. Indeed, the half-bound states $\Phi_0^{\pm}(x, y)$ appear as pole contributions of the continuous spectrum and their presence does not affect the complete sets of eigenfunctions $S^{\pm} = \left[N_{\pm}(x, y, k, l), \{\Phi_j^{+}(x, y), \Phi_j^{-}(x, y)\}_{j=1}^n\right]$. However, the eigenfunctions $M_{\pm}(x, y, k)$ and $N_{\pm}(x, y, k, l)$ are singular at $k = k_0$ according to Eqs. (3.34) and (3.38). As a result, the variation of the scattering data $r_{\pm}(k, l)$ defined by

Eq. (3.93) becomes divergent as $k \to k_0$ if the nongeneric potential $u(x, y)$ is perturbed by a correction $\Delta u(x, y)$,

$$\Delta r_{\pm}(k, l) \to \pm \frac{r_0^{\pm}(l) K_0^{\pm}}{2\pi i (k - (k_0 \mp i0))^2}, \tag{3.102}$$

where $K_0^{\pm} = -K_0$ and

$$K_0 = \iint_R dy dx \Delta u(x, y) |\Phi_0(x, y)|^2. \tag{3.103}$$

Combining Eqs. (3.35) and (3.102), we find that the perturbation $\Delta u(x, y)$ shifts the pole at $k = k_0$ into the complex domain,

$$k_0^{\pm\epsilon} = k_0 \pm \frac{i\epsilon K_0}{2\pi}. \tag{3.104}$$

This shift crosses the real axis if $\text{sign}(\epsilon K_0) > 0$. In this case, the eigenfunctions $M_+(x, y, k)$ and $M_-(x, y, k)$ acquire a new pole in the upper and lower half-plane of k, respectively (cf. Eqs. (3.27) and (3.34)). We prove below that the bifurcation of the embedded eigenvalue into the complex plane occurs under the condition $\text{sign}(\epsilon K_0) > 0$ and Eq. (3.104) gives the leading order of the new eigenvalue. In the opposite case, i.e. when $\text{sign}(\epsilon K_0) < 0$, the analyticity properties of $M_{\pm}(x, y, k)$ in the corresponding domains of k are not affected and we expect that the embedded eigenvalue just disappears. Here we derive asymptotic expansions for the new eigenvalue and associated bound state. The results are applied to the problem of generation of a KPI lump by a localized initial pulse.

3.4.1. Asymptotic expressions for a new eigenvalue and bound state. Suppose that the type II bifurcation occurs under the perturbation $\Delta u(x, y)$. A new bound state $\Phi_{n+1}^{\pm\epsilon}(x, y)$ can be decomposed through the complete sets $S^{\pm}$ according to Eq. (3.72) with the eigenvalue $k = k_{n+1}^{\pm\epsilon}$ such that $\lim_{\epsilon \to 0} k_{n+1}^{\pm\epsilon} = k_0$. If the potential $u(x, y)$ is nongeneric, the homogeneous integral equation (3.73) has a singular kernel at $k' \to k_0$ if $k_{n+1}^{\pm\epsilon} \to k_0$. Solving this equation asymptotically in the limit $\epsilon \to 0$, we derive the following result.

Proposition 3.8. *Under the conditions that the potential $u(x, y)$ exhibits an embedded eigenvalue at $k = k_0$ and the perturbation $\Delta u(x, y)$ satisfies the criterion $K_0 > 0$ (see Eq. (3.103), the potential $u^{\epsilon} = u(x, y) + \epsilon \Delta u(x, y)$ supports a bound state in a neighbourhood of $k = k_0$ for $\epsilon > 0$. The eigenvalue $k_{n+1}^{\pm\epsilon}$ for the new bound state $\Phi_{n+1}^{\pm\epsilon}(x, y)$ is defined by*

$$k_{n+1}^{\pm\epsilon} = k_0 \pm i\epsilon \Delta k + \mathrm{O}(\epsilon^2),$$

where

$$\Delta k = \frac{1}{2\pi} K_0 > 0. \tag{3.105}$$

Proof. We consider an asymptotic solution of Eq. (3.73) at $k, l \to k_0$ and $\epsilon \to 0$. Using Eq. (3.38), we rescale the variables in the problem,

$$K_{\pm}(k, k', l, l') = \frac{r_0^{\mp}(l) r_0^{\mp *}(l') P_{\pm}(k, k', l, l')}{(k - (k_0 \pm i0))(k' - (k_0 \mp i0))}, \quad \alpha_{\pm}(k, l) = \frac{r_0^{\mp}(l) A_{\pm}(k, l)}{k - (k_0 \pm i0)}. \tag{3.106}$$

Then, we evaluate the singular contribution from the pole $k' = k_0$ in the integral of Eq. (3.73) and find the leading order term in the form,

$$A_\pm(k,l) \to \frac{\epsilon P_\pm(k,k_0,l,k_0)A_\pm(k_0,k_0)}{4\pi^2} Q_\pm, \tag{3.107}$$

where

$$Q_\pm = \iint_D \frac{dk'dl'|r_0^\mp(l')|^2}{(k'-k_0)^2(k'-k_0 \mp i\epsilon\Delta k)}.$$

Using the formal expansion (3.41) and the constraint (3.40), we evaluate $Q_\pm$ by means of the residue theorem as

$$Q_\pm = \int_{-\infty}^{\infty} \frac{2dk}{(k-k_0)(k-k_0 \mp i\epsilon\Delta k)} = \frac{2\pi}{\epsilon\Delta k}\mathrm{sign}(\epsilon\Delta k). \tag{3.108}$$

Writing Eq. (3.107) at $k = l = k_0$ and assuming $\epsilon > 0$, we find the simple result,

$$|\Delta k| = \frac{1}{2\pi} P_\pm(k_0,k_0,k_0,k_0) = \frac{1}{2\pi} K_0.$$

The latter equation is self-consistent only for $K_0 > 0$, when the bifurcation occurs and the new eigenvalue has the asymptotic approximation (3.105). The new bound state $\Phi_{n+1}^{\pm\epsilon}(x,y)$ is defined by Eqs. (3.72), (3.106), and (3.107). Using the same approach, we simplify the expression for $\Phi_{n+1}^{\pm\epsilon}(x,y)$ for finite $R = \sqrt{x^2+y^2}$,

$$\Phi_{n+1}^{\pm\epsilon}(x,y) = \pm\frac{iA_\pm(k_0,k_0)}{4\pi^2} Q_\pm \Phi_0^\pm(x,y) + \mathrm{O}(\epsilon). \tag{3.109}$$

Using Eq. (3.108), we satisfy the normalization condition (3.30) by specifying $A_\pm(k_0,k_0) = \mp i\epsilon K_0$. Then, Eqs. (3.72), (3.106), and (3.107) reduce to the asymptotic expression for the new bound state,

$$\Phi_{n+1}^{\pm\epsilon}(x,y) = \mp\frac{i\epsilon}{4\pi^2 K}\iint_D dkdl \frac{r_0^\mp(l)P_\pm(k,k_0,l,k_0)N_\pm(x,y,k,l)}{(k-(k_0 \pm i0))(k-k_0 \mp i\epsilon\Delta k)} + \mathrm{O}(\epsilon),$$

where the integral term is the order of O(1). □

We have thus found that, for the type II bifurcation, a new eigenvalue appears transversely to the real axis in the neighbourhood of the embedded eigenvalue and a new bound state arises from a localized eigenfunction corresponding to the half-bound state.

3.4.2. Example: Generation of a single KPI lump. The multilump potentials of the linear problem (1.2) do not belong to the nongeneric potentials of type II since $r_\pm(k,l)$ are not singular for real k. Indeed, for such potentials, $r_\pm(k,l) = 0$ at $l \neq k$ and

$$N_\pm(x,y,k,l) = M_\pm(x,y,l)e^{i\beta(x,y,k,l)},$$

where

$$M_\pm(x,y,l) = 1 - i\sum_{j=1}^{n}\left[\frac{\Phi_j^+(x,y)}{l-k_j^+} + \frac{\Phi_j^-(x,y)}{l-k_j^-}\right].$$

It is clear from this expression that the embedded eigenvalues at real k are not supported by the multilump potentials. In the particular case $n = 0$, we conclude that the zero

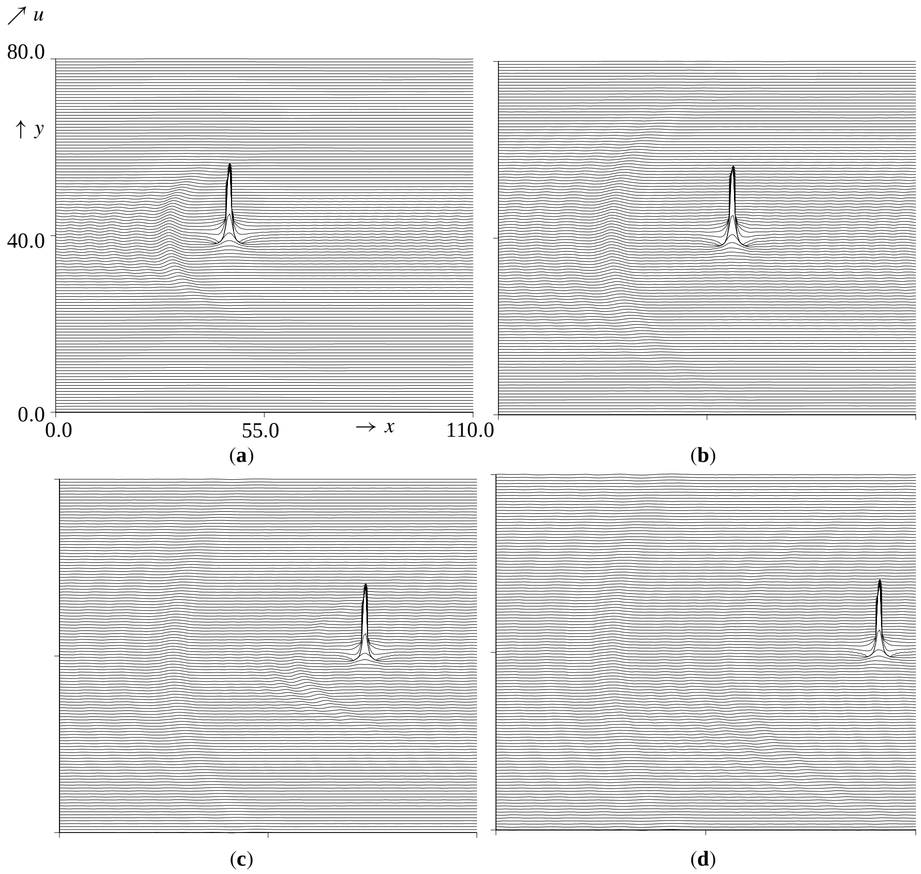

Fig. 3.1. Formation of a new KPI lump for the initial condition (3.110) and $a = 1.5$ at times $t = 5$ **(a)**, $t = 10$ **(b)**, $t = 15$ **(c)**, and $t = 20$ **(d)** (**(a)**–**(d)** in the same coordinates)

background $u(x, y) = 0$ does not exhibit embedded eigenvalues and, therefore, small initial data do not generate new eigenvalues. This implies that there must be a threshold for amplitude of the initial localized pulse to generate a new eigenvalue in the problem (1.2) and an associated lump in the KPI equation (1.1). This result is valid if the initial data $\Delta u(x, y) \sim \mathrm{O}(1)$ as $\epsilon \to 0$. Note that the existence of a threshold follows also from the rigorous paper by Fokas and Sung [19] where a small-norm assumption was used to eliminate lumps from the spectral problem (1.2).

In order to illustrate this result, we reproduce in Fig. 2 the numerical simulations of the KPI equation (1.1) performed by M. He [33]. The initial condition was chosen in the form of the KPI lump (3.98) with $p_1 = 0$, $\kappa_1 = 1/2$ and an arbitrary amplitude a,

$$u(x, y, 0) = 4a \frac{1 + y^2 - x^2}{(1 + x^2 + y^2)^2}. \tag{3.110}$$

If $a = 1$, it coincides with the KPI lump. If the amplitude a is greater or close to the amplitude of the KPI lump, the initial pulse transforms into a steady-state solitary wave. Fig. 2(a–d) shows successive snapshots at various times for the evolution of the initial

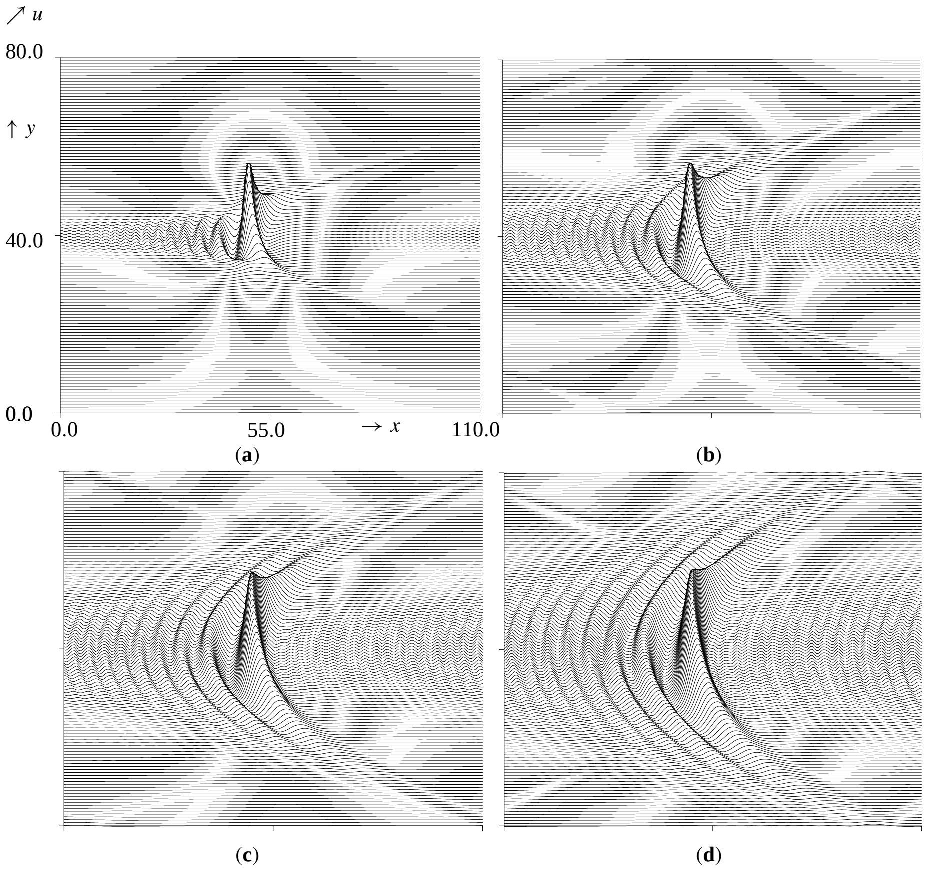

Fig. 3.2. Transformation of an initial pulse to a linear wave packet for the initial condition (3.110) and $a = 0.5$ at times $t = 2$ **(a)**, $t = 4$ **(b)**, $t = 6$ **(c)**, and $t = 8$ **(d)** (**(a)**–**(d)** in the same coordinates)

data (3.110) with $a = 1.5$. It is clearly seen that the initial pulse evolves into a KPI lump. On the other hand, if the amplitude a is small enough, the initial pulse broads up and decays into linear dispersive waves. Fig. 3(a–d) shows the decay of initial data (3.110) with $a = 0.5$. Since the multilump potentials do not support embedded eigenvalues, a small perturbation does not generate new bound states in the spectral problem (1.2). Therefore, similarly to the case $n = 0$, there is a threshold for the amplitude of a perturbation to the multilump potential to generate a new eigenvalue and an associated KPI lump.

4. Discussion

We have presented a complete analysis of the spectral decomposition for the time-independent and time-dependent Schrödinger equations within the RH formalism of inverse scattering. The spectral problems (1.2) and (1.4) are formulated for self-adjoint operators where the spectral decomposition, inner products and completeness relations follow from the spectral theory in Hilbert spaces [4] subject to the assumption that

$u(x, y) \in L^1$. Since the multilump potentials violate this assumption, the discrete spectrum of the time-dependent Schrödinger equation (1.2) does not fit into this theory and the corresponding eigenfunctions diverge exponentially,

$$\varphi = \Phi_j^{\pm}(x, y) e^{-ik_j^{\pm} x - ik_j^{\pm 2} y},$$

where $\Phi_j^{\pm}(x, y)$ are defined by Eq. (3.28). These nonlocalized "bound states" account for resonant poles of the operator resolvent [4]. Spectral decomposition and completeness relations were not derived in this context.

In the framework of the RH formalism of the inverse scattering, we have transformed the self-adjoint spectral problems (1.2) and (1.4) to the non-self-adjoint form (2.1) and (3.1), where new non-standard scalar products were introduced and orthogonality and completeness relations were proved by means of direct computations. Although no rigorous result is available for general non-self-adjoint linear operators, we conjecture that linear problems associated to nonlinear evolution equations within the formalism of inverse scattering always possess a complete basis for the spectral decomposition.

We mention now some results concerning other linear spectral problems considered in the RH formalism of inverse scattering [6].

The ILW equation. This integro-differential equation is related to the scalar (local) RH boundary value problem (1.7) and (1.8). The associated linear problem generalizes Eq. (1.4) and has a standard complete basis of eigenfunctions. The discrete spectrum of this problem is associated to solitons of the ILW equation [45].

The BO equation. This equation is related to the scalar (nonlocal) RH problem (1.7) and (1.8) for $a_{\pm}(k) = 1$. The discrete spectrum is associated to lumps (algebraic solitons) of the BO equation [7]. The spectral decomposition for the associated linear problem was recently analyzed [10].

Equations of the AKNS scheme. These equations are associated to the AKNS spectral problem and include the NLS equation and the modified KdV equation as particular cases [1]. The AKNS spectral problem can be formulated through the vector (local) RH boundary value problem and the discrete spectrum corresponds to solitons of the nonlinear evolution equations [6]. The standard spectral decomposition was proved in Ref. [1].

The DSI system. This system is related to the AKNS spectral problem in two dimensions. The vector (nonlocal) RH boundary value problem can be formulated and has a discrete spectrum associated to dromions of the DSI system [46,47].

In this paper, the spectral decomposition was used to solve the particular problem associated to nonlinear evolution equations, whether or not a small initial disturbance supports propagation of a soliton. Equivalently, this problem concerns the existence of a single eigenvalue for the discrete spectrum of the associated linear problem with a small potential. Extending the results of this paper, we conjecture that spectral problems with nongeneric potentials of type I may possess a single eigenvalue for a small potential while spectral problems with nongeneric potentials of type II have no eigenvalues for small potentials. We present below a table which summarizes the results on soliton generation for the problems solvable by means of inverse scattering.

nonlinear equation	bound states	type of bifurcation	reference
KdV equation	solitons	type I	[35]–[37]
ILW equation	solitons	type I	[14]
BO equation	lumps	type I	[13]
KPI equation	lumps	type II	this paper
AKNS equations	solitons	type II	[48]
DSI system	dromions	?	[49]

Finally, there are also linear problems which possess localized bound states and are related to the $\bar{\partial}$ formalism of inverse scattering rather than to the RH formalism. An example is provided by the DSII system [9]. The eigenfunctions of the continuous spectrum for these linear problems have no simple analytical properties in k and the spectral decomposition and bifurcations of eigenvalues remain open for further studies.

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