

STATIONARY STATES OF THE CUBIC CONFORMAL FLOW ON \mathbb{S}^3

PIOTR BIZOŃ AND DOMINIKA HUNIK-KOSTYRA

Institute of Physics, Jagiellonian University
Kraków, Poland

DMITRY PELINOVSKY

Department of Mathematics, McMaster University
Hamilton, Ontario L8S 4K1, Canada

(Communicated by Pierre Germain)

ABSTRACT. We consider the resonant system of amplitude equations for the conformally invariant cubic wave equation on the three-sphere. Using the local bifurcation theory, we characterize all stationary states that bifurcate from the first two eigenmodes. Thanks to the variational formulation of the resonant system and energy conservation, we also determine variational characterization and stability of the bifurcating states. For the lowest eigenmode, we obtain two orbitally stable families of the bifurcating stationary states: one is a constrained maximizer of energy and the other one is a local constrained minimizer of the energy, where the constraints are due to other conserved quantities of the resonant system. For the second eigenmode, we obtain two local constrained minimizers of the energy, which are also orbitally stable in the time evolution. All other bifurcating states are saddle points of energy under these constraints and their stability in the time evolution is unknown.

1. Introduction. The conformally invariant cubic wave equation on \mathbb{S}^3 (the unit three-dimensional sphere) is a toy model for studying the dynamics of resonant interactions between nonlinear waves on a compact manifold. The long-time behavior of small solutions of this equation is well approximated by solutions of an infinite dimensional time-averaged Hamiltonian system, called the *cubic conformal flow*, that was introduced and studied in [3]. In terms of complex amplitudes $\{\alpha_n(t)\}_{n \in \mathbb{N}}$, this system takes the form

$$i(n+1) \frac{d\alpha_n}{dt} = \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{njkn+j-k} \bar{\alpha}_j \alpha_k \alpha_{n+j-k}, \quad (1.1)$$

where $S_{njkn+j-k} = \min(n, j, k, n+j-k) + 1$ are the interaction coefficients. The cubic conformal flow (1.1) is the Hamiltonian system with the symplectic form

2010 *Mathematics Subject Classification.* 35L70, 35Q75, 37G05, 37K45, 37K50.

Key words and phrases. Conformal flow, stationary states, local bifurcation, variational characterization, orbital stability.

This research was supported by the Polish National Science Centre grant no. 2017/26/A/ST2/00530.

$\sum_{n=0}^{\infty} 2i(n+1)d\bar{\alpha}_n \wedge (-d\alpha_n)$ and the conserved energy function

$$H(\alpha) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{njk, n+j-k} \bar{\alpha}_n \bar{\alpha}_j \alpha_k \alpha_{n+j-k}. \quad (1.2)$$

The attention of [3] has been focused on understanding the patterns of energy transfer between the modes. In particular, a three-dimensional invariant manifold was found on which the dynamics is Liouville-integrable with exactly periodic energy flows. Of special interest are the stationary states for which no transfer of energy occurs. A wealth of explicit stationary states have been found in [3], however a complete classification of stationary states was deemed as an open problem.

The purpose of this paper is to study existence and stability of all stationary states that bifurcate from the first two eigenmodes of the cubic conformal flow. Thanks to the Hamiltonian formulation, we are able to give the variational characterization of the bifurcating families. Among all the families, we identify several particular stationary states which are orbitally stable in the time evolution: one is a constrained maximizer of energy and the other ones are local constrained minimizers of the energy, where the constraints are induced by other conserved quantities of the cubic conformal flow.

The constrained maximizer of energy can be normalized to the form

$$\alpha_n(t) = (1 - p^2)p^n e^{-it}, \quad (1.3)$$

where $p \in (0, 1)$ is a parameter. This solution was labeled as the *ground state* in our previous work [4], where we proved orbital stability of the ground state in spite of its degeneracy with respect to parameter $p \in (0, 1)$.

Two constrained minimizers of energy are given by the exact solutions:

$$\alpha_n(t) = c \left[(1 - p^2)n - \frac{1}{2}(1 + 5p^2 \pm \sqrt{1 - 14p^2 + p^4}) \right] p^n e^{-i\lambda t + i\omega t}, \quad (1.4)$$

where $p \in (0, 2 - \sqrt{3})$ and $c \in (0, \infty)$ are parameters, whereas (λ, ω) are expressed by

$$\lambda = \frac{c^2}{6} \left(\frac{3 - 4p^2}{1 - p^2} \pm \frac{(3 + 4p^2)\sqrt{1 - 14p^2 + p^4}}{(1 - p^2)^2} \right) \quad (1.5)$$

and

$$\omega = \frac{c^2}{12} \frac{1 + p^2 \pm \sqrt{1 - 14p^2 + p^4}}{1 - p^2}. \quad (1.6)$$

The cutoff in the interval $(0, 2 - \sqrt{3})$ for p ensures that $1 - 14p^2 + p^4 > 0$. Since the constrained minimizers of energy are nondegenerate with respect to (λ, ω) , their orbital stability follows from the general stability theory [12].

By using the local bifurcation methods of this paper, we are able to prove that *the stationary states (1.3) and (1.4) are respectively maximizer and two local minimizers of energy constrained by two other conserved quantities in the limit of small p* . For the ground state (1.3), we know from our previous work [4] that it remains a constrained maximizer of energy for any $p \in (0, 1)$. For the stationary states (1.4), we have checked numerically that they remain local constrained minimizers of energy for any $p \in (0, 2 - \sqrt{3})$ and $c > 0$, however, we do not know if any of them is a global constrained minimizer of energy.

In addition, we have proven *existence of another local constrained minimizer of energy bifurcating from the second eigenmode*. The new minimizer is nondegenerate

with respect to (λ, ω) , hence again its orbital stability follows from the general stability theory in [12]. However, we show numerically that this stationary state remains a local constrained minimizer only near the bifurcation point and becomes a saddle point of energy far from the bifurcation point.

Bifurcation analysis of this paper for the first two eigenmodes suggests existence of other local constrained minimizers of energy bifurcating from other eigenmodes. This poses an open problem of characterizing a global constrained minimizer of energy for the cubic conformal flow (1.1). Another open problem is to understand orbital stability of the saddle points of energy, in particular, to investigate if other conserved quantities might contribute to stabilization of saddle points of energy.

The cubic conformal flow (1.1) shares many properties with a cubic resonant system for the Gross-Pitaevskii equation in two dimensions [1, 11]. In particular, the stationary states for the lowest Landau level invariant subspace of the cubic resonant system have been thoroughly studied in [7] by using the bifurcation theory from a simple eigenvalue [6]. In comparison with Section 6.4 in [7], where certain symmetries were imposed to reduce multiplicity of eigenvalues, we develop normal form theory for bifurcations of all distinct families of stationary states from the double eigenvalue without imposing any a priori symmetries.

Another case of a completely integrable resonant system with a wealth of stationary states is the cubic Szegő equation [8, 9]. Classification of stationary states and their stability has been performed for the cubic and quadratic Szegő equations in [15] and [16] respectively.

One more example of a complete classification of all travelling waves of finite energy for the energy-critical half-wave map onto \mathbb{S}^2 is given in [13], where the spectrum of linearization at the travelling waves is studied by using Jacobi operators and conformal transformations. The half-wave map was found to be another integrable system with the Lax pair formulation [10]. It is unclear in the present time if the cubic conformal flow on \mathbb{S}^3 is also an integrable system with the Lax pair formulation.

Organization of the paper. Symmetry, conserved quantities, and some particular stationary states for the cubic conformal flow (1.1) are reviewed in Section 2 based on the previous works [3, 4]. Local bifurcation results including the normal form computations for the lowest eigenmode are contained in Section 3, where we also give a variational characterization of the bifurcating families and the proof of extremal properties for the stationary states (1.3) and (1.4). Similar bifurcation results and variational characterization of the bifurcating states from the second eigenmode are obtained in Section 4. Numerical results confirming local minimizing properties of the stationary states (1.4) for all admissible values of p are reported in Section 5.

Notations. We denote the set of nonnegative integers by \mathbb{N} and the set of positive integers by \mathbb{N}_+ . A sequence $(\alpha_n)_{n \in \mathbb{N}}$ is denoted for short by α . The space of square-summable sequences on \mathbb{N} is denoted by $\ell^2(\mathbb{N})$. It is equipped with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. The weighted space $\ell^{2,2}(\mathbb{N})$ denotes the space of squared integrable sequences with the weight $(1+n)^2$. We write $X \lesssim Y$ to state $X \leq CY$ for some universal (i.e., independent of other parameters) constant $C > 0$. Terms of the Taylor series in (ϵ, μ) of the order $\mathcal{O}(\epsilon^p, \mu^p)$ are denoted by $\mathcal{O}(p)$.

2. Preliminaries. Here we recall from [3, 4] some relevant properties of the cubic conformal flow (1.1) and its stationary states. The cubic conformal flow (1.1) enjoys

the following three one-parameter groups of symmetries:

$$\text{Scaling: } \alpha_n(t) \rightarrow c\alpha_n(c^2t), \quad (2.1)$$

$$\text{Global phase shift: } \alpha_n(t) \rightarrow e^{i\theta}\alpha_n(t), \quad (2.2)$$

$$\text{Local phase shift: } \alpha_n(t) \rightarrow e^{in\phi}\alpha_n(t), \quad (2.3)$$

where c , θ , and ϕ are real parameters. By the Noether theorem, the latter two symmetries give rise to two conserved quantities:

$$Q(\alpha) = \sum_{n=0}^{\infty} (n+1)|\alpha_n|^2, \quad (2.4)$$

$$E(\alpha) = \sum_{n=0}^{\infty} (n+1)^2|\alpha_n|^2. \quad (2.5)$$

It is proven in Theorem 1.2 of [4] that $H(\alpha) \leq Q(\alpha)^2$ and the equality is achieved if and only if $\alpha = cp^n$ for some $c, p \in \mathbb{C}$ with $|p| < 1$.

As is shown in Appendix A of [4] (see also [2] for generalizations), there exists another conserved quantity of the conformal flow (1.1) in the form

$$Z(\alpha) = \sum_{n=0}^{\infty} (n+1)(n+2)\bar{\alpha}_{n+1}\alpha_n. \quad (2.6)$$

This quantity is related to another one-parameter group of symmetries:

$$\alpha_n(t) \rightarrow [e^{sD}\alpha(t)]_n := \sum_{k=0}^{\infty} \frac{s^k}{k!} [D^k\alpha(t)]_n, \quad (2.7)$$

where $s \in \mathbb{R}$ is arbitrary and D is a difference operator given by

$$[D\alpha]_n := n\alpha_{n-1} - (n+2)\alpha_{n+1}. \quad (2.8)$$

Note that the difference operator D is obtained from $Df = \{2i(Z - \bar{Z}), f\}$ acting on any function $f(\alpha, \bar{\alpha})$ on phase space, where the Poisson bracket is defined by

$$\{f, g\} := \sum_{k=0}^{\infty} \frac{1}{2i(k+1)} \left(\frac{\partial f}{\partial \bar{\alpha}_k} \frac{\partial g}{\partial \alpha_k} - \frac{\partial f}{\partial \alpha_k} \frac{\partial g}{\partial \bar{\alpha}_k} \right),$$

thanks to the symplectic structure $\sum_{n=0}^{\infty} 2i(n+1)d\bar{\alpha}_n \wedge (-d\alpha_n)$ of the conformal flow (1.1). The factor $(2i)$ in the definition of D is chosen for convenience. There exists another one-parameter group of symmetries obtained from $\tilde{D}f = \{2(Z + \bar{Z}), f\}$, which is not going to be used in this paper.

Stationary states of the cubic conformal flow (1.1) are obtained by the separation of variables

$$\alpha_n(t) = A_n e^{-i\lambda t + in\omega t}, \quad (2.9)$$

where the complex amplitudes A_n are time-independent, while parameters λ and ω are real. Substituting (2.9) into (1.1), we get a nonlinear system of algebraic equations for the amplitudes:

$$(n+1)(\lambda - n\omega)A_n = \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{njk, n+j-k} \bar{A}_j A_k A_{n+j-k}. \quad (2.10)$$

Stationary states (2.9) with parameters λ and ω are critical points of the functional

$$K(\alpha) = \frac{1}{2}H(\alpha) - \lambda Q(\alpha) - \omega [Q(\alpha) - E(\alpha)], \quad (2.11)$$

where H , Q , and E are given by (1.2), (2.4), and (2.5).

Let $\alpha = A + a + ib$, where A is a real root of the algebraic system (2.10), whereas a and b are real and imaginary parts of the perturbation. Because the stationary solution A is a critical point of K , the first variation of K vanishes at $\alpha = A$ and the second variation of K at $\alpha = A$ can be written as a quadratic form associated with the Hessian operator. In variables above, we obtain the quadratic form in the diagonalized form:

$$K(A + a + ib) - K(A) = \langle L_+ a, a \rangle + \langle L_- b, b \rangle + \mathcal{O}(\|a\|^3 + \|b\|^3), \quad (2.12)$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\ell^2(\mathbb{N})$ and $\|\cdot\|$ is the induced norm. After straightforward computations, we obtain the explicit form for the self-adjoint operators $L_\pm : D(L_\pm) \rightarrow \ell^2(\mathbb{N})$, where $D(L_\pm) \subset \ell^2(\mathbb{N})$ is the maximal domain and L_\pm are unbounded operators given by

$$\begin{aligned} (L_\pm a)_n &= \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{njk, n+j-k} [2A_j A_{n+j-k} a_k \pm A_k A_{n+j-k} a_j] \\ &\quad - (n+1)(\lambda - n\omega) a_n. \end{aligned} \quad (2.13)$$

Some particular solutions $\{A_n\}_{n \in \mathbb{N}}$ of the stationary system (2.10) are reviewed below.

2.1. Single-mode states. The simplest solutions of (2.10) are the *single-mode states* given by

$$A_n = c \delta_{nN}, \quad (2.14)$$

with $\lambda - N\omega = |c|^2$, where $c \in \mathbb{C}$ is an amplitude and $N \in \mathbb{N}$ is fixed. Thanks to the transformations (2.1) and (2.2), we can set $c = 1$ so that $\lambda - N\omega = 1$.

2.2. Invariant subspace of stationary states. As is shown in [3], system (2.10) can be reduced to three nonlinear equations with the substitution

$$A_n = (\beta + \gamma n) p^n, \quad (2.15)$$

where p, β, γ are complex parameters satisfying the nonlinear algebraic system

$$\frac{-\omega p}{(1+y)^2} = \frac{p}{6} (2y|\gamma|^2 + \bar{\beta}\gamma), \quad (2.16)$$

$$\frac{\lambda\gamma}{(1+y)^2} = \frac{\gamma}{6} (5|\beta|^2 + (18y^2 + 4y)|\gamma|^2 + (6y-1)\bar{\beta}\gamma + 10y\bar{\gamma}\beta), \quad (2.17)$$

$$\begin{aligned} \frac{\lambda\beta}{(1+y)^2} &= \beta (|\beta|^2 + (6y^2 + 2y)|\gamma|^2 + 2y\beta\bar{\gamma}) \\ &\quad + \gamma (2y|\beta|^2 + (4y+2)y^2|\gamma|^2 + y^2\bar{\beta}\gamma), \end{aligned} \quad (2.18)$$

and we have introduced $y := |p|^2/(1-|p|^2)$ so that $1/(1+y) = 1-|p|^2$. It is clear from the decay of the sequence (2.15) as $n \rightarrow \infty$ that p must be restricted to the unit disk: $|p| < 1$.

The analysis of solutions to system (2.16)–(2.18) simplifies if we assume that the parameters p, β, γ are real-valued, which can be done without loss of generality. To see this, note that p can be made real-valued by the transformation (2.3). If $\beta = 0$, then equation (2.18) implies $\gamma = 0$ and hence no nontrivial solutions exist. If $\beta \neq 0$, it can be made real by the transformation (2.2). If β is real, then γ is real, because equation (2.16) implies that $\text{Im}(\bar{\beta}\gamma) = 0$. Thus, it suffices to consider system (2.16)–(2.18) with $p \in (0, 1)$ and $\beta, \gamma \in \mathbb{R}$.

There exist exactly four families of solutions to the system (2.16)–(2.18).

2.2.1. *Ground state.* Equation (2.17) is satisfied if $\gamma = 0$. This implies that $\omega = 0$ from equation (2.16) and $\lambda(1-p^2)^2 = \beta^2$ from equation (2.18) with $p \in (0, 1)$ and $\beta \in \mathbb{R}$. Parameterizing $\beta = c$ and bringing all together yield the geometric sequence

$$A_n = cp^n, \quad \lambda = \frac{c^2}{(1-p^2)^2}. \quad (2.19)$$

As $p \rightarrow 0$, this solution tends to the $N = 0$ single-mode state (2.14). Thanks to the transformation (2.1), one can set $c = 1 - p^2$, then $\lambda = 1$ yields the normalized state (1.3).

As is explained above, the geometric sequence (2.19) is a maximizer of $H(\alpha)$ for fixed $Q(\alpha)$, therefore, we call it the *ground state*. Nonlinear stability of the ground state has been proven in [4], where the degeneracy due to the parameter p has been controlled with the use of conserved quantities $E(\alpha)$ and $Z(\alpha)$ in (2.5) and (2.6), respectively.

2.2.2. *Twisted state.* If $\gamma \neq 0$ but $\omega = 0$, equation (2.16) is satisfied with $\gamma = -\beta/(2y)$. Then, equation (2.17) yields $\lambda = \beta^2(1+y)^3/(4y)$, whereas equation (2.18) is identically satisfied with $p \in (0, 1)$ and $\beta \in \mathbb{R}$. Parameterizing $\beta = -2cp$ and bringing all together yield the *twisted state*

$$A_n = cp^{n-1}((1-p^2)n - 2p^2), \quad \lambda = \frac{c^2}{(1-p^2)^2}. \quad (2.20)$$

As $p \rightarrow 0$, this solution tends to the $N = 1$ single-mode state (2.14). Thanks to transformation (2.1), one can set $c = 1 - p^2$, then $\lambda = 1$.

2.2.3. *Pair of stationary states.* If $\gamma \neq 0$ and $\omega \neq 0$, then λ can be eliminated from system (2.17) and (2.18), which results in the algebraic equation

$$(2\gamma y + \beta)(12\gamma^2 y^2 + 6\beta\gamma y + 6\gamma^2 y + \beta^2 + \beta\gamma) = 0. \quad (2.21)$$

The first factor corresponds to the twisted state (2.20). Computing γ from the quadratic equation in the second factor yield two roots:

$$\gamma = -\frac{(1-p^2)\beta}{12p^2(1+p^2)} \left[1 + 5p^2 \mp \sqrt{1 - 14p^2 + p^4} \right]. \quad (2.22)$$

The real roots exist for $1 - 14p^2 + p^4 \geq 0$ which is true for $p^2 \leq 7 - 4\sqrt{3}$, or equivalently, for $p \in [0, 2 - \sqrt{3}]$. The pair of solutions can be parameterized as follows:

$$\gamma_{\pm} = c(1-p^2), \quad \beta_{\pm} = -\frac{1}{2}c(1+5p^2 \pm \sqrt{1-14p^2+p^4}), \quad (2.23)$$

where $c \in \mathbb{R}$ is arbitrary. Using equations (2.16) and (2.17) again, we get

$$\lambda_{\pm} = \frac{c^2}{6} \left(\frac{3-4p^2}{1-p^2} \pm \frac{(3+4p^2)\sqrt{1-14p^2+p^4}}{(1-p^2)^2} \right) \quad (2.24)$$

and

$$\omega_{\pm} = \frac{c^2}{12} \frac{1+p^2 \pm \sqrt{1-14p^2+p^4}}{1-p^2}. \quad (2.25)$$

As $p \rightarrow 0$, the branch with the upper sign converges to the $N = 0$ single-mode state

$$p \rightarrow 0: \quad A_n \rightarrow -c\delta_{n0}, \quad \lambda_+ \rightarrow c^2, \quad \omega_+ \rightarrow \frac{c^2}{6}, \quad (2.26)$$

while the branch with the lower sign converges to the $N = 1$ single-mode state

$$p \rightarrow 0: \quad A_n \rightarrow \tilde{c}\delta_{n1}, \quad \lambda_- \rightarrow \frac{5}{3}\tilde{c}^2, \quad \omega_- \rightarrow \frac{2}{3}\tilde{c}^2, \quad (2.27)$$

where $\tilde{c} = cp = \mathcal{O}(1)$ has been rescaled as $p \rightarrow 0$. The solutions (2.15), (2.23), (2.24), and (2.25) have been cast as the stationary states (1.4) with (1.5) and (1.6). By using our bifurcation analysis, we will prove that these stationary states are local minimizers of $H(\alpha)$ for fixed $Q(\alpha)$ and $E(\alpha)$ for small positive p . We also show numerically that this conclusion remains true for every $p \in (0, 2 - \sqrt{3})$.

We summarize that the four explicit families (2.19), (2.20), and (2.23) with (2.24) and (2.25) form a complete set of stationary states given by (2.15).

2.3. Other stationary states. Other stationary states were constructed in [3] from the generating function $u(t, z)$ given by the power series expansion:

$$u(t, z) = \sum_{n=0}^{\infty} \alpha_n(t) z^n. \quad (2.28)$$

The conformal flow (1.1) can be rewritten for $u(t, z)$ as the integro-differential equation

$$i\partial_t \partial_z (zu) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{d\zeta}{\zeta} \overline{u(t, \zeta)} \left(\frac{\zeta u(t, \zeta) - zu(t, z)}{\zeta - z} \right)^2. \quad (2.29)$$

The stationary solutions expressed by (2.9) yield the generating function in the form

$$u(t, z) = U(z e^{i\omega t}) e^{-i\lambda t}, \quad (2.30)$$

where U is a function of one variable given by $U(z) = \sum_{n=0}^{\infty} A_n z^n$. The family of stationary solutions expressed by (2.15) is generated by the function

$$U(z) = \frac{\beta}{1-pz} + \frac{\gamma pz}{(1-pz)^2}. \quad (2.31)$$

Additionally, any finite Blaschke product

$$U(z) = \prod_{k=1}^N \frac{z - \bar{p}_k}{1 - p_k z} \quad (2.32)$$

yields a stationary state with $\lambda = 1$ and $\omega = 0$ for $p_1, \dots, p_N \in \mathbb{C}$. If $N = 1$ and $p_1 = p \in (0, 1)$, the generating function (2.32) yields a stationary solution of the system (2.10) in the form

$$A_0 = -p, \quad A_{n \geq 1} = p^{-1}(1-p^2)p^n, \quad \lambda = 1, \quad \omega = 0. \quad (2.33)$$

This solution can be viewed as a continuation of the $N = 1$ single-mode state (2.14) in $p \neq 0$.

Another family of stationary states is generated by the function

$$U(z) = \frac{cz^N}{1-p^{N+1}z^{N+1}}, \quad \lambda = \frac{c^2}{(1-p^{2N+2})^2}, \quad \omega = 0, \quad (2.34)$$

where $N \in \mathbb{N}$, $p \in (0, 1)$, and $c \in \mathbb{R}$ is arbitrary. When $N = 0$, the function (2.34) generates the geometric sequence (2.19). When $N = 1$, the function (2.34) generates another stationary solution

$$A_n = c \begin{cases} p^{n-1}, & n \text{ odd,} \\ 0, & n \text{ even,} \end{cases} \quad \lambda = \frac{c^2}{(1-p^4)^2}, \quad \omega = 0, \quad (2.35)$$

which is a continuation of the $N = 1$ single-mode state (2.14) in $p \neq 0$. Thanks to transformation (2.1), one can set $c = 1 - p^4$, then $\lambda = 1$. The new solutions (2.33) and (2.35) appear in the local bifurcation analysis from the $N = 1$ single-mode state (2.14).

3. Bifurcations from the lowest eigenmode. We restrict our attention to the real-valued solutions of the stationary equations (2.10), which satisfy the system of algebraic equations

$$(n+1)(\lambda - n\omega)A_n = \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{njk, n+j-k} A_j A_k A_{n+j-k}. \quad (3.1)$$

Here we study bifurcations of stationary states from the lowest eigenmode given by (2.14) with $N = 0$. Without loss of generality, the scaling transformation (2.1) yields $\lambda = 1$.

3.1. Classification of the bifurcating states. By setting $A_n = \delta_{n0} + a_n$ with real-valued perturbation a , we rewrite the system (3.1) with $\lambda = 1$ in the perturbative form

$$L(\omega)a + N(a) = 0, \quad (3.2)$$

where $L(\omega)$ is a diagonal operator with entries

$$[L(\omega)]_{nn} = \begin{cases} 2, & n = 0, \\ n(n+1)\omega - n + 1, & n \geq 1, \end{cases} \quad (3.3)$$

and $N(a)$ includes quadratic and cubic nonlinear terms:

$$[N(a)]_n = 2 \sum_{j=0}^{\infty} a_j a_{n+j} + \sum_{k=0}^n a_k a_{n-k} + \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{njk, n+j-k} a_j a_k a_{n+j-k}. \quad (3.4)$$

We have the following result on the nonlinear terms.

Lemma 1. *Fix an integer $m \geq 2$. If*

$$a_{m\ell+1} = a_{m\ell+2} = \cdots = a_{m\ell+m-1} = 0, \quad \text{for every } \ell \in \mathbb{N}, \quad (3.5)$$

then

$$[N(a)]_{m\ell+1} = [N(a)]_{m\ell+2} = \cdots = [N(a)]_{m\ell+m-1} = 0, \quad \text{for every } \ell \in \mathbb{N}. \quad (3.6)$$

Proof. Let us inspect $[N(a)]_n$ in (3.4) for $n = m\ell + \nu$ with $\ell \in \mathbb{N}$ and $\nu \in \{1, 2, \dots, m-1\}$. Since $a_j \neq 0$ only if j is multiple of m , then $a_{n+j} = 0$ in the first term of (3.4) for all n which are not a multiple of m . Similarly, since $a_k \neq 0$ only if k is multiple of m , then $a_{n-k} = 0$ in the second term of (3.4). Finally, since $a_j \neq 0$ and $a_k \neq 0$ only if j and k are multiple of m , then $a_{n+j-k} = 0$ in the third term of (3.4). Hence, all terms of (3.4) are identically zero for $n = m\ell + \nu$ with $\ell \in \mathbb{N}$ and $\nu \in \{1, 2, \dots, m-1\}$. \square

Bifurcations from the lowest eigenmode are identified by zero eigenvalues of the diagonal operator $L(\omega)$.

Lemma 2. *There exists a sequence of bifurcations at $\omega \in \{\omega_m\}_{m \in \mathbb{N}_+}$ with*

$$\omega_m = \frac{m-1}{m(m+1)}, \quad m \in \mathbb{N}_+, \quad (3.7)$$

where all bifurcation points are simple except for $\omega_2 = \omega_3 = 1/6$.

Proof. The sequence of values (3.7) yield zero diagonal entries in (3.3). To study if the bifurcation points are simple, we consider solutions of $\omega_n = \omega_m$ for $n \neq m$. This equation is equivalent to $mn = m + n + 1$, which has only two solutions $(m, n) \in \{(2, 3); (3, 2)\}$. Therefore, all bifurcation points are simple except for the double point $\omega_2 = \omega_3 = 1/6$. \square

The standard Crandall–Rabinowitz theory [6] can be applied to study bifurcation from the simple zero eigenvalue.

Theorem 1. *Fix $m = 1$ or an integer $m \geq 4$. There exists a unique branch of solutions $(\omega, A) \in \mathbb{R} \times \ell^2(\mathbb{N})$ to system (3.1) with $\lambda = 1$, which can be parameterized by small ϵ such that (ω, A) is smooth in ϵ and*

$$|\omega - \omega_m| + \sup_{n \in \mathbb{N}} |A_n - \delta_{n0} - \epsilon \delta_{nm}| \lesssim \epsilon^2. \quad (3.8)$$

Proof. Let us consider the decomposition:

$$\omega = \omega_m + \Omega, \quad a_n = \epsilon \delta_{nm} + b_n, \quad n \in \mathbb{N}, \quad (3.9)$$

where ω_m is defined by (3.7), ϵ is arbitrary and $b_m = 0$ is set from the orthogonality condition $\langle b, e_m \rangle = 0$. Let $L_* = L(\omega_m)$. The system (3.2) is decomposed into the invertible part

$$[F(\epsilon, \Omega; b)]_n := [L_*]_{nn} b_n + n(n+1)\Omega b_n + [N(\epsilon e_m + b)]_n = 0, \quad n \neq m, \quad (3.10)$$

and the bifurcation equation

$$G(\epsilon, \Omega; b) := m(m+1)\Omega + \epsilon^{-1} [N(\epsilon e_m + b)]_m = 0. \quad (3.11)$$

Since

$$[L_*]_{nn} = \begin{cases} 2, & n = 0, \\ \frac{(n-m)(nm-m-n-1)}{m(m+1)}, & n \geq 1, \end{cases} \quad (3.12)$$

we have $[L_*]_{nn} \neq 0$ for every $n \neq m$ and $[L_*]_{nn} \rightarrow \infty$ as $n \rightarrow \infty$. Therefore, the Implicit Function Theorem can be applied to

$$F(\epsilon, \Omega; b) : \mathbb{R}^2 \times \ell^{2,2}(\mathbb{N}) \rightarrow \ell^2(\mathbb{N}),$$

where as is defined by (3.10), F is smooth in its variables, $F(0, \Omega; 0) = 0$ for every $\Omega \in \mathbb{R}$, $\partial_b F(0, 0; 0) = L_*$, and $\|F(\epsilon, \Omega; 0)\|_{\ell^2} \lesssim \epsilon^2$. For every small ϵ and small Ω , there exists a unique small solution of system (3.10) in $\ell^2(\mathbb{N})$ such that $\|b\|_{\ell^2} \lesssim \epsilon^2(1 + |\Omega|)$. Denote this solution by $b(\epsilon, \Omega)$.

Since

$$[N(\epsilon e_m)]_m = \epsilon^3 S_{mmmm}, \quad (3.13)$$

one power of ϵ is canceled in (3.11) and the Implicit Function Theorem can be applied to

$$G(\epsilon, \Omega; b(\epsilon, \Omega)) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R},$$

where as is defined by (3.11), G is smooth in its variables, $G(0, 0; b(0, 0)) = 0$, and

$$\partial_\Omega G(0, 0; b(0, 0)) = m(m+1) \neq 0.$$

For every small ϵ , there is a unique small root Ω of the bifurcation equation (3.11) such that $|\Omega| \lesssim \epsilon^2$ thanks to (3.13). \square

Remark 1. The unique branch bifurcating from $\omega_1 = 0$ coincides with the normalized ground state (1.3) and yields the exact result $\Omega = 0$. The small parameter ϵ is defined in terms of the small parameter p by the exact relation $\epsilon := p(1 - p^2)$.

Remark 2. For the unique branch bifurcating from ω_m with $m \geq 4$, we claim that $A_n \neq 0$, $n \in \mathbb{N}$ if and only if n is a multiple of m . Indeed, by Lemma 1, the system (3.2) can be reduced for a new sequence $\{a_{m\ell}\}_{\ell \in \mathbb{N}}$ whereas all other elements are identically zero. By Lemma 2, the value $\omega = \omega_m$ is a simple bifurcation point of this reduced system. By Theorem 1, there exists a unique branch of solutions of this reduced system with the bounds (3.8). Therefore, by uniqueness, the bifurcating solution satisfies the reduction of Lemma 1, that is, $A_n \neq 0$, $n \in \mathbb{N}$ if and only if n is a multiple of m .

Remark 3. There are at least three branches bifurcating from the double point $\omega_2 = \omega_3$. One branch satisfies (3.5) with $\ell = 2$, the other branch satisfies (3.5) with $\ell = 3$, and the third branch is given by the explicit solution (1.4) for the upper sign. We show in Theorem 2 that these are the only branches bifurcating from the double point.

As is well-known [5], normal form equations have to be computed in order to study branches bifurcating from the double zero eigenvalue at the bifurcation point $\omega_* := \omega_2 = \omega_3 = 1/6$.

Theorem 2. Fix $\omega_* := \omega_2 = \omega_3 = 1/6$. There exist exactly three branches of solutions $(\omega, A) \in \mathbb{R} \times \ell^2(\mathbb{N})$ to system (3.1) with $\lambda = 1$, which can be parameterized by small (ϵ, μ) such that (ω, A) is smooth in (ϵ, μ) and

$$|\omega - \omega_*| + \sup_{n \in \mathbb{N}} |A_n - \delta_{n0} - \epsilon \delta_{n2} - \mu \delta_{n3}| \lesssim (\epsilon^2 + \mu^2). \quad (3.14)$$

The three branches are characterized by the following:

- (i) $\epsilon = 0$, $\mu \neq 0$;
- (ii) $\epsilon \neq 0$, $\mu = 0$;
- (iii) $\epsilon < 0$, $|\mu - 2|\epsilon|^{3/2}| \lesssim \epsilon^2$,

and the branch (iii) is double degenerate up to the reflection $\mu \mapsto -\mu$.

Proof. Let us consider the decomposition:

$$\omega = \omega_* + \Omega, \quad a_n = \epsilon \delta_{n2} + \mu \delta_{n3} + b_n, \quad n \in \mathbb{N}, \quad (3.15)$$

where $\omega_* := \omega_2 = \omega_3 = 1/6$, (ϵ, μ) are arbitrary, and $b_2 = b_3 = 0$ are set from the orthogonality condition $\langle b, e_2 \rangle = \langle b, e_3 \rangle = 0$. The system (3.2) is decomposed into the invertible part

$$\begin{cases} 2b_0 + [N(\epsilon e_2 + \mu e_3 + b)]_0 = 0, \\ \frac{1}{6}(n-2)(n-3)b_n + n(n+1)\Omega b_n + [N(\epsilon e_2 + \mu e_3 + b)]_n = 0, \\ n \neq \{0, 2, 3\}, \end{cases} \quad (3.16)$$

and the bifurcation equations

$$\begin{cases} 6\Omega\epsilon + [N(\epsilon e_2 + \mu e_3 + b)]_2 = 0, \\ 12\Omega\mu + [N(\epsilon e_2 + \mu e_3 + b)]_3 = 0. \end{cases} \quad (3.17)$$

By the Implicit Function Theorem (see the proof of Theorem 1), there exists a unique map $\mathbb{R}^3 \ni (\epsilon, \mu, \Omega) \mapsto b \in \ell^2(\mathbb{N})$ for small (ϵ, μ, Ω) such that equations (3.16) are satisfied and $\|b\|_{\ell^2} \lesssim (\epsilon^2 + \mu^2)(1 + |\Omega|)$. Denote this solution by $b(\epsilon, \mu, \Omega)$.

Compared to the proof of Theorem 1, it is now more difficult to consider solutions of the system of two algebraic equations (3.17). In order to resolve the degeneracy of these equations, we have to compute the solution $b(\epsilon, \mu, \Omega)$ up to the cubic terms in (ϵ, μ) under the apriori assumption $|\Omega| \lesssim (\epsilon^2 + \mu^2)$. Substituting the expansion for $b(\epsilon, \mu, \Omega)$ into the system (3.17) and expanding it up to the quartic terms in

(ϵ, μ) , we will be able to confirm the apriori assumption $|\Omega| \lesssim (\epsilon^2 + \mu^2)$ and to obtain all solutions of the system (3.17) for (ϵ, μ, Ω) near $(0, 0, 0)$.

First, we note that if $|\Omega| \lesssim (\epsilon^2 + \mu^2)$, then $b_n = \mathcal{O}(3)$ for every $n \geq 7$, where $\mathcal{O}(3)$ denotes terms of the cubic and higher order in (ϵ, μ) . Thanks to the cubic smallness of b_n for $n \geq 7$, we can compute $[N(\epsilon e_2 + \mu e_3 + b)]_n$ for $n \in \{0, 1, 4, 5, 6\}$ up to and including the cubic order:

$$\begin{cases} [N(\epsilon e_2 + \mu e_3 + b)]_0 = 2(\epsilon^2 + \mu^2) + \mathcal{O}(4), \\ [N(\epsilon e_2 + \mu e_3 + b)]_1 = 2\epsilon\mu + 2\epsilon b_1 + 2\mu b_4 + 2\epsilon^2\mu + \mathcal{O}(4), \\ [N(\epsilon e_2 + \mu e_3 + b)]_4 = \epsilon^2 + 2\epsilon b_6 + 2\mu b_1 + 3\epsilon\mu^2 + \mathcal{O}(4), \\ [N(\epsilon e_2 + \mu e_3 + b)]_5 = 2\epsilon\mu + \mathcal{O}(4), \\ [N(\epsilon e_2 + \mu e_3 + b)]_6 = \mu^2 + 2\epsilon b_4 + \mathcal{O}(4). \end{cases}$$

By using the system (3.16) and the quadratic approximations for (b_1, b_4, b_6) , we obtain $(b_0, b_1, b_4, b_5, b_6)$ up to and including the cubic order:

$$\begin{cases} b_0 = -\epsilon^2 - \mu^2 + \mathcal{O}(4), \\ b_1 = -6\epsilon\mu + 48\epsilon^2\mu + \mathcal{O}(4), \\ b_4 = -3\epsilon^2 + 30\epsilon\mu^2 + \mathcal{O}(4), \\ b_5 = -2\epsilon\mu + \mathcal{O}(4), \\ b_6 = -\frac{1}{2}\mu^2 + 3\epsilon^3 + \mathcal{O}(4). \end{cases}$$

Next, we compute $[N(\epsilon e_2 + \mu e_3 + b)]_n$ for $n \in \{2, 3\}$ up to and including the quartic order:

$$\begin{cases} [N(\epsilon e_2 + \mu e_3 + b)]_2 = 4\epsilon b_0 + 2\epsilon b_4 + 2\mu b_1 + 2\mu b_5 + b_1^2 + 2b_4 b_6 \\ \quad + 4\epsilon\mu b_1 + 3\epsilon^3 + 6\epsilon\mu^2 + 3\mu^2 b_4 + \mathcal{O}(5), \\ [N(\epsilon e_2 + \mu e_3 + b)]_3 = 2\epsilon b_1 + 2\epsilon b_5 + 4\mu b_0 + 2\mu b_6 + 2b_1 b_4 \\ \quad + 2\epsilon^2 b_1 + 6\epsilon^2\mu + 6\epsilon\mu b_4 + 4\mu^3 + \mathcal{O}(5). \end{cases}$$

By using the quadratic and cubic approximations for $(b_0, b_1, b_4, b_5, b_6)$, we rewrite the system (3.17) up to and including the quartic order:

$$\begin{cases} 6\Omega\epsilon = \epsilon(7\epsilon^2 + 14\mu^2 - 162\epsilon\mu^2) + \mathcal{O}(5), \\ 12\Omega\mu = \mu(14\epsilon^2 + \mu^2 - 108\epsilon^3) + \mathcal{O}(5). \end{cases} \quad (3.18)$$

By Lemma 1, if $\epsilon = 0$, then $[N(\mu e_3 + b(0, \mu, \Omega))]_2 = 0$, whereas if $\mu = 0$, then $[N(\epsilon e_2 + b(\epsilon, 0, \Omega))]_3 = 0$. Hence, the system (3.18) can be rewritten in the equivalent form:

$$\begin{cases} 6\Omega\epsilon = \epsilon(7\epsilon^2 + 14\mu^2 - 162\epsilon\mu^2 + \mathcal{O}(4)), \\ 12\Omega\mu = \mu(14\epsilon^2 + \mu^2 - 108\epsilon^3 + \mathcal{O}(4)). \end{cases} \quad (3.19)$$

We are looking for solutions to the system (3.19) with $(\epsilon, \mu) \neq (0, 0)$, because $b(0, 0, \Omega) = 0$ follows from the system (3.16). There exist three nontrivial solutions to the system (3.18):

- (I) $\epsilon = 0$, $\mu \neq 0$, and $\Omega = \frac{1}{12}\mu^2 + \mathcal{O}(4)$;
- (II) $\epsilon \neq 0$, $\mu = 0$, and $\Omega = \frac{7}{6}\epsilon^2 + \mathcal{O}(4)$;
- (III) $\epsilon \neq 0$, $\mu \neq 0$, and

$$\begin{cases} 6\Omega = 7\epsilon^2 + 14\mu^2 - 162\epsilon\mu^2 + \mathcal{O}(4), \\ 12\Omega = 14\epsilon^2 + \mu^2 - 108\epsilon^3 + \mathcal{O}(4). \end{cases} \quad (3.20)$$

All the solutions satisfy the apriori assumption $|\Omega| \lesssim (\epsilon^2 + \mu^2)$. In order to consider persistence of these solutions in the system (3.19), we compute the Jacobian matrix:

$$J(\epsilon, \mu, \Omega) = \begin{bmatrix} -6\Omega + 21\epsilon^2 + 14\mu^2 - 324\epsilon\mu^2 & 28\epsilon\mu - 324\epsilon^2\mu \\ 28\epsilon\mu - 324\epsilon^2\mu & -12\Omega + 14\epsilon^2 + 3\mu^2 - 108\epsilon^3 \end{bmatrix} + \mathcal{O}(4).$$

We proceed in each case as follows:

- (I) The Jacobian is invertible for small (ϵ, μ) admitting the expansion in (I). By the Implicit Function Theorem, there exists a unique continuation of this root in the system (3.18). By Lemma 1 with $m = 3$, the bifurcating solution corresponds to the reduction with $a_{3j+1} = a_{3j+2} = 0$ for every $j \in \mathbb{N}$, hence $\epsilon = 0$ persists beyond all orders of the expansion. This yields the solution (i).
- (II) The Jacobian is invertible for small (ϵ, μ) admitting the expansion in (II). By the Implicit Function Theorem, there exists a unique continuation of this root in the system (3.18). By Lemma 1 with $m = 2$, the bifurcating solution corresponds to the reduction with $a_{2j+1} = 0$ for every $j \in \mathbb{N}$, hence $\mu = 0$ persists beyond all orders of the expansion. This yields the solution (ii).
- (III) Eliminating Ω from the system (3.20) yields the root finding problem:

$$27\mu^2 - 324\epsilon\mu^2 + 108\epsilon^3 + \mathcal{O}(4) = 0,$$

which has only two solutions for (ϵ, μ) near $(0, 0)$ from the two roots of the quadratic equation $\mu^2 + 4\epsilon^3 + \mathcal{O}(4) = 0$. Computing

$$\det(J(\epsilon, \mu, \Omega)) = 3024\epsilon^5 + \mathcal{O}(6) \neq 0$$

verifies that $J(\epsilon, \mu, \Omega)$ is invertible at each of the two roots. By the Implicit Function Theorem, there exists a unique continuation of each of the two roots in the system (3.18). For the root with $\mu > 0$, this yields the solution (iii). Thanks to the symmetry (2.3) with $\phi = \pi$, every solution with $\mu > 0$ can be uniquely reflected to the solution with $\mu < 0$ by the transformation $a_{2k} \mapsto a_{2k}$ and $a_{2k+1} \mapsto -a_{2k+1}$ for $k \in \mathbb{N}$. Hence the two solutions with $\mu > 0$ and $\mu < 0$ are generated from the same branch (iii) up to reflection $\mu \mapsto -\mu$.

No other branches bifurcate from the point $\omega_* = 1/6$. □

Remark 4. The branch (iii) bifurcating from $\omega_* = 1/6$ coincides with the exact solution (2.23) for the upper sign. Indeed, by normalizing $\lambda = 1$ and taking the limit $p \rightarrow 0$ as in (2.26), we obtain $c = -1 - p^2 + \mathcal{O}(p^4)$, $\beta = 1 + \mathcal{O}(p^4)$, and $\gamma = -1 + \mathcal{O}(p^4)$. The parameter ϵ and μ are related to each other by means of the small parameter p with the definitions $\epsilon := -p^2 + \mathcal{O}(p^6)$ and $\mu := -2p^3 + \mathcal{O}(p^7)$, hence $\mu^2 + 4\epsilon^3 = \mathcal{O}(\epsilon^5)$.

Remark 5. Branches (i) and (ii) bifurcating from $\omega_* = 1/6$ can be obtained by the Crandall–Rabinowitz theory [6] by reducing sequences on the constrained subspace of $\ell^2(\mathbb{N})$ by the constraints (3.5) with $\ell = 2$ and $\ell = 3$ respectively. The zero eigenvalue is simple on the constrained subspace of $\ell^2(\mathbb{N})$, which enables application of the theory, as it was done in [7] in a similar context.

3.2. Variational characterization of the bifurcating states. The following lemma gives variational characterization of the lowest eigenmode at the bifurcation points $\{\omega_m\}_{m \in \mathbb{N}_+}$ in Lemma 2. The variational characterization is based on the action functional $K(\alpha)$ in (2.11), its second variation in (2.12), and the linearization operators $L_{\pm} : D(L_{\pm}) \rightarrow \ell^2(\mathbb{N})$ in (2.13).

Lemma 3. *The following is true:*

- For $\omega_1 = 0$, the $N = 0$ single-mode state (2.14) is a degenerate saddle point of K with one positive eigenvalue, zero eigenvalue of multiplicity three, and infinitely many negative eigenvalues bounded away from zero.

- For $\omega_2 = \omega_3 = 1/6$, the $N = 0$ single-mode state (2.14) is a degenerate minimizer of K with zero eigenvalue of multiplicity five and infinitely many positive eigenvalues bounded away from zero.
- For ω_m with $m \geq 4$, the $N = 0$ single-mode state (2.14) is a degenerate saddle point of K with $2(m - 2)$ negative eigenvalues, zero eigenvalue of multiplicity three, and infinitely many positive eigenvalues bounded away from zero.

Proof. By the scaling transformation (2.1), we take $\lambda = 1$ and $A_n = \delta_{n0}$, for which the explicit form of the linearized operators L_{\pm} in (2.13) yields

$$(L_{\pm}a)_n = (1 - n)a_n \pm a_0\delta_{n0} + n(n + 1)\omega a_n, \quad n \in \mathbb{N}. \quad (3.21)$$

We note that $L_+ = L(\omega)$ given by (3.3) and L_- is only different from L_+ at the first diagonal entry at $n = 0$ (which is 0 instead of 2). Because L_{\pm} in (3.21) are diagonal, the assertion of the lemma is proven from explicit computations:

- If $\omega_1 = 0$, then $\sigma(L_+) = \{2, 0, -1, -2, \dots\}$ and $\sigma(L_-) = \{0, 0, -1, -2, \dots\}$, which yields the result.
- If $\omega_* := \omega_2 = \omega_3 = 1/6$, then

$$[L(\omega_*)]_{nn} = \frac{1}{6}(n - 2)(n - 3), \quad n \geq 1,$$

which yields the result.

- If $\omega_m = (m - 1)/(m(m + 1))$ with $m \geq 4$, then

$$[L(\omega_m)]_{nn} = \frac{1}{m(m + 1)}(n - m)(mn - m - n - 1), \quad n \geq 1.$$

For $n = 1$ and every $n > m$, $[L(\omega_m)]_{nn} > 0$. For $2 \leq n \leq m - 1$, $[L(\omega_m)]_{nn} < 0$. For $n = m$, $[L(\omega_m)]_{mm} = 0$. This yields the result. \square

Remark 6. It is shown in Lemma 6.2 of [4] that the normalized ground state (1.3) bifurcating from ω_1 has the same variational characterization as the $N = 0$ single-mode state, hence it is a triple-degenerate constrained maximizer of H subject to fixed Q . The triple degeneracy of the family (2.19) is due to two gauge symmetries (2.2) and (2.3), as well as the presence of the additional parameter p . As is explained in [2], the latter degeneracy is due to the additional symmetry (2.7). Indeed, by applying e^{sD} with $s \in \mathbb{R}$ to $\alpha_n(t) = \delta_{n0}e^{-it}$ and using the general transformation law derived in [2], we obtain another solution in the form:

$$\alpha_n(t) = \frac{(\tanh s)^n}{\cosh^2 s} e^{-it},$$

which coincides with the ground state (1.3) after the definition $p := \tanh s$.

Remark 7. All branches bifurcating from ω_m for $m \geq 2$ cannot be obtained by applying e^{sD} with $s \in \mathbb{R}$ to the $N = 0$ single-mode state $\alpha_n(t) = \delta_{n0}e^{-it}$, because the latter state is independent of the values of ω .

Remark 8. All branches bifurcating from ω_m for $m \geq 4$ have too many negative and positive eigenvalues and therefore, they represent saddle points of H subject to fixed Q and E .

It remains to study the three branches bifurcating from $\omega_* = \omega_2 = \omega_3 = 1/6$. By Lemma 3, there is a chance that some of these three branches represent local constrained minimizers of H subject to fixed Q and E . One needs to consider how

the zero eigenvalue of multiplicity five splits when $\omega \neq \omega_*$ with $|\omega - \omega_*|$ sufficiently small and how the eigenvalues change under the two constraints of fixed Q and E . The following lemma presents the count of negative eigenvalues of the operators L_\pm denoted as $n(L_\pm)$ at the three branches bifurcating from ω_* .

Lemma 4. *Consider the three bifurcating branches in Theorem 2 for $\omega_* := \omega_2 = \omega_3 = 1/6$. For every $\omega \neq \omega_*$ with $|\omega - \omega_*|$ sufficiently small, the following is true:*

- (i) $n(L_+) = 2, n(L_-) = 1$;
- (ii) $n(L_+) = 1, n(L_-) = 1$;
- (iii) $n(L_+) = 1, n(L_-) = 0$.

For each branch, L_- has a double zero eigenvalue, L_+ has no zero eigenvalue, and the rest of the spectrum of L_+ and L_- is strictly positive and is bounded away from zero.

Proof. Substituting $\lambda = 1$ and $\omega = \omega_* + \Omega$ into (2.13) yields

$$(L_\pm a)_n = \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{njkn+j-k} [2A_j A_{n+j-k} a_k \pm A_k A_{n+j-k} a_j] - 2a_n + \frac{1}{6}(n-2)(n-3)a_n + n(n+1)\Omega a_n, \quad (3.22)$$

where $A_n = \delta_{n0} + \epsilon\delta_{n2} + \mu\delta_{n3} + b_n$ with $\langle b, e_2 \rangle = \langle b, e_3 \rangle = 0$ follows from the decomposition (3.15). The correction terms (Ω, b) are uniquely defined by parameters (ϵ, μ) . In what follows, we consider the three branches in Theorem 2 separately.

Case (i). $\epsilon = 0, \mu \neq 0$. Here we have $\Omega = \frac{1}{12}\mu^2 + \mathcal{O}(\mu^4)$, $b_0 = -\mu^2 + \mathcal{O}(\mu^4)$, $b_1 = b_4 = b_5 = 0$, $b_6 = -\frac{1}{2}\mu^2 + \mathcal{O}(\mu^4)$, and $b_n = \mathcal{O}(\mu^3)$ for $n \geq 7$. Substituting these expansions in (3.22), we obtain

$$L_\pm = L_\pm^{(0)} + \mu L_\pm^{(1)} + \mu^2 L_\pm^{(2)} + \mathcal{O}(\mu^3),$$

where

$$\begin{aligned} (L_\pm^{(0)} a)_n &= \frac{1}{6}(n-2)(n-3)a_n \pm a_{-n}, \\ (L_\pm^{(1)} a)_n &= 2(a_{n+3} + a_{n-3}) \pm 2a_{3-n}, \\ (L_\pm^{(2)} a)_n &= \frac{1}{12}n(n+1)a_n + 2(\min(3, n) - 1)a_n - (a_{n+6} + a_{n-6}) \\ &\quad \mp 2a_{-n} \pm \min(3, n, 6-n)a_{6-n}. \end{aligned}$$

Let us represent the first 7-by-7 matrix block of the operator L_\pm and truncate it by up to and including $\mathcal{O}(\mu^2)$ terms. The corresponding matrix blocks denoted by \hat{L}_+ and \hat{L}_- are given respectively by

$$\hat{L}_+ = \begin{bmatrix} 2 - 4\mu^2 & 0 & 0 & 4\mu & 0 & 0 & -\mu^2 \\ 0 & \frac{1}{3} + \frac{1}{6}\mu^2 & 2\mu & 0 & 2\mu & \mu^2 & 0 \\ 0 & 2\mu & \frac{5}{2}\mu^2 & 0 & 2\mu^2 & 2\mu & 0 \\ 4\mu & 0 & 0 & 8\mu^2 & 0 & 0 & 2\mu \\ 0 & 2\mu & 2\mu^2 & 0 & \frac{1}{3} + \frac{17}{3}\mu^2 & 0 & 0 \\ 0 & \mu^2 & 2\mu & 0 & 0 & 1 + \frac{13}{2}\mu^2 & 0 \\ -\mu^2 & 0 & 0 & 2\mu & 0 & 0 & 2 + \frac{15}{2}\mu^2 \end{bmatrix}$$

and

$$\hat{L}_- = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\mu^2 \\ 0 & \frac{1}{3} + \frac{1}{6}\mu^2 & -2\mu & 0 & 2\mu & -\mu^2 & 0 \\ 0 & -2\mu & \frac{5}{2}\mu^2 & 0 & -2\mu^2 & 2\mu & 0 \\ 0 & 0 & 0 & 2\mu^2 & 0 & 0 & 2\mu \\ 0 & 2\mu & -2\mu^2 & 0 & \frac{1}{3} + \frac{17}{3}\mu^2 & 0 & 0 \\ 0 & -\mu^2 & 2\mu & 0 & 0 & 1 + \frac{13}{2}\mu^2 & 0 \\ -\mu^2 & 0 & 0 & 2\mu & 0 & 0 & 2 + \frac{15}{2}\mu^2 \end{bmatrix}$$

Zero eigenvalue of $L_+^{(0)}$ is double and is associated with the subspace spanned by $\{e_2, e_3\}$. There are two invariant subspaces of \hat{L}_+ , one is spanned by $\{e_0, e_3, e_6\}$ and the other one is spanned by $\{e_1, e_2, e_4, e_5\}$. This makes perturbative analysis easier. For the subspace spanned by $\{e_0, e_3, e_6\}$, the eigenvalue problem is given by

$$\begin{cases} (2 - 4\mu^2)x_0 + 4\mu x_3 - \mu^2 x_6 = \lambda x_0, \\ 4\mu x_0 + 8\mu^2 x_3 + 2\mu x_6 = \lambda x_3, \\ -\mu^2 x_0 + 2\mu x_3 + (2 + \frac{15}{2}\mu^2)x_6 = \lambda x_6, \end{cases}$$

The small eigenvalue for small μ is obtained by normalization $x_3 = 1$. Then, we obtain

$$x_0 = -2\mu + \mathcal{O}(\mu^3), \quad x_6 = -\mu + \mathcal{O}(\mu^3),$$

and

$$\lambda = -2\mu^2 + \mathcal{O}(\mu^4). \quad (3.23)$$

For the subspace spanned by $\{e_1, e_2, e_4, e_5\}$, the eigenvalue problem is given by

$$\begin{cases} (\frac{1}{3} + \frac{1}{6}\mu^2)x_1 + 2\mu x_2 + 2\mu x_4 + \mu^2 x_5 = \lambda x_1, \\ 2\mu x_1 + \frac{5}{2}\mu^2 x_2 + 2\mu^2 x_4 + 2\mu x_5 = \lambda x_2, \\ 2\mu x_1 + 2\mu^2 x_2 + (\frac{1}{3} + \frac{17}{2}\mu^2)x_4 = \lambda x_4, \\ \mu^2 x_1 + 2\mu x_2 + (1 + \frac{13}{2}\mu^2)x_5 = \lambda x_5, \end{cases}$$

The small eigenvalue for small μ is obtained by normalization $x_2 = 1$. Then, we obtain

$$x_1 = -6\mu + \mathcal{O}(\mu^3), \quad x_4 = 30\mu^2 + \mathcal{O}(\mu^4), \quad x_5 = -2\mu + \mathcal{O}(\mu^3),$$

and

$$\lambda = -\frac{27}{2}\mu^2 + \mathcal{O}(\mu^3). \quad (3.24)$$

By the perturbation theory, for every $\mu \neq 0$ sufficiently small, L_+ has two simple (small) negative eigenvalues. Other eigenvalues are bounded away from zero for small μ and by Lemma 2, all other eigenvalues of L_+ are strictly positive. Hence, $n(L_+) = 2$.

Zero eigenvalue of $L_-^{(0)}$ is triple and is associated with the subspace spanned by $\{e_0, e_2, e_3\}$. For every $\mu \neq 0$, a double zero eigenvalue of L_- exists due to the two symmetries (2.2) and (2.3). The two eigenvectors for the double zero eigenvalue of L_- are spanned by $\{e_0, e_3, e_6, \dots\}$. There are two invariant subspaces of \hat{L}_- , one is spanned by $\{e_0, e_3, e_6\}$ and the other one is spanned by $\{e_1, e_2, e_4, e_5\}$. Since we only need to compute a shift of the zero eigenvalue of $L_-^{(0)}$, we only consider the subspace of \hat{L}_- spanned by $\{e_1, e_2, e_4, e_5\}$. For this subspace, the eigenvalue

problem for \hat{L}_- is given by

$$\begin{cases} (\frac{1}{3} + \frac{1}{6}\mu^2)x_1 - 2\mu x_2 + 2\mu x_4 - \mu^2 x_5 = \lambda x_1, \\ -2\mu x_1 + \frac{5}{2}\mu^2 x_2 - 2\mu^2 x_4 + 2\mu x_5 = \lambda x_2, \\ 2\mu x_1 - 2\mu^2 x_2 + (\frac{1}{3} + \frac{17}{2}\mu^2)x_4 = \lambda x_4, \\ -\mu^2 x_1 + 2\mu x_2 + (1 + \frac{13}{2}\mu^2)x_5 = \lambda x_5, \end{cases}$$

The small eigenvalue for small μ is obtained by normalization $x_2 = 1$. Then, we obtain

$$x_1 = 6\mu + \mathcal{O}(\mu^3), \quad x_4 = -30\mu^2 + \mathcal{O}(\mu^4), \quad x_5 = -2\mu + \mathcal{O}(\mu^3),$$

and

$$\lambda = -\frac{27}{2}\mu^2 + \mathcal{O}(\mu^3). \quad (3.25)$$

By the perturbation theory, for every $\mu \neq 0$ sufficiently small, L_- has one simple (small) negative eigenvalue and the double zero eigenvalue. Other eigenvalues are bounded away from zero for small μ and by Lemma 2, all other eigenvalues of L_- are strictly positive. Hence, $n(L_-) = 1$.

Case (ii). $\epsilon \neq 0$, $\mu = 0$. Here we have $\Omega = \frac{7}{6}\epsilon^2 + \mathcal{O}(\epsilon^4)$, $b_0 = -\epsilon^2 + \mathcal{O}(\epsilon^4)$, $b_1 = b_5 = 0$, $b_4 = -3\epsilon^2 + \mathcal{O}(\epsilon^4)$, $b_6 = 3\epsilon^3 + \mathcal{O}(\epsilon^4)$, and $b_n = \mathcal{O}(\epsilon^4)$ for $n \geq 7$. Substituting these expansions in (3.22), we obtain

$$L_{\pm} = L_{\pm}^{(0)} + \epsilon L_{\pm}^{(1)} + \epsilon^2 L_{\pm}^{(2)} + \epsilon^3 L_{\pm}^{(3)} + \mathcal{O}(\epsilon^4),$$

where

$$\begin{aligned} (L_{\pm}^{(0)} a)_n &= \frac{1}{6}(n-2)(n-3)a_n \pm a_{-n}, \\ (L_{\pm}^{(1)} a)_n &= 2(a_{n+2} + a_{n-2}) \pm 2a_{2-n}, \\ (L_{\pm}^{(2)} a)_n &= \frac{7}{6}n(n+1)a_n + 2(\min(2, n) - 1)a_n - 6(a_{n+4} + a_{n-4}) \\ &\quad \mp 2a_{-n} \pm (\min(2, n, 4-n) - 5)a_{4-n}, \\ (L_{\pm}^{(3)} a)_n &= -2(a_{n+2} + a_{n-2}) + 6(a_{n+6} + a_{n-6}) \\ &\quad - 6(\min(2, n) + 1)a_{n+2} - 6(\min(2, n-2) + 1)a_{n-2} \\ &\quad \mp 2a_{2-n} \mp 6\min(2, n, 6-n)a_{6-n}. \end{aligned}$$

Let us represent the first 7-by-7 matrix block of the operator L_{\pm} and truncate it by up to and including $\mathcal{O}(\epsilon^3)$ terms. The corresponding matrix blocks denoted by \hat{L}_+ and \hat{L}_- are given respectively by

$$\begin{bmatrix} 2-4\epsilon^2 & 0 & 4\epsilon-10\epsilon^3 & 0 & -11\epsilon^2 & 0 & 6\epsilon^3 \\ 0 & \frac{1}{3}+2\epsilon+\frac{7}{3}\epsilon^2-2\epsilon^3 & 0 & 2\epsilon-4\epsilon^2-14\epsilon^3 & 0 & -6\epsilon^2-6\epsilon^3 & 0 \\ 4\epsilon-10\epsilon^3 & 0 & 6\epsilon^2 & 0 & 2\epsilon-32\epsilon^3 & 0 & -6\epsilon^2 \\ 0 & 2\epsilon-4\epsilon^2-14\epsilon^3 & 0 & 16\epsilon^2-12\epsilon^3 & 0 & 2\epsilon-20\epsilon^3 & 0 \\ -11\epsilon^2 & 0 & 2\epsilon-32\epsilon^3 & 0 & \frac{1}{3}+\frac{7}{3}\epsilon^2 & 0 & 2\epsilon-20\epsilon^3 \\ 0 & -6\epsilon^2-6\epsilon^3 & 0 & 2\epsilon-20\epsilon^3 & 0 & 1+37\epsilon^2 & 0 \\ 6\epsilon^3 & 0 & -6\epsilon^2 & 0 & 2\epsilon-20\epsilon^3 & 0 & 2+51\epsilon^2 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 & -6\epsilon^3 & 0 & -\epsilon^2 & 0 & 6\epsilon^3 \\ 0 & \frac{1}{3}-2\epsilon+\frac{7}{3}\epsilon^2+2\epsilon^3 & 0 & 2\epsilon+4\epsilon^2-14\epsilon^3 & 0 & -6\epsilon^2+6\epsilon^3 & 0 \\ -6\epsilon^3 & 0 & 12\epsilon^2 & 0 & 2\epsilon-8\epsilon^3 & 0 & -6\epsilon^2 \\ 0 & 2\epsilon+4\epsilon^2-14\epsilon^3 & 0 & 16\epsilon^2+12\epsilon^3 & 0 & 2\epsilon-20\epsilon^3 & 0 \\ -\epsilon^2 & 0 & 2\epsilon-8\epsilon^3 & 0 & \frac{1}{3}+\frac{7}{3}\epsilon^2 & 0 & 2\epsilon-20\epsilon^3 \\ 0 & -6\epsilon^2+6\epsilon^3 & 0 & 2\epsilon-20\epsilon^3 & 0 & 1+37\epsilon^2 & 0 \\ 6\epsilon^3 & 0 & -6\epsilon^2 & 0 & 2\epsilon-20\epsilon^3 & 0 & 2+51\epsilon^2 \end{bmatrix}$$

Zero eigenvalue of $L_+^{(0)}$ is double and is associated with the subspace spanned by $\{e_2, e_3\}$. There are two invariant subspaces of \hat{L}_+ , one is spanned by $\{e_0, e_2, e_4, e_6\}$ and the other one is spanned by $\{e_1, e_3, e_5\}$. For the subspace spanned by $\{e_0, e_2, e_4, e_6\}$, the eigenvalue problem is given by

$$\begin{cases} (2 - 4\epsilon^2)x_0 + (4\epsilon - 10\epsilon^3)x_2 - 11\epsilon^2x_4 + 6\epsilon^3x_6 = \lambda x_0, \\ (4\epsilon - 10\epsilon^3)x_0 + 6\epsilon^2x_2 + (2\epsilon - 32\epsilon^3)x_4 - 6\epsilon^2x_6 = \lambda x_2, \\ -11\epsilon^2x_0 + (2\epsilon - 32\epsilon^3)x_2 + (\frac{1}{3} + \frac{76}{3}\epsilon^2)x_4 + (2\epsilon - 20\epsilon^3)x_6 = \lambda x_4, \\ 6\epsilon^3x_0 - 6\epsilon^2x_2 + (2\epsilon - 20\epsilon^3)x_4 + (2 + 51\epsilon^2)x_6 = \lambda x_6. \end{cases}$$

The small eigenvalue for small ϵ is obtained by normalization $x_2 = 1$. Then, we obtain

$$x_0 = -2\epsilon + \mathcal{O}(\epsilon^3), \quad x_4 = -6\epsilon + \mathcal{O}(\epsilon^3), \quad x_6 = 9\epsilon^2 + \mathcal{O}(\epsilon^4),$$

and

$$\lambda = -14\epsilon^2 + \mathcal{O}(\epsilon^4). \quad (3.26)$$

For the subspace spanned by $\{e_1, e_3, e_5\}$, the eigenvalue problem is given by

$$\begin{cases} (\frac{1}{3} + 2\epsilon + \frac{7}{3}\epsilon^2 - 2\epsilon^3)x_1 + (2\epsilon - 4\epsilon^2 - 14\epsilon^3)x_3 - (6\epsilon^2 + 6\epsilon^3)x_5 = \lambda x_1, \\ (2\epsilon - 4\epsilon^2 - 14\epsilon^3)x_1 + (16\epsilon^2 - 12\epsilon^3)x_3 + (2\epsilon - 20\epsilon^3)x_5 = \lambda x_3, \\ -(6\epsilon^2 + 6\epsilon^3)x_1 + (2\epsilon - 20\epsilon^3)x_3 + (1 + 37\epsilon^2)x_5 = \lambda x_5. \end{cases}$$

The small eigenvalue for small ϵ is obtained by normalization $x_3 = 1$. Then, we obtain

$$x_1 = -6\epsilon + 48\epsilon^2 + \mathcal{O}(\epsilon^3), \quad x_5 = -2\epsilon + \mathcal{O}(\epsilon^3),$$

and

$$\lambda = 108\epsilon^3 + \mathcal{O}(\epsilon^4). \quad (3.27)$$

By the perturbation theory, for every $\epsilon \neq 0$ sufficiently small, L_+ has one simple small negative eigenvalue and one simple small positive eigenvalue. Other eigenvalues are bounded away from zero for small ϵ and by Lemma 2, all other eigenvalues of L_+ are strictly positive. Hence, $n(L_+) = 1$.

Zero eigenvalue of $L_-^{(0)}$ is triple and is associated with the subspace spanned by $\{e_0, e_2, e_3\}$. For every $\epsilon \neq 0$, a double zero eigenvalue of L_- exists due to the two symmetries (2.2) and (2.3). The two eigenvectors for the double zero eigenvalue of L_- are spanned by $\{e_0, e_2, e_4, e_6, \dots\}$. There are two invariant subspaces of \hat{L}_- , one is spanned by $\{e_0, e_2, e_4, e_6\}$ and the other one is spanned by $\{e_1, e_3, e_5\}$. Since we only need to compute a shift of the zero eigenvalue of $L_-^{(0)}$, we only consider the subspace of \hat{L}_- spanned by $\{e_1, e_3, e_5\}$. For this subspace, the eigenvalue problem for \hat{L}_- is given by

$$\begin{cases} (\frac{1}{3} - 2\epsilon + \frac{7}{3}\epsilon^2 + 2\epsilon^3)x_1 + (2\epsilon + 4\epsilon^2 - 14\epsilon^3)x_3 + (-6\epsilon^2 + 6\epsilon^3)x_5 = \lambda x_1, \\ (2\epsilon + 4\epsilon^2 - 14\epsilon^3)x_1 + (16\epsilon^2 + 12\epsilon^3)x_3 + (2\epsilon - 20\epsilon^3)x_5 = \lambda x_3, \\ (-6\epsilon^2 + 6\epsilon^3)x_1 + (2\epsilon - 20\epsilon^3)x_3 + (1 + 37\epsilon^2)x_5 = \lambda x_5. \end{cases}$$

The small eigenvalue for small ϵ is obtained by normalization $x_3 = 1$. Then, we obtain

$$x_1 = -6\epsilon - 48\epsilon^2 + \mathcal{O}(\epsilon^3), \quad x_5 = -2\epsilon + \mathcal{O}(\epsilon^3),$$

and

$$\lambda = -108\epsilon^3 + \mathcal{O}(\epsilon^4). \quad (3.28)$$

By the perturbation theory, for every $\epsilon \neq 0$ sufficiently small, L_- has one simple (small) negative eigenvalue and the double zero eigenvalue. Other eigenvalues are

bounded away from zero for small ϵ and by Lemma 2, all other eigenvalues of L_- are strictly positive. Hence, $n(L_-) = 1$.

Case (iii). $\epsilon < 0$, $\mu = 2|\epsilon|^{3/2} + \mathcal{O}(\epsilon^2)$. Here we introduce $\delta := (-\epsilon)^{1/2}$ and write $\Omega = \frac{7}{6}\delta^4 + \frac{28}{3}\delta^6 + \mathcal{O}(\delta^8)$, $b_0 = -\delta^4 - 4\delta^6 + \mathcal{O}(\delta^8)$, $b_1 = 12\delta^5 + \mathcal{O}(\delta^7)$, $b_4 = -3\delta^4 + \mathcal{O}(\delta^8)$, $b_5 = 4\delta^5 + \mathcal{O}(\delta^8)$, $b_6 = -5\delta^6 + \mathcal{O}(\delta^8)$, and $b_n = \mathcal{O}(\delta^7)$ for $n \geq 7$. Substituting these expansions in (3.22), we obtain

$$L_{\pm} = L_{\pm}^{(0)} + \delta^2 L_{\pm}^{(2)} + \delta^3 L_{\pm}^{(3)} + \delta^4 L_{\pm}^{(4)} + \delta^5 L_{\pm}^{(5)} + \delta^6 L_{\pm}^{(6)} + \mathcal{O}(\delta^7),$$

where

$$\begin{aligned} (L_{\pm}^{(0)} a)_n &= \frac{1}{6}(n-2)(n-3)a_n \pm a_{-n}, \\ (L_{\pm}^{(2)} a)_n &= -2(a_{n+2} + a_{n-2}) \mp 2a_{2-n}, \\ (L_{\pm}^{(3)} a)_n &= -4(a_{n+3} + a_{n-3}) \mp 4a_{3-n}, \\ (L_{\pm}^{(4)} a)_n &= \frac{7}{6}n(n+1)a_n + 2(\min(2, n) - 1)a_n - 6(a_{n+4} + a_{n-4}) \\ &\quad \mp 2a_{-n} \pm (\min(2, n, 4-n) - 5)a_{4-n}, \\ (L_{\pm}^{(5)} a)_n &= -20(a_{n+1} + a_{n-1}) - 8(a_{n+5} + a_{n-5}) + 4\min(2, n-1)a_{n-1} \\ &\quad + 4\min(n, 2)a_{n+1} \mp 24a_{1-n} \pm 4(\min(2, n, 5-n) - 1)a_{5-n}, \\ (L_{\pm}^{(6)} a)_n &= \frac{28}{3}n(n+1)a_n + 8(a_{n+2} + a_{n-2}) - 10(a_{n+6} + a_{n-6}) \\ &\quad + 6\min(2, n-2)a_{n-2} \\ &\quad + 6\min(2, n)a_{n+2} + 8(\min(3, n) - 1)a_n \mp 8a_{-n} \pm 2a_{2-n} \\ &\quad \pm 4\min(3, n, 6-n)a_{6-n} \pm 6\min(2, n, 6-n)a_{6-n}. \end{aligned}$$

Let us represent the first 7-by-7 matrix block of the operator L_{\pm} and truncate it by up to and including $\mathcal{O}(\delta^6)$ terms. The corresponding matrix blocks denoted by \hat{L}_+ and \hat{L}_- are given respectively by

$$\hat{L}_+ = \begin{bmatrix} 2 - 4\delta^4 - 16\delta^6 & -44\delta^5 & -4\delta^2 + 10\delta^6 & -8\delta^3 & & & \\ -44\delta^5 & \frac{1}{3} - 2\delta^2 + \frac{7}{3}\delta^4 + \frac{62}{3}\delta^6 & -4\delta^3 - 16\delta^5 & -2\delta^2 - 4\delta^4 + 14\delta^6 & & & \\ -4\delta^2 + 10\delta^6 & -4\delta^3 - 16\delta^5 & 6\delta^4 + 64\delta^6 & -8\delta^5 & & & \\ -8\delta^3 & -2\delta^2 - 4\delta^4 + 14\delta^6 & -8\delta^5 & 16\delta^4 + 152\delta^6 & & & \\ -11\delta^4 & -4\delta^3 & -2\delta^2 + 40\delta^6 & -12\delta^5 & & & \\ -12\delta^5 & -6\delta^4 + 10\delta^6 & -4\delta^3 & -2\delta^2 + 20\delta^6 & & & \\ -10\delta^6 & -8\delta^5 & -6\delta^4 & -4\delta^3 & & & \\ & & & & & & \\ & -11\delta^4 & -12\delta^5 & -10\delta^6 & & & \\ & -4\delta^3 & -6\delta^4 + 10\delta^6 & -8\delta^5 & & & \\ & -2\delta^2 + 40\delta^6 & -4\delta^3 & -6\delta^4 & & & \\ & -12\delta^5 & -2\delta^2 + 20\delta^6 & -4\delta^3 & & & \\ \frac{1}{3} + \frac{76}{3}\delta^4 + \frac{608}{3}\delta^6 & -12\delta^5 & -2\delta^2 + 20\delta^6 & & & & \\ -12\delta^5 & 1 + 37\delta^4 + 296\delta^6 & -12\delta^5 & & & & \\ -2\delta^2 + 20\delta^6 & -12\delta^5 & 2 + 51\delta^2 + 408\delta^3 & & & & \end{bmatrix}$$

and

$$\hat{L}_- = \begin{bmatrix} 20 & 4\delta^5 & 6\delta^6 & 0 \\ 4\delta^5 & \frac{1}{3} + 2\delta^2 + \frac{7}{3}\delta^4 + \frac{50}{3}\delta^6 & 4\delta^3 - 16\delta^5 & -2\delta^2 + 4\delta^4 + 14\delta^6 \\ 6\delta^6 & 4\delta^3 - 16\delta^5 & 12\delta^4 + 64\delta^6 & -16\delta^5 \\ 0 & -2\delta^2 + 4\delta^4 + 14\delta^6 & -16\delta^5 & 16\delta^4 + 104\delta^6 \\ -\delta^4 & -4\delta^3 & -2\delta^2 & -12\delta^5 \\ -4\delta^5 & -6\delta^4 - 10\delta^6 & -4\delta^3 & -2\delta^2 + 20\delta^6 \\ -10\delta^6 & -8\delta^5 & -6\delta^4 & -4\delta^3 \end{bmatrix}$$

$$\begin{bmatrix} -\delta^4 & -4\delta^5 & -10\delta^6 \\ -4\delta^3 & -6\delta^4 - 10\delta^6 & -8\delta^5 \\ -2\delta^2 & -4\delta^3 & -6\delta^4 \\ -12\delta^5 & -2\delta^2 + 20\delta^6 & -4\delta^3 \\ \frac{1}{3} + \frac{76}{3}\delta^4 + \frac{608}{3}\delta^6 & -12\delta^5 & -2\delta^2 + 20\delta^6 \\ -12\delta^5 & 1 + 37\delta^4 + 296\delta^6 & -12\delta^5 \\ -2\delta^2 + 20\delta^6 & -12\delta^5 & 2 + 51\delta^2 + 408\delta^3 \end{bmatrix}$$

Zero eigenvalue of $L_+^{(0)}$ is double and is associated with the subspace spanned by $\{e_2, e_3\}$. Since no invariant subspaces of \hat{L}_+ exist, we have to proceed with full perturbative expansions. As a first step, we express $(x_0, x_1, x_4, x_5, x_6)$ for the subspace spanned by $\{e_0, e_1, e_4, e_5, e_6\}$ in terms of $\{x_2, x_3\}$ for the subspace spanned by $\{e_2, e_3\}$, λ , and δ . We assume that $\lambda = \mathcal{O}(\delta^4)$ for the small eigenvalues and neglect terms of the order $\mathcal{O}(\delta^7)$ and higher. This expansion is given by

$$\begin{cases} x_0 = (2\delta^2 + 32\delta^6 + \lambda\delta^2)x_2 + 4\delta^3x_3, \\ x_1 = (12\delta^3 + 192\delta^5)x_2 + (6\delta^2 + 48\delta^4 + 240\delta^6 + 18\lambda\delta^2)x_3, \\ x_4 = (6\delta^2 - 312\delta^6 + 18\lambda\delta^2)x_2 + 120\delta^5x_3, \\ x_5 = 4\delta^3x_2 + (2\delta^2 - 58\delta^6 + 2\lambda\delta^2)x_3, \\ x_6 = (9\delta^4 - \frac{459}{2}\delta^6)x_2 + (2\delta^3 - 51\delta^5 - 408\delta^6)x_3. \end{cases}$$

Next, we substitute these expansions to the third and fourth equations of the eigenvalue problem for \hat{L}_+ and again neglect terms of the order $\mathcal{O}(\delta^7)$ and higher:

$$\begin{cases} (-14\delta^4 - \lambda)x_2 - 56\delta^5x_3 = 0, \\ -56\delta^5x_2 + (-8\delta^6 - \lambda)x_3 = 0. \end{cases}$$

The reduced eigenvalue problem has two eigenvalues with the expansions

$$\lambda = -14\delta^4 + \mathcal{O}(\delta^6) \quad \text{when} \quad x_3 = (4\delta + \mathcal{O}(\delta^3))x_2 \quad (3.29)$$

and

$$\lambda = 216\delta^6 + \mathcal{O}(\delta^8) \quad \text{when} \quad x_2 = (-4\delta + \mathcal{O}(\delta^3))x_3. \quad (3.30)$$

By the perturbation theory, for every $\delta \neq 0$ sufficiently small, L_+ has one simple small negative eigenvalue and one simple small positive eigenvalue. Other eigenvalues are bounded away from zero for small δ and by Lemma 2, all other eigenvalues of L_+ are strictly positive. Hence, $n(L_+) = 1$.

Zero eigenvalue of $L_-^{(0)}$ is triple and is associated with the subspace spanned by $\{e_0, e_2, e_3\}$. We proceed again with full perturbative expansions. First, we express (x_1, x_4, x_5, x_6) for the subspace spanned by $\{e_1, e_4, e_5, e_6\}$ in terms of $\{x_0, x_2, x_3\}$ for the subspace spanned by $\{e_0, e_2, e_3\}$, λ , and δ . We assume that $\lambda = \mathcal{O}(\delta^6)$

for the small eigenvalues and neglect terms of the order $\mathcal{O}(\delta^7)$ and higher. This expansion is given by

$$\begin{cases} x_1 = -12\delta^5 x_0 + (-12\delta^3 + 192\delta^5)x_2 + (6\delta^2 - 48\delta^4 + 240\delta^6)x_3, \\ x_4 = 3\delta^4 x_0 + (6\delta^2 - 546\delta^6)x_2 + 120\delta^5 x_3, \\ x_5 = 4\delta^5 x_0 + 4\delta^3 x_2 + (2\delta^2 - 58\delta^6)x_3, \\ x_6 = 8\delta^6 x_0 + 9\delta^4 x_2 + 2\delta^3 x_3. \end{cases}$$

Next, we substitute these expansions to the first, third and fourth equations of the eigenvalue problem for \tilde{L}_- and again neglect terms of the order $\mathcal{O}(\delta^7)$ and higher:

$$\begin{cases} \lambda x_0 = 0, \\ \lambda x_2 = 0, \\ (216\delta^6 - \lambda)x_3 = 0. \end{cases}$$

The double zero eigenvalue persists due to two gauge symmetries, whereas one eigenvalue is expanded by

$$\lambda = 216\delta^6 + \mathcal{O}(\delta^8). \quad (3.31)$$

By the perturbation theory, for every $\delta \neq 0$ sufficiently small, L_- has one simple (small) negative eigenvalue and the double zero eigenvalue. Other eigenvalues are bounded away from zero for small δ and by Lemma 2, all other eigenvalues of L_- are strictly positive. Hence, $n(L_-) = 0$. \square

3.3. Change of negative eigenvalues due to two constraints. We consider the two constraints related to the fixed values of Q and E . The constraints may change the number of negative eigenvalues of the linearization operator L_+ constrained by the following two orthogonality conditions

$$[X_c]^\perp := \{a \in \ell^2(\mathbb{N}) : \langle MA, a \rangle = \langle M^2 A, a \rangle = 0\}, \quad (3.32)$$

where A denotes a real-valued solution of system (2.10) and $M = \text{diag}(1, 2, 3, \dots)$. The constrained space $[X_c]^\perp$ is a *symplectically orthogonal* subspace of $\ell^2(\mathbb{N})$ to $X_0 = \text{span}\{A, MA\} \subset \ell^2(\mathbb{N})$, the two-dimensional subspace associated with the double zero eigenvalue of L_- related to the phase rotation symmetries (2.2) and (2.3). Alternatively, the constrained space arises when the perturbation a does not change at the linear approximation the conserved quantities Q and E defined by (2.4) and (2.5). Note that the constraints in (3.32) are only imposed on the real part of the perturbation a .

Let \tilde{A} denote the stationary state of the stationary equation (2.10) continued with respect to two parameters (λ, ω) . Let $n(L_+)$ and $z(L_+)$ denote the number of negative and zero eigenvalues of L_+ in $\ell^2(\mathbb{N})$ counted with their multiplicities, where L_+ is the linearized operator at \tilde{A} . Let $n_c(L_+)$ and $z_c(L_+)$ denote the number of negative and zero eigenvalues of L_+ constrained in $[X_c]^\perp$. Assume non-degeneracy of the stationary state \tilde{A} in the sense that $z(L_+) = 0$. By Theorem 4.1 in [14], we have

$$n_c(L_+) = n(L_+) - p(D) - z(D), \quad z_c(L_+) = z(D), \quad (3.33)$$

where $p(D)$ and $z(D)$ are the number of positive and zero eigenvalues of the 2×2 matrix

$$D := \begin{bmatrix} \frac{\partial Q}{\partial \lambda} & \frac{\partial Q}{\partial \omega} \\ \frac{\partial(Q-\mathcal{E})}{\partial \lambda} & \frac{\partial(Q-\mathcal{E})}{\partial \omega} \end{bmatrix}, \quad (3.34)$$

with $Q(\lambda, \omega) = Q(\tilde{A})$ and $\mathcal{E}(\lambda, \omega) = E(\tilde{A})$ evaluated at the stationary solution \tilde{A} as a function of the two parameters (λ, ω) .

Remark 9. For the pair of stationary states (1.4), it was computed in [3] that

$$\mathcal{Q}(\lambda, \omega) = \frac{6}{7}(\lambda + \omega), \quad \mathcal{E}(\lambda, \omega) = 6\omega, \quad (3.35)$$

where λ and ω are related to the parameters c and p in (1.5) and (1.6). Substituting (3.35) into (3.34) yields

$$D = \begin{bmatrix} 6/7 & 6/7 \\ 6/7 & -36/7 \end{bmatrix}, \quad (3.36)$$

hence, D has one positive and one negative eigenvalue. If $z(L_+) = 0$ holds, then $n_c(L_+) = n(L_+) - 1$ and $z_c(L_+) = 0$ by (3.33). We will show in Lemma 5 below that the same count is true for all three branches of Theorem 2 bifurcating from ω_* .

By the scaling transformation (2.1), if the stationary state is given by (2.9) with real A , then the stationary state is continued with respect to parameter $c > 0$ as

$$\tilde{\alpha}_n(t) = cA_n e^{-ic^2\lambda t + inc^2\omega t},$$

hence $\tilde{A} = cA$, $\tilde{\lambda} = c^2\lambda$, and $\tilde{\omega} = c^2\omega$. Substituting these relations into $\tilde{\mathcal{Q}}(\tilde{\lambda}, \tilde{\omega}) = \mathcal{Q}(\tilde{A})$ and $\tilde{\mathcal{E}}(\tilde{\lambda}, \tilde{\omega}) = \mathcal{E}(\tilde{A})$ for $\lambda = 1$ yields

$$\tilde{\mathcal{Q}}(\tilde{\lambda}, \tilde{\omega}) = c^2 \mathcal{Q}_0(\omega) = \tilde{\lambda} \mathcal{Q}_0(\tilde{\omega} \tilde{\lambda}^{-1})$$

and

$$\tilde{\mathcal{E}}(\tilde{\lambda}, \tilde{\omega}) = c^2 \mathcal{E}_0(\omega) = \tilde{\lambda} \mathcal{E}_0(\tilde{\omega} \tilde{\lambda}^{-1}),$$

where $\mathcal{Q}_0(\omega) = \mathcal{Q}(1, \omega)$ and $\mathcal{E}_0(\omega) = \mathcal{E}(1, \omega)$. Substituting these representations into (3.34), evaluating derivatives, and setting $c = 1$ yield the computational formula

$$D = \begin{bmatrix} \mathcal{Q}_0(\omega) - \omega \mathcal{Q}'_0(\omega) & \mathcal{Q}'_0(\omega) \\ \mathcal{Q}_0(\omega) - \mathcal{E}_0(\omega) - \omega [\mathcal{Q}'_0(\omega) - \mathcal{E}'_0(\omega)] & \mathcal{Q}'_0(\omega) - \mathcal{E}'_0(\omega) \end{bmatrix}, \quad (3.37)$$

which can be used to compute D for the normalized stationary state A with $\lambda = 1$. The following lemma gives the variational characterization of the three branches in Theorem 2 bifurcating from ω_* as critical points of H subject to fixed Q and E .

Lemma 5. *Consider the three bifurcating branches in Theorem 2 for $\omega_* := \omega_2 = \omega_3 = 1/6$. For every $\omega \neq \omega_*$ with $|\omega - \omega_*|$ sufficiently small, the following is true:*

- (i) *The branch is a saddle point of H subject to fixed Q and E with $n_c(L_+) = 1$ and $n(L_-) = 1$;*
- (ii) *The branch is a saddle point of H subject to fixed Q and E with $n_c(L_+) = 0$ and $n(L_-) = 1$;*
- (iii) *The branch is a local minimizer of H subject to fixed Q and E with $n_c(L_+) = 0$ and $n(L_-) = 0$.*

The critical points are degenerate only with respect to the two phase rotations (2.2) and (2.3) resulting in $z(L_+) = 0$ and $z(L_-) = 2$.

Proof. Case (i). $\epsilon = 0$, $\mu \neq 0$. We compute $\mathcal{Q}_0(\omega)$ and $\mathcal{E}_0(\omega)$ as powers of μ with the following relation between ω and μ :

$$\omega = \frac{1}{6} + \frac{1}{12}\mu^2 + \mathcal{O}(\mu^4).$$

Then it follows that

$$\begin{cases} \mathcal{Q}_0(\omega) = A_0^2 + 4A_3^2 + 7A_6^2 + \dots = 1 + 2\mu^2 + \mathcal{O}(\mu^4), \\ \mathcal{E}_0(\omega) = A_0^2 + 4^2A_3^2 + 7^2A_6^2 + \dots = 1 + 14\mu^2 + \mathcal{O}(\mu^4), \end{cases}$$

so that

$$D = \begin{bmatrix} -3 + \mathcal{O}(\mu^2) & 24 + \mathcal{O}(\mu^2) \\ 24 + \mathcal{O}(\mu^2) & -144 + \mathcal{O}(\mu^2) \end{bmatrix}$$

has one positive and one negative eigenvalue. Since $n(L_+) = 2$ and $z(L_+) = 0$ by Lemma 4, we have $n_c(L_+) = n(L_+) - 1 = 1$ by (3.33). At the same time, $n(L_-) = 1$ and $z(L_-) = 2$ by Lemma 4.

Case (ii). $\epsilon \neq 0$, $\mu = 0$. We compute $\mathcal{Q}_0(\omega)$ and $\mathcal{E}_0(\omega)$ as powers of ϵ with the following relation between ω and ϵ :

$$\omega = \frac{1}{6} + \frac{7}{6}\epsilon^2 + \mathcal{O}(\epsilon^4).$$

Then it follows that

$$\begin{cases} \mathcal{Q}_0(\omega) = A_0^2 + 3A_2^2 + 5A_4^2 + \dots = 1 + \epsilon^2 + \mathcal{O}(\epsilon^4), \\ \mathcal{E}_0(\omega) = A_0^2 + 3^2A_2^2 + 5^2A_4^2 + \dots = 1 + 7\epsilon^2 + \mathcal{O}(\epsilon^4), \end{cases}$$

so that

$$D = \begin{bmatrix} 6/7 + \mathcal{O}(\epsilon^2) & 6/7 + \mathcal{O}(\epsilon^2) \\ 6/7 + \mathcal{O}(\epsilon^2) & -36/7 + \mathcal{O}(\epsilon^2) \end{bmatrix}$$

has one positive and one negative eigenvalue. Since $n(L_+) = 1$ and $z(L_+) = 0$ by Lemma 4, we have $n_c(L_+) = n(L_+) - 1 = 0$ by (3.33). At the same time, $n(L_-) = 1$ and $z(L_-) = 2$ by Lemma 4.

Case (iii). $\epsilon < 0$, $\mu = 2|\epsilon|^{3/2} + \mathcal{O}(\epsilon^2)$. We compute $\mathcal{Q}_0(\omega)$ and $\mathcal{E}_0(\omega)$ as powers of ϵ with the following relation between ω and ϵ :

$$\omega = \frac{1}{6} + \frac{7}{6}\epsilon^2 - \frac{28}{3}\epsilon^3 + \mathcal{O}(\epsilon^4).$$

Then it follows that

$$\begin{cases} \mathcal{Q}_0(\omega) = A_0^2 + 2A_1^2 + 3A_2^2 + 4A_3^2 + 5A_4^2 + \dots = 1 + \epsilon^2 + \mathcal{O}(\epsilon^3), \\ \mathcal{E}_0(\omega) = A_0^2 + 2^2A_1^2 + 3^2A_2^2 + 4^2A_3^2 + 5^2A_4^2 + \dots = 1 + 7\epsilon^2 + \mathcal{O}(\epsilon^3). \end{cases}$$

The only difference in these expansions compared to the case (ii) is the remainder term as large as $\mathcal{O}(\epsilon^3)$ compared to $\mathcal{O}(\epsilon^4)$. This changes the remainder terms in D to $\mathcal{O}(\epsilon)$ compared to $\mathcal{O}(\epsilon^2)$ but does not affect the conclusion on D . Moreover, we can see that the computation of D agrees with the exact expression (3.36) in Remark 9. The count $n_c(L_+) = n(L_+) - 1 = 0$, $n(L_-) = 0$, $z(L_+) = 0$, and $z(L_-) = 2$ follows by Lemma 4. \square

Remark 10. The presence of conserved quantity $Z(\alpha)$ in (2.6) does not modify the variational characterization of the stationary states with $\omega \neq 0$ because $Z(\alpha) = 0$ if α is the stationary state (2.9) with $\omega \neq 0$. This follows from the fact that $Z(\alpha)$ is independent of t , which is impossible if $Z(\alpha) \neq 0$ and $\omega \neq 0$. Hence, any stationary state (2.9) must satisfy the constraint:

$$\omega \neq 0 : \quad \sum_{n=0}^{\infty} (n+1)(n+2)A_n A_{n+1} = 0,$$

which can be verified for all states in Theorem 1 and 2 bifurcating from ω_m for $m \geq 2$.

4. Bifurcation from the second eigenmode. Here we study bifurcations of stationary states in the system of algebraic equations (3.1) from the second eigenmode given by (2.14) with $N = 1$. Without loss of generality, the scaling transformation (2.1) yields $\lambda - \omega = 1$.

4.1. Classification of the bifurcating states. By setting $A_n = \delta_{n1} + a_n$ with real-valued perturbation a , we rewrite the system (3.1) with $\lambda = 1 + \omega$ in the perturbative form (3.2), where $L(\omega)$ is a block-diagonal operator with the diagonal entries

$$[L(\omega)]_{nn} = \begin{cases} 1 - \omega, & n = 0, \\ 4, & n = 1, \\ 1 + 3\omega, & n = 2, \\ (n^2 - 1)\omega - n + 3, & n \geq 3, \end{cases} \quad (4.1)$$

and the only nonzero off-diagonal entries $[L(\omega)]_{02} = [L(\omega)]_{20} = 1$, whereas the nonlinear terms are given by

$$\begin{aligned} [N(a)]_n &= 2 \sum_{j=0}^{\infty} S_{1nj, n+j-1, 1} a_j a_{n+j-1} + \sum_{k=0}^{n+1} S_{1nk, n+1-k} a_k a_{n+1-k} \\ &\quad + \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{njk, n+j-k} a_j a_k a_{n+j-k}. \end{aligned}$$

We have the following result on the nonlinear terms.

Lemma 6. *Fix an integer $m \geq 2$. If $a_0 = 0$ and*

$$a_{m\ell+2} = a_{m\ell+3} = \cdots = a_{m\ell+m} = 0, \quad \text{for every } \ell \in \mathbb{N}, \quad (4.2)$$

then $[N(a)]_0 = 0$ and

$$[N(a)]_{m\ell+2} = [N(a)]_{m\ell+3} = \cdots = [N(a)]_{m\ell+m} = 0, \quad \text{for every } \ell \in \mathbb{N}. \quad (4.3)$$

Proof. The argument repeats the proof of Lemma 1. Under the conditions (4.2), every term in $[N(a)]_n$ for $n = m\ell + \iota$ with $\ell \in \mathbb{N}$, and $\iota \in \{2, 3, \dots, m\}$ is inspected and shown to be zero. \square

Bifurcations from the second eigenmode are identified by zero eigenvalues of the diagonal operator $L(\omega)$.

Lemma 7. *There exists a sequence of bifurcations at $\omega \in \{\omega_m\}_{m \in \mathbb{N}_+}$ with $\omega_1 = 0$, $\omega_2 = 2/3$, and*

$$\omega_m = \frac{m-3}{m^2-1}, \quad m \in \{3, 4, \dots\}. \quad (4.4)$$

All bifurcation points are simple except for the three double points $\omega_1 = \omega_3 = 0$, $\omega_4 = \omega_{11} = 1/15$, and $\omega_5 = \omega_7 = 1/12$.

Proof. The diagonal terms $[L(\omega)]_{nn}$ for $n \in \{3, 4, \dots\}$ vanish at the sequence (4.4). In addition, the double block for $n = 0$ and $n = 2$ given by

$$\begin{bmatrix} 1 - \omega & 1 \\ 1 & 1 + 3\omega \end{bmatrix}$$

has zero eigenvalues if and only if $\omega_1 = 0$ or $\omega_2 = 2/3$. Therefore, $\omega_1 = \omega_3 = 0$ is a double bifurcation point. To study other double bifurcation points, we consider solutions of $\omega_m = \omega_2 = 2/3$ for $m \geq 3$ and $\omega_n = \omega_m$ for $n \neq m \geq 4$. Equation $\omega_m = \omega_2 = 2/3$ is equivalent to $2m^2 - 3m + 7 = 0$, which has no integer solutions.

Equation $\omega_n = \omega_m$ for $n \neq m$ is equivalent to $mn = 3(m+n) - 1$, which can be solved for m in terms of n

$$m = M(n) := \frac{3n-1}{n-3} = 3 + \frac{8}{n-3}.$$

Since the right-hand side is monotonically decreasing in n and $M(12) < 4$, we can find all integer solutions for n in the range from 4 to 11. There exists only two pairs of integer solutions in this range, which give two double points $\omega_4 = \omega_{11} = 1/15$ and $\omega_5 = \omega_7 = 1/12$. All the remaining bifurcation points are simple. \square

Simple bifurcation points can be investigated similarly to the proof of Theorem 1. This yields the following theorem.

Theorem 3. *Fix $m = 2$. There exists a unique branch of solutions $(\omega, A) \in \mathbb{R} \times \ell^2(\mathbb{N})$ to system (3.1) with $\lambda - \omega = 1$, which can be parameterized by small ϵ such that (ω, A) is smooth in ϵ and*

$$|\omega - \omega_2| + \sup_{n \in \mathbb{N}} |A_n - \delta_{n1} - \epsilon(3\delta_{n0} - \delta_{n2})| \lesssim \epsilon^2. \quad (4.5)$$

Fix an integer $m \geq 6$ with $m \neq 7$ and $m \neq 11$. There exists a unique branch of solutions $(\omega, A) \in \mathbb{R} \times \ell^2(\mathbb{N})$ to system (3.1) with $\lambda - \omega = 1$, which can be parameterized by small ϵ such that (ω, A) is smooth in ϵ and

$$|\omega - \omega_m| + \sup_{n \in \mathbb{N}} |A_n - \delta_{n1} - \epsilon\delta_{nm}| \lesssim \epsilon^2. \quad (4.6)$$

Proof. The proof of the second assertion repeats the proof of Theorem 1 verbatim. The proof of the first assertion is based on the block-diagonalization of the singular matrix for $[L(\omega_2)]_{jk}$ with $j, k \in \{0, 2\}$:

$$\begin{bmatrix} 1/3 & 1 \\ 1 & 3 \end{bmatrix}.$$

The null space is spanned by the vector $(3, -1)^T$ and the vector $b \in \ell^2(\mathbb{N})$ in the decomposition (3.9) must satisfy the constraint $3b_0 - b_2 = 0$. The rest of the proof repeats the proof of Theorem 1 after a simple observation that $[N(\epsilon(3e_0 - e_2))]_{0,2} = \mathcal{O}(\epsilon^3)$ as $\epsilon \rightarrow 0$. \square

Remark 11. The unique branch bifurcating from $\omega_2 = 2/3$ coincides with the exact solution (2.23) for the lower sign. Indeed, by normalizing $\lambda - \omega = 1$ and taking the limit $p \rightarrow 0$ as in (2.27), we obtain $c = p^{-1} + \mathcal{O}(p)$, $\beta = -6p + \mathcal{O}(p^3)$, and $\gamma = p^{-1} + \mathcal{O}(p)$. The small parameter ϵ is defined in terms of the small parameter p by $\epsilon := -2p + \mathcal{O}(p^3)$.

Remark 12. For the unique branch bifurcating from ω_m with $m \geq 6$, $m \neq 7$, and $m \neq 11$, Lemma 6 implies that $A_n \neq 0$, $n \in \mathbb{N}$ if and only if $n - 1$ is a multiple of $m - 1$, see Remark 2.

The three double bifurcation points in Lemma 7 have to be checked separately. In order to characterize branches bifurcating from the double point $\omega_5 = \omega_7 = 1/12$, we note the following symmetry. If $u(t, z)$ is a generating function for the conformal flow (1.1) given by the power series (2.28), so is $zu(t, z^2)$. It follows from this symmetry that if $\{A_n\}_{n \in \mathbb{N}}$ satisfies the system (3.1) with parameters (λ, ω) , then the transformed state

$$\tilde{A}_n = \begin{cases} A_m, & n = 2m + 1, \\ 0, & n = 2m, \end{cases} \quad n \in \mathbb{N} \quad (4.7)$$

also satisfies the system (3.1) with parameters

$$\tilde{\lambda} = \lambda + \frac{\omega}{2}, \quad \tilde{\omega} = \frac{\omega}{2}.$$

By Theorem 2, three branches of solutions bifurcate from the lowest eigenmode at the double bifurcation point $\omega_2 = \omega_3 = 1/6$. Applying the transformation (4.7) yields three branches bifurcating from the second eigenmode at the double bifurcation point $\omega_5 = \omega_7 = 1/12$. By computing the normal form similar to the proof of Theorem 2, we have checked that no other solutions bifurcate from this double point. The corresponding computations are omitted here, while the result is formulated in the following theorem.

Theorem 4. *Fix $\omega_* := \omega_5 = \omega_7 = 1/12$. There exist exactly three branches of solutions $(\omega, A) \in \mathbb{R} \times \ell^2(\mathbb{N})$ to system (3.1) with $\lambda - \omega = 1$, which can be parameterized by small (ϵ, μ) such that (ω, A) is smooth in (ϵ, μ) and*

$$|\omega - \omega_*| + \sup_{n \in \mathbb{N}} |A_n - \delta_{n1} - \epsilon \delta_{n5} - \mu \delta_{n7}| \lesssim (\epsilon^2 + \mu^2). \quad (4.8)$$

The three branches are characterized by the following:

- (i) $\epsilon = 0, \mu \neq 0$;
- (ii) $\epsilon \neq 0, \mu = 0$;
- (iii) $\epsilon < 0, |\mu - 2|\epsilon|^{3/2}| \lesssim \epsilon^2$,

and the branch (iii) is double degenerate up to the reflection $\mu \mapsto -\mu$.

Branches bifurcating from the double point $\omega_4 = \omega_{11} = 1/15$ can be investigated by computing the normal form. The following theorem represents the main result.

Theorem 5. *Fix $\omega_* := \omega_4 = \omega_{11} = 1/15$. There exist exactly two branches of solutions $(\omega, A) \in \mathbb{R} \times \ell^2(\mathbb{N})$ to system (3.1) with $\lambda - \omega = 1$, which can be parameterized by small (ϵ, μ) such that (ω, A) is smooth in (ϵ, μ) and*

$$|\omega - \omega_*| + \sup_{n \in \mathbb{N}} |A_n - \delta_{n1} - \epsilon \delta_{n4} - \mu \delta_{n11}| \lesssim (\epsilon^2 + \mu^2). \quad (4.9)$$

The two branches are characterized by the following:

- (i) $\epsilon = 0, \mu \neq 0$;
- (ii) $\epsilon \neq 0, \mu = 0$.

Proof. The proof follows the computations of normal form in Theorem 2. For the double bifurcation point ω_* , we write the decomposition

$$\omega = \omega_* + \Omega, \quad a_n = \epsilon \delta_{n4} + \mu \delta_{n11} + b_n, \quad n \in \mathbb{N},$$

where (ϵ, μ) are arbitrary and $b_4 = b_{11} = 0$ are set from the orthogonality condition $\langle b, e_4 \rangle = \langle b, e_{11} \rangle = 0$. By performing routine computations and expanding the bifurcation equations at $n = 4$ and $n = 11$ up to and including the cubic order, we obtain

$$\begin{cases} 15\Omega\epsilon = \epsilon (7\epsilon^2 + 14\mu^2 + \mathcal{O}(3)), \\ 120\Omega\mu = \mu (14\epsilon^2 + \frac{56}{17}\mu^2 + \mathcal{O}(3)). \end{cases} \quad (4.10)$$

We are looking for solutions to the system (4.10) with $(\epsilon, \mu) \neq (0, 0)$. There exist two nontrivial solutions to the system (4.10):

- (I) $\epsilon = 0, \mu \neq 0$, and $\Omega = \frac{7}{255}\mu^2 + \mathcal{O}(3)$;

(II) $\epsilon \neq 0$, $\mu = 0$, and $\Omega = \frac{7}{15}\epsilon^2 + \mathcal{O}(3)$;

whereas no solution exists with both $\epsilon \neq 0$ and $\mu \neq 0$. The Jacobian matrix of the system (4.10) is given by

$$J(\epsilon, \mu, \Omega) = \begin{bmatrix} -15\Omega + 21\epsilon^2 + 14\mu^2 & 28\epsilon\mu \\ 28\epsilon\mu & -120\Omega + 14\epsilon^2 + \frac{168}{17}\mu^2 \end{bmatrix} + \mathcal{O}(3).$$

For both branches (I) and (II), the Jacobian is invertible for small (ϵ, μ) , hence the two solutions are continued uniquely with respect to parameters (ϵ, μ) and yield branches (i) and (ii). By Lemma 6 with $m = 10$, branch (i) corresponds to the reduction (4.2) with $m = 10$, hence $\epsilon = 0$ persists beyond all orders of the expansion. By Lemma 6 with $m = 3$, branch (ii) corresponds to the reduction (4.2) with $m = 3$, hence $\mu = 0$ persists beyond all orders of the expansion. \square

Remark 13. Branch (ii) of Theorem 5 can be obtained by the symmetry transformation (4.7) from the branch bifurcating from the lowest eigenstate at $\omega_5 = 2/15$.

For the remaining double point $\omega_1 = \omega_3 = 0$, bifurcation of stationary state is more complicated. If we compute the normal form up to and including the cubic order, we obtain a trivial normal form, which is satisfied identically if $\omega = 0$. This outcome of the normal form computations suggests that there exists a two-parameter family of solutions $A \in \ell^2(\mathbb{N})$ to the system (3.1) with $\lambda = 1$ and $\omega = 0$. Indeed, the two eigenvectors for the null space of $L(0)$ are given by $e_0 - e_2$ and e_3 so that we can introduce the decomposition

$$a_n = \epsilon(\delta_{n0} - \delta_{n2}) + \mu\delta_{n3} + b_n, \quad n \in \mathbb{N},$$

subject to the orthogonality conditions $b_0 - b_2 = 0$ and $b_3 = 0$. By computing power expansions for small (ϵ, μ) with MAPLE, we can extend it to any polynomial order with the first terms given by

$$\begin{cases} A_0 = \epsilon + \epsilon\mu + \left(\epsilon\mu^2 - \frac{\epsilon^3}{2}\right) + (\epsilon^3\mu + 2\epsilon\mu^3) + \left(-\frac{5\epsilon^5}{4} + \frac{3\epsilon^3\mu^2}{2} + 3\epsilon\mu^4\right) + \mathcal{O}(6), \\ A_1 = 1 - (\epsilon^2 + \mu^2) + \epsilon^2\mu + (8\epsilon^4\mu + 3\epsilon^2\mu^3) + (-\epsilon^4 - 6\epsilon^2\mu^2 - 2\mu^4) + \mathcal{O}(6), \\ A_2 = -\epsilon + \epsilon\mu + \left(-\frac{\epsilon^3}{2} + \epsilon\mu^2\right) + (\epsilon^3\mu + 2\epsilon\mu^3) + \left(-\frac{5\epsilon^5}{4} + \frac{3\epsilon^3\mu^2}{2} + 3\epsilon\mu^4\right) + \mathcal{O}(6), \\ A_3 = \mu, \\ A_4 = -2\epsilon\mu + (\epsilon^3 + \epsilon\mu^2) - (4\epsilon^3\mu + \epsilon\mu^3) + \left(\frac{5\epsilon^5}{2} + \frac{11\epsilon^3\mu^2}{2} + 3\epsilon\mu^4\right) + \mathcal{O}(6), \\ A_5 = \mu^2 + \epsilon^2\mu + (3\epsilon^4\mu + \epsilon^2\mu^3) + (-\epsilon^4 + \epsilon^2\mu^2 + \mu^4) + \mathcal{O}(6), \\ A_6 = -3\epsilon\mu^2 + (2\epsilon^3\mu + \epsilon\mu^3) - \left(\frac{21}{2}\epsilon^3\mu^2 + 5\epsilon\mu^4\right) + \mathcal{O}(6). \end{cases}$$

The three explicit solutions (2.20), (2.33), and (2.35) are particular solutions of the two-parameter family of stationary states for small p . Indeed, the twisted state (2.20) corresponds to $\epsilon = -2p + \mathcal{O}(p^3)$ and $\mu = 3p^2 + \mathcal{O}(p^4)$, the Blaschke state (2.33) corresponds to $\epsilon = -p + \mathcal{O}(p^3)$ and $\mu = p^2 + \mathcal{O}(p^4)$, and the additional state (2.35) corresponds to $\epsilon = 0$ and $\mu = p^2 + \mathcal{O}(p^6)$.

Remark 14. We have shown by using the general transformation law derived in [2] that the twisted state (2.20) can be obtained from the $N = 1$ single-mode state (2.14) by applying e^{sD} with $s \in \mathbb{R}$ in the symmetry transformation (2.7) with $p = \tanh s$. In the present time, we do not know how to obtain the two-parameter branch of the stationary states above from the $N = 1$ single-mode state (2.14).

4.2. Variational characterization of the bifurcating states. The following lemma gives variational characterization of the second eigenmode at the bifurcation points $\{\omega_m\}_{m \in \mathbb{N}_+}$ in Lemma 7. The variational characterization is based on the action functional $K(\alpha)$ in (2.11), its second variation in (2.12), and the linearization operators $L_{\pm} : D(L_{\pm}) \rightarrow \ell^2(\mathbb{N})$ in (2.13).

Lemma 8. *The following is true:*

- For $\omega_1 = \omega_3 = 0$, the $N = 1$ single-mode state (2.14) is a degenerate saddle point of K with three positive eigenvalues, zero eigenvalue of multiplicity three, and infinitely many negative eigenvalues bounded away from zero.
- For $\omega_2 = 2/3$ and $\omega_6 = 3/35$, the $N = 1$ single-mode state (2.14) is a degenerate minimizer of K with zero eigenvalue of multiplicity three and infinitely many positive eigenvalues bounded away from zero.
- For ω_m with $m \geq 4$ and $m \neq 6$, the $N = 1$ single-mode state (2.14) is a degenerate saddle point of K with an even number of negative eigenvalues, zero eigenvalue of odd multiplicity, and infinitely many positive eigenvalues bounded away from zero.

Proof. By the scaling transformation (2.1), we take $\lambda - \omega = 1$ and $A_n = \delta_{n1}$, for which the explicit form of the linearized operators L_{\pm} in (2.13) yields

$$(L_{\pm}a)_n = [1 - n + 2 \min(n, 1)]a_n \pm [1 + \min(n, 1, 2 - n)]a_{2-n} + (n^2 - 1)\omega a_n. \quad (4.11)$$

We note that $L_+ = L(\omega)$ given by (4.1) and L_- is only different from L_+ at the diagonal entry at $n = 1$ (which is 0 instead of 4) and for the off-diagonal entries at $n = 0$ and $n = 2$ (which are -1 instead of $+1$).

For $L(\omega)$, the 2×2 block at $n = 0$ and $n = 2$,

$$\begin{bmatrix} 1 - \omega & 1 \\ 1 & 1 + 3\omega \end{bmatrix},$$

is positive definite for all $\{\omega_m\}_{m \in \mathbb{N}_+}$ with a simple zero eigenvalue only arising for $\omega_1 = \omega_3 = 0$ and $\omega_2 = 2/3$. The diagonal entries of $L(\omega)$ for $n \geq 3$ are given by

$$(n^2 - 1)\omega + 3 - n, \quad n \geq 3.$$

These entries are negative if $\omega_1 = \omega_3 = 0$ with a simple zero eigenvalue and strictly positive if $\omega_2 = 2/3$. For $\omega_m = (m - 3)/(m^2 - 1)$ with $m \geq 4$, the diagonal entries are simplified in the form:

$$(n^2 - 1)\omega_m + 3 - n = \frac{(n - m)(nm - 3n - 3m + 1)}{m^2 - 1}, \quad n \geq 3.$$

For $m = 4$ and $m = 11$, two eigenvalues are zero, six are negative, and all others are positive. For $m = 5$ and $m = 7$, two eigenvalues are zero, one is negative, and all others are positive. For $m = 6$, one eigenvalue is zero and all others are positive. For $m = 8, 9, 10$ and $m \geq 12$, one eigenvalue is zero, finitely many are negative, and all others are positive.

Multiplying these counts for $L_+ = L(\omega)$ by a factor of 2 due to the matrix operator L_- and adding an additional zero entry of L_- at $n = 1$ yields the assertion of the lemma. \square

Among all stationary states bifurcating from the second eigenmode, we shall only consider the potential minimizers of energy H subject to fixed Q and E . By Lemma 8, this includes only two branches bifurcating from ω_2 and ω_6 . In both cases, we are

able to compute the number of negative eigenvalues of the operators L_{\pm} denoted by $n(L_{\pm})$.

Lemma 9. *Consider the two bifurcating branches in Theorem 3 for ω_2 and ω_6 . For every small $\epsilon \neq 0$, we have $n(L_+) = 1$ and $n(L_-) = 0$. For each branch, L_- has a double zero eigenvalue, L_+ has no zero eigenvalue, and the rest of the spectrum of L_+ and L_- is strictly positive and is bounded away from zero.*

Proof. By the second item of Lemma 8, the corresponding operators L_+ and L_- at the bifurcation point $\epsilon = 0$ have respectively the simple and double zero eigenvalue, whereas the rest of their spectra is strictly positive and is bounded away from zero. By the two symmetries (2.2) and (2.3), the double zero eigenvalue of L_- is preserved for $\epsilon \neq 0$ and the assertion of the lemma for L_- follows by the perturbation theory.

On the other hand, the simple zero eigenvalue of L_+ is not preserved for $\epsilon \neq 0$ and will generally shift to either negative or positive values. We will show that it shifts to the negative values for $\epsilon \neq 0$, hence $n(L_+) = 1$ in both cases and the assertion of the lemma for L_+ follows by the perturbation theory.

For ω_2 , the Lyapunov–Schmidt decomposition of Theorem 3 yields power expansion

$$\omega = \frac{2}{3} - \frac{7}{3}\epsilon^2 + \mathcal{O}(\epsilon^4)$$

and $A_0 = 3\epsilon + \mathcal{O}(\epsilon^3)$, $A_1 = 1 - 4\epsilon^2 + \mathcal{O}(\epsilon^4)$, $A_2 = -\epsilon + \mathcal{O}(\epsilon^3)$, $A_3 = \frac{3}{4}\epsilon^2 + \mathcal{O}(\epsilon^4)$, and $A_n = \mathcal{O}(\epsilon^3)$ for $n \geq 4$. Similarly to the proof of Lemma 4, we compute the 4-by-4 block of the operator L_+ at $n \in \{0, 1, 2, 3\}$ and truncate it up to and including $\mathcal{O}(\epsilon^2)$ terms. The corresponding matrix block denoted by \hat{L}_+ is given by

$$\hat{L}_+ = \begin{bmatrix} \frac{1}{3} + \frac{46}{3}\epsilon^2 & 10\epsilon & 1 - \frac{37}{2}\epsilon^2 & -2\epsilon \\ 10\epsilon & 4 - 32\epsilon^2 & -2\epsilon & 2\epsilon^2 \\ 1 - \frac{37}{2}\epsilon^2 & -2\epsilon & 3 - 9\epsilon^2 & 2\epsilon \\ -2\epsilon & 2\epsilon^2 & 2\epsilon & \frac{16}{3} - \frac{80}{3}\epsilon^2 \end{bmatrix}$$

We are looking for a small eigenvalue of $\hat{L}x = \lambda x$, where $x = (x_0, x_1, x_2, x_3)^T$. Assuming $\lambda = \mathcal{O}(\epsilon^2)$ and expressing $\{x_1, x_3\}$ in terms of $\{x_0, x_2\}$ yield

$$\begin{cases} x_1 = -\frac{\epsilon}{2}(5x_0 - x_2) + \mathcal{O}(\epsilon^3), \\ x_3 = \frac{3\epsilon}{8}(x_0 - x_2) + \mathcal{O}(\epsilon^3). \end{cases}$$

Substituting these expressions into the eigenvalue problem $\hat{L}x = \lambda x$ and truncating it up to and including the order of $\mathcal{O}(\epsilon^2)$, we obtain

$$\begin{cases} \left(\frac{1}{3} - \frac{125}{12}\epsilon^2 - \lambda\right)x_0 + \left(1 - \frac{51}{4}\epsilon^2\right)x_2 = 0, \\ \left(1 - \frac{51}{4}\epsilon^2\right)x_0 + \left(3 - \frac{43}{4}\epsilon^2 - \lambda\right)x_2 = 0. \end{cases}$$

The small eigenvalue of this reduced problem is given by the expansion

$$\lambda = -\frac{14}{5}\epsilon^2 + \mathcal{O}(\epsilon^4). \quad (4.12)$$

Hence $n(L_+) = 1$ due to the shift of the zero eigenvalue of L_+ at ϵ to $\lambda < 0$ for $\epsilon \neq 0$.

For ω_6 , the Lyapunov–Schmidt decomposition of Theorem 3 yields power expansion

$$\omega = \frac{3}{35} + \frac{9}{70}\epsilon^2 + \mathcal{O}(\epsilon^4),$$

with $A_1 = 1 - \epsilon^2 + \mathcal{O}(\epsilon^4)$, $A_6 = \epsilon$, $A_{11} = -\frac{7}{8}\epsilon^2 + \mathcal{O}(\epsilon^4)$, $A_n = \mathcal{O}(\epsilon^3)$ for $n \geq 16$, where $A_n = 0$ for every $n \neq 5\ell + 1$, $\ell \in \mathbb{N}$ by Lemma 6. Similarly to the proof of

Lemma 4, we compute the 3-by-3 block of the operator L_+ at $n = 1$, $n = 6$, and $n = 11$ and truncate it up to and including $\mathcal{O}(\epsilon^2)$ terms. The corresponding matrix block denoted by \hat{L}_+ is given by

$$\hat{L}_+ = \begin{bmatrix} 4 - 8\epsilon^2 & 8\epsilon & -7\epsilon^2 \\ 8\epsilon & 14\epsilon^2 & 4\epsilon \\ -5\epsilon^2 & 4\epsilon & \frac{16}{7} + \frac{150}{7}\epsilon^2 \end{bmatrix}$$

Looking at the small eigenvalue of $\hat{L}x = \lambda x$, where $x = (x_1, x_6, x_{11})^T$, we can normalize $x_6 = 1$ and obtain

$$x_1 = -2\epsilon + \mathcal{O}(\epsilon^3), \quad x_{11} = -\frac{7}{4}\epsilon + \mathcal{O}(\epsilon^3),$$

and

$$\lambda = -9\epsilon^2 + \mathcal{O}(\epsilon^4). \quad (4.13)$$

Hence $n(L_+) = 1$ due to the shift of the zero eigenvalue of L_+ at ϵ to $\lambda < 0$ for $\epsilon \neq 0$. \square

4.3. Change of negative eigenvalues due to two constraints. We consider the two constraints related to the fixed values of Q and E by using the same computational formulas (3.33) and (3.34). Let \tilde{A} denote the stationary state of the stationary equation (2.10) continued with respect to two parameters (λ, ω) . By the scaling transformation (2.1), if the stationary state is given by (2.9) with real A , then the stationary state is continued with respect to parameter $c > 0$ as

$$\tilde{\alpha}_n(t) = cA_n e^{-ic^2\lambda t + inc^2\omega t},$$

hence $\tilde{A} = cA$, $\tilde{\lambda} = c^2\lambda$, and $\tilde{\omega} = c^2\omega$. Substituting these relations into $\tilde{\mathcal{Q}}(\tilde{\lambda}, \tilde{\omega}) = Q(\tilde{A})$ and $\tilde{\mathcal{E}}(\tilde{\lambda}, \tilde{\omega}) = E(\tilde{A})$ for $\lambda - \omega = 1$ yields

$$\tilde{\mathcal{Q}}(\tilde{\lambda}, \tilde{\omega}) = c^2\mathcal{Q}_0(\omega) = (\tilde{\lambda} - \tilde{\omega})\mathcal{Q}_0(\tilde{\omega}(\tilde{\lambda} - \tilde{\omega})^{-1})$$

and

$$\tilde{\mathcal{E}}(\tilde{\lambda}, \tilde{\omega}) = c^2\mathcal{E}_0(\omega) = (\tilde{\lambda} - \tilde{\omega})\mathcal{E}_0(\tilde{\omega}(\tilde{\lambda} - \tilde{\omega})^{-1}),$$

where $\mathcal{Q}_0(\omega) = \mathcal{Q}(1 + \omega, \omega)$ and $\mathcal{E}_0(\omega) = \mathcal{E}(1 + \omega, \omega)$. Substituting these representations into (3.34), evaluating derivatives, and setting $c = 1$ yield the computational formula

$$D = \begin{bmatrix} \mathcal{Q}_0(\omega) - \omega\mathcal{Q}'_0(\omega) & -\mathcal{Q}_0(\omega) + (1 + \omega)\mathcal{Q}'_0(\omega) \\ \mathcal{Q}_0(\omega) - \mathcal{E}_0(\omega) - \omega[\mathcal{Q}'_0(\omega) - \mathcal{E}'_0(\omega)] & -\mathcal{Q}_0(\omega) + \mathcal{E}_0(\omega) + (1 + \omega)[\mathcal{Q}'_0(\omega) - \mathcal{E}'_0(\omega)] \end{bmatrix},$$

which can be used to compute D for the normalized stationary state A with $\lambda - \omega = 1$. The following lemma confirms that the two branches in Lemma 9 are indeed local minimizers of H subject to fixed Q and E .

Lemma 10. *Consider the two bifurcating branches in Theorem 3 for ω_2 and ω_6 . For every small $\epsilon \neq 0$, the two branches are local minimizers of H subject to fixed Q and E with $n_c(L_+) = 0$ and $n(L_-) = 0$.*

Proof. For ω_2 , we use power expansions in Lemma 9 and compute

$$\begin{cases} \mathcal{Q}_0(\omega) = A_0^2 + 2A_1^2 + 3A_2^2 + \dots = 2 - 4\epsilon^2 + \mathcal{O}(\epsilon^4), \\ \mathcal{E}_0(\omega) = A_0^2 + 4A_1^2 + 9A_2^2 + \dots = 4 - 14\epsilon^2 + \mathcal{O}(\epsilon^4), \end{cases}$$

so that

$$D = \begin{bmatrix} 6/7 + \mathcal{O}(\epsilon^2) & 6/7 + \mathcal{O}(\epsilon^2) \\ 6/7 + \mathcal{O}(\epsilon^2) & -36/7 + \mathcal{O}(\epsilon^2) \end{bmatrix}.$$

Note that the expression for D agrees with the exact computations in Remark 9. Therefore, D has one positive and one negative eigenvalue, so that $n_c(L_+) = n(L_+) - 1 = 0$ by (3.33).

For ω_6 , we use power expansions in Lemma 9 and compute

$$\begin{cases} \mathcal{Q}_0(\omega) = 2A_1^2 + 7A_6^2 + \dots = 2 + 3\epsilon^2 + \mathcal{O}(\epsilon^4), \\ \mathcal{E}_0(\omega) = 4A_1^2 + 49A_6^2 + \dots = 4 + 41\epsilon^2 + \mathcal{O}(\epsilon^4), \end{cases}$$

so that

$$D = \begin{bmatrix} \mathcal{O}(\epsilon^2) & 70/3 + \mathcal{O}(\epsilon^2) \\ 70/3 + \mathcal{O}(\epsilon^2) & -2726/81 + \mathcal{O}(\epsilon^2) \end{bmatrix}$$

has again one positive and one negative eigenvalue, hence $n_c(L_+) = n(L_+) - 1 = 0$ by (3.33). \square

5. Numerical approximations. We confirm numerically that the stationary states (1.4) have the same variational characterization in the entire existence interval for p in $(0, 2 - \sqrt{3})$. Therefore, they remain local constrained minimizers of H for fixed Q and E for every $p \in (0, 2 - \sqrt{3})$.

Figure 1 shows the smallest eigenvalues of L_{\pm} computed at the stationary state (1.4) for the upper sign normalized by $\lambda = 1$. In agreement with item (iii) in Lemma 4 for small p , we have $n(L_+) = 1$ and $n(L_-) = 0$ for every $p \in (0, 2 - \sqrt{3})$. The small parameter p is related to the small parameter δ in Lemma 4 by $p = \delta + \mathcal{O}(\delta^5)$, see Remark 4 with $\epsilon = -\delta^2$. The dashed line shows the asymptotic dependencies (3.29), (3.30), and (3.31) for the small eigenvalues of L_+ and L_- .

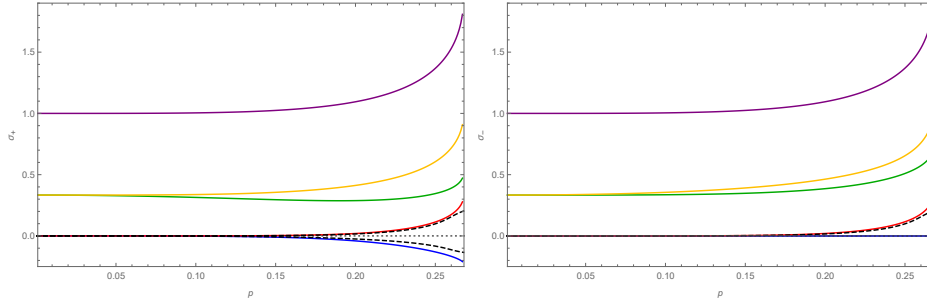


FIGURE 1. The smallest eigenvalues of L_+ (left) and L_- (right) for the stationary state (1.4) for the upper sign normalized by $\lambda = 1$.

Figure 2 shows the smallest eigenvalues of L_{\pm} computed at the stationary state (1.4) for the lower sign normalized by $\lambda - \omega = 1$. In agreement with Lemma 9 for small p , we have $n(L_+) = 1$ and $n(L_-) = 0$ for every $p \in (0, 2 - \sqrt{3})$. The small parameter p is related to the small parameter ϵ in Lemma 9 by $p = -\epsilon/2 + \mathcal{O}(\epsilon^3)$, see Remark 11. The dashed line shows the asymptotic dependence (4.12) for the small eigenvalue of L_+ .

Figure 3 shows the smallest eigenvalues of L_{\pm} computed at the stationary state bifurcating from the second eigenmode at $\omega_6 = 3/35$. We use here parameter ϵ for continuation of the stationary state as in Lemma 9. In agreement with Lemma 9, we have $n(L_+) = 1$ and $n(L_-) = 0$ for small ϵ . However, this result does not hold for larger values of ϵ far from the bifurcation point because additional eigenvalues of L_+ and L_- become negative eigenvalue for $\epsilon \approx 0.04$. Therefore, the stationary

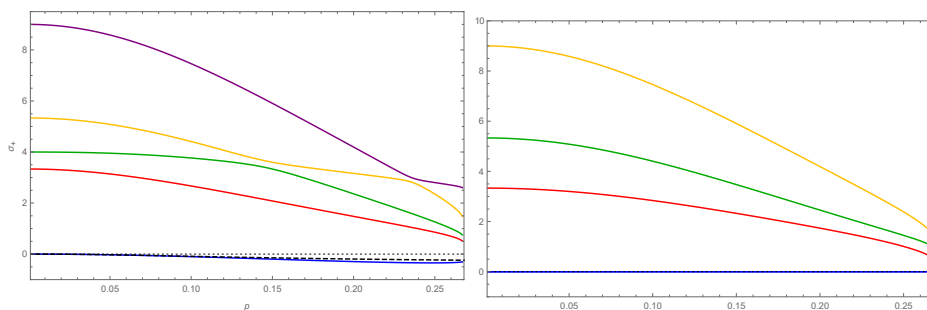


FIGURE 2. The smallest eigenvalues of L_+ (left) and L_- (right) for the stationary state (1.4) for the lower sign normalized by $\lambda - \omega = 1$.

state becomes a saddle point of H for fixed Q and E when $\epsilon \gtrsim 0.04$. The dashed line shows the asymptotic dependence (4.13) for the small eigenvalue of L_+ .

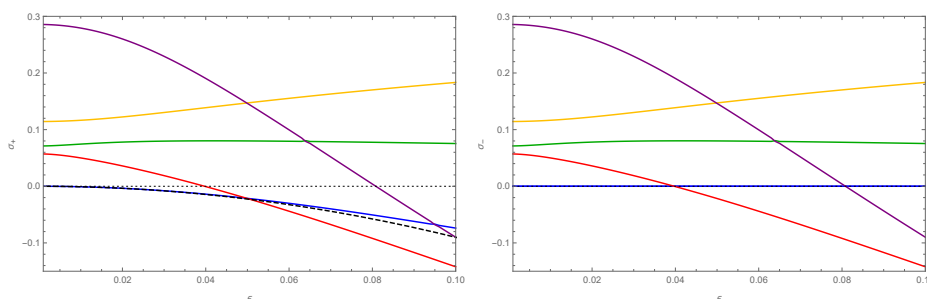


FIGURE 3. The smallest eigenvalues of L_+ (left) and L_- (right) for the branch bifurcating from the second eigenmode at $\omega_6 = 3/35$ with normalization $\lambda - \omega = 1$.

Remark 15. The presence of zero eigenvalue in the spectrum of L_+ at $\epsilon \approx 0.04$ on Figure 3 signals a new bifurcation of the stationary state along the branch. We have checked that the numerical results are stable with respect to truncation. In the present time, it is not clear how to identify new solution branches which may branch off at one or both sides of the bifurcation point.

REFERENCES

- [1] A. Biasi, P. Bizoń, B. Craps and O. Evnin, Exact lowest-Landau-level solutions for vortex precession in Bose-Einstein condensates, *Phys. Rev. A*, **96** (2017), 053615.
- [2] A. Biasi, P. Bizoń and O. Evnin, Solvable cubic resonant systems, *Commun. Math. Phys.*, **369** (2019), 433–456.
- [3] P. Bizoń, B. Craps, O. Evnin, D. Hunik, V. Luyten and M. Maliborski, Conformal flow on S^3 and weak field integrability in AdS_4 , *Commun. Math. Phys.*, **353** (2017), 1179–1199.
- [4] P. Bizoń, D. Hunik–Kostyra and D. Pelinovsky, Ground state of the conformal flow on S^3 , *Commun. Pure Appl. Math.*, **72** (2019), 1123–1151.
- [5] S. N. Chow and J. K. Hale, *Methods of Bifurcation Theory*, Undergraduate Texts in Mathematics, **251**, Springer-Verlag, New York, 1982.
- [6] M. G. Crandall and P. H. Rabinowitz, Bifurcation from simple eigenvalues, *J. Funct. Anal.*, **8** (1971), 321–340.

- [7] P. Gérard, P. Germain and L. Thomann, [On the cubic lowest Landau level equation](#), *Arch. Rat. Mech. Appl.*, **231** (2019), 1073–1128.
- [8] P. Gérard and S. Grellier, [The cubic Szegő equation](#), *Ann. Scient. Éc. Norm. Sup.*, **43** (2010), 761–810.
- [9] P. Gérard and S. Grellier, [Invariant tori for the cubic Szegő equation](#), *Invent. Math.*, **187** (2012), 707–754.
- [10] P. Gérard and E. Lenzmann, [A Lax pair structure for the half-wave maps equation](#), *Lett. Math. Phys.*, **108** (2018), 1635–1648.
- [11] P. Germain, Z. Hani and L. Thomann, [On the continuous resonant equation for NLS: I. Deterministic analysis](#), *J. Math. Pur. App.*, **105** (2016), 131–163.
- [12] M. Grillakis, J. Shatah and W. Strauss, [Stability theory of solitary waves in the presence of symmetry. II](#), *J. Funct. Anal.*, **94** (1990), 308–348.
- [13] E. Lenzmann and A. Schikorra, [On energy-critical half-wave maps to \$S^2\$](#) , *Invent. Math.*, **213** (2018), 1–82.
- [14] D. E. Pelinovsky, [Localization in Periodic Potentials: From Schrödinger Operators to the Gross-Pitaevskii Equation](#), LMS Lecture Note Series, **390**, Cambridge University Press, Cambridge, 2011.
- [15] O. Pocovnicu, [Traveling waves for the cubic Szegő equation on the real line](#), *Anal. PDE*, **4** (2011), 379–404.
- [16] J. Thirouin, [Classification of traveling waves for a quadratic Szegő equation](#), *Discr. Cont. Dynam. Syst.*, **39** (2019), 3099–3122.

Received for publication August 2018.

E-mail address: bizon@th.if.uj.edu.pl

E-mail address: dominika.hunik@uj.edu.pl

E-mail address: dmpeli@math.mcmaster.ca