Generation of large-amplitude solitons in the extended Korteweg–de Vries equation

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We study the extended Korteweg–de Vries equation, that is, the usual Korteweg–de Vries equation but with the inclusion of an extra cubic nonlinear term, for the case when the coefficient of the cubic nonlinear term has an opposite polarity to that of the coefficient of the linear dispersive term. As this equation is integrable, the number and type of solitons formed can be determined from an appropriate spectral problem. For initial disturbances of small amplitude, the number and type of solitons generated is similar to the well-known situation for the Korteweg–de Vries equation. However, our interest here is in initial disturbances of larger amplitude, for which there is the possibility of the generation of large-amplitude ''table-top'' solitons as well as small-amplitude solitons similar to the solitons of the Korteweg–de Vries equation. For this case, and in contrast to some earlier results which assumed that an initial disturbance in the shape of a rectangular box would be typical, we show that the number and type of solitons formed depend crucially on the disturbance shape, and change drastically when the initial disturbance is changed from a rectangular box to a ''sech''-profile. [DOI: 10.1063/1.1521391]

It is well-known that for the classical Korteweg–de Vries equation, the long-time outcome from a localized initial condition is a set of rank-ordered solitons. This is most readily established by considering the associated Schrödinger scattering problem, whose eigenvalues in the discrete spectrum determine the solitons. Here we consider the analogous problem for an extended Korteweg–de Vries equation, which has an additional cubic term whose coefficient has the opposite polarity to the coefficient of the linear dispersive term. This equation supports a family of solitons, ranging from small-amplitude ones similar to those of the Korteweg–de Vries equation, to large-amplitude "table-top" solitons. Like the Korteweg–de Vries equation, this extended equation is also integrable, and has a Lax operator and an associated scattering problem, whose discrete eigenvalues again determine the solitons which can emerge from a localized initial condition. However, unlike the Korteweg–de Vries equation, for which rectangular and "sech"-profile initial disturbances produce qualitatively similar solitons, we show that for this extended Korteweg–de Vries equation, the solitons also depend critically on the shape of the initial disturbance. In particular, the number of "table-top" solitons formed depends on whether initial shape is a rectangular box or of the "sech"-profile type.

I. INTRODUCTION

Groups of large-amplitude flat ''table-top'' internal solitary waves are very often observed in the coastal zones of the World Ocean (see, for instance Refs. 1–3). It is well-known that the Korteweg–de Vries (KdV) equation, extended to include a cubic nonlinear term, with the coefficient of this cubic nonlinear term of the opposite polarity to that of the linear dispersive term, can support a ''table-top'' soliton as the limiting-amplitude member of the soliton family.[4] For a localized initial disturbance, the inverse scattering technique predicts that, for the KdV equation, the long-time outcome is a finite number of solitons, irrespective of whether the initial disturbance is a rectangular box or of a ''sech''-profile. However, according to the prevailing theory for the extended KdV (eKdV) equation, only one large-amplitude ''table-top'' soliton can appear from an energetic initial disturbance (see, for instance, Ref. 5). Here we reexamine the initial-value problem for the extended KdV equation using the inverse scattering technique.

The initial-value problem for the eKdV equation with a rectangular-box initial disturbance was considered by Miles,[5] who reduced a spectral problem whose discrete eigenvalues determine the solitons emitted, to a transcendental equation and analyzed the roots of this equation. It was shown that a small-amplitude rectangular-box disturbance generates many solitons, while a large-amplitude disturbance generates only one soliton, which is close to the large-amplitude''table-top'' soliton. We shall confirm this result for a rectangular-box initial disturbance, but we will also show that this result

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changes for a smooth "sech"-profile initial disturbance. We will show that a smooth initial disturbance generates many large-amplitude wide solitons, accompanied by small-amplitude solitons of smaller width. Thus we will show that the rectangular-box initial disturbance is very special for the extended KdV equation.

The fully extended Korteweg–de Vries equation contains higher-order linear and nonlinear dispersive terms as well as the cubic nonlinear term. In the context of internal solitary waves, such an equation was first derived for interfacial waves in a two-layer system[6] (see also Refs. 7–9), but was later shown to hold also for arbitrary oceanic stratification in density and shear flow.[10–13] For certain background conditions, when the coefficient of the quadratic nonlinear term is close to zero (for instance, when the pycnocline lies close to the middle depth in a two-layer model of density stratification), the higher-order linear and nonlinear dispersive terms can be omitted, and then the fully extended equation takes the form of the so-called Gardner equation, denoted here also as the eKdV equation,

$$\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}+\alpha_1 u^2\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0, \tag{1}$$

where $u(x,t)$ is the amplitude of the relevant wave mode (e.g., u may be the vertical displacement of the pycnocline), x is a horizontal coordinate, and t is time. The coefficients, α, α_1, and β are determined by the background density and shear flow stratification (see the review paper, Ref. 4). It is important to note that the dispersive coefficient, β, is always positive, but that the nonlinear coefficients, α and α_1, can have either sign.

The Gardner equation (1) is completely integrable. Its soliton solutions depend on the sign of coefficient of the cubic nonlinear term, α_1. The "table-top" soliton exists only if α_1 is negative, that is, the cubic nonlinear term has opposite polarity to the linear dispersive term. These solitons are limited in amplitude and speed, such that limiting "table-top" soliton corresponds to the maximum amplitude and speed. Such wide solitons occur, for instance, in two-layer shear flows.[10,13] The polarity of these solitons is determined by the sign of the quadratic nonlinear term, α. For instance, when the pycnocline is close to the bottom (top) the soliton has positive (negative) polarity, that is α is positive (negative).

Using a simple scaling transformation, Eq. (1) with $\alpha>0$, $\alpha_1<0$, and $\beta>0$ can be transformed to the normalized form,

$$\frac{\partial u}{\partial t}+6u(1-u)\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0. \tag{2}$$

The soliton solutions of Eq. (2) are given by

$$u_s(x,t)=\frac{4\kappa^2}{1+\sqrt{1-4\kappa^2}\cosh(2\kappa(x-4\kappa^2 t))}, \tag{3}$$

where the parameter κ, $0<\kappa<1/2$ characterizes the soliton amplitude,

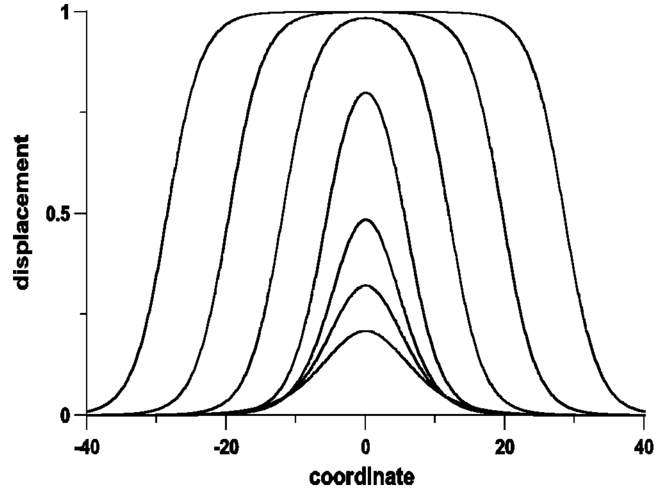

FIG. 1. The solitary wave solution $u(x)$ of the extended KdV equation (2).

$$a=\frac{4\kappa^2}{1+\sqrt{1-4\kappa^2}}, \tag{4}$$

with $0<a<1$. The soliton shape is shown in Fig. 1. The mass M and momentum P of Eq. (2), evaluated for this soliton solution (3), are given by

$$M=\int_{-\infty}^{\infty}u_s\,dx=4\tanh^{-1}\sqrt{\frac{a}{2-a}}, \tag{5}$$

and

$$P=\int_{-\infty}^{\infty}u_s^2\,dx=2\tanh^{-1}\sqrt{\frac{a}{2-a}}-\sqrt{a(2-a)}. \tag{6}$$

The functions $M(a)$ and $P(a)$ are monotonic functions of the soliton amplitude tending to infinity for the "table-top" soliton. We describe the soliton length, L_s, by the value of $L_s=M(a)/a$. The function $L_s(a)$ has a minimum $L_s\approx 5.3$ at $a\approx 0.58$ ($\kappa\approx 0.45$), see Fig. 2. Solitons with $a>0.6$ can be considered as "wide" solitons, while solitons with $a>0.99$ have the "table-top" shape.

Multisoliton solutions of Eq. (2) can also be found, see Refs. 14 and 15. In particular, the two-soliton interaction is

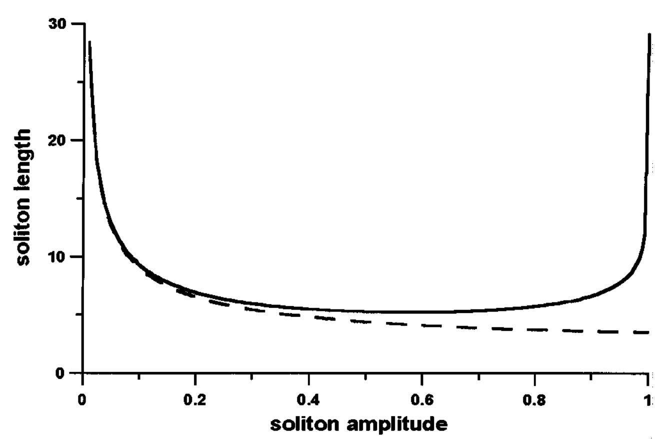

FIG. 2. The length of the solitary wave L_s as a function of its amplitude a (solid line—eKdV equation, dashed line—KdV equation).

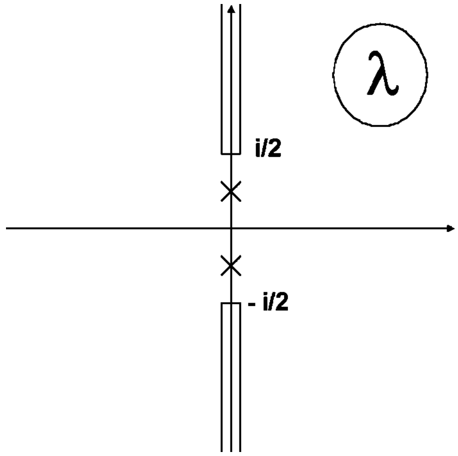

FIG. 3. Location of the spectrum of the Lax operator (9).

elastic but the soliton of small amplitude moving through a ''table-top'' soliton changes its polarity twice. The dynamics of large-amplitude solitons under the action of weak perturbations (dissipation, forcing, variable environment) has been studied in Refs. 16–18. Here we consider the initial-value problem for the eKdV equation (2) using the AKNS spectral problem.[19]

II. SPECTRAL PROBLEM FOR EXTENDED KDV EQUATION

Equation (2) can be obtained from the modified KdV (mKdV) equation

$$\frac{\partial w}{\partial t}-6w^2\frac{\partial w}{\partial x}+\frac{\partial^3 w}{\partial x^3}=0 \tag{7}$$

by using the Lorentz transformation:

$$w=-\tfrac{1}{2}+u(x+\tfrac{3}{2}t,\,t). \tag{8}$$

The Lax operator for the eKdV equation (2) can then be obtained from that for the mKdV equation (7), see Ref. 19,

$$\mathcal{L}=\begin{Bmatrix} -\partial_x & -\frac{1}{2}+u(x) \\ \frac{1}{2}-u(x) & \partial_x \end{Bmatrix}. \tag{9}$$

The spectrum of the Lax operator,

$$\mathcal{L}\boldsymbol{\Psi}=\lambda\boldsymbol{\Psi}, \tag{10}$$

determines the inverse scattering transform for the eKdV equation. Since $\mathcal{L}$ is a skew-symmetric operator: $\mathcal{L}^+=-\mathcal{L}$, the spectrum of $\mathcal{L}$ lies on the imaginary axis of λ. The continuous spectrum can be found from Eq. (10) by letting $u(x)\to 0$, so that

$$\boldsymbol{\Psi}(x)=\boldsymbol{\Psi}_0 e^{ikx}:\quad \mathrm{Im}(\lambda)=\pm\tfrac{1}{2}\sqrt{1+4k^2}. \tag{11}$$

The discrete spectrum corresponds to localized eigenfunctions at infinity, that is,

$$\boldsymbol{\Psi}(x)\to\boldsymbol{\Psi}_\infty e^{-\text{æ}|x|},\quad \mathrm{Im}(\lambda)=\pm\tfrac{1}{2}\sqrt{1-4\text{æ}^2}. \tag{12}$$

Thus the discrete spectrum is located in the gap of the continuous spectrum, $-1/2<\mathrm{Im}(\lambda)<1/2$, see Fig. 3.

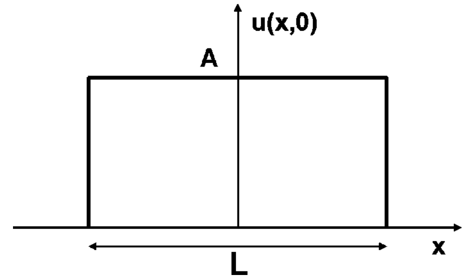

FIG. 4. Rectangular box initial data $u(x)$.

A different form of the Lax operator was suggested by Miles:[20]

$$\mathcal{M}=\begin{Bmatrix} -\partial_x & u(x) \\ u(x)-1 & \partial_x \end{Bmatrix}, \tag{13}$$

with the spectral problem,

$$\mathcal{M}\boldsymbol{\Psi}=\mu\boldsymbol{\Psi}. \tag{14}$$

These spectral problems are equivalent, provided that

$$\mu^2=\tfrac{1}{4}+\lambda^2. \tag{15}$$

Indeed,

$$\mathcal{L}^2=\begin{Bmatrix} \partial_x^2-(u-\frac{1}{2})^2 & -u'(x) \\ -u'(x) & \partial_x^2-(u-\frac{1}{2})^2 \end{Bmatrix}, \tag{16}$$

$$\mathcal{M}^2=\begin{Bmatrix} \partial_x^2+u-u^2 & -u'(x) \\ -u'(x) & \partial_x^2+u-u^2 \end{Bmatrix}, \tag{17}$$

and

$$\mathcal{M}^2\boldsymbol{\Psi}=(\mathcal{L}^2+\tfrac{1}{4})\boldsymbol{\Psi}=(\lambda^2+\tfrac{1}{4})\boldsymbol{\Psi}=\mu^2\boldsymbol{\Psi}. \tag{18}$$

The spectral problem (14) is more convenient for numerical computations than the spectral problem (10) because its discrete spectrum is real with $\mu=\kappa$, which is the soliton parameter for the exact solution (3).

We solve the eigenvalue problem (14) for positive initial data $u(x)$ of various forms, such that $u(x)\to 0$ as $|x|\to\infty$.

Rectangular box. Let the initial disturbance $u(x)$ be a rectangular box of amplitude A and width L, as shown in Fig. 4. The eigenfunction $\boldsymbol{\Psi}(x)$ of Eq. (14) with such potential $u(x)$ should be continuous at $|x|=L/2$ and vanish at infinity ($|x|\to\infty$). Such an eigenfunction $\boldsymbol{\Psi}(x)$ exists if $\mu(=\kappa)$ satisfies the transcendental equation

$$\tan(\sigma L)=\frac{2\kappa\sigma(1-A)}{\sigma^2-(1-2A)\kappa^2},\quad \sigma^2=A(1-A)-\kappa^2 \tag{19}$$

for $0<\kappa<\sqrt{A(1-A)}$, and

$$\tanh(\chi L)=\frac{2\kappa\chi(A-1)}{\chi^2+(1-2A)\kappa^2}, \quad \chi^2=\kappa^2-A(1-A) \tag{20}$$

for $\kappa>\sqrt{A(1-A)}$. The behavior of the eigenvalues is different for weak disturbances ($A<1/2$), moderate disturbances ($1/2<A<1$), and strong disturbances ($A>1$).

For weak disturbances the eigenvalues are bounded by

$$\kappa<\sqrt{A(1-A)}, \tag{21}$$

while the number of eigenvalues depends on the width L of the initial box. A new eigenvalue appears whenever L exceeds the bifurcation value:

$$L>\frac{\pi n}{\sqrt{A}(1-A)}, \tag{22}$$

where $n=1,2,\ldots$. The eigenvalues $\mu(=\kappa)$ as a function of L are shown in Figs. 5(a)–5(c) for $A=0.45$ (weak), $A=0.8$ (moderate), and $A=1.2$ (strong). For weak disturbances ($0<A<1/2$) the pattern is similar to that for the KdV equation, when the eigenvalues are determined by the Schrödinger equation with a square-wall potential. If $A\to 1/2$ and $L\to\infty$, the eigenvalues correspond to many "wide" solitons with near-limiting amplitude $a\approx 1$ when $\kappa\approx 1/2$.

For moderate disturbances ($1/2<A<1$) the behavior of the eigenvalues is different from the KdV-case; one eigenvalue tends to the limiting value $\kappa=1/2$ as L is large and it corresponds to the "table-top" soliton. All other eigenvalues are bounded away from the limiting value $\kappa=1/2$ and they correspond to "KdV-type" solitons of small amplitude [see Fig. 5(b) for $A=0.8$]. Their amplitudes and numbers are still bounded by the constants (21) and (22). When $A\to 1$, the amplitudes of these solitons tend to zero, and the wave momentum concentrates into just one "table-top" soliton.

For strong disturbances ($A>1$), there is always just one eigenvalue determined from Eq. (20), see Fig. 5(c). This implies that a large initial disturbance evolves into just one soliton and a dispersive tail. If the initial disturbance is sufficiently wide, i.e., $L>2$ for $A=1.2$, the soliton has the "table-top" crest. Similar results were obtained by Miles.[5]

Smooth initial disturbance. Let the initial disturbance $u(x)$ be given by

$$u(x,0)=A\,\mathrm{sech}(2x/L), \tag{23}$$

where A is the amplitude and L is the width. Such disturbances have the same momentum $P=A^2L$ as the "box" disturbance, but a slightly larger mass $M=\pi AL/2$. The AKNS system (14) is here integrated numerically, using a finite difference method. The eigenvalues $\mu(=\kappa)$ found numerically are shown in Figs. 6(a) and 6(b) for $A=0.8$ (below the limiting soliton amplitude) and for $A=1.2$ (above the limiting soliton amplitude). For this smooth initial disturbance, the number of eigenvalues grows with L even for large amplitudes A. For $L\gg 1$, the eigenvalues are distributed uniformly from zero to the limiting value, 1/2. This means that the number of the "wide" solitons can be large if the initial disturbance is large enough.

Figure 7 shows eigenvalues $\mu(=\kappa)$ as function of the amplitude A, for $L=10$. For all except the first eigenvalue,

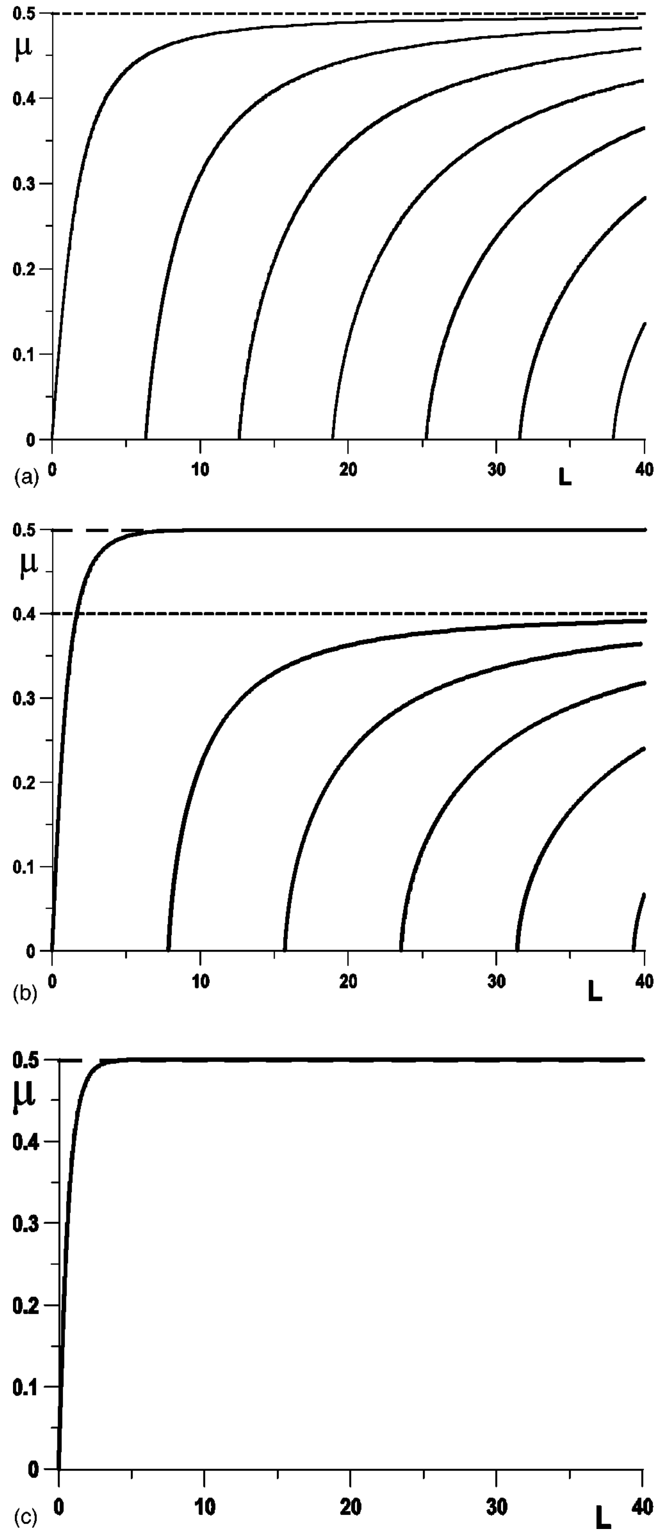

FIG. 5. Discrete spectrum for the rectangular box initial data vs L for (a) $A=0.48$, (b) $A=0.8$, and (c) $A=1.2$. Dotted horizontal lines in (a) and (b) show the limiting amplitude defined by Eq. (21). Dashed horizontal lines in (b) and (c) show the limiting amplitude for the "table-top" solitons.

the eigenvalues occur in closely positioned pairs, such that the distance between the two adjacent eigenvalues decreases with larger values of A. The first eigenvalue slightly increases as the amplitude A increases and it tends to the limiting value $\kappa=1/2$ as A gets large. All other eigenvalues de-

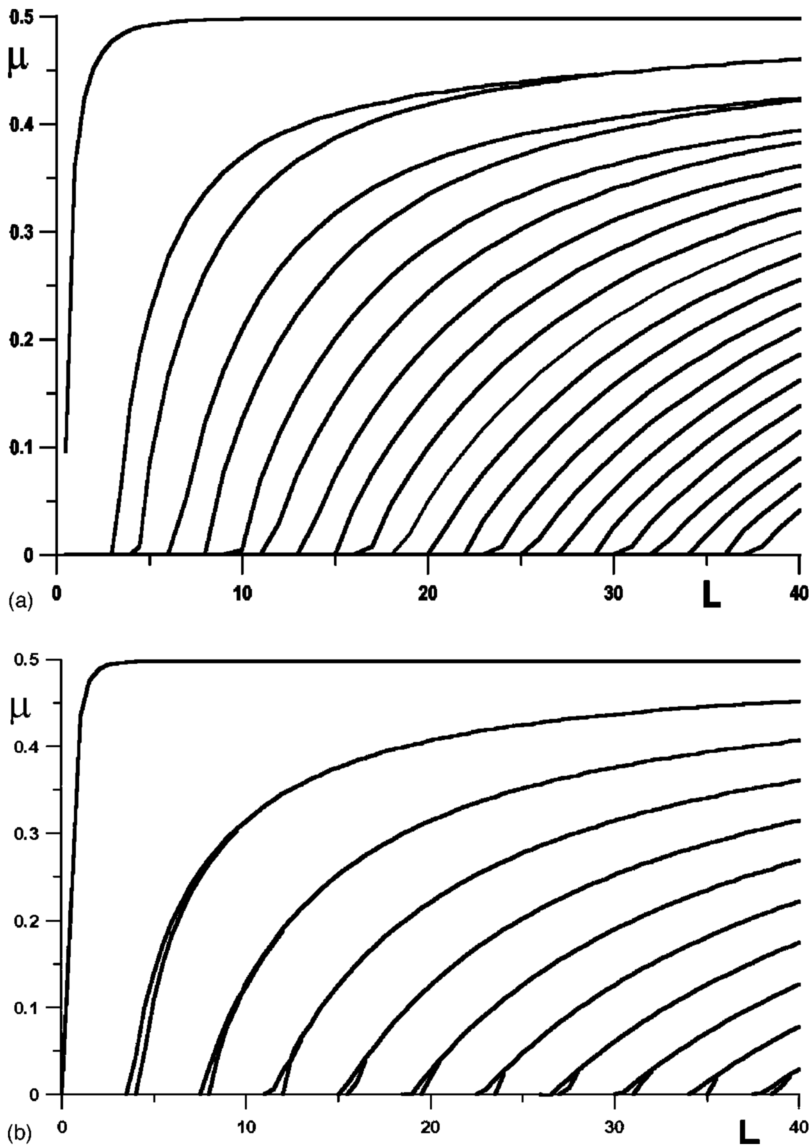

FIG. 6. Discrete spectrum for a smooth sech-profile initial data (23) vs L for (a) $A=0.8$ and (b) $A=1.2$.

crease slightly with larger amplitudes A. For small-amplitude smooth initial disturbances, all eigenvalues are different and do not appear in closely positioned pairs, see Fig. 7.

We conclude that the smooth initial disturbance (23) of large amplitude A produces several eigenvalues, whereas the rectangular-box initial disturbance on Fig. 4 of large amplitude A produces only one eigenvalue. Our analysis confirms the results of numerical simulations of the eKdV equation.[15] The initial disturbance (23) with $A=1.2$ and $L=10$ evolves into one ''table-top'' soliton and a group of small solitons with amplitudes, $a=0.225$ and $a=0.035$ for $t=150$. According to the spectral problem (14), the eigenvalues $\mu=0.32$ and $\mu=0.13$ correspond to solitons with amplitudes $a=0.225$ and $a=0.035$, as obtained in the numerical simulations. These additional eigenvalues $\mu=0.32$ and $\mu=0.13$ are double (see Fig. 6), with a difference in the eigenvalue μ and soliton amplitude a being only a few percent.

III. DISCUSSION

In nonlinear wave physics, the rectangular-box potential is often used to obtain explicit solutions, and typically it characterizes the discrete spectrum of a spectral problem quite well. The main properties of the solitons generated from an initial disturbance are explained quite well with the use of a rectangular box initial disturbance for the KdV equation,[21] the modified KdV equation, and the nonlinear Schrödinger equation.[22,23] For the eKdV equation with a negative coefficient of the cubic nonlinear term, we have shown that the rectangular box potential is not typical, and does not reflect the general properties of the discrete spectrum. To explain this, we reduce the spectral problems (10) and (14) for $\mathbf{\Psi}=(\Psi_1,\Psi_2)^T$ to the Schrödinger spectral problem,

$$[\partial_x^2+v(x)]\Phi=\gamma\Phi, \tag{24}$$

where

$$v(x)=u(x)-u^2(x)-u'(x), \tag{25}$$

$\Phi(x)=\Psi_1(x)+\Psi_2(x)$, and $\gamma=\mu^2$. We note that the relation (25) is the Miura transformation[20] between the eKdV equation (2) and the KdV equation for $v(x,t)$:

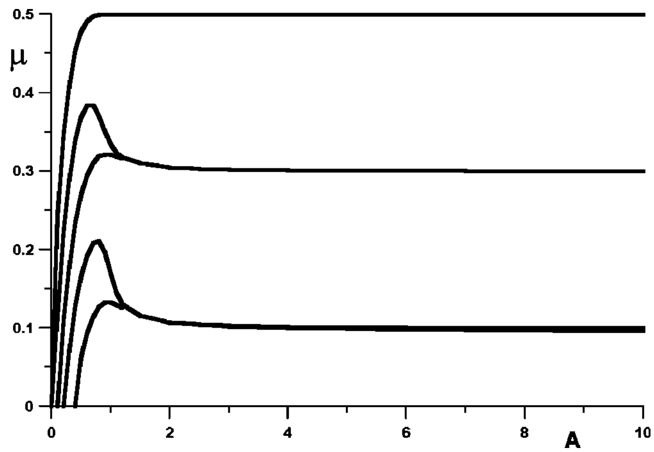

FIG. 7. Eigenvalues of the discrete spectrum for a smooth sech-profile initial data (23) vs A for $L=10$.

$$\frac{\partial v}{\partial t}+6v\frac{\partial v}{\partial x}+\frac{\partial^3 v}{\partial x^3}=0. \tag{26}$$

The potential in the Schrödinger equation $V(x)=-v(x)$ is expressed through $u(x)$ by virtue of Eq. (25). In particular, the rectangular box initial disturbance transforms into the potential shown in Fig. 8 for $A>1$ (a) and for $A<1$ (b). The potential $V(x)$ contains two parts, namely, two δ functions of opposite polarities at $x=\pm L/2$, and a box for $-L/2<x<L/2$ (positive for $A>1$ and negative for $A<1$). For the Schrödinger equation (24), a nonempty discrete spectrum is related with negative parts of the potential $V(x)$. For the case $A>1$ there is only one negative well, a δ function at $x=L/2$, see Fig. 8(a). The δ-function negative potential always traps a single discrete eigenvalue, see for instance Ref. 21. As a result, there is always only one eigenvalue $\gamma=\mu^2$ for a large-amplitude initial box with $A>1$. For the case $A<1$, there is an additional negative square well potential, see Fig. 8(b). The square well potential traps additional discrete eigenvalues, depending on its height and width (related to A and L). This qualitative analysis confirms the results of direct calculations: a large-amplitude rectangular box evolves into only one soliton, while a small-amplitude box evolves into several solitons, see Figs. 5(a) and 5(c).

For smooth initial profiles $u(x)$ of the eKdV equation (2), the corresponding potential $V(x)$ for the Schrödinger equation (24) contains an antisymmetric part equivalent to the pair of the δ functions in the box potential due to the term $u'(x)$, and a symmetric part due to the term $u(1-u)$ in Eq. (25). The potential $V(x)$ for the sech-profile initial disturbance (23) is shown in Fig. 9 for $A>1$ (a) and $A<1$ (b). The antisymmetric part of $V(x)$ may trap eigenvalues γ in the negative δ-function-like potential. The symmetric part of $V(x)$ for $A>1$ displays two wells separated by a crest, see Fig. 9(a). The two symmetric wells support double eigenvalues of the discrete spectrum, depending on the well's height and width. The double eigenvalues are split due to the separation of these same wells in space and the quantum tunneling effect. The symmetric part of $V(x)$ for $A<1$ displays a single well, see Fig. 9(b). The single well generates a finite set of single eigenvalues similar to that for the KdV equation. This qualitative analysis confirms the behavior of the eigenvalues μ in the numerical calculations, see Figs. 6 and 7. We believe that this behavior is typical for any smooth initial profiles $u(x)$. Numerical calculations of the spectrum of the AKNS system (9), performed with a combination of rectangular boxes (a small-amplitude wide box and a large-amplitude narrow box) and with a trapezoidal large-amplitude box, confirm these conclusions: new eigenvalues appear for smooth large-amplitude profiles $u(x)$ (the details of these calculations are not given here).

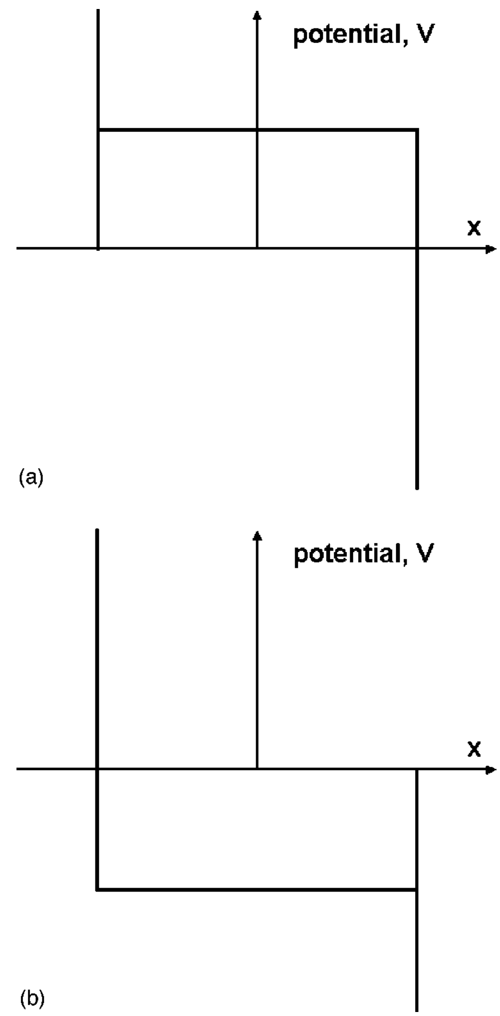

FIG. 8. The potential $V(x)$ of the Schrödinger equation (24) for the rectangular box in Fig. 4 for (a) $A>1$ and (b) $A<1$.

We conclude that the large-amplitude rectangular box $u(x)$ is not a typical initial disturbance for the spectral problems (10) and (14) and it generates only one soliton of the eKdV equation (2), while a smooth disturbance $u(x)$ generates a group of several solitons, depending on the height and width of the initial profile.

IV. CONCLUSION

We have studied the initial-value problem for the eKdV equation (2) with a negative cubic nonlinear term. As this equation is integrable, we have used the Lax operator method to find the number of solitons generated from an initial data $u(x,0)$. The spectral AKNS problems (10) and

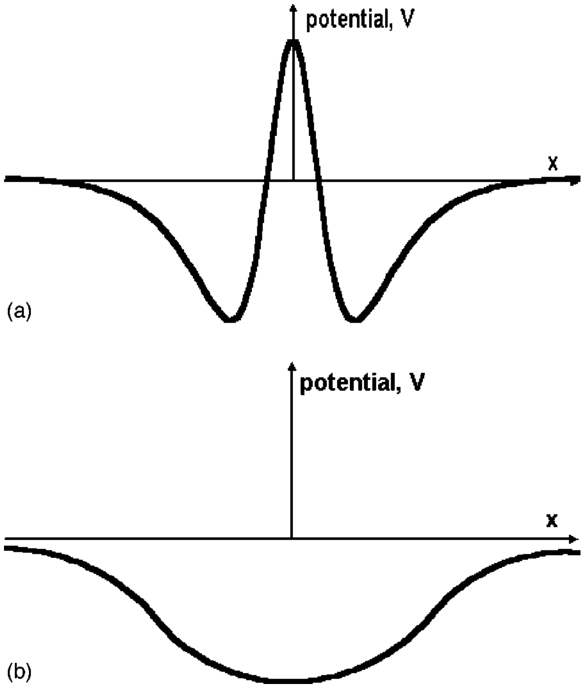

FIG. 9. The potential $V(x)$ of the Schrödinger equation (24) for a smooth sech-profile initial data (23) for (a) $A>1$ and (b) $A<1$.

(14) are solved for both smooth and rectangular box potentials. We show that a nonempty discrete spectrum exists for any positive data $u(x,0)$. The previous result by Miles[5] for the rectangular box disturbances is confirmed and explained by analyzing the Miura transformation (25), which transforms the spectral problems (10) and (14) into the classical Schrödinger equation (24). A rectangular box initial data leads to a singular potential in the Schrödinger equation, and the number of eigenvalues for large-amplitude rectangular box does not exceed one. For a smooth initial data of large amplitude, the potential in the Schrödinger equation contains at least two wells and the number of eigenvalues depends on the height and width of these wells. Thus the generation of large-amplitude ''table-top'' solitons depends both on the intensity of the initial disturbance, and on its shape. This result is sensitive to the initial form when it is close to the box-like form, and this is the main difference from other associated spectral problems analyzed in the classical soliton theories.

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