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The asymptotic stability of solitons in the cubic NLS equation on the line

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We use the inverse scattering transform, the auto-Bäcklund transformation, and the steepest descent method of Deift and Zhou to obtain the asymptotic stability of the solitons in the cubic NLS (nonlinear Schrödinger) equation.

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1. Introduction

We consider the Cauchy problem for the cubic focusing NLS (nonlinear Schrödinger) equation on $\mathbb{R}$:

$$\mathrm{i}u_t + u_{xx} + 2|u|^2 u = 0, \quad u(0) = u_0. \tag{1.1}$$

The Cauchy problem (1.1) is globally well posed in $L^2(\mathbb{R})$, by the following result due to Tsutsumi [1].

THEOREM 1.1 *Given $u_0 \in L^2(\mathbb{R})$, then there exists a unique solution*

$$u(t) \in C^0(\mathbb{R}, L^2(\mathbb{R})) \cap L^4_{loc}(\mathbb{R}, L^\infty(\mathbb{R}))$$

of the integral equation

$$u(t) = e^{\mathrm{i}t\partial_x^2} u_0 + 2\mathrm{i} \int_0^t e^{\mathrm{i}(t-t')\partial_x^2} |u(t')|^2 u(t') dt'. \tag{1.2}$$

We have $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. Furthermore, if $u_{0n} \to u_0$ as $n \to \infty$ in $L^2(\mathbb{R})$ and $u_n(t)$ is the solution of the NLS equation with $u_n(0) = u_{0n}$, then, as $n \to \infty$, for any $t \in \mathbb{R}$ we have $u_n(t) \to u(t)$ in $L^2(\mathbb{R})$.

Remark 1.2 An important class of solutions of the NLS equation is the solitons, defined by

$$\varphi_{\omega,\gamma,v}(t, x - x_0) := \omega e^{\mathrm{i}xv + \mathrm{i}(\omega^2 - v^2)t + \mathrm{i}\gamma} \operatorname{sech}(\omega(x - 2vt - x_0)). \tag{1.3}$$

We are interested here to the question of their asymptotic stability, when u_0 is close to $\varphi_{\omega,\gamma,v}$ for a particular (ω, γ, v).

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Notations The following Hilbert spaces are the closures of the space $C_0^\infty(\mathbb{R})$ with respect to the following norms, where $\langle x\rangle := \sqrt{1+|x|^2}$:

$L^{2,s}(\mathbb{R})$ defined with $\|u\|_{L^{2,s}(\mathbb{R})} := \|\langle x\rangle^s u\|_{L^2(\mathbb{R})}$;
$\dot{H}^s(\mathbb{R})$ defined with $\|u\|_{\dot{H}^s(\mathbb{R})} := \||x|^s \widehat{u}\|_{L^2(\mathbb{R})}$ where $\widehat{u}$ is the Fourier transform;
$H^s(\mathbb{R})$ defined with $\|u\|_{H^s(\mathbb{R})} := \|\langle x\rangle^s \widehat{u}\|_{L^2(\mathbb{R})}$, such that $H^s(\mathbb{R}) = \dot{H}^s(\mathbb{R}) \cap L^2(\mathbb{R})$.

We set $\Sigma_s := H^s(\mathbb{R}) \cap L^{2,s}(\mathbb{R})$. Our aim is to prove the following result.

THEOREM 1.3 *Fix $s \in (1/2, 1]$. Consider the NLS soliton $\varphi_{\omega_0,\gamma_0,v_0}(0, x-x_0)$. Then, there exist positive constants $\varepsilon_0 = \varepsilon_0(\omega_0, v_0)$, $T = T(\omega_0, v_0)$, and $C = C(\omega_0, v_0)$ such that if $u_0 \in L^{2,s}(\mathbb{R})$ and if*

$$\epsilon := \|\varphi_{\omega_0,\gamma_0,v_0}(0, \cdot - x_0) - u_0\|_{L^{2,s}(\mathbb{R})} < \varepsilon_0, \tag{1.4}$$

then there exist two ground states $\varphi_{\omega_1,\gamma_\pm,v_1}(t, x - x_\pm)$ such that for the solution of the Cauchy problem (1.1) provided by Theorem 1.1, we have

$$|(\omega_1, \gamma_\pm, v_1, x_\pm) - (\omega_0, \gamma_0, v_0, x_0)| < C\epsilon \tag{1.5}$$

and, for all $\pm t \geq T$,

$$\|u(t, \cdot) - \varphi_{\omega_1,\gamma_\pm,v_1}(t, \cdot - x_\pm)\|_{L^\infty(\mathbb{R})} < C\epsilon|t|^{-\frac{1}{2}}. \tag{1.6}$$

In general the two ground states $\varphi_{\omega_1,\gamma_\pm,v_1}(t, x - x_\pm)$ are distinct, see Lemma 4.5 at the end of the paper.

A key ingredient for the above result comes from the methods of the inverse scattering transform (IST) theory, found in references [2–6]. In particular, we use the steepest descent method and the auto-Bäcklund transformation discussed in [7]. Theorem 1.3 is an analog of the results about the asymptotic behavior of solutions decaying to 0, obtained in [4–8]. Compared to these references, we do not reproduce in Theorem 1.3 the asymptotic expansions of the solution u for large values of t, but we ease the restrictions on the initial data by allowing $u_0 \in L^{2,s}(\mathbb{R})$ for $s \in (1/2, 1]$ and not just for $s = 1$.

Theorem 1.3 should be contrasted to the results for nonintegrable systems, where the orbits of the solitons which attract the solution $u(t)$ are presumably not the same as $t \to +\infty$ and $t \to -\infty$, see [9–11] for early results. In the case of the cubic NLS equation, it turns out that the selected asymptotic soliton is simply defined by the eigenvalue of a spectral problem supported by the initial datum u_0 but it has a different reflection coefficient, which is zero for the solitons (1.3) and nonzero for a generic u_0.

Another feature of nonintegrable systems is that the rate of decay in the right hand side of (1.6) is generally slower, because of metastable states which are not present for the cubic NLS equation. The theory how to treat these metastable states was initiated in [12,13] and for recent developments and further references, we refer to [14–16]. Obviously, the absence of metastable states for the cubic NLS equation simplifies the discussion. Notice that [17] conjectures the nonexistence of metastable states in integrable systems.

Theorem 1.3 appears to be out of reach of the perturbative methods initiated in [9–11] and developed in a number of papers using a similar framework. This is because of the "strength" of the cubic nonlinearity in the cubic NLS equation. This strength is responsible for the fact that the classical result in [18] on the dispersion of small solutions of L^2 subcritical equations does not apply to the cubic NLS equation, although it was proved also for the cubic NLS equation a decade later in [8], with an approach similar to [18] but with an additional normal form argument. The results in [8,18] are based on *invariant fields* which exploit symmetries of the equations not present in the case of the linearization of an NLS at a soliton. And while [18] has been partially extended to settings without translation symmetry in [19], so far the approach in [8,18] has never been applied directly to the problem of asymptotic stability of the solitons.

Mizumachi and Pelinovsky [20] proposed to treat the orbital stability of solitons of the cubic NLS equation by using an auto-Bäcklund transformation which transforms a soliton in the zero solution and preserves the equation. They proved that the Bäcklund transformation is a homeomorphism in L^2. The Bäcklund transformation can then be used to transfer Theorem 1.1 into a statement about solutions close to the soliton in $L^2(\mathbb{R})$, in particular proving that solitons of the cubic NLS equation are orbitally stable in $L^2(\mathbb{R})$, thus transposing to $L^2(\mathbb{R})$ the classical result of orbital stability proved for the space $H^1(\mathbb{R})$ in [21]. In [20], a discussion was initiated on the possible use of the same transformation to transfer the dispersion scattering result for small solutions in [8] to an asymptotic stability result for solitons in Σ_1. However, it is an open question whether or not the Bäcklund transformation in [20] is a homeomorphism in Σ_1.

The inspiration for the present paper comes from a paper by Deift and Park [7], where there is a particularly simple and explicit Bäcklund transformation, see (4.3) later. Using the steepest descent method of Deift and Zhou [22], it is possible to bound all the terms of formula (4.3) and prove Theorem 1.3. Specifically, by means of direct scattering, it is possible to derive the spectral data associated with the solution u of the Cauchy problem (1.1). Then, from mapping properties of the inverse scattering transform proved in [3,5], which are similar to mapping properties of the inverse Fourier transform, the solution u is expressed by means of the transformation formula (4.3) as the sum of a pure radiation solution $\widetilde{u}$ and an appropriate fraction of Jost functions associated to the potential $\widetilde{u}$. The results in [5–7] are applied directly to the pure radiation solution $\widetilde{u}$. Also Jost functions and their fraction can be easily analyzed using other results from [5–7]. This yields Theorem 1.3. Notice that Theorem 3.1 in Section 3 extends the result in [5,6] to initial data in $L^{2,s}(\mathbb{R})$ for all $s \in (1/2, 1]$.

We do not make any particular claim of originality, since Theorem 1.3 is a natural corollary of the previous works [5–7,23]. Nonetheless, we feel that it is important that Theorem 1.3 be stated explicitly and proved.

The paper is organized as follows. Section 2 gives details of the direct and inverse scattering transforms for the cubic NLS equation. Section 3 contains a review of the asymptotic scattering theory for the pure radiation solution. Section 4 explains the arguments needed to prove the asymptotic stability of solitons formulated in Theorem 1.3.

2. Direct and inverse scattering transforms

The Cauchy problem (1.1) for the cubic NLS equation can be solved through the direct and inverse scattering transform.

Consider a function $u(x) \in L^1(\mathbb{R})$ and recall that $L^{2,s}(\mathbb{R})$ is embedded into $L^1(\mathbb{R})$ for any $s > \frac{1}{2}$. The spectral system associated with the cubic NLS equation takes the form:

$$\psi_x = -\mathrm{i}z\sigma_3\psi + Q(u(x))\psi, \tag{2.1}$$

where

$$Q(u(x)) := \begin{pmatrix} 0 & u(x) \\ -\overline{u}(x) & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Set $e_1 = (1,0)^T$ and $e_2 = (0,1)^T$. According to the direct scattering theory[24], for any fixed $z \in \mathbb{C}_+$ (i.e. $\operatorname{Im} z > 0$) there exists a unique $\mathbb{C}^2$ valued solution $\phi(x,z)$ of the spectral system (2.1) such that

$$\lim_{x\to-\infty} \phi(x,z)e^{\mathrm{i}xz} = e_1 \quad \text{and} \quad \lim_{x\to+\infty} \phi(x,z)e^{\mathrm{i}xz} = a(z)e_1, \tag{2.2}$$

where $a(z)$ is an analytic function in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$ with $\lim_{z\to\infty} a(z) = 1$. We call $a(z)$ the *scattering function*. The following result is well known (see, i.e. [2,24]).

Lemma 2.1 *There exists an open dense set $\mathcal{G} \subset L^1(\mathbb{R})$ such that, for $u \in \mathcal{G}$, the scattering function $a(z)$ has at most a finite number of zeros forming a set $\mathcal{Z}_+ = \{z_1, \ldots, z_n\}$ in $\mathbb{C}_+$, with $a(z) \neq 0$ for all $z \in \mathbb{R}$ and $a'(z_k) \neq 0$ for all k. The cardinality $u \to \sharp\mathcal{Z}_+$ is locally constant near u in $\mathcal{G}$ and the map $\mathcal{G} \ni u \to (z_1, \ldots, z_n) \in \mathbb{C}_+^n$ is locally Lipschitz.*

We denote by $\mathcal{G}_n$ the open subset of $\mathcal{G}$ formed by the elements such that $\sharp\mathcal{Z}_+ = n$.

Remark 2.2 We call the potentials in $\mathcal{G}$ *generic*. $\mathcal{G}_1$ contains the solitons (1.3). Notice that small L^1-perturbations to the solitons are in $\mathcal{G}_1$. See [25] for other integrable equations where this is not true.

There exists a unique $\mathbb{C}^2$ valued solution $\psi(x,z)$ of the spectral system (2.1) satisfying

$$\lim_{x\to+\infty} \psi(x,z)e^{-\mathrm{i}xz} = e_2. \tag{2.3}$$

If $u \in \mathcal{G}$ and $\mathcal{Z}_+ \neq \emptyset$, then for each $z_k \in \mathcal{Z}_+$ we have $\phi(x,z_k) = \gamma_k\psi(x,z_k)$ for some $\gamma_k \in \mathbb{C}_* := \mathbb{C}\backslash\{0\}$. Set $c_k = \gamma_k/a'(z_k)$ and call it *the norming constant*.

For $z \in \mathbb{R}$, the solution of the spectral system (2.1) with the boundary value

$$\lim_{x\to-\infty} \phi(x,z)e^{\mathrm{i}xz} = e_1 \tag{2.4}$$

satisfies the scattering problem

$$\lim_{x\to+\infty} \left[\phi(x,z)e^{\mathrm{i}xz} - a(z)e_1 - e^{2\mathrm{i}xz}b(z)e_2\right] = 0, \tag{2.5}$$

where $b(z)$ is a continuous function on $\mathbb{R}$. Set $r(z) := \frac{b(z)}{a(z)}$ and call it the *reflection coefficient*.

We consider now the Jost functions defined by the Volterra integral equations, see [24],

$$\begin{aligned} \mathfrak{m}_1^{\pm}(x,z) &= e_1 + \int_{\pm\infty}^{x} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\mathrm{i}(x-y)z} \end{pmatrix} Q(u(y))\mathfrak{m}_1^{\pm}(y,z)dy, \\ \mathfrak{m}_2^{\pm}(x,z) &= e_2 + \int_{\pm\infty}^{x} \begin{pmatrix} e^{-2\mathrm{i}(x-y)z} & 0 \\ 0 & 1 \end{pmatrix} Q(u(y))\mathfrak{m}_2^{\pm}(y,z)dy. \end{aligned} \tag{2.6}$$

The functions $\mathfrak{m}_1^-(x,z)$ and $\mathfrak{m}_2^+(x,z)$ are analytic for $z \in \mathbb{C}_+$, whereas the functions $\mathfrak{m}_2^-(x,z)$ and $\mathfrak{m}_1^+(x,z)$ are analytic for $z \in \mathbb{C}_-$, [24].

Remark 2.3 In terms of functions ϕ and ψ introduced in (2.2) and (2.3), we have $\mathfrak{m}_1^-(x,z) = \phi(x,z)e^{ixz}$ and $\mathfrak{m}_2^+(x,z) = \psi(x,z)e^{-\mathrm{i}xz}$ for $z \in \mathbb{C}_+$.

From the scattering problem (2.5) and the Wronskian identities for the spectral system (2.1), we have for $z \in \mathbb{R}$,

$$a(z) = \det[\mathfrak{m}_1^-(x,z), \mathfrak{m}_2^+(x,z)] \tag{2.7}$$

and

$$b(z) = \det[\mathfrak{m}_1^+(x,z), e^{-2\mathrm{i}xz}\mathfrak{m}_1^-(x,z)], \tag{2.8}$$

where matrices $[\cdot,\cdot]$ are defined in the sense of column vectors and the Wronskian determinants are x-independent. The following result is obtained with a minor modification of the argument in Theorem 3.2 [5].

Lemma 2.4 *Let $s \in \left(\frac{1}{2}, 1\right]$. For $u \in L^{2,s}(\mathbb{R}) \cap \mathcal{G}$ we have $r \in H^s(\mathbb{R})$. Furthermore, the map $L^{2,s}(\mathbb{R}) \cap \mathcal{G} \ni u \to r \in H^s(\mathbb{R})$ is locally Lipschitz.*

Proof We make the following claim: for any fixed $\kappa_0 > 0$ there exists a positive constant C such that if $\|u\|_{L^{2,s}(\mathbb{R})} \le \kappa_0$, then we have for $j = 1, 2$:

$$\begin{aligned} &\|\mathfrak{m}_j^-(x,z) - e_j\|_{H_z^s(\mathbb{R})} \le C\|u\|_{L^{2,s}(\mathbb{R})} \quad \text{for all } x \le 0 \\ &\|\mathfrak{m}_j^+(x,z) - e_j\|_{H_z^s(\mathbb{R})} \le C\|u\|_{L^{2,s}(\mathbb{R})} \quad \text{for all } x \ge 0. \end{aligned} \tag{2.9}$$

Let us assume (2.9) for a moment. Then $b \in H^s(\mathbb{R})$ because

$$\begin{aligned} b(z) &= \det\left[\mathfrak{m}_1^+(0,z), \mathfrak{m}_1^-(0,z)\right] \\ &= \det\left[\mathfrak{m}_1^+(0,z) - e_1, \mathfrak{m}_1^-(0,z) - e_1\right] \\ &\quad + \det\left[\mathfrak{m}_1^+(0,z) - e_1, e_1\right] + \det\left[e_1, \mathfrak{m}_1^-(0,z) - e_1\right], \end{aligned} \tag{2.10}$$

where we recall that $H^s(\mathbb{R})$ is a Banach algebra with respect to pointwise multiplication for any $s > \frac{1}{2}$. Similarly, $(a-1) \in H^s(\mathbb{R})$ because

$$\begin{aligned} a(z) &= \det\left[\mathfrak{m}_1^-(0,z), \mathfrak{m}_2^+(0,z)\right] \\ &= \det\left[\mathfrak{m}_1^-(0,z) - e_1, \mathfrak{m}_2^+(0,z)\right] + \det\left[e_1, \mathfrak{m}_2^+(0,z) - e_2\right] + \det\left[e_1, e_2\right] \\ &= 1 + \det\left[\mathfrak{m}_1^-(0,z) - e_1, \mathfrak{m}_2^+(0,z) - e_2\right] \\ &\quad + \det\left[e_1, \mathfrak{m}_2^+(0,z) - e_2\right] + \det\left[\mathfrak{m}_1^-(0,z) - e_1, e_2\right]. \end{aligned} \tag{2.11}$$

We conclude that if $u \in L^{2,s}(\mathbb{R}) \cap \mathcal{G}$ then $r \in H^s(\mathbb{R})$. So this shows that we have a map $L^{2,s}(\mathbb{R}) \cap \mathcal{G} \ni u \to r \in H^s(\mathbb{R})$. We skip the proof of the fact that the map $L^{2,s}(\mathbb{R}) \cap \mathcal{G} \ni u \to r \in H^s(\mathbb{R})$ is locally Lipschitz.

We now prove (2.9). It suffices to consider the case $j = 1$ and the minus sign only. The proof is based on the fact that if there is $s \in (0, 1]$ such that for an $f \in L^2(\mathbb{R})$ we have

$$\|f(\cdot + h) - f(\cdot)\|_{L^2(\mathbb{R})} \le C|h|^s, \quad \forall h \in \mathbb{R}, \tag{2.12}$$

then $f \in H^s(\mathbb{R}) = \dot{H}^s(\mathbb{R}) \cap L^2(\mathbb{R})$ and there is a positive constant c independent of f, such that $\|f\|_{\dot{H}^s(\mathbb{R})} \le c\, C$.

Let us define

$$Kf(x,z) := \int_{-\infty}^{x} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\mathrm{i}(x-y)z} \end{pmatrix} Q(u(y)) f(y,z) dy.$$

By Theorem 3.2 [5] (see also [24]), we have

$$\|Ke_2\|_{L^\infty_x(\mathbb{R}, L^2_z(\mathbb{R}))} \leq \|u\|_{L^2(\mathbb{R})}$$

and for any $x_0 \leq +\infty$,

$$\left\|(1-K)^{-1}\right\|_{L^\infty_x((-\infty,x_0),L^2_z(\mathbb{R}))\to L^\infty_x((-\infty,x_0),L^2_z(\mathbb{R}))} \leq e^{\sqrt{2}\|u\|_{L^1}}.$$

Furthermore, for $x \leq 0$ we have

$$\begin{aligned} \|\mathfrak{m}_1^-(x,z) - e_2\|_{L^2_z(\mathbb{R})} &\leq e^{\sqrt{2}\|u\|_{L^1}} \left\| \int_{-\infty}^{x} e^{2\mathrm{i}(x-y)z} \overline{u}(y) dy \right\|_{L^2_z(\mathbb{R})} \\ &\leq e^{\sqrt{2}\|u\|_{L^1}} \left(\int_{-\infty}^{x} \langle y \rangle^{2s} |u(y)|^2 dy \right)^{1/2} \langle x \rangle^{-s} \\ &\leq e^{\sqrt{2}\|u\|_{L^1}} \|u\|_{L^{2,s}(\mathbb{R})} \langle x \rangle^{-s}. \end{aligned} \tag{2.13}$$

To complete the proof of (2.9) for $j=1$ and the minus sign it is enough to prove and estimate of the form (2.12) with $C \lesssim \|u\|_{L^{2,s}(\mathbb{R})}$. Define $n(x,z) := \mathfrak{m}_1^-(x,z+h) - \mathfrak{m}_1^-(x,z)$ for $h \in \mathbb{R}$. We have

$$\begin{aligned} (1-K)n(x,z) &= \int_{-\infty}^{x} \begin{pmatrix} 0 & 0 \\ 0 & e^{2\mathrm{i}(x-y)(z+h)} - e^{2\mathrm{i}(x-y)z} \end{pmatrix} Q(u(y))(\mathfrak{m}_1^-(y,z) - e_1) dy \\ &+ \int_{-\infty}^{x} \begin{pmatrix} 0 \\ \left(e^{2\mathrm{i}(x-y)(z+h)} - e^{2\mathrm{i}(x-y)z}\right) \overline{u}(y) \end{pmatrix} dy. \end{aligned} \tag{2.14}$$

Using the Fourier transform $\mathcal{F}$, we have for $x \leq 0$,

$$\begin{aligned} &\left\| \int_{-\infty}^{x} \left(e^{2\mathrm{i}(x-y)(z+h)} - e^{2\mathrm{i}(x-y)z}\right) \overline{u}(y) dy \right\|_{L^2_z} \\ &= \|\mathcal{F}^*[u(\cdot+x)\chi_{\mathbb{R}_-}](z+h) - \mathcal{F}^*[u(\cdot+x)\chi_{\mathbb{R}_-}](z)\|_{L^2_z} \\ &\leq C\|\mathcal{F}^*[u(\cdot+x)\chi_{\mathbb{R}_-}](z)\|_{H^s_z(\mathbb{R})}|h|^s = \|u(y+x)\|_{L^{2,s}_y(\mathbb{R}_-)}|h|^s \leq \|u\|_{L^{2,s}(\mathbb{R})}|h|^s \end{aligned} \tag{2.15}$$

and, using estimate (2.13),

$$\|\text{first term r.h.s. (2.14)}\|_{L^2_z} \leq 2^{1-s}|h|^s \int_{-\infty}^{x} |y|^s |u(y)| \, \|\mathfrak{m}_1^-(y,z) - e_2\|_{L^2_z(\mathbb{R})} dy \tag{2.16}$$

$$\leq 2^{1-s}|h|^s e^{\|u\|_{L^1}} \|u\|_{L^{2,s}(\mathbb{R})} \int_{-\infty}^{x} |y|^s \langle y \rangle^{-s} |u(y)| dy \leq 2^{1-s}|h|^s e^{\|u\|_{L^1}} \|u\|_{L^{2,s}(\mathbb{R})} \|u\|_{L^1}.$$

Then, by (2.14)–(2.16) we get $\|\mathfrak{m}_1^-(x,z+h) - \mathfrak{m}_1^-(x,z)\|_{L^2_z(\mathbb{R})} \leq C|h|^s\|u\|_{L^{2,s}(\mathbb{R})}$ for $x \leq 0$, where C is a fixed constant for $\|u\|_{L^{2,s}(\mathbb{R})} \leq \kappa_0$, for a preassigned bound κ_0. This implies that for all $x \leq 0$ we have $\|\mathfrak{m}_1^-(x,z) - e_1\|_{\dot{H}^s_z(\mathbb{R})} \leq C\|u\|_{L^{2,s}(\mathbb{R})}$ for some positive constant C. Combined with (2.13) this yields the claim (2.9) for $j=1$ and for the minus sign. The other cases are similar. □

Lemma 2.4 provides the direct scattering information we need. Now we recall a number of facts about inverse scattering. The spectral data in the space

$$\mathcal{S}(s,n) := \left\{ r(z) \in H^s(\mathbb{R}), \quad (z_1,\ldots,z_n) \in \mathbb{C}^n_+, \quad (c_1,\ldots,c_n) \in \mathbb{C}^n_* \right\} \tag{2.17}$$

are used to recover the potential u in matrix $Q(u)$ of the spectral system (2.1). Set

$$V_x(z) := \begin{pmatrix} 1+|r(z)|^2 & e^{-2\mathrm{i}xz}\overline{r}(z) \\ e^{2\mathrm{i}xz}r(z) & 1 \end{pmatrix} \tag{2.18}$$

and consider the following Riemann–Hilbert (RH) problem:

(i) $m(x,\cdot)$ is meromorphic in $\mathbb{C}\backslash\mathbb{R}$;

(ii) $m(x,\cdot)$ has continuous boundary values $m_\pm(x,\cdot)$ on $\mathbb{R}$ satisfying

$$m_+(x,z) = m_-(x,z)V_x(z);$$

(iii) $\lim_{z\to\infty} m(x,z) = 1$;

(iv) $m(x,z)$ has simple poles in $\mathcal{Z} = \mathcal{Z}_+ \cup \mathcal{Z}_-$, where $\mathcal{Z}_- = \{\bar{z}_1,\ldots,\bar{z}_n\}$ in $\mathbb{C}_-$, and for each $z_k \in \mathcal{Z}_+$ and $\bar{z}_k \in \mathcal{Z}_-$, we have

$$\begin{aligned} \mathrm{Res}_{z=z_k} m(x,z) &= \lim_{z\to z_k} m(x,z)V_x(z_k), \\ \mathrm{Res}_{z=\bar{z}_k} m(x,z) &= \lim_{z\to \bar{z}_k} m(x,z)V_x(\bar{z}_k), \end{aligned} \tag{2.19}$$

with

$$V_x(z_k) := \begin{pmatrix} 0 & 0 \\ e^{2\mathrm{i}xz_k}c_k & 0 \end{pmatrix}, \quad V_x(\bar{z}_k) := \begin{pmatrix} 0 & -e^{-2\mathrm{i}x\bar{z}_k}\bar{c}_k \\ 0 & 0 \end{pmatrix}. \tag{2.20}$$

From the solution of the RH problem (i)–(iv), the potential u in the matrix $Q(u)$ is found by means of *the reconstruction formula*:

$$u(x) := 2\mathrm{i} \lim_{z\to\infty} z\, m_{12}(x,z). \tag{2.21}$$

Remark 2.5 In terms of the analytic functions $\mathfrak{m}^\pm_{1,2}$ introduced from the Volterra integral Equations (2.6), we have the equalities

$$m_+(x,z) = \left[a(z)^{-1}\mathfrak{m}_1^-(x,z), \mathfrak{m}_2^+(x,z) \right], \quad m_-(x,z) = \left[\mathfrak{m}_1^+(x,z), \overline{a(\overline{z})^{-1}}\mathfrak{m}_2^-(x,z) \right].$$

We introduce now the Cauchy operator $C_\mathbb{R}$ acting on functions $h(z) \in L^2(\mathbb{R})$,

$$(C_\mathbb{R} h)(z) = \frac{1}{2\pi\mathrm{i}} \int_\mathbb{R} \frac{h(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{2.22}$$

with the boundary values

$$(C^\pm_\mathbb{R} h)(z) = \lim_{\varepsilon\searrow 0} \frac{1}{2\pi\mathrm{i}} \int_\mathbb{R} \frac{h(\zeta)}{\zeta - (z\pm\mathrm{i}\varepsilon)} d\zeta, \quad z \in \mathbb{R}.$$

The solution $m(x,z)$ of the RH problem (i)–(iv) is given by the following formula:

$$m(x,z) = 1 - \sum_{\zeta\in\mathcal{Z}} \frac{M_x(\zeta)V_x(\zeta)}{\zeta - z} + \frac{1}{2\pi\mathrm{i}} \int_\mathbb{R} \frac{M_x(\zeta)(V_x(\zeta)-1)}{\zeta - z} d\zeta, \tag{2.23}$$

where $M_x(z)$ is defined for $z \in \mathbb{R} \cup \mathcal{Z}$ in the space $M_{2\times 2}(\mathbb{C})$ of complex 2×2 matrices and satisfies system (2.24)–(2.25) written below.

Lemma 2.6 below implies that the map $\mathcal{G}_n \cap L^{2,s}(\mathbb{R}) \to \mathcal{S}(s,n)$, which is defined by Lemmas 2.1 and 2.4, is one-to-one. This result is due to Zhou [3], but we prove it for completeness, following the argument in Lemma 5.2 [7].

Lemma 2.6 *Let $r \in H^s(\mathbb{R})$ with $s > 1/2$. Then, for any $x \in \mathbb{R}$ there exists and is unique a solution $M_x : \mathbb{R} \cup \mathcal{Z} \to M_{2\times 2}(\mathbb{C})$ of the following system of integral and algebraic equations:*

$$M_x(z) = 1 - \sum_{\zeta \in \mathcal{Z}} \frac{M_x(\zeta)V_x(\zeta)}{\zeta - z} + \lim_{\varepsilon \searrow 0} \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{M_x(\zeta)(V_x(\zeta) - 1)}{\zeta - (z - \mathrm{i}\varepsilon)} d\zeta, \quad z \in \mathbb{R} \tag{2.24}$$

and

$$M_x(z) = 1 - \sum_{\zeta \in \mathcal{Z}\setminus\{z\}} \frac{M_x(\zeta)V_x(\zeta)}{\zeta - z} + \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{M_x(\zeta)(V_x(\zeta) - 1)}{\zeta - z} d\zeta, \quad z \in \mathcal{Z} \tag{2.25}$$

such that $(M_x(z) - 1) \in L^2_z(\mathbb{R})$.

Proof For the operator C_{V_x} defined by $C_{V_x}h := C^-(h(V_x - 1))$, the system of integral and algebraic Equations (2.24) and (2.25) reduces to

$$(1 - C_{V_x})(M_x - 1) + \sum_{\zeta \in \mathcal{Z}} \frac{M_x(\zeta)V_x(\zeta)}{\zeta - z} = C_{V_x}1, \quad z \in \mathbb{R} \tag{2.26}$$

and

$$\begin{aligned} M_x(z) &+ \sum_{\zeta \in \mathcal{Z}\setminus\{z\}} \frac{M_x(\zeta)V_x(\zeta)}{\zeta - z} - \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{(M_x(\zeta) - 1)(V_x(\zeta) - 1)}{\zeta - z} d\zeta \\ &= 1 + \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{(V_x(\zeta) - 1)}{\zeta - z} d\zeta, \quad z \in \mathcal{Z}. \end{aligned} \tag{2.27}$$

By Lemma 5.2 [7], there exists a fixed c s.t. for $\|r\|_{L^\infty(\mathbb{R})} = \rho$ the operator $1 - C_{V_x}$ is invertible in $L^2(\mathbb{R})$ and $\|(1 - C_{V_x})^{-1}\|_{L^2 \to L^2} \leq c\langle \rho \rangle^2$. It is easy to conclude by the Fredholm alternative that the inhomogeneous system (2.26)–(2.27) admits exactly one solution if and only if $f = 0$ is the only solution $f : \mathbb{R} \cup \mathcal{Z} \to M_{2\times 2}(\mathbb{C})$ with $f_{|\mathbb{R}} \in L^2(\mathbb{R})$ of

$$\begin{aligned} &(1 - C_{V_x})f + \sum_{\zeta \in \mathcal{Z}} \frac{f(\zeta)V_x(\zeta)}{\zeta - z} = 0, \quad z \in \mathbb{R} \\ &f(z) + \sum_{\zeta \in \mathcal{Z}\setminus\{z\}} \frac{f(\zeta)V_x(\zeta)}{\zeta - z} - \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{f(\zeta)(V_x(\zeta) - 1)}{\zeta - z} d\zeta = 0, \quad z \in \mathcal{Z}. \end{aligned} \tag{2.28}$$

We set for $z \in \mathbb{C}\setminus(\mathcal{Z} \cup \mathbb{R})$

$$m(z) := -\sum_{\zeta \in \mathcal{Z}} \frac{f(\zeta)V_x(\zeta)}{\zeta - z} + \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{f(\zeta)(V_x(\zeta) - 1)}{\zeta - z} d\zeta. \tag{2.29}$$

Notice that

$$m_-(z) = -\sum_{\zeta\in\mathcal{Z}} \frac{f(\zeta)V_x(\zeta)}{\zeta - z} + C^-(f(V_x-1)). \tag{2.30}$$

By $C^-(f(V_x-1)) = C_{V_x} f$ and (2.28) we get $m_-(z) = f(z)$ for $z \in \mathbb{R}$. We have

$$\begin{aligned} m_+(z) &= -\sum_{\zeta\in\mathcal{Z}} \frac{f(\zeta)V_x(\zeta)}{\zeta - z} + C^+(f(V_x-1)) \\ &= -\sum_{\zeta\in\mathcal{Z}} \frac{f(\zeta)V_x(\zeta)}{\zeta - z} + C^-(f(V_x-1))(z) + f(z)(V_x(z)-1) \\ &= m_-(z) + m_-(z)(V_x - 1) = m_-(z)V_x(z). \end{aligned} \tag{2.31}$$

We have

$$\begin{aligned} 0 &= \int_{\mathbb{R}} C^+(f(V_x-1))\left(C^-(f(V_x-1))\right)^* \\ &= \int_{\mathbb{R}} \left(m_+(z) + \sum_{\zeta\in\mathcal{Z}} \frac{f(\zeta)V_x(\zeta)}{\zeta - z}\right)\left(m_-(z) + \sum_{\zeta\in\mathcal{Z}} \frac{f(\zeta)V_x(\zeta)}{\zeta - z}\right)^* dz \\ &= \int_{\mathbb{R}} m_+ m_-^* dz + \sum_{\zeta\in\mathcal{Z}} \int_{\mathbb{R}} \frac{m_+(z)dz}{\bar{\zeta} - z} V_x^*(\zeta) f^*(\zeta) + \left(\sum_{\zeta\in\mathcal{Z}} \int_{\mathbb{R}} \frac{m_-(z)dz}{\bar{\zeta} - z} V_x^*(\zeta) f^*(\zeta)\right)^* \\ &\quad + \sum_{A\in\{+,-\}} \sum_{\zeta,\xi\in\mathcal{Z}_A} f(\zeta)V_x(\zeta)V_x^*(\xi)f^*(\xi) \int_{\mathbb{R}} \frac{dz}{(\zeta - z)(\bar{\xi} - z)}. \end{aligned} \tag{2.32}$$

The $A = +$ term in the last line cancels with the following:

$$\begin{aligned} \sum_{\zeta\in\mathcal{Z}_+} \int_{\mathbb{R}} \frac{m_+(z)dz}{\bar{\zeta} - z} V_x^*(\zeta) f^*(\zeta) &= \sum_{\zeta\in\mathcal{Z}_+} \int_{\mathbb{R}} \frac{C^+(f(V_x-1))(z)dz}{\bar{\zeta} - z} V_x^*(\zeta) f^*(\zeta) \\ &\quad - \sum_{\zeta\in\mathcal{Z}_+} \sum_{\xi\in\mathcal{Z}} \int_{\mathbb{R}} \frac{dz}{(\bar{\zeta} - z)(\xi - z)} f(\xi)V_x(\xi)V_x^*(\zeta)f^*(\zeta) \\ &= -\sum_{\zeta,\xi\in\mathcal{Z}_+} \int_{\mathbb{R}} \frac{dz}{(\bar{\zeta} - z)(\xi - z)} f(\xi)V_x(\xi)V_x^*(\zeta)f^*(\zeta). \end{aligned}$$

We have, by $\mathrm{Res}(m, \bar{\zeta})V_x^*(\zeta) = f(\bar{\zeta})V_x(\bar{\zeta})V_x^*(\zeta) = -f(\bar{\zeta})V_x^*(\zeta)V_x^*(\zeta) = 0$,

$$\begin{aligned} \sum_{\zeta\in\mathcal{Z}_-} \int_{\mathbb{R}} \frac{m_+(z)dz}{\bar{\zeta} - z} V_x^*(\zeta) f^*(\zeta) &= \sum_{\zeta\in\mathcal{Z}_-} \int_{\mathbb{R}} \left(m_+(z) - \frac{\mathrm{Res}(m, \bar{\zeta})}{z - \bar{\zeta}}\right) \frac{V_x^*(\zeta)f^*(\zeta)}{\bar{\zeta} - z} dz \\ &= -2\pi \mathrm{i} \sum_{\zeta\in\mathcal{Z}_-} f(\bar{\zeta})V_x^*(\zeta)f^*(\zeta) + \sum_{\zeta\in\mathcal{Z}_-} \sum_{\xi\in\mathcal{Z}_+\setminus\{\bar{\zeta}\}} f(\xi) \underbrace{V_x(\xi)V_x^*(\zeta)}_{0} f^*(\zeta) \frac{2\pi \mathrm{i}}{\bar{\zeta} - \xi} \\ &= -2\pi \mathrm{i} \sum_{\zeta\in\mathcal{Z}_-} f(\bar{\zeta})V_x^*(\zeta)f^*(\zeta). \end{aligned} \tag{2.33}$$

Here, we have used the fact that for all $\zeta \in \mathcal{Z}$ by (2.28)–(2.29) we have

$$\lim_{z\to\zeta}\left(m(z)-\frac{\mathrm{Res}(m,\zeta)}{z-\zeta}\right)=C(f(V_x-1))(\zeta)-\sum_{\zeta'\in\mathcal{Z}\setminus\{\zeta\}}\frac{f(\zeta')V_x(\zeta')}{\zeta'-\zeta}=f(\zeta).$$

The following term cancels with the $A=-$ term in the last line of (2.32):

$$\begin{aligned}\left(\sum_{\zeta\in\mathcal{Z}_-}\int_{\mathbb{R}}\frac{m_-(z)dz}{\overline{\zeta}-z}V_x^*(\zeta)f^*(\zeta)\right)^* &= \sum_{\zeta\in\mathcal{Z}_-} f(\zeta)V_x(\zeta)\left[\int_{\mathbb{R}}\frac{(C^-(f(V_x-1)))^*}{\zeta-z}dz\right.\\ &\quad\left.-\sum_{\xi\in\mathcal{Z}}\int_{\mathbb{R}}\frac{dz}{(\zeta-z)(\overline{\xi}-z)}f(\zeta)V_x(\zeta)V_x^*(\xi)f^*(\xi)\right]\\ &=-\sum_{\zeta,\xi\in\mathcal{Z}_-}f(\zeta)V_x(\zeta)V_x^*(\xi)f^*(\xi)\int_{\mathbb{R}}\frac{dz}{(\zeta-z)(\overline{\xi}-z)}.\end{aligned}\tag{2.34}$$

We have

$$\begin{aligned}&\left(\sum_{\zeta\in\mathcal{Z}_+}\int_{\mathbb{R}}\frac{m_-(z)dz}{\overline{\zeta}-z}V_x^*(\zeta)f^*(\zeta)\right)^*\\ &=\left(\sum_{\zeta\in\mathcal{Z}_+}\int_{\mathbb{R}}\left(m_-(z)-\frac{\mathrm{Res}(m,\overline{\zeta})}{z-\overline{\zeta}}\right)\frac{V_x^*(\zeta)f^*(\zeta)}{\overline{\zeta}-z}dz\right)^*\\ &=\left(2\pi\mathrm{i}\sum_{\zeta\in\mathcal{Z}_+}f(\overline{\zeta})V_x^*(\zeta)f^*(\zeta)\right)^*=-2\pi\mathrm{i}\sum_{\zeta\in\mathcal{Z}_+}f(\zeta)V_x(\zeta)f^*(\overline{\zeta}).\end{aligned}\tag{2.35}$$

The terms from (2.33) and (2.35) cancel out in (2.32) because of $V_x(\overline{\zeta})=-V_x^*(\zeta)$. Then, by $m_+=m_-V_x$, (2.32) yields

$$0=\int_{\mathbb{R}}m_-(z)V_x(z)m_-^*(z)dz.\tag{2.36}$$

Since $V_x(z)$ is strictly positive, this implies $m_-(z)=f(z)=0$ for $z\in\mathbb{R}$. But then by (2.30) we have also $f(\zeta)V_x(\zeta)=0$ for $\zeta\in\mathcal{Z}$. Then $f(z)=0$ for $z\in\mathcal{Z}$ by (2.28). So, we have completed the proof that if f solves (2.28) then $f=0$. □

We now recall another result due to Zhou [3] on the inverse scattering, which we state in Lemma 2.7 below. This result is only stated for the case of pure radiation solutions of the cubic NLS equation with $n=0$. We need Lemma 2.7 in order to establish the fact that the map $\mathcal{G}_0\cap L^{2,s}(\mathbb{R})\to\mathcal{S}(s,0)$ is not only one-to-one but also onto.

Lemma 2.7 *Let $r\in H^s(\mathbb{R})$, $\mathcal{Z}=\emptyset$, and consider the potential u defined by the reconstruction formula (2.21). Then $u\in L^{2,s}(\mathbb{R})$. Furthermore, for any positive κ_0, there is a constant C such that for $\|r\|_{L^\infty(\mathbb{R})}\le\kappa_0$, we have $\|u\|_{L^{2,s}(\mathbb{R})}\le C\|r\|_{H^s(\mathbb{R})}$.*

Proof We only sketch the argument, referring to references [3,5,7] for more information and details. We first sketch $u(x) \in L^{2,s}(\mathbb{R}_+)$. We factorize the matrix in (2.18) writing $V_x(z) = V_{x-}^{-1} V_{x+}$, where

$$V_{x+}(z) := \begin{pmatrix} 1 & 0 \\ e^{2\mathrm{i}xz} r(z) & 1 \end{pmatrix}, \; V_{x-}(z) := \begin{pmatrix} 1 & -e^{-2\mathrm{i}xz}\overline{r}(z) \\ 0 & 1 \end{pmatrix}. \tag{2.37}$$

Set now $C_{w_x} h := C^+(h w_{x-}) + C^-(h w_{x+})$ for $w_{x\pm} := \pm(V_{x\pm} - 1)$. Then we consider a function $\mu_x \in 1 + L^2(\mathbb{R})$ such that

$$(1 - C_{w_x})(\mu_x)(z) = 1. \tag{2.38}$$

For $w_x(\zeta) := V_{x+}(\zeta) - V_{x-}(\zeta)$ we get that the $m(x,z)$ in (2.23) (in the case when all the $c_j = 0$) can be expressed also as

$$m(x,z) = 1 + \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{\mu_x(\zeta) w_x(\zeta)}{\zeta - z} d\zeta. \tag{2.39}$$

For $x \geq 0$, by the argument in Lemma 3.4 [5] for a fixed c_s we have

$$\|C^{\pm}(1 - V_{x\mp}(z))\|_{L^2_z(\mathbb{R})} \leq c_s \langle x \rangle^{-s} \|r\|_{H^s(\mathbb{R})}$$

(notice that $x \leq 0$ in Lemma 3.4 [5], because of the different definition of the operator in (2.1)). This implies immediately

$$\|C_{w_x} 1\|_{L^2_z(\mathbb{R})} \leq 2c_s \langle x \rangle^{-s} \|r\|_{H^s(\mathbb{R})}.$$

We consider

$$\mu_x - 1 = \left(1 - C_{w_x}\right)^{-1} C_{w_x} 1$$

and correspondingly

$$\|\mu_x - 1\|_{L^2_z} \leq \| \left(1 - C_{w_x}\right)^{-1} \|_{L^2_z \to L^2_z} \|C_{w_x} 1\|_{L^2_z}.$$

We have $\| \left(1 - C_{w_x}\right)^{-1} \|_{L^2_z \to L^2_z} \leq c\langle \rho \rangle^2$ by Lemma 5.2 [7] for a fixed c, where $\rho := \|r\|_{L^\infty(\mathbb{R})}$. We conclude that for $x \geq 0$ and for any κ_0 there is a constant C such that

$$\|\mu_x - 1\|_{L^2_z} \leq C \langle x \rangle^{-s} \|r\|_{H^s(\mathbb{R})}$$

for $\rho \leq \kappa_0$. Finally, the argument in Theorem 3.5 [5] yields $\|u\|_{L^{2,s}(\mathbb{R}_+)} \leq C\|r\|_{H^s(\mathbb{R})}$.

In order to prove $u(x) \in L^{2,s}(\mathbb{R}_-)$ we consider instead the decomposition

$$V_x = \begin{pmatrix} 1 & 0 \\ \frac{e^{2\mathrm{i}xz} r(z)}{1+|r(z)|^2} & 1 \end{pmatrix} \begin{pmatrix} 1 + |r(z)|^2 & 0 \\ 0 & \frac{1}{1+|r(z)|^2} \end{pmatrix} \begin{pmatrix} 1 & \frac{e^{-2\mathrm{i}xz}\overline{r}(z)}{1+|r(z)|^2} \\ 0 & 1 \end{pmatrix}.$$

We then consider the RH problem with matrix $\widetilde{V}_x := \delta_-^{\sigma_3} V_x \delta_+^{-\sigma_3}$ for $\delta(z)$ the solution of the problem (3.1) with $z_0 = +\infty$ introduced later in Proposition 3.3. Correspondingly, we get estimates $\|\widetilde{u}\|_{L^{2,s}(\mathbb{R}_-)} \leq C\|\widetilde{r}\|_{H^s(\mathbb{R})} \leq c\|r\|_{H^s(\mathbb{R})}$ for a function $\widetilde{u}$ associated to $\widetilde{r} := r\delta_+\delta_-$ and for fixed c when $\rho \leq \kappa_0$, by proceeding as above. Finally, $\widetilde{u} = u$. For more details see [5]. □

We now discuss the representation of the solutions of the Cauchy problem (1.1) in terms of the inverse scattering transform. We recall the following result, see [26].

THEOREM 2.8 *Given $u_0 \in H^1(\mathbb{R})$, then there exists a unique solution $u(t) \in C^0(\mathbb{R}, H^1(\mathbb{R})) \cap L^4_{loc}(\mathbb{R}, L^\infty(\mathbb{R}))$ of the integral Equation (1.2).*

The solution of Theorem 2.8 is the same of Theorem 1.1.

Suppose that $u_0 \in H^1(\mathbb{R}) \cap L^{2,s}(\mathbb{R})$ for fixed $s \in \left(\frac{1}{2}, 1\right]$. Then the solution remains in $u(t) \in H^1(\mathbb{R}) \cap L^{2,s}(\mathbb{R})$ for all $t \in \mathbb{R}$, by standard arguments (see p.1072 in [18], which can be extended to noninteger s by Lemma 2.3 in [8]). For the solution of the cubic NLS Equation (1.1) with $u_0 \in H^1(\mathbb{R}) \cap L^{2,s}(\mathbb{R})$, the time evolution of the scattering data is well defined, according to the following result (see p.39 in [24]):

LEMMA 2.9 *For an initial datum $u_0 \in H^1(\mathbb{R}) \cap L^{2,s}(\mathbb{R}) \cap \mathcal{G}$ we have $u(t) \in H^1(\mathbb{R}) \cap L^{2,s}(\mathbb{R}) \cap \mathcal{G}$ for all $t \in \mathbb{R}$ and the spectral data $\mathcal{S}(s,n)$ in (2.17) evolve as follows:*

$$e^{4iz^2t} r(z) \in H^s(\mathbb{R}), \quad (z_1, \ldots, z_n) \in \mathbb{C}^n_+, \quad (e^{4iz_1^2t} c_1, \ldots, e^{4iz_n^2t} c_n) \in \mathbb{C}^n_*. \tag{2.40}$$

Remark 2.10 To recover the solitons (1.3), we take the spectral data:

$$r = 0, \quad z_1 = \alpha_1 + i\beta_1 \in \mathbb{C}_+, \quad e^{-4iz_1^2t} c_1 \in \mathbb{C}_*. \tag{2.41}$$

Then, we obtain

$$u(x,t) = -2i\beta_1 e^{-2i\alpha_1 x - 4it(\alpha_1^2 - \beta_1^2) - i\psi_0} \operatorname{sech}(2\beta_1 x + 8t\alpha_1\beta_1 - \delta_0), \tag{2.42}$$

where $\delta_0 := \log\left(\frac{|c_1|}{2\beta_1}\right)$ and $\psi_0 := \arg(c_1)$. Note the correspondence: $\omega = 2\beta_1$ and $v = -2\alpha_1$, for solitons in (1.3).

3. Dispersion for pure radiation solutions

Elements of $\mathcal{G}$ such that $\mathcal{Z}_+ = \emptyset$ generate pure radiation solutions of the cubic NLS equation. These solutions satisfy the following asymptotic behavior.

THEOREM 3.1 *Fix $s \in (1/2, 1]$. Let $u_0 \in \mathcal{G} \cap L^{2,s}(\mathbb{R})$ such that $\mathcal{Z} = \emptyset$. Then there exist constants $C(u_0) > 0$ and $T(u_0) > 0$ such that the solution of the cubic NLS Equation (1.1) satisfies*

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \le C(u_0)|t|^{-\frac{1}{2}} \text{ for all } |t| \ge T(u_0).$$

There are furthermore constants $C_0 > 0$, $T_0 > 0$ and small $\varepsilon_0 > 0$ such that for $\|u_0\|_{L^{2,s}(\mathbb{R})} < \varepsilon_0$, we can take $C(u_0) = C_0\|u_0\|_{L^{2,s}(\mathbb{R})}$ and $T(u_0) = T_0$.

Remark 3.2 In [8], the result of Theorem 3.1 is proved with $L^{2,s}(\mathbb{R})$ replaced by Σ_s for any $s > \frac{1}{2}$, only in the case of small u_0 with $\|u_0\|_{\Sigma_s} < \varepsilon_0$. In the case of the defocusing NLS Equation (1.1) (that is, with $+2|u|^2u$ replaced by $-2|u|^2u$), Theorem 3.1 for $s = 1$ is proved in [4,5]. For the focusing NLS Equation (1.1), Theorem 3.1 for $s = 1$ is proved in [7]. Notice also that all these references contain proofs of the asymptotic expansions for the solution u at large t, which we do not discuss here.

In the rest of Section 3 we prove Theorem 3.1. With minor modifications, we follow closely the proof in [6], which involves the $\overline{\partial}$ operator, where $\overline{\partial} := \frac{1}{2}(\partial_x + i\partial_y)$. Here we extend the result in [6], valid for $s = 1$, to any $s \in (1/2, 1]$.

3.1. *Proof of Theorem 3.1*

The proof starts by assuming additionally that $u_0 \in H^1(\mathbb{R})$. Fix $z_0 \in \mathbb{R}$. First of all we consider the scalar RH problem

$$\begin{cases} \delta_+(z) = \delta_-(z)(1+|r(z)|^2) & \text{for z} < \text{z}_0 \\ \delta_+(z) = \delta_-(z) & \text{for } z > z_0 \end{cases} \tag{3.1}$$

with $\delta(z)$ holomorphic in $\mathbb{C}\backslash\mathbb{R}$ and $\delta(z) \to 1$ as $z \to \infty$. The following statement is in Proposition 5.1 [7] and is elementary to prove.

Proposition 3.3 *We have*

$$\delta(z) = e^{\gamma(z)}, \quad \gamma(z) := \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{z_0} \frac{\log(1+|r(\varsigma)|^2)}{\varsigma - z} d\varsigma.$$

For $z \not\leq z_0$ *we have* $\delta(z) = \overline{\delta(\overline{z})}$ *and* $\langle\rho\rangle^{-1} \leq |\delta(z)| \leq \langle\rho\rangle$ *where* $\rho := \|r\|_{L^\infty(\mathbb{R})}$; *for* $\mp \operatorname{Im} z > 0$ *we have* $|\delta^{\pm 1}(z)| \leq 1$.

The function $\gamma(z)$ has an expansion

$$\begin{aligned} \gamma(z) &= \mathrm{i}\nu(z_0)\log(z-z_0) + \mathrm{i}\nu(z_0)(z-z_0)\log(z-z_0) \\ &\quad - \mathrm{i}\nu(z_0)(z-z_0+1)\log(z-z_0+1) + \beta(z,z_0) \end{aligned} \tag{3.2}$$

where in the r.h.s. the main term is the first, and where

$$\nu(z_0) := -\frac{1}{2\pi}\log(1+|r(z_0)|^2) \text{ and for } \chi(\zeta,z_0) = \chi_{[z_0-1,z_0]}(\zeta)(\zeta - z_0 + 1)$$

$$\beta(z,z_0) = \int_{-\infty}^{z_0} \left\{\log(1+|r(\zeta)|^2) - \log(1+|r(z_0)|^2)\chi(\zeta,z_0)\right\} \frac{d\zeta}{2\pi\mathrm{i}(\zeta - z)}. \tag{3.3}$$

Let

$$\theta(z) := 2(z-z_0)^2 - 2z_0^2 \text{ with } z_0 := -\frac{x}{4t}. \tag{3.4}$$

Then we consider the RH problem (i)–(iii) with

$$v_{t,x}(z) := \begin{pmatrix} 1+|r(z)|^2 & e^{-2\mathrm{i}t\theta}\overline{r}(z) \\ e^{2\mathrm{i}t\theta} r(z) & 1 \end{pmatrix}.$$

We factorize

$$v_{t,x} = \begin{cases} W_L W_R & \text{for } z > z_0 \\ U_L U_0 U_R & \text{for } z < z_0 \end{cases}$$

for

$$W_L = \begin{pmatrix} 1 & e^{-2\mathrm{i}t\theta}\overline{r}(z) \\ 0 & 1 \end{pmatrix}, \ W_R = \begin{pmatrix} 1 & 0 \\ e^{2\mathrm{i}t\theta} r(z) & 1 \end{pmatrix}, \ U_L = \begin{pmatrix} 1 & 0 \\ \frac{e^{2\mathrm{i}t\theta} r(z)}{1+|r(z)|^2} & 1 \end{pmatrix}$$

$$U_R = \begin{pmatrix} 1 & \frac{e^{-2\mathrm{i}t\theta}\overline{r}(z)}{1+|r(z)|^2} \\ 0 & 1 \end{pmatrix}, \ U_0 = \begin{pmatrix} 1+|r(z)|^2 & 0 \\ 0 & \frac{1}{1+|r(z)|^2} \end{pmatrix}. \tag{3.5}$$

We end Section 3.1 with an estimate on the function $\beta(z,z_0)$.

LEMMA 3.4 *Let $L_\phi = z_0 + e^{-\mathrm{i}\phi}\mathbb{R} = \{z = z_0 + e^{-\mathrm{i}\phi}u : u \in \mathbb{R}\}$. Consider the $s \in (1/2, 1]$ in Theorem 1.3. Then there is a fixed $C(\rho, s)$ s.t. for any $z_0 \in \mathbb{R}$ and any $\phi \in (0, \pi)$*

$$\|\beta(e^{-\mathrm{i}\phi}\cdot, z_0)\|_{H^s(\mathbb{R})} \le C(\rho, s)\|r\|_{H^s(\mathbb{R})} \tag{3.6}$$

$$|\beta(z, z_0) - \beta(z_0, z_0)| \le C(\rho, s)\|r\|_{H^s(\mathbb{R})}|z - z_0|^{s-\frac{1}{2}} \quad \text{for all } z \in L_\phi. \tag{3.7}$$

Proof First of all these estimates hold for $s = 1$, and are a consequence of $\|C_{\mathbb{R}} f\|_{H^\tau(L_\phi)} \le C_\tau \|f\|_{H^\tau(\mathbb{R})}$ for $\tau = 0, 1$, which are proved in Lemma 23.3 [27]. We obtain (3.6) for $\tau = s$ when $s \in (0, 1)$ by interpolation. The estimate (3.7) is a consequence of (3.6) and of the following elementary estimate when $s \in (1/2, 1]$:

$$|f(x) - f(y)| \le C_s \|f\|_{H^s(\mathbb{R})}|x - y|^{s-\frac{1}{2}} \quad \text{for all } x, y \in \mathbb{R} \text{ and } f \in H^s(\mathbb{R}) \text{ for a fixed } C_s. \tag{3.8}$$

This is an elementary consequence of $f(x+h) - f(x) = \frac{1}{\sqrt{2\pi}} \int e^{\mathrm{i}x\xi}(e^{\mathrm{i}h\xi} - 1)\widehat{f}(\xi)d\xi$ for $y = x + h$. Then for any $\kappa > 0$ we have for a fixed C_s

$$\begin{aligned} |f(x+h) - f(x)| &\le \frac{|h|}{\sqrt{2\pi}} \left(\int_{|\xi| \le \kappa} |\xi|^{2-2s} d\xi \right)^{\frac{1}{2}} \|f\|_{H^s} + \frac{1}{\sqrt{2\pi}} \left(\int_{|\xi| \ge \kappa} |\xi|^{-2s} d\xi \right)^{\frac{1}{2}} \|f\|_{H^s} \\ &\le C_s \left(|h|\kappa^{\frac{3-2s}{2}} + \kappa^{\frac{1-2s}{2}} \right) \|f\|_{H^s}. \end{aligned}$$

The r.h.s. equals $2C_s|h|^{s-\frac{1}{2}}\|f\|_{H^s}$ for $\kappa = |h|^{-1}$. □

3.1.1. *The model RH problem*

We consider the RH problem

$$\begin{cases} P \text{analytic} \quad \text{in } \mathbb{C}\backslash\Sigma_P \\ P(\zeta) = 1 + \frac{P_1}{\zeta} + O(\zeta^{-2}) \quad \text{as } \zeta \to \infty \\ P_+(\zeta) = P_+(\zeta) \quad V_P(\zeta) \text{ in } \Sigma_P \end{cases} \tag{3.9}$$

where $\Sigma_P := \overline{\cup_{n=1}^4 \Sigma_P^n}$ with $\Sigma_P^1 = e^{\mathrm{i}\frac{\pi}{4}}\mathbb{R}_+$, $\Sigma_P^2 = e^{-\mathrm{i}\frac{\pi}{4}}\mathbb{R}_-$, $\Sigma_P^3 = e^{\mathrm{i}\frac{\pi}{4}}\mathbb{R}_-$ and $\Sigma_P^4 = e^{-\mathrm{i}\frac{\pi}{4}}\mathbb{R}_+$ inheriting the orientations of $\mathbb{R}_\pm$. The matrix $V_P(\zeta)$ is defined by

$$V_P(\zeta) := \begin{cases} \begin{pmatrix} 1 & 0 \\ r_0\zeta^{-2\mathrm{i}\nu_0}e^{\mathrm{i}\zeta^2/2} & 1 \end{pmatrix} & \text{for } \zeta \in \Sigma_P^1 \\ \begin{pmatrix} 1 & \frac{\overline{r}_0}{1+|r_0|^2}\zeta^{2\mathrm{i}\nu_0}e^{-\mathrm{i}\zeta^2/2} \\ 0 & 1 \end{pmatrix} & \text{for } \zeta \in \Sigma_P^2 \\ \begin{pmatrix} 1 & 0 \\ \frac{r_0}{1+|r_0)|^2}\zeta^{-2\mathrm{i}\nu_0}e^{\mathrm{i}\zeta^2/2} & 1 \end{pmatrix} & \text{for } \zeta \in \Sigma_P^3 \\ \begin{pmatrix} 1 & \overline{r}_0\zeta^{2\mathrm{i}\nu_0}e^{-\mathrm{i}\zeta^2/2} \\ 0 & 1 \end{pmatrix} & \text{for } \zeta \in \Sigma_P^4 \end{cases} \tag{3.10}$$

where r_0 is a free parameter that we fix in (3.15) and $\nu_0 = \nu(z_0)$. The solution of this RH problem can be worked out following word by word [4,p.54–57]. Set

$$k_1 := \frac{-\mathrm{i}\sqrt{2\pi}e^{\mathrm{i}\pi/4}e^{-\pi\nu_0/2}}{r_0\Gamma(-\mathrm{i}\nu_0)}, \quad k_2 := \frac{\nu_0}{k_1}. \tag{3.11}$$

Consider for $\mathrm{Im}\,\zeta > 0$ the matrix $\Psi^+(\zeta)$ with

$$\Psi^+_{11}(\zeta) = e^{-3\pi\nu_0/4} D_{\mathrm{i}\nu_0}(e^{-3\mathrm{i}\pi/4}\zeta)\,, \quad \Psi^+_{22}(\zeta) = e^{\pi\nu_0/4} D_{-\mathrm{i}\nu_0}(e^{-\mathrm{i}\pi/4}\zeta)\,,$$
$$\Psi^+_{12}(\zeta) = e^{\pi\nu_0/4}(-\mathrm{i}k_2)^{-1}\left(\partial_\zeta(D_{-\mathrm{i}\nu_0}(e^{-\mathrm{i}\pi/4}\zeta)) - \frac{\mathrm{i}\zeta}{2} D_{-\mathrm{i}\nu_0}(e^{-\mathrm{i}\pi/4}\zeta)\right)\,,$$
$$\Psi^+_{21}(\zeta) = e^{-3\pi\nu_0/4}(\mathrm{i}k_1)^{-1}\left(\partial_\zeta(D_{\mathrm{i}\nu_0}(e^{-\mathrm{i}3\pi/4}\zeta)) + \frac{\mathrm{i}\zeta}{2} D_{\mathrm{i}\nu_0}(e^{-\mathrm{i}3\pi/4}\zeta)\right)\,.$$

Consider for $\mathrm{Im}\,\zeta < 0$ the matrix $\Psi^-(\zeta)$ with

$$\Psi^-_{11}(\zeta) = e^{\pi\nu_0/4} D_{\mathrm{i}\nu_0}(e^{\mathrm{i}\pi/4}\zeta)\,, \quad \Psi^-_{22}(\zeta) = e^{-3\pi\nu_0/4} D_{\mathrm{i}\nu_0}(e^{3\mathrm{i}\pi/4}\zeta)\,,$$
$$\Psi^-_{12}(\zeta) = e^{-3\pi\nu_0/4}(-\mathrm{i}k_2)^{-1}\left(\partial_\zeta(D_{-\mathrm{i}\nu_0}(e^{\mathrm{i}3\pi/4}\zeta)) - \frac{\mathrm{i}\zeta}{2} D_{-\mathrm{i}\nu_0}(e^{\mathrm{i}3\pi/4}\zeta)\right)\,,$$
$$\Psi^-_{21}(\zeta) = e^{\pi\nu/4}(\mathrm{i}k_1)^{-1}\left(\partial_\zeta(D_{\mathrm{i}\nu_0}(e^{\mathrm{i}\pi/4}\zeta)) + \frac{\mathrm{i}\zeta}{2} D_{\mathrm{i}\nu_0}(e^{\mathrm{i}\pi/4}\zeta)\right)\,.$$

Here $D_a(\zeta)$ is the unique entire function solving

$$\frac{d^2}{d\zeta^2} D_a(\zeta) + \left(\frac{1}{2} - \frac{\zeta^2}{4} + a\right) D_a(\zeta) = 0$$

that for $|\arg(\zeta)| < 3\pi/4$ satisfies

$$D_a(\zeta) \sim e^{-\zeta^2/4}\zeta^a\left(1 + \sum_{n=1}^{\infty}(-1)^n 2^n \frac{\prod_{j=1}^n\left(\frac{a}{2} - (j-1)\right)\left(\frac{a}{2} - (j-1/2)\right)}{n!\zeta^{2n}}\right)$$

and such that $D'_a(\zeta) + \frac{\zeta}{2} D_a(\zeta) = a D_{a-1}(\zeta)$, (3.12)

see Chapter 16 [28]. If we introduce the angular sectors

$$\Omega_1 = \{\zeta : \arg\zeta \in (0, \pi/4\}\,, \quad \Omega_2 = \{\zeta : \arg\zeta \in (\pi/4, 3\pi/4)\}\,,$$
$$\Omega_3 = \{\zeta : \arg\zeta \in (3\pi/4, \pi)\}\,, \quad \Omega_4 = \{\zeta : \arg\zeta \in (\pi, 5\pi/4)\}\,,$$
$$\Omega_5 = \{\zeta : \arg\zeta \in (\pi + 5\pi/4, 7\pi/4)\}\,, \quad \Omega_6 = \{\zeta : \arg\zeta \in (7\pi/4, 2\pi)\}\,,$$

then, following,[4] see also [7], we have

$$P(\zeta) = \Psi^+(\zeta)\begin{pmatrix} 1 & 0 \\ -r_0 & 1 \end{pmatrix}\zeta^{-\mathrm{i}\nu_0\sigma_3} e^{\mathrm{i}\zeta^2\sigma_3/4} \quad \text{for } \zeta \in \Omega_1\,,$$
$$P(\zeta) = \Psi^+(\zeta)\zeta^{-\mathrm{i}\nu_0\sigma_3} e^{\mathrm{i}\zeta^2\sigma_3/4} \quad \text{for } \zeta \in \Omega_2\,,$$
$$P(\zeta) = \Psi^+(\zeta)\begin{pmatrix} 1 & \frac{-\overline{r}_0}{1+|r_0|^2} \\ 0 & 1 \end{pmatrix}\zeta^{-\mathrm{i}\nu_0\sigma_3} e^{\mathrm{i}\zeta^2\sigma_3/4} \quad \text{for } \zeta \in \Omega_3\,,$$
$$P(\zeta) = \Psi^-(\zeta)\begin{pmatrix} 1 & 0 \\ \frac{r_0}{1+|r_0|^2} & 1 \end{pmatrix}\zeta^{-\mathrm{i}\nu_0\sigma_3} e^{\mathrm{i}\zeta^2\sigma_3/4} \quad \text{for } \zeta \in \Omega_4\,,$$
$$P(\zeta) = \Psi^-(\zeta)\zeta^{-\mathrm{i}\nu_0\sigma_3} e^{\mathrm{i}\zeta^2\sigma_3/4} \quad \text{for } \zeta \in \Omega_5\,,$$
$$P(\zeta) = \Psi^-(\zeta)\begin{pmatrix} 1 & \overline{r}_0 \\ 0 & 1 \end{pmatrix}\zeta^{-\mathrm{i}\nu_0\sigma_3} e^{\mathrm{i}\zeta^2\sigma_3/4} \quad \text{for } \zeta \in \Omega_6\,. \tag{3.13}$$

The fact that $P(\zeta)$ satisfies the model RH problem (3.9) can be seen by direct computation (specifically, it solves an equivalent RH with an additional jump matrix 1 over $\mathbb{R}$: this fact can be checked directly by exploiting the fact that $\Psi^+(\zeta) = \Psi^-(\zeta)\begin{pmatrix} 1+|r_0|^2 & \overline{r}_0 \\ r_0 & 1 \end{pmatrix}$ and the monodromy properties of z^ν like in [4,p.48]).

By elementary computations which use (3.13) and (3.12), see [4], we have

$$\lim_{\mathbb{C}_\pm \ni \zeta \to \infty} [\Psi^\pm(\zeta)\zeta^{-\mathrm{i}\nu_0\sigma_3} e^{\mathrm{i}\zeta^2\sigma_3/4} - 1]\zeta = P_1 \text{ with } P_1 := \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix}. \tag{3.14}$$

Exploiting the rapid convergence to 1 as $\zeta \to \infty$ of the extension of $(V_P)|_{\Sigma_P^1}$ to Ω_1, of $(V_P)|_{\Sigma_P^2}$ to Ω_3, of $(V_P)|_{\Sigma_P^3}$ to Ω_4 and of $(V_P)|_{\Sigma_P^4}$ to Ω_6, it is easy to conclude that $\lim_{\zeta\to\infty}\zeta(P(\zeta)-1) = P_1$ in each sector Ω_j. In each sector we have $\det P(\zeta) = 1$, see [4, p.54].

With respect to the analysis in [4], we need to add few more remarks of quantitative nature on $P(\zeta)$. We fix

$$r_0 := \widehat{r}_0 e^{\mathrm{i}\nu_0 \log(8t) - 4\mathrm{i}t z_0^2} \quad \text{and} \quad \widehat{r}_0 := r(z_0)e^{-2\mathrm{i}\nu(z_0) - 2\beta(z_0,z_0)}. \tag{3.15}$$

By $|r_0| = |r(z_0)| \le C_s \|r\|_{H^s(\mathbb{R})} \le C(u_0)$ there is a $C(u_0)$ such that by (3.3) and (3.11) we get $|k_1| + |k_2| \le C(u_0)$. Furthermore the following is true.

Lemma 3.5 *Let $\rho = \|r\|_{L^\infty(\mathbb{R})}$. For any ρ_0 there exists a C such that for $\rho \le \rho_0$ we have*

$$|P(\zeta)| \le C \quad \text{for all } \zeta \notin \mathbb{R} \text{ and} \tag{3.16}$$

$$|P(\zeta) - 1 - P_1/\zeta| \le C\,\rho|\zeta|^{-2} \quad \text{if also } |\zeta| \ge 1. \tag{3.17}$$

Proof We focus only on (3.17), since (3.16) follows by (3.17) and by the fact that $D_a(\zeta)$ is an entire function in (a, ζ). The proof of (3.17) is based on formulas for $D_{\mathrm{i}\nu_0}(\zeta)$ for which we refer to Chapter 16 [28].

Recall that $D_{\mathrm{i}\nu_0}(\zeta) = 2^{\frac{\mathrm{i}\nu_0}{2}+\frac14}\zeta^{-\frac12} W_{\frac{\mathrm{i}\nu_0}{2}+\frac14,-\frac14}(\frac{\zeta^2}{2})$, where for $|\arg(z)| < 3\pi/2$ we have

$$W_{\frac{\mathrm{i}\nu_0}{2}+\frac14,-\frac14}(z) = e^{-z/2} z^{\mathrm{i}\frac{\nu_0}{2}+\frac14}\left[1 - z^{-1}\frac{\Gamma\left(\frac32 - \mathrm{i}\frac{\nu_0}{2}\right)\Gamma\left(1-\mathrm{i}\frac{\nu_0}{2}\right)}{\Gamma\left(\frac12 - \mathrm{i}\frac{\nu_0}{2}\right)\Gamma\left(-\mathrm{i}\frac{\nu_0}{2}\right)}\right.$$
$$\left. + \frac{1}{\Gamma\left(\frac12 - \mathrm{i}\frac{\nu_0}{2}\right)\Gamma\left(-\mathrm{i}\frac{\nu_0}{2}\right)}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty-\frac32}^{+\mathrm{i}\infty-\frac32} z^{\varsigma}\Gamma(\varsigma)\Gamma\left(-\varsigma+\frac12-\mathrm{i}\frac{\nu_0}{2}\right)\Gamma\left(-\varsigma-\mathrm{i}\frac{\nu_0}{2}\right)d\varsigma\right].$$

To bound the integral we use:

$|z^{\varsigma}| = |z|^{\mathrm{Re}(\varsigma)}e^{-t\arg(z)}$ for $\varsigma = \mathrm{Re}(\varsigma) + \mathrm{i}t$;

$|\Gamma(z)| \le \sqrt{2\pi}|z^{z-\frac12}|e^{\frac{K}{\mathrm{Re}(z)}}$ for $\mathrm{Re}\, z > 0$ and for $K > 0$ the constant in [28,p.249];

$\Gamma(\varsigma) = \dfrac{\Gamma(\varsigma+2)}{\varsigma(\varsigma+1)}$.

Then the absolute value of the integral is bounded by

$$C_1|z|^{-\frac{3}{2}}\int_{\mathbb{R}} e^{-t\arg(z)}e^{-t\arg\left(\frac{1}{2}+\mathrm{i}t\right)}\left|\frac{3}{2}-\mathrm{i}t\right|^{-1}\left|\frac{1}{2}-\mathrm{i}t\right|^{-1}$$
$$\times\, e^{\left(t+\frac{\nu_0}{2}\right)\arg\left(2-\mathrm{i}\left(t+\frac{\nu_0}{2}\right)\right)}\left|2-\mathrm{i}\left(t+\frac{\nu_0}{2}\right)\right|^{\frac{3}{2}} e^{\left(t+\frac{\nu_0}{2}\right)\arg\left(\frac{3}{2}-\mathrm{i}\left(t+\frac{\nu_0}{2}\right)\right)}\left|\frac{3}{2}-\mathrm{i}\left(t+\frac{\nu_0}{2}\right)\right|dt$$
$$\le C_2|z|^{-\frac{3}{2}}\int_{\mathbb{R}} e^{-t\arg(z)-|t|\frac{3}{2}\pi}\langle t\rangle^{\frac{1}{2}}dt \le C_3|z|^{-\frac{3}{2}}\left(\frac{3}{2}\pi-|\arg(z)|\right)^{-\frac{3}{2}}$$

for fixed constants which depend on ρ_0 and for $|\arg(z)| < \frac{3}{2}\pi$. This and the identity (3.12) yield inequality (3.17) if ζ is outside a union of preassigned small cones containing Σ_P. Near the cones we can proceed by estimating similarly the r.h.s.'s of the identities

$$D_{\mathrm{i}\nu_0}(\zeta) = e^{-\nu_0\pi}D_{\mathrm{i}\nu_0}(-\zeta) + \frac{\sqrt{2\pi}}{\Gamma(-\mathrm{i}\nu_0)}e^{\frac{\mathrm{i}}{2}(\mathrm{i}\nu_0+1)\pi}D_{-\mathrm{i}\nu_0-1}(-\mathrm{i}\zeta)\,,$$
$$D_{\mathrm{i}\nu_0}(\zeta) = e^{\nu_0\pi}D_{\mathrm{i}\nu_0}(-\zeta) + \frac{\sqrt{2\pi}}{\Gamma(-\mathrm{i}\nu_0)}e^{-\frac{\mathrm{i}}{2}(\mathrm{i}\nu_0+1)\pi}D_{-\mathrm{i}\nu_0-1}(\mathrm{i}\zeta)\,.$$

This completes the proof of Lemma 3.5. □

3.1.2. *The $\overline{\partial}$ argument*

We follow closely the argument of Dieng and McLaughlin [6] which have a simpler discussion than in [4,5,7] as to how to localize the RH to the model RH problem. We modify slightly [6] to allow the case $s \in (1/2, 1)$ in Theorem 3.1.

We fix a smooth cut-off function of compact support, with $\chi(x) \ge 0$ for any x and $\int \chi dx = 1$. For $\varepsilon \ne 0$ let $\chi_\varepsilon(x) = \varepsilon^{-1}\chi(\varepsilon^{-1}x)$. For $z \in \mathbb{C}$ and for the convolution $f * g(x) = \int f(x-y)g(y)dy$, we define $\mathbf{r}(z)$ as follows:

$$\mathbf{r}(z) = \begin{cases} r(\operatorname{Re} z) & \text{for } \operatorname{Im} z = 0 \\ \chi_{\operatorname{Im} z} * r(\operatorname{Re} z) & \text{for } \operatorname{Im} z \ne 0, \end{cases} \tag{3.18}$$

The first step is the following proposition.

Proposition 3.6 *Set* $\widehat{r}_0 = r(z_0)e^{-2\mathrm{i}\nu(z_0)-2\beta(z_0,z_0)}$ *as in* (3.15). *Fix* $\lambda_0 > 0$ *and assume* $\|r\|_{H^s} < \lambda_0$ *for a preassigned* $s \in (1/2, 1]$. *Then there exist functions* R_j *defined in* $\overline{\Omega}_j$ *for* $j = 1, 3, 4, 6$ *and a constant* c *such that the following properties hold:*

$$\begin{cases} R_1(z) = r(z) & \textit{for } z - z_0 \in \mathbb{R}_+, \\ R_1(z) = f_1(z-z_0) := \widehat{r}_0(z-z_0)^{-2\mathrm{i}\nu(z_0)}\delta^2(z) & \textit{for } z - z_0 \in e^{\mathrm{i}\frac{\pi}{4}}\mathbb{R}_+; \end{cases}$$

$$\begin{cases} R_3(z) = \frac{\overline{r}(z)}{1+|r(z)|^2} & \textit{for } z - z_0 \in \mathbb{R}_-, \\ R_3(z) = f_3(z-z_0) := \frac{\overline{\widehat{r}_0}}{1+|r(z_0)|^2}(z-z_0)^{2\mathrm{i}\nu(z_0)}\delta^{-2}(z) & \textit{for } z - z_0 \in e^{3\mathrm{i}\frac{\pi}{4}}\mathbb{R}_+; \end{cases}$$

$$\begin{cases} R_4(z) = \frac{r(z)}{1+|r(z)|^2} & \textit{for } z - z_0 \in \mathbb{R}_-, \\ R_4(z) = f_4(z-z_0) := \frac{\widehat{r}_0}{1+|r(z_0)|^2}(z-z_0)^{-2\mathrm{i}\nu(z_0)}\delta^2(z) & \textit{for } z - z_0 \in e^{5\mathrm{i}\frac{\pi}{4}}\mathbb{R}_+; \end{cases}$$

$$\begin{cases} R_6(z) = \overline{r}(z) & \text{for } z - z_0 \in \mathbb{R}_+, \\ R_6(z) = f_6(z-z_0) := \overline{\widehat{r}}_0(z-z_0)^{-2i\nu(z_0)}\delta^{-2}(z) & \text{for } z - z_0 \in e^{-i\frac{\pi}{4}}\mathbb{R}_+; \end{cases}$$

$\forall\, j \in \{1, 3, 4, 6\}$, $\forall\, z \in \Omega_j + z_0$ *and for* $\varphi(x) = -\chi(x) - x\chi'(x)$, *we have for a fixed* c

$$|\overline{\partial} R_j(z)| \le c\|r\|_{H^s(\mathbb{R})}|z-z_0|^{s-\frac{3}{2}} + c|\partial_{\mathrm{Re}\, z}\mathbf{r}(z)| + c|(\mathrm{Im}\, z)^{-1}\varphi_{\mathrm{Im}\, z} * r(\mathrm{Re}\, z)| \tag{3.19}$$

Proof The $R_j(z)$ can be defined explicitly. For $j = 1, 3$ in particular, we set for $z - z_0 = u + i\nu$ and $b(x) = \cos(2x)$,

$$\begin{aligned} R_1(z) &= b(\arg(u+i\nu))\mathbf{r}(z) + (1 - b(\arg(u+i\nu)))f_1(u+i\nu), \\ R_3(z) &= \cos(2(\arg(z-z_0)-\pi))\frac{\overline{\mathbf{r}}(z)}{1+|\mathbf{r}(z)|^2} \\ &\quad + (1-\cos(2(\arg(z-z_0)-\pi)))f_3(u+i\nu). \end{aligned} \tag{3.20}$$

The other $R_j(z)$'s can be defined similarly. This yields functions with the desired boundary values. Now we prove the bounds, and for definiteness we consider case $j = 1$ only. We have

$$\overline{\partial} R_1 = (\mathbf{r} - f_1)\overline{\partial} b + \frac{b}{2}(\chi_{\mathrm{Im}\, z} * r'(\mathrm{Re}\, z) + i(\mathrm{Im}\, z)^{-1}\varphi_{\mathrm{Im}\, z} * r(\mathrm{Re}\, z)),$$

with $\varphi(x) = -\chi(x) - x\chi'(x)$. Notice that $\widehat{\varphi}(0) = 0$. Then we have the bound

$$\begin{aligned} |\overline{\partial} R_1| &\le |\chi_{\mathrm{Im}\, z} * r'(\mathrm{Re}\, z)| + |(\mathrm{Im}\, z)^{-1}\varphi_{\mathrm{Im}\, z} * r(\mathrm{Re}\, z)| \\ &\quad + \frac{c}{|z-z_0|}\left(|r(z) - r(z_0)| + |f_1(z) - r(z_0)|\right). \end{aligned}$$

To obtain the desired estimate for $|\overline{\partial} R_1|$ we need to bound the last line. By (3.8) we have $|r(z) - r(z_0)| \le C|z-z_0|^{s-\frac{1}{2}}\|r\|_{H^s}$. Next, we have

$$\begin{aligned} f_1(z) - r(z_0) = r(z_0) \times [&\exp(2i\nu(z_0)((z-z_0)\log(z-z_0) - (z-z_0+1)\log(z-z_0+1)) \\ &+ 2(\beta(z,z_0) - \beta(z_0,z_0))) - 1]. \end{aligned}$$

By Lemma 3.4 we have $\beta(z, z_0) - \beta(z_0, z_0) = C(\rho, s)\|r\|_{H^s}|z-z_0|^{s-\frac{1}{2}}$. Since for z close to z_0 both $(z-z_0)\log(z-z_0)$ and $(z-z_0+1)\log(z-z_0+1)$ are $O(|z-z_0|^{s-\frac{1}{2}})$, we get the desired estimate for $|\overline{\partial} R_1|$. □

We now extend as follows the matrices in (3.5):

$$\begin{aligned} W_R &= \begin{pmatrix} 1 & 0 \\ e^{2it\theta}R_1 & 1 \end{pmatrix} \ \text{in } \Omega_1 + z_0, \quad U_R = \begin{pmatrix} 1 & e^{-2it\theta}R_3 \\ 0 & 1 \end{pmatrix} \ \text{in } \Omega_3 + z_0, \\ U_L &= \begin{pmatrix} 1 & 0 \\ e^{2it\theta}R_4 & 1 \end{pmatrix} \ \text{in } \Omega_4 + z_0, \quad W_L = \begin{pmatrix} 1 & e^{-2it\theta}R_6 \\ 0 & 1 \end{pmatrix} \ \text{in } \Omega_6 + z_0. \end{aligned} \tag{3.21}$$

We set

$$A := \begin{cases} mW_R^{-1} & \text{in } \Omega_1 + z_0, \\ m & \text{in } (\Omega_2 \cup \Omega_5) + z_0, \\ mU_R^{-1} & \text{in } \Omega_3 + z_0, \\ mU_L & \text{in } \Omega_4 + z_0, \\ mW_L & \text{in } \Omega_6 + z_0. \end{cases} \tag{3.22}$$

We set $B := A\delta^{-\sigma_3}$, obtaining a new function with jump relations $B_+(z) = B_-(z)V_B(z)$ with jump matrix defined by

$$V_B(z) := \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{2it\theta}R_1(z)\delta^{-2}(z) & 1 \end{pmatrix} & \text{for } z \in z_0 + e^{i\pi/4}\mathbb{R}_+, \\ \begin{pmatrix} 1 & -e^{-2it\theta}R_3(z)\delta^{-2}(z) \\ 0 & 1 \end{pmatrix} & \text{for } z \in z_0 + e^{3i\pi/4}\mathbb{R}_+, \\ \begin{pmatrix} 1 & 0 \\ e^{2it\theta}R_4(z)\delta^{-2}(z) & 1 \end{pmatrix} & \text{for } z \in z_0 + e^{5i\pi/4}\mathbb{R}_+, \\ \begin{pmatrix} 1 & -e^{-2it\theta}R_6(z)\delta^{2}(z) \\ 0 & 1 \end{pmatrix} & \text{for } z \in z_0 + e^{-i\pi/4}\mathbb{R}_+. \end{cases}$$

Set now $E(z) := B(z)P^{-1}(\sqrt{8t}(z - z_0))$. By the choice (3.15) of the parameter r_0 in (3.10), the jump matrices of $B(z)$ and of $P(\sqrt{8t}(z - z_0))$ coincide. This is elementary to check and holds for the same reasons of [6]. As a consequence, $E(z)$ does not have jump discontinuities. We now reverse the construction, we define E using Corollary 3.8 below and define $B(z)$ by $B(z) = E(z)P(\sqrt{8t}(z - z_0))$. First though, we have the following auxiliary lemma, see [6].

Lemma 3.7 *Let $\|r\|_{H^s} \le \lambda_0$ for a preassigned $s \in (1/2, 1]$. Consider the following operator*

$$JH(z) := \frac{1}{\pi}\int_{\mathbb{C}} \frac{H(\varsigma)W(\varsigma)}{\varsigma - z}dA(\varsigma) \tag{3.23}$$

with, for $\zeta = \sqrt{8t}(z - z_0)$,

$$W(z) := \begin{cases} P(\zeta)\begin{pmatrix} 0 & 0 \\ e^{2it\theta}\delta^{-2}(z)\bar\partial R_1(z) & 0 \end{pmatrix}P^{-1}(\zeta) & \text{for } z \in \Omega_1, \\ P(\zeta)\begin{pmatrix} 0 & -e^{-2it\theta}\delta^{-2}(z)\bar\partial R_3(z) \\ 0 & 0 \end{pmatrix}P^{-1}(\zeta) & \text{for } z \in \Omega_3, \\ P(\zeta)\begin{pmatrix} 0 & 0 \\ e^{2it\theta}\delta^{-2}(z)\bar\partial R_4(z) & 0 \end{pmatrix}P^{-1}(\zeta) & \text{for } z \in \Omega_4, \\ P(\zeta)\begin{pmatrix} 0 & -e^{-2it\theta}\delta^{2}(z)\bar\partial R_6(z) \\ 0 & 0 \end{pmatrix}P^{-1}(\zeta) & \text{for } z \in \Omega_6, \\ (0 \quad \text{for } z \in \Omega_2 \cup \Omega_5). \end{cases}$$

Then, we have $J : L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C}) \cap C^0(\mathbb{C})$ and there exists a $C = C(\lambda_0)$ s.t.

$$\|J\|_{L^\infty(\mathbb{C})\to L^\infty(\mathbb{C})} \le Ct^{\frac{1-2s}{4}} \quad \text{for all } t > 0. \tag{3.24}$$

Proof For definiteness let $H \in L^\infty(\Omega_1)$. Then

$$\pi|JH(z)| \le \|H\|_{L^\infty}\|\delta^{-2}\|_{L^\infty(\Omega_1)}\int_{\Omega_1}\frac{|\bar\partial R_1(\varsigma)e^{2it\theta}|}{|\varsigma - z|}dA(\varsigma). \tag{3.25}$$

We have $\|\delta^{-2}\|_{L^\infty(\Omega_1)} \leq 1$ by Proposition 3.3. By (3.19) to bound (3.25) it is enough to bound I_j for $j = 1, 2, 3$ with

$$I_j = \int_{\Omega_1} \frac{|X_j(\zeta)e^{2it\theta}|}{|\varsigma - z|} dA(\varsigma), \quad X_1(z) := \partial_{\mathrm{Re}\, z}\mathbf{r}(z),$$
$$X_2(z) := \|r\|_{H^s(\mathbb{R})}|z - z_0|^{s-\frac{3}{2}}, \quad X_3(z) := (\mathrm{Im}\, z)^{-1}\varphi_{\mathrm{Im}\, z} * r(\mathrm{Re}\, z). \tag{3.26}$$

The estimates are like those in Section 2.4 [6]. We have, for $\varsigma - z_0 = u + i\nu$ and for $z - z_0 = \alpha + i\beta$,

$$I_1 = \int_{\Omega_1} \frac{|\partial_u \mathbf{r}(\varsigma)|e^{-8tu\nu}}{|\varsigma - z|} du d\nu \leq \int_0^\infty d\nu \int_\nu^\infty \frac{|\partial_u \mathbf{r}(\varsigma)|e^{-8tu\nu}}{|\varsigma - z|} du$$
$$\leq \int_0^\infty d\nu e^{-8t\nu^2} \|\partial_u \mathbf{r}(u,\nu)\|_{L^2_u} \|((u-\alpha)^2 + (\nu-\beta)^2)^{-1}\|_{L^2_u(\nu,\infty)}. \tag{3.27}$$

By elementary computation we have $\|((u-\alpha)^2 + (\nu-\beta)^2)^{-1}\|_{L^2_u(\nu,\infty)} \leq C|\nu - \beta|^{-\frac{1}{2}}$, see (3.33) below. By Plancherel we have for fixed C

$$\|\partial_u \mathbf{r}(u,\nu)\|_{L^2_u} = \|\partial_u \int_{\mathbb{R}} \nu^{-1}\chi(\nu^{-1}(u-t))r(t)dt\|_{L^2_u} = \|\xi \widehat{\chi}(\nu\xi)\widehat{r}(\xi)\|_{L^2}$$
$$\leq \nu^{s-1}\|\xi^{1-s}\widehat{\chi}(\xi)\|_{L^\infty}\|r\|_{H^s} \leq C\nu^{s-1}\|r\|_{H^s}. \tag{3.28}$$

So

$$I_1 \leq C\|r\|_{H^s} \int_{\mathbb{R}} d\nu e^{-8t\nu^2}|\nu|^{s-1}|\nu - \beta|^{-\frac{1}{2}}$$
$$\leq Ct^{\frac{1-2s}{4}}\|r\|_{H^s} \int_{\mathbb{R}} d\nu e^{-8\nu^2}(|\nu|^{s-\frac{3}{2}} + |\nu - \sqrt{t}\beta|^{s-\frac{3}{2}})$$
$$\leq (3C \int_{\mathbb{R}} e^{-8\nu^2}|\nu|^{s-\frac{3}{2}} d\nu)\|r\|_{H^s} t^{\frac{1-2s}{4}}. \tag{3.29}$$

For the last inequality we used the fact that for any $c \in \mathbb{R}$

$$\int_{\mathbb{R}} e^{-8\nu^2}|\nu - c|^{s-\frac{3}{2}} d\nu \leq \int_{|\nu|\leq|\nu-c|} e^{-8\nu^2}|\nu|^{s-\frac{3}{2}} d\nu + \int_{|\nu|\geq|\nu-c|} e^{-8(\nu-c)^2}|\nu - c|^{s-\frac{3}{2}} d\nu$$
$$\leq 2\int_{\mathbb{R}} e^{-8\nu^2}|\nu|^{s-\frac{3}{2}} d\nu. \tag{3.30}$$

The estimate for I_3 is similar after replacing (3.28) with

$$\|\nu^{-2}\int \varphi(\nu^{-1}(u-t))r(t)dt\|_{L^2_u} = \|\nu^{-1}\xi^{-s}\widehat{\varphi}(\nu\xi)\xi^s\widehat{r}(\xi)\|_{L^2}$$
$$\leq \nu^{s-1}\|\xi^{-s}\widehat{\varphi}(\xi)\|_{L^\infty}\|r\|_{H^s} \leq C\nu^{s-1}\|r\|_{H^s}, \tag{3.31}$$

where the latter bound holds since $\widehat{\varphi}$ is a fixed Schwartz function with $\widehat{\varphi}(0) = 0$. Proceeding like in (3.27), we finally consider

$$I_2 \leq \int_0^\infty e^{-8t\nu^2} d\nu \||\varsigma - z_0|^{s-\frac{3}{2}}\|_{L^p(\nu,\infty)} \||\varsigma - z|^{-1}\|_{L^q(\nu,\infty)} \tag{3.32}$$

with an appropriate pair $1/p + 1/q = 1$. By [6] we have

$$\| |\varsigma - z|^{-1} \|_{L^q(\nu,\infty)} \le C|\nu - \beta|^{\frac{1}{q}-1} \tag{3.33}$$

and

$$\begin{aligned} \| |\varsigma - z_0|^{s-\frac{3}{2}} \|_{L^p(\nu,\infty)} &= \left(\int_\nu^\infty |u + i\nu|^{p(s-\frac{3}{2})} du \right)^{\frac{1}{p}} = \left(\int_\nu^\infty (u^2 + \nu^2)^{p\frac{2s-3}{4}} du \right)^{\frac{1}{p}} \\ &= \nu^{\frac{2s-3}{2}+\frac{1}{p}} \left(\int_1^\infty (u^2 + 1)^{p\frac{2s-3}{4}} du \right)^{\frac{1}{p}}. \end{aligned} \tag{3.34}$$

So by (3.32) and using again (3.30), we obtain

$$I_2 \le C' \int_0^\infty e^{-8t\nu^2} \nu^{\frac{2s-3}{2}+\frac{1}{p}} |\nu - \beta|^{\frac{1}{q}-1} d\nu \le 3C' \int_0^\infty e^{-8t\nu^2} \nu^{\frac{2s-3}{2}} d\nu \le C t^{\frac{1-2s}{4}}. \tag{3.35}$$

The proof that $J(L^\infty) \subset C^0$ can be seen by the above estimates using standard facts, like dominated convergence, is skipped here. □

Taking E as solution of $E = 1 + J(E)$ we obtain the following result.

Corollary 3.8 *Fix $\lambda_0 > 0$ and assume $\|r\|_{H^s} < \lambda_0$. Then there exist a constant T such that for $t \ge T$ there exists a $E(z)$ continuous in $\mathbb{C}$ and satisfying the following additional properties:*

(1) *$E(z)$ is continuous in $\mathbb{C}$,*
(2) *E solves the system $\overline{\partial} E = EW$,*
(3) *$E(z) \to 1$ for $z \to \infty$.* □

Claim (3) in Corollary 3.8 can be replaced by the following sharper result.

Lemma 3.9 *There exists $\varepsilon_0 > 0$ such that for $\|r\|_{H^s} < \varepsilon_0$ there exist constants T and c such that for $t \ge T$ and for $z \in \Omega_2 \cup \Omega_5$*

$$\begin{aligned} &E(z) = 1 + \frac{E_1}{z} + O(z^{-2}) \\ &|E_1| \le c\|u_0\|_{L^{2,s}} t^{-\frac{2s+1}{4}} \text{ for } t \ge T. \end{aligned} \tag{3.36}$$

Proof We have $E_1 = \frac{1}{\pi} \int_{\mathbb{C}} EW dA$, so $|E_1| \le \frac{\|E\|_\infty}{\pi} \sum_j \int_{\Omega_j} |W| dA$. We bound the integrals using a decomposition as in (3.26) and for definiteness we consider only $j = 1$. For $\ell = 1, 3$ we have by (3.28) and (3.31) and starting as in (3.27), using

$$\|e^{-8tu\nu}\|_{L^q_u(\nu,\infty)} = \left(\int_\nu^\infty e^{-8qtu\nu} du \right)^{\frac{1}{q}} = (8qt\nu)^{-\frac{1}{q}} e^{-8qt\nu^2},$$

we have

$$\int_{\Omega_1} |X_\ell(\zeta)e^{2\mathrm{i}t\theta}|dA \le \|r\|_{H^s} \int_0^\infty v^{s-1}\|e^{-8tuv}\|_{L^2_u(v,\infty)}dv$$
$$\le C' t^{-\frac{1}{2}} \int_0^\infty v^{s-\frac{3}{2}} e^{-tv^2} dv \|r\|_{H^s} = C_s t^{-\frac{2s+1}{4}} \|r\|_{H^s}. \tag{3.37}$$

For $\ell = 2$ we use (3.34) and the elementary bound

$$\int_{\Omega_1} |X_2(\varsigma)e^{2\mathrm{i}t\theta}|dA \le \|r\|_{H^s} \int_0^\infty \||\varsigma - z_0|^{s-\frac{3}{2}}\|_{L^p(v,\infty)} \|e^{-8tuv}\|_{L^q_u(v,\infty)} dv$$
$$\le C\|r\|_{H^s} t^{-\frac{1}{q}} \int_0^\infty v^{\frac{2s-3}{2}+\frac{1}{p}-\frac{1}{q}} e^{-tv^2} dv \le C_s t^{-\frac{2s+1}{4}} \|r\|_{H^s}. \tag{3.38}$$

Then we get (3.36) by $\|r\|_{H^s} \le C\|u_0\|_{L^{2,s}}$ for a fixed C by the Lipschitz continuity of Lemma 2.4. □

Theorem 3.1 follows by $m(t,x,z) = E(z)P(\sqrt{8t}(z - z_0))\delta^{\sigma_3}(z)$ in $\Omega_3 + z_0$, with

$$m(z) = 1 + \frac{m_1}{z} + O(z^{-2}) \text{ with } m_1 = E_1 + \frac{P_1}{\sqrt{8t}} + \begin{pmatrix} \delta_1 & 0 \\ 0 & -\delta_1 \end{pmatrix},$$

where by (2.21) and Proposition 3.8 for $t \ge T(s, \lambda_0)$ and a fixed $C = C(s, \lambda_0)$ we have

$$\left| u(t,x) - 2\mathrm{i}\frac{k_1}{\sqrt{8t}} \right| \le C|t|^{-\frac{2s+1}{4}} \text{ and } |u(t,x)| \le C|t|^{-\frac{1}{2}},$$

where we recall that we have fixed $s \in (1/2, 1]$. The time reversibility of the NLS (1.1) (see also later in Lemma 4.5) yields the same estimates also $\forall\, t \le -T(\lambda_0)$. This proves Theorem 3.1 for $u_0 \in H^1(\mathbb{R}) \cap L^{2,s}(\mathbb{R})$.

Consider $u_0 \in L^{2,s}(\mathbb{R})$ but $u_0 \notin H^1(\mathbb{R})$. Let $u(t)$ be the solution, provided by Theorem 1.1, of the corresponding Cauchy problem (1.1). Consider a sequence $u_{0n} \in L^{2,s}(\mathbb{R}) \cap H^1(\mathbb{R})$ such that $u_{0n} \to u_0$ in $L^{2,s}(\mathbb{R})$. Then for the reflection coeffic ients we have $r_n \to r$ in H^s by Lemma 2.4.

We can assume $\|r_n\|_{H^s} \le 2\|r\|_{H^s}$ for all n. By the discussion developed so far, there is a fixed C, which depends only on λ_0, where $\lambda_0 \ge \|r\|_{H^s}$, such that for $|t| \ge T(\lambda_0)$ we have $|u_n(t,x)| \le C|t|^{-\frac{1}{2}}$ for almost any x. By Theorem 1.1 we know that for any t we have $u_n(t) \to u(t)$ in $L^2(\mathbb{R})$. This implies that for almost any x we have $u_n(t,x) \to u(t,x)$. In turn, we can conclude that $|u(t,x)| \le C|t|^{-\frac{1}{2}}$ for almost any x. This completes the proof of the statement in Theorem 3.1 also in the case when $u_0 \in L^{2,s}(\mathbb{R})$ but $u_0 \notin H^1(\mathbb{R})$.

3.1.3. *Several remarks*

Lemma 3.7 yields $\|E - 1\|_{L^\infty(\mathbb{C})} \le Ct^{\frac{1-2s}{4}}$. However we will need the following lemma.

Lemma 3.10 *Let $z_1 \in \mathbb{C}_+$. Assume $\|u_0\|_{L^{2,s}(\mathbb{R})} < \varepsilon_0$. Then there are a $\varepsilon_0 > 0$, a $c > 0$ and a $T > 0$ such that*

$$|1 - E(z_1)| \le c\, t^{-\frac{2s+1}{4}} \|u_0\|_{L^{2,s}(\mathbb{R})} \quad \textit{for } t \ge T. \tag{3.39}$$

Proof The argument is like in Lemma 3.9. We have $|E-1| \le \frac{\|E\|_\infty}{\pi} \sum_j \int_{\Omega_j} \frac{|W|}{|\varsigma - z_1|} dA$. Once again, we estimate only the term with $j = 1$. Using the notation in (3.26) and proceeding like in (3.27), for $z_1 = \alpha_1 + i\beta_1$ we have for $\ell = 1, 3$

$$\int_{\Omega_1} \frac{|X_\ell(\varsigma) e^{2it\theta}|}{|\varsigma - z_1|} dA \le \|r\|_{H^s}[A_1 + A_2], \; A_1 := \int_0^{\frac{\beta_1}{2}} v^{s-1} \| \frac{e^{-8tuv}}{|\varsigma - z_1|} \|_{L^2_u(v,\infty)} dv$$

$$A_2 := \int_{\frac{\beta_1}{2}}^{\infty} v^{s-1} \left\| \frac{e^{-8tuv}}{|\varsigma - z_1|} \right\|_{L^2_u(v,\infty)} dv. \tag{3.40}$$

As in (3.37) we have

$$A_1 = \int_0^{\frac{\beta_1}{2}} v^{s-1} \left\| \frac{e^{-8tuv}}{\sqrt{(u + \frac{x}{4t} - \alpha_1)^2 + (v - \beta_1)^2}} \right\|_{L^2_u(v,\infty)} dv$$
$$\le C'(\beta_1) \int_0^{\frac{\beta_1}{2}} v^{s-1} \|e^{-8tuv}\|_{L^2_u(v,\infty)} dv \le C(s, \beta_1) t^{-\frac{2s+1}{4}}.$$

By (3.33), using $t \ge 1$ and $e^{-8tv^2} \le e^{-t\gamma_1^2} e^{-4v^2}$ for $v \ge \frac{\beta_1}{2}$, and using bounds similar to those for (3.29), we have

$$A_2 \le \int_{\frac{\beta_1}{2}}^{\infty} e^{-8tv^2} v^{s-1} \| |\varsigma - z_1|^{-1} \|_{L^2_u(v,\infty)} dv \le C \int_{\frac{\beta_1}{2}}^{\infty} e^{-8tv^2} v^{s-1} |v - \beta_1|^{-\frac{1}{2}} dv$$
$$\le C e^{-t\beta_1^2} \int_0^{\infty} e^{-4v^2} v^{s-1} |v - \beta_1|^{-\frac{1}{2}} dv \le C' e^{-t\beta_1^2}. \tag{3.41}$$

Turning to the case $\ell = 2$, we similarly have

$$\int_{\Omega_1} \frac{|X_2(\varsigma) e^{2it\theta}|}{|\varsigma - z_1|} dA \le \|r\|_{H^s}[B_1 + B_2],$$

$$B_1 := \int_0^{\frac{\beta_1}{2}} \int_v^{\infty} |\varsigma - z_0|^{s-\frac{3}{2}} \frac{e^{-8tuv}}{|\varsigma - z_1|} dv, \; B_2 := \int_{\frac{\beta_1}{2}}^{\infty} \int_v^{\infty} |\varsigma - z_0|^{s-\frac{3}{2}} \frac{e^{-8tuv}}{|\varsigma - z_1|} dv.$$

Then $B_1 \le C(\beta_1, s) t^{-\frac{2s+1}{4}}$ by (3.38) and by $|\varsigma - z_1| \ge \beta_1/2$. We have by (3.32)–(3.35) and using $t \ge 1$

$$B_2 \le \int_{\frac{\beta_1}{2}}^{\infty} e^{-8tv^2} \| |\varsigma - z_0|^{s-\frac{3}{2}} \|_{L^p(v,\infty)} \| |\varsigma - z_1|^{-1} \|_{L^q(v,\infty)} dv$$
$$\le C e^{-t\beta_1^2} \int_0^{\infty} e^{-4v^2} v^{\frac{2s-3}{2} + \frac{1}{p}} |v - \beta_1|^{\frac{1}{q} - 1} dv \le C_s e^{-t\gamma_1^2}. \tag{3.42}$$

□

Lemma 3.11 *Fix $z_1 = \alpha_1 + i\beta_1$ with $\beta_1 > 0$. There is ε_0 sufficiently small such that for $\|u_0\|_{L^{2,s}(\mathbb{R})} < \varepsilon_0$ there is a constant C such that*

$$|1 - W_R(z_1)| \le C e^{-t8\beta_1^2} \|u_0\|_{L^{2,s}(\mathbb{R})} \text{ if } z_1 \in \Omega_1 + z_0$$
$$|1 - U_R^{-1}(z_1)| \le C e^{-t8\beta_1^2} \|u_0\|_{L^{2,s}(\mathbb{R})} \text{ if } z_1 \in \Omega_3 + z_0. \tag{3.43}$$

Proof By (3.20) we have that $\|R_j\|_{L^\infty(\Omega_j+z_0)} \le C'\|r\|_{H^s(\mathbb{R})} \le C\|u_0\|_{L^{2,s}(\mathbb{R})}$ for $j = 1, 3$. If $z_1 \in \Omega_1 + z_0$ we have $\alpha_1 + \frac{x}{4t} \ge \beta_1$ and so $|e^{-2it\theta}| \le e^{-8t(\alpha_1+\frac{x}{4t})\beta_1} \le e^{-t8\beta_1^2}$. If $z_1 \in \Omega_3 + z_0$ we have similarly $|e^{2it\theta}| \le e^{-t8\beta_1^2}$. This yields (3.43). □

LEMMA 3.12 *Fix $z_1 = \alpha_1 + i\beta_1$ with $\beta_1 > 0$. Fix $\rho_0 > 0$. Let $\rho := \|r\|_{L^\infty(\mathbb{R})}$ and assume $\rho < \rho_0$. Then there exists a constant C independent from z_0 such that*

$$|\delta(z_1) - \Delta(z_1)| \le C\|r\|_{L^2}^2$$
$$\text{where } \Delta(z_1) := \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\alpha_1} \frac{\log(1+|r(\varsigma)|^2)}{\varsigma - z_1} d\varsigma\right). \tag{3.44}$$

Fix $K > 0$. Then for $|z_0 - \alpha_1| \le K/\sqrt{t}$ there exists a constant C such that

$$|\delta(z_1) - \Delta(z_1)| \le \frac{C}{\sqrt{t}\beta_1}\log(1+\rho^2). \tag{3.45}$$

Proof By Proposition 3.3, we have for a fixed c

$$\left|\gamma(z_1) - \frac{1}{2\pi i}\int_{-\infty}^{\alpha_1} \frac{\log(1+|r(\varsigma)|^2)}{\varsigma - z_1} d\varsigma\right| = \frac{1}{2\pi}\left|\int_{\alpha_1}^{z_0} \frac{\log(1+|r(\varsigma)|^2)}{\varsigma - \alpha_1 - i\beta_1} d\varsigma\right| \le \frac{c}{\beta_1}\|r\|_{L^2}^2.$$

This yields (3.44) since the bound $|\delta(z)| \le (1+\rho^2)$ is independent from z_0. Similarly (3.45) follows from

$$\left|\gamma(z_1) - \frac{1}{2\pi i}\int_{-\infty}^{\alpha_1} \frac{\log(1+|r(\varsigma)|^2)}{\varsigma - z_1} d\varsigma\right| = \frac{1}{2\pi}\left|\int_{\alpha_1}^{z_0} \frac{\log(1+|r(\varsigma)|^2)}{\varsigma - \alpha_1 - i\beta_1} d\varsigma\right|$$
$$\le \frac{|z_0 - \alpha_1|}{\beta_1}\log(1+\rho^2).$$

These yield Lemma 3.12. □

We will use the inequalities in Section 3.1.3 for the proof of Theorem 1.3. Notice that similar inequalities are also in Lemmas 5.18–5.21 [7].

4. Proof of Theorem 1.3

Recall by Remark 2.10 that solitons (1.3) belong to $\mathcal{G}_1$ (see under Lemma 2.1). Since $\mathcal{G}_1$ is an open subset of $L^1(\mathbb{R})$, see Lemma 2.1, if the value of $\varepsilon_0 > 0$ in the bound (1.4) is small enough, then the initial datum u_0 belongs to $\mathcal{G}_1$. Notice also that the positive constant ε_0 can be taken independent of (γ_0, x_0). Indeed, when we replace $u_0(x)$ with $u_0(x - x_0)$, their scattering function $a(z)$ is the same, while $e^{i\gamma_0}u_0(x)$ describes a compact set in $L^{2,s}(\mathbb{R})$ as γ_0 varies in $\mathbb{R}$.

We consider now an initial datum u_0 satisfying the bound (1.4). The scattering datum associated with the initial datum u_0, which by Lemma 2.1 and Remark 2.10 belongs to the space $\mathcal{S}(1, 1)$ defined in (2.17), is close to those of the soliton $\varphi_{\omega_0,\gamma_0,v_0}(0, x - x_0)$ by Lemmas 2.1 and 2.4. By Lemma 2.4, we know that $u_0 \in L^{2,s}(\mathbb{R})$ implies $r \in H^s(\mathbb{R})$. Furthermore, by the Lipschitz continuity of $u_0 \to r$ and the fact that the soliton has $r \equiv 0$, we have $\|r\|_{H^s(\mathbb{R})} \le C\epsilon$, with $C = C(\omega_0, v_0)$ and the value of ϵ is given in (1.4).

We define now a map

$$\mathcal{G}_1 \times \mathbb{C}_+ \times \mathbb{C}_* \ni (u_0, z_1, c_1) \mapsto \widetilde{u}_0 \in \mathcal{G}_0 \tag{4.1}$$

by means of the transformation

$$\widetilde{r}(z) := r(z)\frac{z - z_1}{z - \overline{z}_1}. \tag{4.2}$$

By its definition, $\widetilde{r} \in H^s(\mathbb{R})$ if $r \in H^s(\mathbb{R})$ and there is $C > 0$ such that $\|\widetilde{r}\|_{H^s(\mathbb{R})} \leq C\|r\|_{H^s(\mathbb{R})}$. We then define $\widetilde{u}_0 \in \mathcal{G}_0 \cap L^{2,s}(\mathbb{R})$ by the reconstruction formula (2.21), after the corresponding RH problem (i)–(iii) is solved for the scattering datum in $\mathcal{S}(1, 0) = \{\widetilde{r} \in H^s(\mathbb{R})\}$, see (2.17). By Lemma 2.7, we know that $\widetilde{u}_0 \in \mathcal{G}_0 \cap L^{2,s}(\mathbb{R})$ with norm $\|\widetilde{u}_0\|_{L^{2,s}(\mathbb{R})} \leq C\|\widetilde{r}\|_{H^s(\mathbb{R})} \leq C\epsilon$.

We now assume also that $u_0 \in H^1(\mathbb{R})$, to define the time evolution of the scattering data in $\mathcal{S}(1, 1)$ and $\mathcal{S}(1, 0)$. Let $\widetilde{u}(t) \in \mathcal{G}_0 \cap L^{2,s}(\mathbb{R}) \cap H^1(\mathbb{R})$ be the solution of the cubic NLS equation with the initial datum $\widetilde{u}_0$ and $u(t) \in \mathcal{G}_1 \cap L^{2,s}(\mathbb{R}) \cap H^1(\mathbb{R})$ be the solution of the cubic NLS equation with the initial datum u_0.

Denote the solution of the RH problem (i)–(iv) associated to $\widetilde{u}(t)$ by $m(t, x, z)$. The two solutions $u(t)$ and $\widetilde{u}(t)$ are related by the auto-Bäcklund transformation formula, which we state now.

Lemma 4.1 *We have*

$$u(t, x) = \widetilde{u}(t, x) + \boldsymbol{B}, \quad \boldsymbol{B} := 4\,\mathrm{Im}(z_1)\frac{\mathfrak{b}_1\bar{\mathfrak{b}}_2}{|\mathfrak{b}_1|^2 + |\mathfrak{b}_2|^2}, \tag{4.3}$$

where

$$\mathfrak{b}_1 := e^{-\mathrm{i}xz_1}m_{11}(t, x, z_1) - \frac{c_1 m_{12}(t, x, z_1)e^{\mathrm{i}xz_1 + 4\mathrm{i}tz_1^2}}{2\mathrm{i}\,\mathrm{Im}(z_1)}, \tag{4.4}$$

$$\mathfrak{b}_2 := e^{-\mathrm{i}xz_1}m_{21}(t, x, z_1) - \frac{c_1 m_{22}(t, x, z_1)e^{\mathrm{i}xz_1 + 4\mathrm{i}tz_1^2}}{2\mathrm{i}\,\mathrm{Im}(z_1)}. \tag{4.5}$$

Proof Note that $(\mathfrak{b}_1, \mathfrak{b}_2)^T$ is a solution of the spectral system (2.1), and hence the transformation formula (4.3) is a particular example of the general auto-Bäcklund transformation formula used in [20] (after the transformation $\tilde{u} \to -\tilde{u}$ and $\mathfrak{b}_2 \to -\mathfrak{b}_2$, which leaves (2.1) invariant). The particular expressions (4.4)–(4.5) were used in [7] and we give a sketch of the proof of this transformation formula from Appendix A in [7].

We denote by m (resp. $\mathbf{m}$) the solution of the RH problem (i)–(iv) associated to $\widetilde{u}$ (resp. u). We set $\psi = me^{-\mathrm{i}\sigma_3 xz}$. Then consider the function $\widehat{\psi}(x, z)$

$$\widehat{\psi}(x, z) := \mathfrak{a}(x)\mu(z)\mathfrak{a}^{-1}(x)\psi(x, z)\mu^{-1}(z),$$

where

$$\mu(z) := \begin{pmatrix} z - z_1 & 0 \\ 0 & z - \overline{z}_1 \end{pmatrix}$$

and $\mathfrak{a} = [\mathfrak{a}_1, \mathfrak{a}_2]$ with

$$\mathfrak{a}_1(x) := \psi(x, z_1)\begin{pmatrix} 1 \\ \frac{-c_1}{z_1 - \overline{z}_1} \end{pmatrix}, \quad \mathfrak{a}_2(x) := \psi(x, z_1)\begin{pmatrix} \frac{-\overline{c}_1}{z_1 - \overline{z}_1} \\ 1 \end{pmatrix}.$$

By symmetries of the spectral system (2.1) we have $\mathfrak{a}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{\mathfrak{a}}_1$. Notice that $\mathfrak{a}_1 = (\mathfrak{b}_1, \mathfrak{b}_2)^T$ is given by (4.4) and (4.5).

The function $\widehat{\psi}(x, z)$ has poles only at z_1 and $\overline{z}_1$; $\widehat{m}(x, z) := \widehat{\psi}(x, z)e^{\mathrm{i}\sigma_3 xz}$ satisfies (2.19)–(2.20) for $k = 1$. Furthermore, $\widehat{m}(x, z)$ satisfies (i)–(iv) of the RH problem involving $\widehat{V}_x(z) = e^{-\mathrm{i}\sigma_3 xz}(z)\widehat{V}(z)e^{\mathrm{i}\sigma_3 xz}$ with

$$\widehat{V}(z) = \widehat{\psi}_-^{-1}(x, z)\widehat{\psi}_+(x, z) = \mu(z)\psi_-^{-1}(x, z)\psi_+(x, z)\mu^{-1}(z) = \begin{pmatrix} 1 + |\widehat{r}(z)|^2 & \overline{\widehat{r}(z)} \\ \widehat{r}(z) & 1 \end{pmatrix},$$

where

$$\widehat{r}(z) := r(z)\frac{z - \overline{z}_1}{z - z_1}.$$

All these formulas are in [7], with a different notation (our reflection coefficient $r(z)$ is equivalent to $\overline{r}(z)$ in [7], whereas our z is $-z/2$ in [7]). It is clear by the uniqueness of the inverse problem that $\mathbf{m} = \widehat{m}$.

We have expansions $\mathbf{m}(x, z) = 1 + \frac{\mathbf{m}_1(x)}{z} + o(z^{-1})$ and $m(x, z) = 1 + \frac{m_1(x)}{z} + o(z^{-1})$. By an elementary computation, we have $\mathbf{m}_1 = m_1 - \mathfrak{a}\mu_1\mathfrak{a} + \mu_1$, where

$$\mu_1 := \begin{pmatrix} z_1 & 0 \\ 0 & \overline{z}_1 \end{pmatrix}.$$

Therefore, the reconstruction formula (2.21) yields

$$u = \mathrm{i}[\sigma_3, \mathbf{m}_1]_{12} = \mathrm{i}[\sigma_3, m_1 - \mathfrak{a}\mu_1\mathfrak{a}]_{12},$$

which proves (4.3). □

Remark 4.2 The soliton in Remark 2.10 is obtained for $\widetilde{u} = 0$ and $m(x, z) = I$, when

$$\mathfrak{b}_1 = e^{-\mathrm{i}xz_1} \quad \text{and} \quad \mathfrak{b}_2 = -\frac{c_1}{2\mathrm{i}\,\mathrm{Im}(z_1)}e^{\mathrm{i}xz_1 + 4\mathrm{i}tz_1^2}.$$

By Theorem 3.1, we know that there exist constants $C_0 > 0$ and $T > 0$ such that for all $|t| \geq T$, we have

$$\|\widetilde{u}(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C_0\epsilon|t|^{-\frac{1}{2}},$$

since there is a constant $C > 0$ such that $\|\widetilde{u}_0\|_{L^{2,s}(\mathbb{R})} \leq C\epsilon$.

To prove Theorem 1.3 we need to focus only on $\mathbf{B}$. From the proof, we will see that the (ω_1, v_1) of the statement of Theorem 1.3 are those of the soliton with spectral data (z_1, c_1).

We will consider only positive times, focusing on $t \gg 1$. We know that

$$m(t, x, z_1) = \begin{cases} E(z_1)P(\sqrt{8t}(z_1 - z_0))\delta^{\sigma_3}(z_1)W_R(z_1) & \text{if } z_1 \in \Omega_1 + z_0\,, \\ E(z_1)P(\sqrt{8t}(z_1 - z_0))\delta^{\sigma_3}(z_1) & \text{if } z_1 \in \Omega_2 + z_0\,, \\ E(z_1)P(\sqrt{8t}(z_1 - z_0))\delta^{\sigma_3}(z_1)U_R^{-1}(z_1) & \text{if } z_1 \in \Omega_3 + z_0\,. \end{cases} \tag{4.6}$$

We have the following estimate.

LEMMA 4.3 *Fix $\lambda_0 > 0$. Then there is a $C > 0$ and a $T > 0$ such that for $\|\widetilde{r}\|_{H^s(\mathbb{R})} < \lambda_0$ we have for $t \geq T$*

$$|m_{11}(t,x,z_1) - \delta(z_1)| + |m_{22}(t,x,z_1) - \delta^{-1}(z_1)| \leq C\|\widetilde{r}\|_{H^1(\mathbb{R})} t^{-\frac{1}{2}} (\|\widetilde{r}\|_{H^s(\mathbb{R})} + t^{-\frac{2s-1}{4}})$$
$$\left| m_{12}(t,x,z_1) - \frac{\delta^{-1}(z_1)k_1}{\sqrt{8t}(z_1 - z_0)} \right| + \left| m_{21}(t,x,z_1) - \frac{\delta(z_1)k_2}{\sqrt{8t}(z_1 - z_0)} \right| \leq C\|\widetilde{r}\|_{H^s(\mathbb{R})} t^{-\frac{2s+1}{4}}.$$

Proof By Lemma 3.10, we have $E(z_1) = 1 + O(\|\widetilde{r}\|_{H^s(\mathbb{R})} t^{-\frac{2s+1}{4}})$. By Lemma 3.11, we have similar expansions for $W_R(z_1)$ and $U_R(z_1)$. We furthermore know by Proposition 3.3 that $|\delta^{\pm}(z_1)| \leq 1 + \rho^2$ for $\rho = \|\widetilde{r}\|_{L^\infty(\mathbb{R})}$. From Section 3.1.1, we recall the expansion

$$P(\sqrt{8t}(z_1 - z_0)) = 1 + \frac{P_1}{\sqrt{8t}(z_1 - z_0)} + O(\|\widetilde{r}\|_{H^s(\mathbb{R})} t^{-1}),$$

where the O-term depends on a fixed $C = C(\lambda_0)$ and P_1 is given in (3.14). We also recall that $|k_1| + |k_2| < C\|\widetilde{r}\|_{H^s(\mathbb{R})}$. These observations yield Lemma 4.3. □

Now we start to analyze the term **B** in (4.3). Consider the following inequalities:

$$|e^{-\mathrm{i}xz_1} m_{11}(t,x,z_1)| > 10 \left| \frac{c_1 m_{12}(t,x,z_1) e^{\mathrm{i}xz_1 + 4\mathrm{i}tz_1^2}}{2\mathrm{i}\,\mathrm{Im}(z_1)} \right|, \tag{4.7}$$

$$10|e^{-\mathrm{i}xz_1} m_{21}(t,x,z_1)| < \left| \frac{c_1 m_{22}(t,x,z_1) e^{\mathrm{i}xz_1 + 4\mathrm{i}tz_1^2}}{2\mathrm{i}\,\mathrm{Im}(z_1)} \right|. \tag{4.8}$$

LEMMA 4.4 *Given $\varepsilon_0 > 0$ small, there exist $T(\varepsilon_0) > 0$ and $C > 0$ such that, if $\|\widetilde{r}\|_{H^s(\mathbb{R})} < \varepsilon_0$ and if (t,x) is such that at least one of (4.7)–(4.8) is false, then we have $|\boldsymbol{B}| < Ct^{-\frac{1}{2}}\epsilon$ for $t \geq T(\varepsilon_0)$.*

Proof Let us start by assuming that for (t,x) inequality (4.7) is false. We are only interested to the case when t is large. For the ϵ of (1.4) and $\rho = \|\widetilde{r}\|_{L^\infty(\mathbb{R})}$, Lemma 4.3 implies for $t \geq T$,

$$\begin{aligned} |m_{12}| &\leq (1+\rho^2)|k_1| t^{-\frac{1}{2}} + C\epsilon t^{-\frac{2s+1}{4}} \\ &\leq t^{-\frac{1}{2}} \epsilon K \left(\frac{1}{2}(1+\rho^2)^{-1} - Ct^{\frac{1-2s}{4}} - C\epsilon \right) \\ &\leq t^{-\frac{1}{2}} \epsilon K |m_{22}|, \end{aligned} \tag{4.9}$$

for a fixed and sufficiently large constant K. Then, if (4.7) is false and $t \geq T$, both terms in (4.7) are bounded from above by

$$\left| \frac{c_1 m_{22}(t,x,z_1) e^{\mathrm{i}xz_1 + 4\mathrm{i}tz_1^2}}{2\mathrm{i}\,\mathrm{Im}(z_1)} \right|.$$

For $t \geq T$ by the same argument of (4.9) we have also

$$|e^{-\mathrm{i}xz_1} m_{21}(t,x,z_1)| \leq t^{-\frac{1}{2}} \epsilon K |e^{-\mathrm{i}xz_1} m_{11}(t,x,z_1)|. \tag{4.10}$$

We conclude that for $t \geq T$ and if (t, x) is in the domain where (4.7) is false, we have for some fixed K

$$|\mathbf{B}| \leq K \frac{\left|m_{12} e^{\mathrm{i}xz_1 - 4\mathrm{i}tz_1^2} \overline{m}_{22} e^{-\mathrm{i}x\overline{z}_1 - 4\mathrm{i}t\overline{z}_1^2}\right|}{|m_{22} e^{\mathrm{i}xz_1 + 4\mathrm{i}tz_1^2}|^2} = K \frac{|m_{12}|}{|m_{22}|} \leq \frac{CK}{\sqrt{t}} \epsilon. \tag{4.11}$$

So now we assume that (t, x) is such that (4.7) is true. Notice that by (4.10) and (4.7) we have for a fixed K

$$\frac{|\mathfrak{b}_1 e^{\mathrm{i}x\overline{z}_1} \overline{m_{21}}|}{\|\mathfrak{b}\|^2} \leq K \frac{|e^{-\mathrm{i}xz_1} m_{11} e^{\mathrm{i}x\overline{z}_1} \overline{m_{21}}|}{|e^{-\mathrm{i}xz_1} m_{11}|^2} = K \frac{|m_{21}|}{|m_{11}|} \leq \frac{CK}{\sqrt{t}} \epsilon. \tag{4.12}$$

Since we are assuming that (t, x) is such that (4.7)–(4.8) are not both true, we assume now that (4.7) is true and (4.8) is false. Then by (4.12), for a fixed K

$$|\mathbf{B}| \leq 4 |\operatorname{Im} z_1| \frac{|\mathfrak{b}_1 \mathfrak{b}_2|}{\|\mathfrak{b}\|^2} \leq K \frac{|\mathfrak{b}_1 e^{\mathrm{i}x\overline{z}_1} \overline{m_{21}}|}{\|\mathfrak{b}\|^2} \leq \frac{CK}{\sqrt{t}} \epsilon. \tag{4.13}$$

The above inequalities prove Lemma 4.4 for values of (t, x), for which (4.7)–(4.8) are not both true. □

We assume now that (4.7)–(4.8) are true. Then, by the last inequality in (4.11) and by (4.12), up to terms bounded by $Ct^{-\frac{1}{2}}\epsilon$, what is left is the analysis of

$$-2\mathrm{i} \frac{e^{-\mathrm{i}xz_1} m_{11} \overline{c}_1 \overline{m_{22}} e^{-\mathrm{i}x\overline{z}_1 - 4\mathrm{i}t\overline{z}_1^2}}{\|\mathfrak{b}\|^2}. \tag{4.14}$$

Set now

$$b^2 := |e^{-\mathrm{i}xz_1} m_{11}|^2 + \left| \frac{c_1 m_{22} e^{\mathrm{i}xz_1 + 4\mathrm{i}tz_1^2}}{2\mathrm{i} \operatorname{Im}(z_1)} \right|^2$$

and expand

$$\|\mathfrak{b}\|^2 = b^2 \left(1 + O\left(b^{-1} \left|c_1 m_{12} e^{\mathrm{i}xz_1 + 4\mathrm{i}tz_1^2}\right|\right) + O\left(b^{-1} \left|m_{21} e^{-\mathrm{i}xz_1}\right|\right)\right).$$

Then the quantity in (4.14) is of the form

$$\begin{aligned} &- 2\mathrm{i} e^{-\mathrm{i}xz_1} m_{11} \frac{\overline{c}_1 \overline{m_{22}} e^{-\mathrm{i}x\overline{z}_1 - 4\mathrm{i}t\overline{z}_1^2}}{b^2} \\ &\times \left(1 + O\left(b^{-1} \left|c_1 m_{12} e^{\mathrm{i}xz_1 + 4\mathrm{i}tz_1^2}\right|\right) + O\left(b^{-1} \left|m_{21} e^{-\mathrm{i}xz_1}\right|\right)\right). \end{aligned} \tag{4.15}$$

We claim that the quantity in (4.15) equals

$$-2\mathrm{i} \frac{e^{-\mathrm{i}xz_1} \delta(z_1) (\overline{\delta}(z_1))^{-1} \overline{c}_1 e^{-\mathrm{i}x\overline{z}_1 - 4\mathrm{i}t\overline{z}_1^2}}{|e^{-\mathrm{i}xz_1} \delta(z_1)|^2 + \left| \frac{c_1 e^{\mathrm{i}xz_1 + 4\mathrm{i}tz_1^2}}{2\mathrm{i} \operatorname{Im}(z_1)} \delta(z_1)^{-1} \right|^2} (1 + O(\epsilon t^{-\frac{1}{2}})). \tag{4.16}$$

To prove this claim, we observe that since $m_{ii} = \delta^{-(-1)^i}(z_1) + O(\epsilon t^{-\frac{1}{2}})$ and $|\delta^{\pm 1}(z_1)| \geq \langle \rho \rangle^{-2}$, we have

$$b^2 = |e^{-\mathrm{i}xz_1}\delta(z_1)|^2 + \left| \frac{c_1 e^{\mathrm{i}xz_1 + 4\mathrm{i}tz_1^2}}{2\mathrm{i}\,\mathrm{Im}(z_1)} \delta(z_1)^{-1} \right|^2 (1 + O(\epsilon t^{-\frac{1}{2}})).$$

We have $O\left(b^{-1}\left|c_1 m_{12} e^{\mathrm{i}xz_1+4\mathrm{i}tz_1^2}\right|\right) = O(\epsilon t^{-\frac{1}{2}})$ by

$$b^{-1}\left|c_1 m_{12} e^{\mathrm{i}xz_1+4\mathrm{i}tz_1^2}\right| \leq \frac{\left|m_{12} e^{\mathrm{i}xz_1+4\mathrm{i}tz_1^2}\right|}{\left|m_{22} e^{\mathrm{i}xz_1+4\mathrm{i}tz_1^2}\right|} = \frac{|m_{12}|}{|m_{22}|} \leq C\epsilon t^{-\frac{1}{2}}.$$

We have $O\left(b^{-1}\left|m_{21} e^{\mathrm{i}x\bar{z}_1}\right|\right) = O(\epsilon t^{-\frac{1}{2}})$ by

$$b^{-1}\left|m_{21} e^{-\mathrm{i}xz_1}\right| \leq \frac{\left|m_{21} e^{-\mathrm{i}xz_1}\right|}{\left|m_{11} e^{-\mathrm{i}xz_1}\right|} = \frac{|m_{21}|}{|m_{11}|} \leq C\epsilon t^{-\frac{1}{2}}.$$

Hence (4.16) is proved.

Now we look at the term in (4.16). For $z_1 = \alpha_1 + \mathrm{i}\beta_1$, $d_1 = \log(\frac{|c_1|}{2\beta_1})$ and $\vartheta_1 = \arg(c_1)$, dropping the factor $(1 + O(\epsilon t^{-\frac{1}{2}}))$, for $\Delta(z_1)$ defined in (3.44) and inserting trivial factors $\Delta/\Delta = 1$ and $\overline{\Delta}/\overline{\Delta} = 1$, the expression in (4.16) equals

$$\frac{-4\mathrm{i}\beta_1 e^{-2\mathrm{i}\alpha_1 x - 4\mathrm{i}t(\alpha_1^2 - \beta_1^2) - \mathrm{i}\vartheta_1} \frac{\delta(z_1)}{\Delta(z_1)} \frac{\overline{\Delta}(z_1)}{\overline{\delta}(z_1)} \frac{\Delta(z_1)}{\overline{\Delta}(z_1)}}{e^{2\beta_1 x + 8t\alpha_1\beta_1 - d_1} |\frac{\delta(z_1)}{\Delta(z_1)}|\, |\Delta(z_1)| + e^{-(2\beta_1 x + 8t\alpha_1\beta_1 - d_1)} |\frac{\Delta(z_1)}{\delta(z_1)}|\, |\Delta(z_1)|^{-1}}. \tag{4.17}$$

Fix now a constant $\kappa > 0$. Then (4.17) differs from the soliton solution

$$-2\mathrm{i}\beta_1 e^{-2\mathrm{i}\alpha_1 x - 4\mathrm{i}t(\alpha_1^2 - \beta_1^2) - \mathrm{i}\vartheta_1 + 2\mathrm{i}\arg(\Delta(z_1))} \mathrm{sech}(2\beta_1 x + 8t\alpha_1\beta_1 - d_1 + \log(|\Delta(z_1)|)) \tag{4.18}$$

by less than $c\kappa t^{-\frac{1}{2}}\epsilon$. To prove this claim we observe that the difference of (4.17) and (4.18) can be bounded, up to a constant factor $C = C(\omega_0, v_0)$, by the sum of the following two error terms:

$$\frac{\left| \frac{\delta(z_1)}{\Delta(z_1)} \frac{\overline{\Delta}(z_1)}{\overline{\delta}(z_1)} - 1 \right|}{e^{8\beta_1 t|z_0 - \alpha_1|}(1 + \|\widetilde{r}\|^2_{L^\infty(\mathbb{R})})^{-1}} \tag{4.19}$$

and

$$\left| \mathrm{sech}\,(8\beta t(-z_0 + \alpha_1) - d_1 + \log(|\Delta(z_1)|)) \right. \\ \left. -\mathrm{sech}\left(8\beta_1 t(-z_0 + \alpha_1) - d_1 + \log(|\Delta(z_1)|) + \log\left(\frac{|\delta(z_1)|}{|\Delta(z_1)|}\right)\right)\right|. \tag{4.20}$$

We bound first (4.19). For $|z_0 - \alpha_1| \geq \kappa t^{-\frac{1}{2}}$ formula (4.19) is bounded by $Ce^{-8\beta_1\kappa\sqrt{t}}\epsilon$ by (3.44). For $|z_0 - \alpha_1| \leq \kappa t^{-\frac{1}{2}}$, for a fixed K and using (3.45) we bound (4.19) by

$$\left(1 + \|\widetilde{r}\|^2_{L^\infty(\mathbb{R})}\right) \left| \frac{\delta(z_1)}{\Delta(z_1)} \frac{\overline{\Delta}(z_1)}{\overline{\delta}(z_1)} - 1 \right| \leq 4\frac{C}{\sqrt{t}} \log\left(1 + \|\widetilde{r}\|^2_{L^\infty(\mathbb{R})}\right) \leq Kt^{-\frac{1}{2}}\epsilon^2.$$

By Lagrange Theorem, (4.20) is bounded by

$$C\operatorname{sech}\left(8\beta_1 t(-z_0+\alpha_1)-d_1+\log(|\Delta(z_1)|)+c\log\left(\frac{|\delta(z_1)|}{|\Delta(z_1)|}\right)\right)\left|\log\left(\frac{|\delta(z_1)|}{|\Delta(z_1)|}\right)\right|$$

for some $c \in (0,1)$. This satisfies bounds similar to those satisfied by (4.19).

To complete the proof of Theorem 1.3 when $u_0 \in H^1(\mathbb{R}) \cap L^{2,s}(\mathbb{R})$, we need to show that when one of (4.7)–(4.8) is false, then the function in (4.18) is $O(\epsilon t^{-\frac{1}{2}})$. By Lemma 4.3 the fact that (4.7), resp. (4.8), false means that for a fixed $C = C(\rho_0) > 0$ we have

$$|e^{-2\mathrm{i}xz_1-4\mathrm{i}t\overline{z}_1^2}| = e^{2(\beta_1 x+4t\alpha_1\beta_1)} \le C\epsilon t^{-\frac{1}{2}}$$

and

$$|e^{2\mathrm{i}x\overline{z}_1+4\mathrm{i}tz_1^2}| = e^{-2(\beta_1 x+4t\alpha_1\beta_1)} \le C\epsilon t^{-\frac{1}{2}}.$$

Any of these yields our claim that the function in (4.18) is $O(\epsilon t^{-\frac{1}{2}})$.

This completes the proof of Theorem 1.3 for $u_0 \in H^1(\mathbb{R}) \cap L^{2,s}(\mathbb{R})$. Notice that for $t \ge T(\varepsilon_0)$ the soliton in formula (1.6) is given by formula (4.18).

When $u_0 \in L^{2,s}(\mathbb{R})$ but $u_0 \notin H^1(\mathbb{R})$, we consider a sequence $u_n \in H^1(\mathbb{R}) \cap L^{2,s}(\mathbb{R})$ with $u_n \to u_0$ as $n \to \infty$ in $L^{2,s}(\mathbb{R})$. Then the sequence of spectral data from $\{u_n\}$ converges to the spectral datum of u_0. This implies that for $t \ge T(\varepsilon_0)$ we have

$$\|u_n(t,\cdot) - \varphi_{\omega_n,\gamma_+^{(n)},v_n}(t,\cdot - x_+^{(n)})\|_{L^\infty(\mathbb{R})} < C\epsilon t^{-\frac{1}{2}}, \tag{4.21}$$

with a fixed constant C, since C can be made to depend only on values of ε_0 and (ω_0, v_0) in Theorem 1.3. The sequence $\{(\omega_n, v_n)\}$ converges to the parameters of the soliton with spectral datum (z_1, c_1) obtained from the spectral datum (z_1, c_1, r) of u_0. Finally, $\{(\gamma_+^{(n)}, x_+^{(n)})\}$ is a convergent sequence, as can be seen in (4.18) from their continuous dependence on the spectral data. This means that for almost any x and for any $t \ge T(\varepsilon_0)$, we have

$$\lim_{n\to\infty}\left(u_n(t,x) - \varphi_{\omega_n,\gamma_+^{(n)},v_n}(t,x-x_+^{(n)})\right) = u(t,x) - \varphi_{\omega_1,\gamma_+,v_1}(t,x-x_+).$$

Hence, bound (4.21) implies that for any $t \ge T(\varepsilon_0)$, we have

$$\|u(t,\cdot) - \varphi_{\omega_1,\gamma_+,v_1}(t,\cdot - x_+)\|_{L^\infty(\mathbb{R})} \le C\epsilon t^{-\frac{1}{2}}.$$

The proof of Theorem 1.3 is complete. □

We end the paper explaining the remark that the ground states $\varphi_{\omega_1,\gamma_\pm,v_1}(t, x - x_\pm)$ in the statement of Theorem 1.3 are in general distinct. The + ground state has been computed explicitly in (4.18).

Lemma 4.5 *The − ground state is given by formula (4.18) but with $\Delta(z_1)$ replaced by*

$$\Lambda(z_1) = \exp\left(\frac{1}{2\pi\mathrm{i}}\int_{\alpha_1}^{\infty}\frac{\log(1+|r(\varsigma)|^2)}{\varsigma - z_1}d\varsigma\right). \tag{4.22}$$

Proof We know that if $u(t,x)$ solves (1.1) then $v(t,x) := \overline{u}(-t,x)$ solves the NLS with initial value $\overline{u}_0(x)$. By standard arguments which can be derived from (2.6), if $(r(z), z_1, c_1)$ are the spectral data of $u_0 \in \mathcal{G}_1$, then we have $\overline{u}_0 \in \mathcal{G}_1$ with spectral data $(\overline{r}(-z), -\overline{z}_1, -\overline{c}_1)$. Using the latter, by (4.18) we then get for $-t \nearrow \infty$

$$v(-t,x) \sim 2\mathrm{i}\beta_1 e^{2\mathrm{i}\alpha_1 x+4\mathrm{i}t(\alpha_1^2-\beta_1^2)+\mathrm{i}\vartheta_1-2\mathrm{i}\arg(\Lambda(z_1))}\operatorname{sech}(2\beta_1 x+8t\alpha_1\beta_1-d_1+\log(|\Lambda(z_1)|))$$

with $\Lambda(z_1)$ defined in terms of its complex conjugate (the following is simply (3.44) for the spectral data of $\overline{u}_0$)

$$\overline{\Lambda(z_1)} := \exp\left(\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{-\alpha_1}\frac{\log(1+|r(-\varsigma)|^2)}{\varsigma+\alpha_1-\mathrm{i}\beta_1}d\varsigma\right).$$

Then (4.22) is true. Using $u(t,x)=\overline{v}(-t,x)$ and so taking the complex conjugate of the above formula, we obtain for $t\to-\infty$

$$u(t,x)\sim -2\mathrm{i}\beta_1 e^{-2\mathrm{i}\alpha_1 x-4\mathrm{i}t(\alpha_1^2-\beta_1^2)-\mathrm{i}\vartheta_1+2\mathrm{i}\arg(\Lambda(z_1))}\mathrm{sech}(2\beta_1 x+8t\alpha_1\beta_1-d_1+\log(|\Lambda(z_1)|))$$

thus completing the proof of Lemma 4.5. □

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