Periodic oscillations of dark solitons in parabolic potentials

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Abstract. We reformulate the Gross–Pitaevskii equation with a parabolic potential as a discrete dynamical system using the basis of Hermite functions. We consider small amplitude stationary solutions with a single zero, known as dark solitons, and examine their existence and linear stability. Furthermore we prove, under appropriate conditions, the persistence of a periodic motion in a neighborhood of such solutions when the parabolic potential is perturbed by a small bounded and spatially decaying potential. Our results on existence, stability and nonlinear dynamics of the relevant solutions are corroborated by numerical computations.

1. Introduction

We study the Gross-Pitaevskii (GP) equation with parabolic and bounded potentials in the form

$$iU_T = -\frac{1}{2}U_{XX} + \gamma^2 X^2 U + \nu V(X)U + \sigma |U|^2 U, \tag{1.1}$$

where the solution $U(X,T) : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{C}$ decays to zero as $|X| \to \infty$, the potential $V(X)$ is bounded and decaying, the strength constants γ and ν are real-valued, and the nonlinear parameter σ is normalized to $\sigma = 1$ ($\sigma = -1$) for the defocusing (focusing) cubic nonlinearity. This equation is of particular interest in the context of Bose-Einstein condensates, i.e., dilute alkali vapors at near-zero temperatures, where many recent papers have addressed the dynamics of localized dips in the ground state trapped by a magnetically induced parabolic potential, see, e.g., the recent review [**12**]. The question that we study concerns whether the localized density dips oscillate periodically near the center point $X = 0$ of the parabolic potential. If the motion of a localized dip is truly periodic, the frequency of periodic oscillations is of interest [**4**], while if the periodic oscillations are destroyed due to emission of radiation, the gradual change in the amplitude of oscillations is to be understood [**17**]. Numerical simulations show that solutions experience radiation and amplitude changes if the confining parabolic potential with $\gamma \neq 0$ is perturbed

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by a bounded periodic potential with $\nu \neq 0$, while they exhibit no radiation and time-periodic oscillations in the case of purely parabolic confinement with $\gamma \neq 0$ and $\nu = 0$ [**19**].

If $\sigma = 1$ and γ, ν are small, a localized dip on the ground state of the GP equation (1.1) is approximated near the center by the so-called dark soliton of the defocusing nonlinear Schrödinger (NLS) equation, which is the reason why we use the term "dark soliton" for this localized solution. In our previous paper [**18**] we studied the persistence and stability of a dark soliton in the presence of an exponentially decaying potential, where methods of Lyapunov–Schmidt reductions, Evans functions and the stability theory in Pontryagin space were employed. These methods can not be applied to the GP equation (1.1) with small non-zero γ since the parabolic potential deforms drastically the spectrum of the linearized problem: the continuous spectral band at $\gamma = 0$ becomes an infinite sequence of isolated eigenvalues for $\gamma \neq 0$. Therefore, we do not consider here the limit $\gamma \to 0$. Moreover, we transform the GP equation (1.1) to the γ-independent form

$$iu_t = -\frac{1}{2}u_{xx} + \frac{1}{2}x^2 u + \delta W(x) u + \sigma |u|^2 u, \tag{1.2}$$

through the rescaling transformation $x = \lambda X$, $t = \lambda^2 T$, $u(x,t) = \lambda^{-1} U(X,T)$, $\delta = \lambda^{-2}\nu$, $W(x) = V(\lambda^{-1}x)$, and $\lambda = 2^{1/4}\gamma^{1/2}$.

The substitution $u(x,t) = e^{-\frac{i}{2}t - i\mu t}\phi(x)$ reduces equation (1.2) to the second-order non-autonomous ODE

$$-\frac{1}{2}\phi''(x) + \frac{1}{2}x^2\phi(x) + \delta W(x)\phi(x) + \sigma\phi^3(x) = \left(\mu + \frac{1}{2}\right)\phi(x), \tag{1.3}$$

where $\phi : \mathbb{R} \mapsto \mathbb{R}$. A strong solution of the ODE (1.3) is called a dark soliton if $\phi(x)$ has a single zero on $x \in \mathbb{R}$ and it decays to zero sufficiently rapidly as $|x| \to \infty$. If $\delta = 0$, a classification of all localized solutions of the second-order ODE (1.3) with a shooting method is suggested in recent work [**2**]. Construction of stationary solutions with the Hermite–Gaussian modes is considered in [**11**].

We consider solutions of the GP equation (1.2) with $W \in L^2(\mathbb{R})$ in space

$$\mathcal{H}_1(\mathbb{R}) = \{u \in H^1(\mathbb{R}) : \quad xu \in L^2(\mathbb{R})\} \tag{1.4}$$

equipped with the squared norm

$$\|u\|^2_{\mathcal{H}_1} = \int_{\mathbb{R}} \left(|u'(x)|^2 + (x^2+1)|u(x)|^2\right) dx. \tag{1.5}$$

Global existence of solutions of the GP equation (1.2) with $W \in L^2(\mathbb{R})$ has been proved in space $u \in \mathcal{H}_1(\mathbb{R})$ for all $t \in \mathbb{R}_+$ (see Proposition 2.2 in [**5**]).

If $\delta = 0$, a stationary solution of the GP equation (1.2) can be extended to the time-periodic solution using the explicit transformation

$$u(x,t) = e^{ip(t)x - \frac{i}{2}p(t)q(t) - \frac{i}{2}t - i\mu t - i\theta}\phi(x - q(t)), \tag{1.6}$$

where $\dot{q} = p$, $\dot{p} = -q$, and θ is an arbitrary parameter. The system of time-evolution equations for (p,q) is equivalent to the harmonic oscillator equation $\ddot{q} + q = 0$, which has the explicit solution

$$q(t) = s\cos(t+\varphi), \qquad p(t) = -s\sin(t+\varphi), \tag{1.7}$$

where parameters (s,φ) are arbitrary. The periodic solution (1.6)–(1.7) persists for any values of $\mu \in \mathbb{R}$, for which the stationary solution $\phi(x)$ exists.

Our work is devoted to the study of periodic solutions of the GP equation (1.2) in a local neighborhood of the stationary solutions of the ODE (1.3) for parameter values (μ,δ) near the point $(1,0)$. The special value $\mu = 1$ corresponds to the second eigenvalue of the linear Schrödinger operator

$$\mathcal{L} = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 - \frac{1}{2} \tag{1.8}$$

with the eigenfunction $\phi(x) = \varepsilon x e^{-x^2/2}$. Here ε is a parameter of the family of stationary solutions of the nonlinear ODE (1.3), which bifurcates from the small-amplitude eigenmode by means of a standard local bifurcation [**7**]. We will show that, under some assumptions on $W(x)$, a periodic solution (1.6)–(1.7) persists along a bifurcation curve $\delta = \delta_*(\mu)$ near the point $(\mu,\delta) = (1,0)$. The period T of the δ-perturbed periodic solution is close to $T_0 = 2\pi$.

To formulate the main result of this paper, we introduce a linearized problem associated to the stationary solution $u(x,t) = e^{-\frac{i}{2}t - i\mu t}\phi(x)$ of the GP equation (1.2) in the form

$$\mathcal{L}_+ v = \Omega w, \quad \mathcal{L}_- w = \Omega v, \tag{1.9}$$

where $\mathcal{L}_\pm$ are Schrödinger operators,

$$\left\{ \begin{array}{rcl} \mathcal{L}_+ & = & -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 - \frac{1}{2} - \mu + \delta W(x) + 3\sigma\phi^2(x), \\ \mathcal{L}_- & = & -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 - \frac{1}{2} - \mu + \delta W(x) + \sigma\phi^2(x), \end{array} \right. \tag{1.10}$$

which admit closed self-adjoint extensions in $L^2(\mathbb{R})$ with the domain in $\mathcal{H}_2(\mathbb{R})$, where

$$\mathcal{H}_2(\mathbb{R}) = \{u \in H^2(\mathbb{R}) : \; xu' \in L^2(\mathbb{R}), \; x^2 u \in L^2(\mathbb{R})\}. \tag{1.11}$$

Our main result is now formulated as follows.

Theorem 1.1. *Assume that $W \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. There exists $\varepsilon_0 > 0$ and $\delta_0 > 0$, such that the ODE (1.3) admits a unique family of solutions for any $\varepsilon \in [0,\varepsilon_0)$ and $\delta \in [0,\delta_0)$ with the property*

$$\|\phi - \varepsilon x e^{-x^2/2}\|_{\mathcal{H}_1} \le C_1 \varepsilon \left(\delta + \varepsilon^2\right), \quad |\mu - 1| \le C_2 \left(\delta + \varepsilon^2\right), \tag{1.12}$$

for some (ε,δ)-independent constants $C_1, C_2 > 0$. Moreover, assuming that the conditions (3.17) and (3.18) below are met, there exists a curve $\delta = \delta_*(\varepsilon)$ in a neighborhood of the point $(\varepsilon,\delta) = (0,0)$, such that $|\delta_*(\varepsilon)| \le C_3\varepsilon^2$ for some $C_3 > 0$, along which the linearized problem (1.9) admits an L^2-normalized solution (v_0, w_0) for eigenvalue $\Omega_0 = 1$ and the GP equation (1.2) admits a family of time-periodic space-localized solutions in the form $u(x,t) = e^{-\frac{i}{2}t - i\mu t - i\theta} v(x,t)$ with the properties $v \in \mathcal{H}_1(\mathbb{R})$ for any $t \in \mathbb{R}$, $v\left(x, t + \frac{2\pi}{\Omega}\right) = v(x,t)$ for all $(x,t) \in \mathbb{R}^2$, $|\Omega - 1| \le C_0 \varepsilon^2 s^2$, and

$$\|v(\cdot,t) - \phi(x) - \varepsilon s v_0(x)\cos(\Omega t + \varphi) + i\varepsilon s w_0(x)\sin(\Omega t + \varphi)\|_{\mathcal{H}_1} \le C\varepsilon s^2, \tag{1.13}$$

where $s \in [0, s_0)$, θ and φ are arbitrary parameters, and C_0, C are (ε, s)-independent positive constants.

Remark 1.2. Similarly to the explicit solution (1.6)–(1.7), the periodic solution of Theorem 1.1 is parametrized by θ, φ, and s, in addition to the small parameter ε. Parameters θ and φ can be set to zero because of the two obvious symmetries of the GP equation (1.2): the gauge invariance $u(x,t) \mapsto u(x,t)e^{i\theta_0}$, $\forall \theta_0 \in \mathbb{R}$ and the translation invariance $u(x,t) \mapsto u(x,t-t_0)$, $\forall t_0 \in \mathbb{R}$. The parameter s measures the small amplitude of periodic oscillations.

REMARK 1.3. Theorem 1.1 holds also if $\delta = 0$ with $v_0 = \phi'(x)$ and $w_0 = -x\phi(x)$. In this case, the exact solution (1.6)–(1.7) exists and shows that the periodic solution can be continued for any $\varepsilon \in \mathbb{R}$ and $s \in \mathbb{R}$. In addition, the exact solution shows that $C_0 = 0$, such that $\Omega = 1$ for any $\varepsilon \in \mathbb{R}$ and $s \in \mathbb{R}$.

Although the explicit solution (1.6)–(1.7) persists for all $\mu \in \mathbb{R}$, the persistence of periodic orbits of the GP equation (1.2) with $\delta \neq 0$ is proved in Theorem 1.1 only near $\mu = 1$ along the bifurcation curve $\delta = \delta_*(\mu)$. This is due to the fact that the curve $\delta = \delta_*(\mu)$ in the neighborhood of the point $(\mu, \delta) = (1,0)$ is the only curve (except the line $\delta = 0$), where we can prove rigorously the non-resonance conditions $n\Omega_0 \neq \Omega_m$, $\forall n, m \in \mathbb{N}$, where $\Omega_0 = 1$ is the eigenvalue of the eigenmode (v_0, w_0) in the linearized problem (1.9) and $\Omega_m \neq 1$, $m \in \mathbb{N}$ are other eigenvalues of the linearized problem (1.9) near the point $(\mu, \delta) = (1,0)$. We do not have control of the non-resonance conditions far from the point $(\mu, \delta) = (1,0)$ and away from the curve $\delta = \delta_*(\mu)$. Although the proof of Theorem 1.1 is a modification of the proof of the Lyapunov Center Theorem for persistence of periodic orbits in a neighborhood of an elliptic stationary point [**15**, Chapter II], it is, nevertheless, complicated by the presence of translational eigenmodes associated with the double zero eigenvalue of the linearized problem and by the infinite-dimensional setting of the problem.

Our strategy for the proof of Theorem 1.1 is to use a complete set of Hermite functions and to reformulate the evolution problem for the GP equation (1.2) as an infinite-dimensional discrete dynamical system for coefficients of the decomposition (Section 2). This technical trick is motivated by the fact that the spectrum of the linearized problem associated with the parabolic potential is purely discrete, such that the components of the decomposition are normal modes of the linearized system. Existence of stationary solutions $\phi(x)$ of the ODE (1.3) and spectral stability of stationary solutions in the linearized problem are studied in the framework of the discrete dynamical system (Section 3). The proof of existence of periodic solutions of the GP equation (1.2) relies on the construction of periodic orbits in the discrete dynamical system (Section 4). The analytical results are verified with numerical approximations of solutions of the ODE (1.3), eigenvalues of the linearized problem (1.9) and solutions of the GP equation (1.2) (Section 5). We would like to note that the same results can be obtained with the PDE formulation of the time-evolution problem, if the convergence of the series of eigenfunctions of the linearized problem (1.9) can be proven.

Our main result is in agreement with Theorem 2.1 in [**8**], where the Newton particle equation is obtained in a more general context of multi-dimensional confining potentials and arbitrary nonlinear functions of the GP equation

$$i\dot{\psi} = -\nabla^2 \psi + V(x)\psi - f(\psi). \tag{1.14}$$

Newton's equation is derived for parameters (a, b) of the solitary wave solution of the unperturbed equation (1.14) with $V(x) \equiv 0$ and it takes the form

$$\dot{a} = 2b, \qquad \dot{b} = -\nabla V(a). \tag{1.15}$$

Adopting our notations for the time variable and the potential function $V(a) = a^2 + \delta W(a)$, we rewrite Newton's equation (1.15) in the explicit form

$$\ddot{a} + a = \frac{\delta}{2} W'(a),$$

which recovers the bound $|\Omega - 1| \lesssim \delta$ for the frequency of the periodic solution of Theorem 1.1. If $|\delta_*(\varepsilon)| \lesssim \varepsilon^2$, then $|\Omega - 1| \lesssim \varepsilon^2$, in agreement with Theorem 1.1.

There are several differences between results of Theorem 2.1 in [**8**] and our Theorem 1.1. First, Newton's equation (1.15) is valid on finite time intervals. Second, its derivation is carried out in the limit where the localization length of the stationary solution $\phi(x)$ is much smaller than the confinement length of the potential $V(x)$. Third, the exact periodicity is not guaranteed by the periodic solutions of Newton's equation (1.15) because of the remainder terms. In our case, the result of Theorem 1.1 is valid for all time intervals, the localization and confinement lengths are of the same order, and the exact periodicity is guaranteed for all times. On the other hand, our results are valid only near the linear limit of the GP equation (1.2).

We recall that the 2π-oscillations of dark solitons in the GP equation (1.2) with $\delta = 0$ were predicted from the Ehrenfest Theorem in much earlier papers (see references in [**4**] and [**8**]). However, it was argued that the period $T_0 = 2\pi$ is not observed in numerical simulations of oscillations of dark solitons for $\sigma = 1$ and small γ [**4, 17, 19**]. It was suggested in this work (see review in [**12**]) that dark solitons oscillate with a larger period $T_1 = 2\sqrt{2}\pi$. (From a qualitative point of view, $\Omega_0 = 1$ corresponds to a frequency of oscillations of the ground state supporting a dark soliton, while $\Omega_1 = \frac{1}{\sqrt{2}}$ corresponds to a frequency of oscillations of the dark soliton near the center of the ground state.) Our numerical results show that both eigenvalues $\Omega_0 = 1$ and $\Omega_1 = \frac{1}{\sqrt{2}}$ occur in the spectrum of the linearized problem (1.9) as $\mu \to \infty$ but the non-resonance conditions $n\Omega_1 \neq \Omega_m$ for $n, m \geq 2$ are violated for the frequency $\Omega_1 = \frac{1}{\sqrt{2}}$ as $n, m \to \infty$. (The resonance conditions are also violated for $\Omega_0 = 1$ in the limit $\mu \to \infty$ but the existence of a periodic solution with the frequency $\Omega_0 = 1$ is guaranteed by the exact solution (1.6)–(1.7).) Therefore, at the present time, we cannot construct periodic solutions with period $T_1 = 2\sqrt{2}\pi$ in the limit $\mu \to \infty$.

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2. Formalism of the discrete dynamical system

The set of Hermite functions is defined by the standard expressions [**1**, Chapter 22]:

$$\phi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-x^2/2}, \qquad \forall n = 0, 1, 2, 3, ..., \tag{2.1}$$

where $H_n(x)$ denote the Hermite polynomials, e.g. $H_0 = 1$, $H_1 = 2x$, $H_2 = 4x^2 - 2$, $H_3 = 8x^3 - 12x$, etc. Since the Hermite functions are eigenfunctions of the linear Schrödinger equation

$$-\frac{1}{2}\phi_n''(x) + \frac{1}{2}x^2\phi_n(x) = \left(n + \frac{1}{2}\right)\phi_n(x), \tag{2.2}$$

the Sturm–Liouville theory implies that the set of Hermite functions $\{\phi_n(x)\}_{n=0}^{\infty}$ forms an orthogonal basis in $L^2(\mathbb{R})$. The normalization coefficients in the expressions (2.1) ensure that the Hermite functions satisfy orthonormality conditions

$$(\phi_n, \phi_m) = \delta_{n,m}. \tag{2.3}$$

We represent a solution $u(x,t)$ of the GP equation (1.2) by the series of eigenfunctions

$$u(x,t) = e^{-\frac{i}{2}t} \sum_{n=0}^{\infty} a_n(t)\phi_n(x) \tag{2.4}$$

where the components $(a_0, a_1, a_2, ...)$ form a vector $\mathbf{a}$ on $\mathbb{N}$. When the series representation (2.4) is substituted to the GP equation (1.2), the PDE problem is converted to the discrete dynamical system

$$i\dot{a}_n = na_n + \delta \sum_{m=0}^{\infty} W_{n,m} a_m + \sigma \sum_{n_1,n_2,n_3=0}^{\infty} K_{n,n_1,n_2,n_3} a_{n_1} \bar{a}_{n_2} a_{n_3}, \tag{2.5}$$

where $W_{n,m} = (\phi_n, W\phi_m)$ and $K_{n,n_1,n_2,n_3} = (\phi_n, \phi_{n_1}\phi_{n_2}\phi_{n_3})$. Let $l_s^2(\mathbb{N})$ be a weighted discrete l^2-space equipped with the squared norm

$$\|\mathbf{a}\|_{l_s^2}^2 = \sum_{n=0}^{\infty} (1+n)^{2s} |a_n|^2 < \infty, \tag{2.6}$$

for some $s \in \mathbb{R}$. Since the set $\{\phi_n(x)\}_{n=0}^{\infty}$ forms an orthonormal basis in $L^2(\mathbb{R})$, we note the isometry $\|u\|_{L^2}^2 = \|\mathbf{a}\|_{l^2}^2$, so that $u \in L^2(\mathbb{R})$ if and only if $\mathbf{a} \in l^2(\mathbb{N})$. The quantity

$$Q = \|u\|_{L^2}^2 = \|\mathbf{a}\|_{l^2}^2 \tag{2.7}$$

is constant in the time evolution of the GP equation (1.2) and the discrete dynamical system (2.5). Both systems are also Hamiltonian with the time-conserved Hamiltonian function in the form

$$\begin{aligned} E &= \frac{1}{2} \int_{\mathbb{R}} \left(|u_x|^2 + x^2|u|^2 + |u|^2 + 2\delta W(x)|u|^2 + \sigma |u|^4 \right) dx \\ &= \sum_{n=0}^{\infty} n|a_n|^2 + \delta \sum_{n,m=0}^{\infty} W_{n,m} a_n \bar{a}_m + \sigma \sum_{n,n_1,n_2,n_3=0}^{\infty} K_{n,n_1,n_2,n_3} a_n a_{n_1} \bar{a}_{n_2} \bar{a}_{n_3}. \end{aligned}$$

The Hamiltonian function E is bounded if $W \in L^\infty(\mathbb{R})$ and $u \in \mathcal{H}_1(\mathbb{R})$. The following results establish the equivalence between $\mathcal{H}_1(\mathbb{R})$ for u and $l_{1/2}^2(\mathbb{N})$ for $\mathbf{a}$ and determine the phase space for the discrete dynamical system (2.5).

LEMMA 2.1. *Let* $u(x) = \sum_{m=0}^{\infty} a_n \phi_n(x)$. *Then* $u \in \mathcal{H}_1(\mathbb{R})$ *if and only if* $\mathbf{a} \in l_{1/2}^2(\mathbb{N})$.

PROOF. It follows directly that

$$\begin{aligned}\|u\|^2_{\mathcal{H}_1} &= \int_{\mathbb{R}} \left(|u'(x)|^2 + (x^2+1)|u(x)|^2\right) dx \\ &= \sum_{n_1,n_2=0}^{\infty} a_{n_1}\bar{a}_{n_2} \int_{\mathbb{R}} \left[\phi'_{n_1}(x)\phi'_{n_2}(x) + (x^2+1)\phi_{n_1}(x)\phi_{n_2}(x)\right] dx \\ &= 2\sum_{n_1,n_2=0}^{\infty} a_{n_1}\bar{a}_{n_2}(1+n_2)(\phi_{n_1},\phi_{n_2}) = 2\|\mathbf{a}\|^2_{l^2_{1/2}},\end{aligned}$$

where the orthogonality relations (2.3) have been used. □

REMARK 2.2. By the same method, we can prove that $u \in \mathcal{H}_2(\mathbb{R})$ if and only if $\mathbf{a} \in l^2_1(\mathbb{N})$. A more general correspondence between $\mathcal{H}_n(\mathbb{R})$ and $l^2_{n/2}(\mathbb{N})$ for positive integer n was recently obtained in [**20**].

LEMMA 2.3. *Assume that $W \in L^2(\mathbb{R})$. The vector field of the dynamical system (2.5) maps $l^2_{1/2}(\mathbb{N})$ to $l^2_{-1/2}(\mathbb{N})$*

PROOF. The vector field of the dynamical system (2.5) is decomposed into three parts represented by functions $\mathbf{f}(\mathbf{a})$, $\delta\mathbf{g}(\mathbf{a})$ and $\sigma\mathbf{h}(\mathbf{a})$, where

$$f_n = n a_n, \quad g_n = \sum_{m=0}^{\infty} W_{n,m} a_m, \quad h_n = \sum_{n_1,n_2,n_3=0}^{\infty} K_{n,n_1,n_2,n_3} a_{n_1}\bar{a}_{n_2} a_{n_3}.$$

The first part satisfies the estimate

$$\|\mathbf{f}(\mathbf{a})\|^2_{l^2_s} = \sum_{n=0}^{\infty}(1+n)^{2s} n^2 |a_n|^2 \le \|\mathbf{a}\|^2_{l^2_{s+1}},$$

such that $\mathbf{f} : l^2_{s+1}(\mathbb{N}) \mapsto l^2_s(\mathbb{N})$ for all $s \in \mathbb{R}$. If $\mathbf{a} \in l^2_{1/2}(\mathbb{N})$, then $s = -\frac{1}{2}$. Since

$$|W_{n,m}| \le \|W\|_{L^2}\|\phi_n\phi_m\|_{L^2} \le \|W\|_{L^2}\|\phi_n\|_{L^4}\|\phi_m\|_{L^4},$$

the second part satisfies the estimate

$$\begin{aligned}\|\mathbf{g}(\mathbf{a})\|^2_{l^2_s} &= \sum_{n=0}^{\infty}(1+n)^{2s} \sum_{m_1,m_2=0}^{\infty} W_{n,m_1}W_{n,m_2} a_{m_1}\bar{a}_{m_2} \\ &\le \|W\|^2_{L^2}\left(\sum_{n=0}^{\infty}(1+n)^{2s}\|\phi_n\|^2_{L^4}\right)\left(\sum_{m=0}^{\infty}\|\phi_m\|_{L^4}|a_m|\right)^2 \\ &\le \|W\|^2_{L^2}\left(\sum_{n=0}^{\infty}(1+n)^{2s}\|\phi_n\|^2_{L^4}\right)\left(\sum_{m=0}^{\infty}(1+m)^{-2(s+1)}\|\phi_m\|^2_{L^4}\right)\|\mathbf{a}\|^2_{l^2_{s+1}}.\end{aligned}$$

By the main result of [**6**], there exists a constant $C > 0$ such that

$$\|\phi_n\|^4_{L^4} \le C\frac{\log(2+n)}{\sqrt{1+n}}, \qquad \forall n \in \mathbb{N}. \tag{2.8}$$

Therefore, the series $\sum_{n=0}^{\infty}(1+n)^{2s}\|\phi_n\|^2_{L^4}$ and $\sum_{m=0}^{\infty}(1+m)^{-2(s+1)}\|\phi_m\|^2_{L^4}$ converge for all $-\frac{5}{8} < s < -\frac{3}{8}$, such that $\mathbf{g} : l^2_{s+1}(\mathbb{N}) \mapsto l^2_s(\mathbb{N})$ for all $-\frac{5}{8} < s < -\frac{3}{8}$.

The value $s = -\frac{1}{2}$ is a middle point of this interval. Finally, the last part of the vector field satisfies the estimate

$$\begin{aligned}
\|\mathbf{h}(\mathbf{a})\|^2_{l^2_s} &= \sum_{n=0}^{\infty}(1+n)^{2s} \sum_{n_1,n_2,n_3=0}^{\infty} \sum_{m_1,m_2,m_3=0}^{\infty} K_{n,n_1,n_2,n_3} K_{n,m_1,m_2,m_3} a_{n_1}\bar{a}_{n_2}a_{n_3}\bar{a}_{m_1}a_{m_2}\bar{a}_{m_3} \\
&= \sum_{n=0}^{\infty}(1+n)^{2s} \left|\left(\phi_n u, |u|^2\right)\right|^2 \leq \left(\sum_{n=0}^{\infty}(1+n)^{2s}\|u\phi_n\|^2_{L^2}\right)\|u\|^4_{L^4} \\
&\leq \left(\sum_{n=0}^{\infty}(1+n)^{2s}\|\phi_n\|^2_{L^4}\right)\|u\|^6_{L^4},
\end{aligned}$$

where $u(x) = \sum\limits_{n=0}^{\infty} a_n\phi_n(x)$ and the series $\sum_{n=0}^{\infty}(1+n)^{2s}\|\phi_n\|^2_{L^4}$ converges if $s = -\frac{1}{2}$. By the Sobolev embedding, we have

$$\|u\|^4_{L^4} \leq \|u\|^2_{L_\infty}\|u\|^2_{L^2} \leq C_0\|u\|^4_{H^1} \leq C_0\|u\|^4_{\mathcal{H}_1},$$

for some $C_0 > 0$. Since the norm in $\mathcal{H}_1(\mathbb{R})$ for the function $u(x)$ is equivalent to the norm in $l^2_{1/2}(\mathbb{Z})$ for the vector $\mathbf{a}$ by Lemma 2.1, we obtain that $\mathbf{h} : l^2_{1/2}(\mathbb{N}) \mapsto l^2_{-1/2}(\mathbb{N})$. The triangle inequality applied to the nonlinear vector field $\mathbf{f}(\mathbf{a})+\delta\mathbf{g}(\mathbf{a})+\sigma\mathbf{h}(\mathbf{a})$ concludes the proof of the lemma. □

THEOREM 2.4. *Assume that $W \in L^2(\mathbb{R})$. There exists a global solution $\mathbf{a}(t) \in C^1(\mathbb{R}, l^2_{1/2}(\mathbb{N}))$ of the discrete dynamical system (2.5).*

PROOF. By Proposition 2.2 in [**5**], there exists a global solution

$$u(t) \in C^1(\mathbb{R}, \mathcal{H}_1(\mathbb{R}))$$

of the GP equation (1.2) if $W \in L^2(\mathbb{R})$. By Lemma 2.1, the trajectory $u(t) \in \mathcal{H}_1(\mathbb{R})$ is equivalent to the trajectory $\mathbf{a}(t) \in l^2_{1/2}(\mathbb{N})$ on $t \in \mathbb{R}$. By Lemma 2.3, the vector field of the discrete dynamical system (2.5) acts on the phase space $l^2_{1/2}(\mathbb{N}) \subset l^2(\mathbb{N})$. The system is hence equivalent to the GP equation (1.2) due to standard orthogonal projections in $L^2(\mathbb{R})$. □

3. Existence and stability of stationary solutions

Stationary solutions of the dynamical system (2.5) take the form $\mathbf{a}(t) = \mathbf{A}e^{-i\mu t}$, where $\mathbf{A}$ is a time-independent vector and μ is a parameter of the solution. By Lemma 2.1, if $\mathbf{A} \in l^2_{1/2}(\mathbb{N})$ and $\phi(x) = \sum\limits_{n=0}^{\infty} A_n\phi_n(x)$, then $\phi \in \mathcal{H}_1(\mathbb{R})$ is a stationary solution of the GP equation (1.2), that is $\phi(x)$ satisfies the ODE (1.3). By Lemma 2.3, the vector $\mathbf{A}$ is found as a root of the infinite-dimensional cubic vector field $\mathbf{F} : l^2_{1/2}(\mathbb{N}) \times \mathbb{R}^2 \mapsto l^2_{-1/2}(\mathbb{N})$, where the n-th component of $\mathbf{F}(\mathbf{A}; \mu, \delta)$ is given by

$$F_n = (\mu - n)A_n - \delta\sum_{m=0}^{\infty} W_{n,m}A_m - \sigma \sum_{n_1,n_2,n_3=0}^{\infty} K_{n,n_1,n_2,n_3}A_{n_1}\bar{A}_{n_2}A_{n_3}. \tag{3.1}$$

The Jacobian operator $D_\mathbf{A}\mathbf{F}(\mathbf{0}; \mu, 0)$ is a diagonal matrix with entries $\mu - n$ and it admits a one-dimensional kernel if $\mu = n_0$ for any non-negative integer n_0. The corresponding eigenvector is $\mathbf{e}_{n_0}$, the unit vector in $l^2(\mathbb{N})$. According to local bifurcation theory [**7**], each eigenvector of $D_\mathbf{A}\mathbf{F}(\mathbf{0}; n_0, 0)$ can be uniquely continued in a local neighborhood of the point $\mathbf{A} = \mathbf{0} \in l^2_{1/2}(\mathbb{N})$ and $(\mu, \delta) = (n_0, 0) \in \mathbb{R}^2$. We are particularly interested in the second eigenvalue $n_0 = 1$ with the second eigenvector $\mathbf{e}_1$, which corresponds to the *dark soliton* $\phi(x)$ with a single zero. (Other bifurcations of stationary localized solutions $\phi(x)$ are considered in [**2, 10, 11**].) Details of this bifurcation are given in the following proposition.

PROPOSITION 1. Assume that $W \in L^2(\mathbb{R})$ and consider real-valued roots $\mathbf{A} \in l^2_{1/2}(\mathbb{N})$ of the vector field $\mathbf{F}(\mathbf{A};\mu,\delta)$. There exists $\varepsilon_0 > 0$ and $\delta_0 > 0$, such that the solution set of $\mathbf{F}(\mathbf{A};\mu,\delta) = \mathbf{0}$ includes a unique family of solutions for any $\varepsilon \in [0,\varepsilon_0)$ and $\delta \in [0,\delta_0)$ with the property
(3.2)
$$\|\mathbf{A} - \varepsilon \mathbf{e}_1\|_{l^2_{1/2}} \leq C_1 \varepsilon \left(\delta + \varepsilon^2\right), \quad \left|\mu - 1 - \varepsilon^2 \sigma K_{1,1,1,1} - \delta W_{1,1}\right| \leq C_2 \left(\delta + \varepsilon^2\right)^2,$$
for some (ε,δ)-independent constants $C_1, C_2 > 0$. Moreover, if $\sigma \neq 0$, the solution $\mathbf{A}$ is smooth with respect to μ for sufficiently small (ε,δ) and $\frac{d}{d\mu}\|\mathbf{A}\|^2_{l^2} \neq 0$.

PROOF. Both $\mathbf{F}(\mathbf{A};\mu,\delta)$ and $D_{\mathbf{A}}\mathbf{F}(\mathbf{A};\mu,\delta)$ are continuous in a local neighborhood of $\mathbf{A} = \mathbf{0} \in l^2_{1/2}(\mathbb{N})$ and $(\mu,\delta) = (1,0) \in \mathbb{R}^2$. At the point $\mathbf{A} = \mathbf{0}$ and $(\mu,\delta) = (1,0)$, the operator has a one-dimensional kernel with the eigenvector $\mathbf{e}_1 \in l^2(\mathbb{N})$. By using the method of Lyapunov–Schmidt reductions [**7**], we set $\mathbf{A} = \varepsilon \left[\mathbf{e}_1 + \tilde{\mathbf{A}}\right]$ and $\mu = 1 + \tilde{\mu}$, where $(\tilde{\mathbf{A}}, \mathbf{e}_1) = 0$, that is $\tilde{A}_1 = 0$. The orthogonal projection of equation (3.1) to $\mathbf{e}_1$ gives a bifurcation equation for $\tilde{\mu}$
$$\begin{aligned}\tilde{\mu} \;=\; & \delta\left(W_{1,1} + \sum_{m=0}^{\infty} W_{1,m}\tilde{A}_m\right) + \sigma\varepsilon^2\left[K_{1,1,1,1} + 3\sum_{n_1=0}^{\infty} K_{1,1,1,n}\tilde{A}_{n_1}\right.\\ & \left. +3\sum_{n_1,n_2=0}^{\infty} K_{1,1,n_1,n_2}\tilde{A}_{n_1}\tilde{A}_{n_2} + \sum_{n_1,n_2,n_3=0}^{\infty} K_{1,n_1,n_2,n_3}\tilde{A}_{n_1}\tilde{A}_{n_2}\tilde{A}_{n_3}\right].\end{aligned}$$
Let P_1 be an orthogonal projection to the complement of $\mathbf{e}_1$ in $l^2(\mathbb{N})$. Then, the inverse of $P_1 D_{\mathbf{A}}\mathbf{F}(\mathbf{0};1,0)P_1$ exists and is a bounded operator from $l^2_{-1/2}(\mathbb{N})$ to $l^2_{1/2}(\mathbb{N})$. By the Implicit Function Theorem in space $l^2_{1/2}(\mathbb{N}) \times \mathbb{R}^2$, there exists a unique smooth solution $\tilde{\mathbf{A}}$ in the neighborhood of $\tilde{\mathbf{A}} = \mathbf{0} \in l^2_{1/2}(\mathbb{N})$ parameterized by (ε,δ) in the neighborhood of $(0,0) \in \mathbb{R}^2$, such that $\|\tilde{\mathbf{A}}\|_{l^2_{1/2}} \leq C_1(\delta+\varepsilon^2)$ for some $C_1 > 0$. By the Implicit Function Theorem in $\mathbb{R}\times\mathbb{R}^2$, there exists a unique smooth solution $\tilde{\mu}$ of the bifurcation equation in the neighborhood of $\tilde{\mu} = 0$ for (ε,δ) near the point $(0,0) \in \mathbb{R}^2$, such that $|\tilde{\mu} - \varepsilon^2\sigma K_{1,1,1,1} - \delta W_{1,1}| \leq C_2(\delta+\varepsilon^2)^2$ for some $C_2 > 0$. Since $\|\mathbf{A}\|^2_{l^2} = \varepsilon^2 + \mathrm{O}(\varepsilon^2(\varepsilon^2+\delta))$ and $\mu - 1 = \frac{3\sigma\varepsilon^2}{4\sqrt{2\pi}} + \delta W_{1,1} + \mathrm{O}(\varepsilon^2+\delta)^2$, where the value $K_{1,1,1,1} = \frac{3}{4\sqrt{2\pi}}$ is computed in Table I, then $\frac{d}{d\mu}\|\mathbf{A}\|^2_{l^2} \neq 0$ near $(\mu,\delta) = (1,0)$ for $\sigma \neq 0$. □

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$K_{n,n,n,n}$	$\frac{1}{\sqrt{2\pi}}$	$\frac{3}{4\sqrt{2\pi}}$	$\frac{41}{64\sqrt{2\pi}}$	$\frac{147}{256\sqrt{2\pi}}$	$\frac{8649}{16384\sqrt{2\pi}}$	$\frac{32307}{65536\sqrt{2\pi}}$
$K_{1,n,n,1}$	$\frac{1}{2\sqrt{2\pi}}$	$\frac{3}{4\sqrt{2\pi}}$	$\frac{7}{16\sqrt{2\pi}}$	$\frac{11}{32\sqrt{2\pi}}$	$\frac{75}{256\sqrt{2\pi}}$	$\frac{133}{512\sqrt{2\pi}}$
$K_{0,1,1,n}$	$\frac{1}{2\sqrt{2\pi}}$	0	$\frac{1}{8\sqrt{\pi}}$	0	$-\frac{3\sqrt{3}}{32\sqrt{\pi}}$	0

Table I: Numerical values for $K_{n,n,n,n} = \|\phi_n\|^4_{L^4}$, $K_{1,n,n,1} = (\phi_1^2, \phi_n^2)$, and $K_{0,1,1,n} = (\phi_0\phi_n, \phi_1^2)$.

Spectral stability of the stationary solution $e^{-i\mu t}\mathbf{A}$ is studied with the expansion

$$\mathbf{a}(t) = e^{-i\mu t}\left[\mathbf{A} + (\mathbf{B}-\mathbf{C})\,e^{i\Omega t} + \left(\bar{\mathbf{B}}+\bar{\mathbf{C}}\right)e^{-i\bar{\Omega} t} + \mathrm{O}(\|\mathbf{B}\|^2 + \|\mathbf{C}\|^2)\right], \tag{3.3}$$

where the spectral parameter $\Omega \in \mathbb{C}$ and the eigenvector $(\mathbf{B},\mathbf{C}) \in l^2(\mathbb{N},\mathbb{C}^2)$ satisfy the linear problem

$$L_+\mathbf{B} = \Omega\mathbf{C}, \qquad L_-\mathbf{C} = \Omega\mathbf{B}, \tag{3.4}$$

associated with matrix operators $L_\pm$. Their n-th components are defined in the form

$$\left\{\begin{array}{lcl} (L_+\mathbf{B})_n & = & (n-\mu)B_n + \delta\sum_{m=0}^\infty W_{n,m}B_m + 3\sigma\sum_{m=0}^\infty V_{n,m}B_m, \\ (L_-\mathbf{C})_n & = & (n-\mu)C_n + \delta\sum_{m=0}^\infty W_{n,m}C_m + \sigma\sum_{m=0}^\infty V_{n,m}C_m, \end{array}\right.$$

where $V_{n,m} = \sum_{n_2,n_3=0}^\infty K_{n,m,n_2,n_3}A_{n_2}A_{n_3}$. We have used here the symmetry of the coefficients K_{n,n_1,n_2,n_3} with respect to the interchange of (n_1,n_2,n_3).

Lemma 3.1. *Assume that $W \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and let $\mathbf{A} \in l^2_{1/2}(\mathbb{N})$ be a real-valued root of the vector field $\mathbf{F}(\mathbf{A};\mu,\delta)$. Operators L_+ and L_- admit closed self-adjoint extensions in $l^2(\mathbb{N})$ with the domain in $l^2_1(\mathbb{N})$.*

Proof. The diagonal unbounded part of $L_\pm$ maps $l^2_1(\mathbb{N})$ to $l^2(\mathbb{N})$. We need to show that the non-diagonal parts of $L_\pm$ represent bounded perturbations from $l^2(\mathbb{N})$ to $l^2(\mathbb{N})$ if $\mathbf{A} \in l^2_{1/2}(\mathbb{N})$ and $W \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. This is done by using computations similar to those in the proof of Lemma 2.3:

$$\begin{aligned} \sum_{n=0}^\infty \left|\sum_{m=0}^\infty W_{n,m}B_m\right|^2 &= \sum_{n=0}^\infty |(\phi_n, Wv)|^2 \\ &= \|Wv\|^2_{L^2} \leq \|W\|^2_{L^\infty}\|v\|^2_{L^2} = \|W\|^2_{L^\infty}\sum_{n=0}^\infty |B_n|^2 \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^\infty \left|\sum_{m=0}^\infty V_{n,m}B_m\right|^2 &= \sum_{n=0}^\infty |(\phi_n, u^2 v)|^2 \\ &= \|u^2 v\|^2_{L^2} \leq \|u\|^4_{L^\infty}\|v\|^2_{L^2} \leq C\|u\|^4_{\mathcal{H}_1}\sum_{n=0}^\infty |B_n|^2, \end{aligned}$$

where $u = \sum_{n=0}^\infty A_n\phi_n(x) \in \mathcal{H}_1(\mathbb{R})$, $v = \sum_{n=0}^\infty B_n\phi_n(x) \in L^2(\mathbb{R})$, and $C > 0$. □

Remark 3.2. The result of Lemma 3.1 is obvious from the equivalence between the space $\mathcal{H}_2(\mathbb{R})$ for the function $v(x)$ and the space $l^2_1(\mathbb{N})$ for the vector $\mathbf{B}$, see Remark 2.2. The matrix operators $L_\pm$ represent the action of differential operators $\mathcal{L}_\pm$ given by (1.10) on the basis of Hermite functions (2.1) in $\mathcal{H}_2(\mathbb{R})$.

Remark 3.3. The linear problem (3.4) has eigenvalue $\Omega = 0$ of geometric multiplicity one and algebraic multiplicity two due to the exact solution

$$L_-\mathbf{A} = \mathbf{0}, \qquad L_+\partial_\mu\mathbf{A} = \mathbf{A}, \tag{3.5}$$

where the smoothness of $\mathbf{A}$ with respect to μ near $\mu = 1$ is guaranteed by Proposition 1. Eigenvectors for the double zero eigenvalue are represented by the expansions

$$\mathbf{A} = \varepsilon \mathbf{e}_1 + \mathrm{O}(\varepsilon(\varepsilon^2 + \delta)), \qquad \partial_\mu \mathbf{A} = \frac{4\sqrt{2\pi}}{3\sigma\varepsilon} \mathbf{e}_1 + \mathrm{O}(\varepsilon^{-1}(\varepsilon^2 + \delta)), \tag{3.6}$$

where ε and δ are sufficiently small. As a result, $\langle \mathbf{A}, \partial_\mu \mathbf{A} \rangle = \frac{4\sqrt{2\pi}}{3\sigma} + \mathrm{O}(\varepsilon^2 + \delta)$.

When $\mathbf{A} = \mathbf{0}$ and $(\mu, \delta) = (1, 0)$, the spectrum of the eigenvalue problem (3.4) is known in the explicit form. It consists of eigenvalues $\Omega = 0$ and $\Omega = \pm 1$ of geometric and algebraic multiplicities two and simple eigenvalues $\Omega = \pm m$ for all $m = 2, 3, ...$ The double zero eigenvalue persists for any ε and δ according to the exact solution (3.5). Let us enumerate the non-zero eigenvalues as $\{\pm \Omega_m\}_{m=0}^{\infty}$ such that $\Omega_0 = \Omega_1 = 1$ and $\Omega_m = m$, $m \geq 2$ for $\varepsilon = 0$ and $\delta = 0$. Splitting of non-zero eigenvalues in a local neighborhood of $\mathbf{A} = \mathbf{0}$ and $(\mu, \delta) = (1, 0)$ is described by the following proposition.

Proposition 2. Let $\mathbf{A} \in l^2_{1/2}(\mathbb{N})$ be defined by Proposition 1 for sufficiently small ε and δ and for $W \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Non-zero real eigenvalues $\{\Omega_m\}_{m=2}^{\infty}$ of the linear problem (3.4) for any $\varepsilon \in [0, \varepsilon_0)$ and $\delta \in [0, \delta_0)$ satisfy the bound

$$\left| \Omega_m - m + \varepsilon^2 \sigma \left(K_{1,1,1,1} - 2K_{m+1,1,1,m+1} \right) + \delta \left(W_{1,1} - W_{m+1,m+1} \right) \right| \leq C_2 \left(\varepsilon^2 + \delta \right)^2, \tag{3.7}$$

for some (ε,δ,m)-independent constant $C_2 > 0$. If $\delta = 0$, then $\Omega_0 = 1$ and Ω_1 satisfies the bound

$$\left| \Omega_1 - 1 + \frac{\varepsilon^2 \sigma}{8\sqrt{2\pi}} \right| \leq C_1 \varepsilon^4 \tag{3.8}$$

for some ε-independent constant $C_1 > 0$. If $W_{0,0} \neq W_{1,1} \neq W_{2,2}$, there exists a curve $\delta = \delta_*(\varepsilon)$ with the property

$$\left| \delta_*(\varepsilon) - \frac{\varepsilon^2 \sigma}{8\sqrt{2\pi}} \frac{3W_{1,1} - 2W_{2,2} - W_{0,0}}{(W_{1,1} - W_{0,0})(W_{1,1} - W_{2,2})} \right| \leq C_0 \varepsilon^4, \tag{3.9}$$

such that $\Omega_0 = 1$ and Ω_1 satisfies the bound

$$\left| \Omega_1 - 1 + \frac{\varepsilon^2 \sigma}{8\sqrt{2\pi}} \left(1 + \frac{(3W_{1,1} - 2W_{2,2} - W_{0,0})(W_{0,0} - W_{2,2})}{(W_{1,1} - W_{0,0})(W_{1,1} - W_{2,2})} \right) \right| \leq C_1 \varepsilon^4 \tag{3.10}$$

for some ε-independent constants $C_0, C_1 > 0$.

Proof. Since the essential spectrum of the matrix operators $L_\pm$ is empty and the potential terms are bounded perturbations to the unbounded diagonal terms, isolated eigenvalues split according to the standard perturbation theory for isolated eigenvalues [**9**, Chapter 7]. The eigenvectors for a simple eigenvalue $\Omega = m$ with $m = 2, 3, ...$ are expanded as follows

$$\mathbf{B} = \mathbf{e}_{m+1} + \tilde{\mathbf{B}}, \quad \mathbf{C} = \mathbf{e}_{m+1} + \tilde{\mathbf{C}}, \quad \Omega = m + \tilde{\Omega}.$$

Projections to the component $n = m + 1$ gives the linear system at principal order

$$\begin{aligned} m \left(\tilde{B}_{m+1} - \tilde{C}_{m+1} \right) &= \delta \left(W_{1,1} - W_{m+1,m+1} \right) + \sigma\varepsilon^2 \left(K_{1,1,1,1} - 3K_{m+1,1,1,m+1} \right) + \tilde{\Omega}, \\ m \left(\tilde{C}_{m+1} - \tilde{B}_{m+1} \right) &= \delta \left(W_{1,1} - W_{m+1,m+1} \right) + \sigma\varepsilon^2 \left(K_{1,1,1,1} - K_{m+1,1,1,m+1} \right) + \tilde{\Omega}. \end{aligned}$$

The linear system has a solution if and only if $\tilde{\Omega} = \sigma\varepsilon^2 \left(2K_{m+1,1,1,m+1} - K_{1,1,1,1} \right) +$ $\delta \left(W_{m+1,m+1} - W_{1,1} \right)$. By Lemma 3.1, the potentials of the linear operators $L_\pm$ are bounded perturbations from $l^2(\mathbb{N})$ to $l^2(\mathbb{N})$. Therefore, there exists a constant $C_2 > 0$ uniformly in m, such that the eigenvalues $\{\Omega_m\}_{m=2}^{\infty}$ persist according to the expansion (3.7).

The eigenvectors for the double eigenvalue $\Omega = 1$ are expanded as follows

$$\mathbf{B} = \alpha \mathbf{e}_0 + \beta \mathbf{e}_2 + \tilde{\mathbf{B}}, \quad \mathbf{C} = -\alpha \mathbf{e}_0 + \beta \mathbf{e}_2 + \tilde{\mathbf{C}}, \quad \Omega = 1 + \tilde{\Omega},$$

where (α, β) are arbitrary parameters. Projections to the components $n = 0$ and $n = 2$ give the linear system at principal order

$$\begin{aligned}
\left(\tilde{B}_0 + \tilde{C}_0\right) &= \delta\left(W_{0,0}\alpha + W_{0,2}\beta - W_{1,1}\alpha\right) + \sigma\varepsilon^2\left(3K_{0;1,1,0}\alpha + 3K_{0,1,1,2}\beta - K_{1,1,1,1}\alpha\right) + \tilde{\Omega}\alpha, \\
-\left(\tilde{B}_0 + \tilde{C}_0\right) &= \delta\left(W_{0,0}\alpha - W_{0,2}\beta - W_{1,1}\alpha\right) + \sigma\varepsilon^2\left(K_{0,1,1,0}\alpha - K_{0,1,1,2}\beta - K_{1,1,1,1}\alpha\right) + \tilde{\Omega}\alpha, \\
\left(\tilde{B}_2 - \tilde{C}_2\right) &= \delta\left(W_{1,1}\beta - W_{2,0}\alpha - W_{2,2}\beta\right) + \sigma\varepsilon^2\left(K_{1,1,1,1}\beta - 3K_{2,1,1,0}\alpha - 3K_{2,1,1,2}\beta\right) + \tilde{\Omega}\beta, \\
-\left(\tilde{B}_2 - \tilde{C}_2\right) &= \delta\left(W_{1,1}\beta + W_{2,0}\alpha - W_{2,2}\beta\right) + \sigma\varepsilon^2\left(K_{1,1,1,1}\beta + K_{2,1,1,0}\alpha - K_{2,1,1,2}\beta\right) + \tilde{\Omega}\beta.
\end{aligned}$$

The linear system has a solution if and only if (α, β) satisfies a homogeneous system

$$\begin{aligned}
\delta\left(W_{1,1} - W_{0,0}\right)\alpha + \sigma\varepsilon^2\left(K_{1,1,1,1}\alpha - 2K_{0,1,1,0}\alpha - K_{0,1,1,2}\beta\right) &= \tilde{\Omega}\alpha, \\
\delta\left(W_{2,2} - W_{1,1}\right)\beta + \sigma\varepsilon^2\left(-K_{1,1,1,1}\beta + K_{2,1,1,0}\alpha + 2K_{2,1,1,2}\beta\right) &= \tilde{\Omega}\beta.
\end{aligned}$$

The homogeneous system for (α, β) has a non-zero solution if and only if $\tilde{\Omega}$ satisfies a quadratic equation

$$\begin{aligned}
&\tilde{\Omega}^2 + \tilde{\Omega}\left(\delta(W_{0,0} - W_{2,2}) + 2\sigma\varepsilon^2(K_{0,1,1,0} - K_{2,1,1,2})\right) + \delta^2(W_{0,0} - W_{1,1})(W_{1,1} - W_{2,2}) \\
&\quad + \delta\sigma\varepsilon^2\left[(W_{0,0} - W_{1,1})(K_{1,1,1,1} - 2K_{2,1,1,2}) + (W_{1,1} - W_{2,2})(2K_{0,1,1,0} - K_{1,1,1,1})\right] \\
&\quad\quad + \sigma^2\varepsilon^4\left[(K_{1,1,1,1} - 2K_{2,1,1,2})(2K_{0,1,1,0} - K_{1,1,1,1}) + K_{0,1,1,2}K_{2,1,1,0}\right] = 0.
\end{aligned}$$

Using the explicit values from Table I, we rewrite the quadratic equation above in the explicit form

$$\begin{aligned}
&\tilde{\Omega}^2 + \tilde{\Omega}\left(\delta(W_{0,0} - W_{2,2}) + \frac{\sigma\varepsilon^2}{8\sqrt{2\pi}}\right) + \delta^2(W_{0,0} - W_{1,1})(W_{1,1} - W_{2,2}) \\
&\quad + \frac{\delta\sigma\varepsilon^2}{8\sqrt{2\pi}}\left(3W_{1,1} - 2W_{2,2} - W_{0,0}\right) = 0.
\end{aligned} \tag{3.11}$$

If $\delta = 0$, one root is zero, while the other root satisfies the asymptotic expansion (3.8). The zero root persists for any $\varepsilon \in \mathbb{R}$ according to the exact eigenvector of the linear problem (3.4) with elements $B_n = (\phi_n, \phi')$ and $C_n = -(\phi_n, x\phi)$, see Remark 1.3. If $\delta \neq 0$, one root of the quadratic equation (3.11) is zero at the curve $\delta = \delta_*(\varepsilon)$, where $\delta_*(\varepsilon)$ satisfies the expansion (3.9). Persistence of the curve $\delta = \delta_*(\varepsilon)$ is proved with the Implicit Function Theorem and the perturbation theory for isolated eigenvalues. The other non-zero root of the quadratic equation (3.11) satisfies the asymptotic expansion (3.10). □

Corollary 1. Let $(\mathbf{B}_m, \mathbf{C}_m)$ be an eigenvector of the linear problem (3.4) for the eigenvalue $\Omega_m \in \mathbb{R}_+$, $m = 0, 1, 2, 3, ...$ in Proposition 2. If $\delta = 0$, the eigenvalue Ω_0 has positive signature of $\langle \mathbf{B}_0, L_+\mathbf{B}_0 \rangle$, the eigenvalue Ω_1 has negative signature of $\langle \mathbf{B}_1, L_+\mathbf{B}_1 \rangle$, while all other eigenvalues Ω_m with $m = 2, 3, ...$ have positive signature of $\langle \mathbf{B}_m, L_+\mathbf{B}_m \rangle$. If $\delta = \delta_*(\varepsilon)$, the signature of Ω_0 and Ω_1 remains the same if

$$\frac{W_{1,1} - W_{0,0}}{\sqrt{2}(W_{1,1} - W_{2,2})} > 1 \tag{3.12}$$

and it is opposite, otherwise.

PROOF. If $\delta = 0$, the homogeneous system for (α, β) has a one-parameter family of solutions with $\beta = -\sqrt{2}\alpha$ for $\tilde{\Omega} = 0$ and $\alpha = -\sqrt{2}\beta$ for $\tilde{\Omega} = -\frac{\sigma\varepsilon^2}{8\sqrt{2\pi}}$. Therefore, $\langle \mathbf{B}_0, L_+\mathbf{B}_0 \rangle = -|\alpha|^2 + |\beta|^2 + \mathrm{O}(\varepsilon^2) > 0$ and $\langle \mathbf{B}_1, L_+\mathbf{B}_1 \rangle = -|\alpha|^2 + |\beta|^2 + \mathrm{O}(\varepsilon^2) < 0$ for sufficiently small ε. For all other eigenvalues, $\langle \mathbf{B}_m, L_+\mathbf{B}_m \rangle = m + \mathrm{O}(\varepsilon^2)$ for $m = 2, 3, ...$ If $\delta = \delta_*(\varepsilon)$, we obtain the solution of the homogeneous system for (α, β) in the form

$$\beta = \frac{W_{1,1} - W_{0,0}}{\sqrt{2}(W_{1,1} - W_{2,2})}\alpha$$

for $\tilde{\Omega} = 0$ and

$$\beta = \frac{\sqrt{2}(W_{1,1} - W_{2,2})}{W_{1,1} - W_{0,0}}\alpha$$

for

$$\tilde{\Omega} = -\frac{\varepsilon^2\sigma}{8\sqrt{2\pi}}\left(1 + \frac{(3W_{1,1} - 2W_{2,2} - W_{0,0})(W_{0,0} - W_{2,2})}{(W_{1,1} - W_{0,0})(W_{1,1} - W_{2,2})}\right).$$

This finishes the proof of the corollary. □

LEMMA 3.4. *Let $\{(\mathbf{B}_m, \mathbf{C}_m)\}_{m=0}^{\infty}$ be a set of real-valued eigenvectors of the linear problem (3.4) for the set of positive eigenvalues $\{\Omega_m\}_{m=0}^{\infty}$. If all positive eigenvalues are distinct, the set of eigenvectors is symplectically orthogonal such that*

$$\langle \mathbf{B}_{m'}, \mathbf{C}_m \rangle = 0, \ \forall m' \neq m \qquad \langle \mathbf{B}_m, \mathbf{C}_m \rangle \neq 0, \ \forall m = 0, 1, 2, 3... \tag{3.13}$$

In addition, two eigenvectors $\{(\mathbf{0}, \mathbf{A}), (\partial_\mu \mathbf{A}, \mathbf{0})\}$ for the double zero eigenvalue $\Omega = 0$ are symplectically orthogonal to other eigenvectors and $\langle \mathbf{A}, \partial_\mu \mathbf{A} \rangle \neq 0$ for sufficiently small ε and δ. The set of eigenvectors

$$\{(\mathbf{B}_m, \mathbf{C}_m)\}_{m=0}^{\infty} \oplus \{(\mathbf{B}_m, -\mathbf{C}_m)\}_{m=0}^{\infty} \oplus \{(\mathbf{0}, \mathbf{A}), (\partial_\mu \mathbf{A}, \mathbf{0})\} \tag{3.14}$$

forms an orthogonal basis in $l^2(\mathbb{N}, \mathbb{R}^2)$ with respect to the symplectic projections (3.13).

PROOF. Since $L_\pm$ are self-adjoint in $l^2(\mathbb{N})$ and Ω_m is a real eigenvalue, then the eigenvector $(\mathbf{B}_m, \mathbf{C}_m)$ of the linear problem (3.4) can be chosen to be real-valued. The orthogonality relations (3.13) follow by direct computations from the linear problem (3.4) for distinct eigenvalues $\Omega_{m'} \neq \Omega_m$ for all $m' \neq m$. By the Fredholm Alternative Theorem, $\langle \mathbf{B}_m, \mathbf{C}_m \rangle \neq 0$ if the eigenvalue Ω_m is simple. By Remark 3.3, $\langle \mathbf{A}, \partial_\mu \mathbf{A} \rangle \neq 0$ for sufficiently small ε and δ. By Proposition 2, the eigenvectors of the set (3.14) are represented for sufficiently small ε and δ by the standard basis $\{\mathbf{e}_m\}_{m=0}^{\infty} \oplus \{\mathbf{e}_m\}_{m=0}^{\infty}$ perturbed by a bounded perturbation in $l^2(\mathbb{N})$ of the order $\mathrm{O}(\varepsilon^2 + \delta)$. In addition

$$\Omega_m = m + \mathrm{O}(\varepsilon^2 + \delta), \quad \langle \mathbf{B}_m, \mathbf{C}_m \rangle = \frac{\langle \mathbf{B}_m, L_+\mathbf{B}_m \rangle}{\Omega_m} = 1 + \mathrm{O}(m^{-1}(\varepsilon^2 + \delta)), \tag{3.15}$$

for all $m \geq 2$ and sufficiently small ε and δ, uniformly in m. Since no other eigenvalues exist and the essential spectrum is empty, the set of linearly independent eigenvectors (3.14) is complete in $l^2(\mathbb{N}, \mathbb{R}^2)$. According to the Banach Theorem for non-self-adjoint operators [**16**], the set is a basis if and only if the spectral projections are bounded from below by a non-zero constant in the limit $m \to \infty$. The latter condition follows from the uniform asymptotic distribution (3.15). Therefore, the set (3.14) is a basis in $l^2(\mathbb{N}, \mathbb{R}^2)$. □

Lemma 3.5. *If $\delta = 0$, the eigenvalue $\Omega_0 = 1$ is non-resonant with respect to the eigenvalues $\{\Omega_m\}_{m\in\mathbb{N}}$ for sufficiently small ε in the sense that there exists a ε-independent constant $C > 0$ such that*

$$|\Omega_m - n| \geq C\varepsilon^2 \tag{3.16}$$

uniformly in $m, n \in \mathbb{N}$. If $\delta = \delta_*(\varepsilon)$ and

$$\frac{3W_{1,1} - 2W_{2,2} - W_{0,0}}{(W_{1,1} - W_{0,0})(W_{1,1} - W_{2,2})} > 0, \qquad W_{1,1} > W_{m+1,m+1}, \quad m = 2, 3, 4, ... \tag{3.17}$$

and

$$1 + \frac{(3W_{1,1} - 2W_{2,2} - W_{0,0})(W_{0,0} - W_{2,2})}{(W_{1,1} - W_{0,0})(W_{1,1} - W_{2,2})} \neq 0, \tag{3.18}$$

then the uniform bound (3.16) is also valid for sufficiently small ε.

Proof. If $\delta = 0$, the uniform bound (3.16) follows from the bounds (3.7) and (3.8) if $K_{1,1,1,1} - 2K_{m+1,1,1,m+1} > 0$ for $m = 2, 3, ...$ Indeed, the first few values are found from Table I to be positive and monotonically increasing, e.g.

$$\begin{aligned} K_{1,1,1,1} - 2K_{3,1,1,3} &= \frac{1}{16\sqrt{2\pi}}, \\ K_{1,1,1,1} - 2K_{4,1,1,4} &= \frac{21}{128\sqrt{2\pi}}, \\ K_{1,1,1,1} - 2K_{5,1,1,5} &= \frac{59}{256\sqrt{2\pi}}. \end{aligned}$$

According to the main theorem in [**6**], the sequence $\{\|\phi_n\|_{L^4}\}_{n\in\mathbb{N}}$ is monotonically decreasing to zero with the bound (2.8). Since $K_{m+1,1,1,m+1} \leq \|\phi_1\|^2_{L^4}\|\phi_{m+1}\|^2_{L^4}$, then

$$K_{1,1,1,1} - 2K_{m+1,1,1,m+1} \geq \|\phi_1\|^2_{L^4}\left(\|\phi_1\|^2_{L^4} - 2\|\phi_{m+1}\|^2_{L^4}\right), \qquad \forall m = 2, 3, ...$$

Since $\|\phi_{m+1}\|^2_{L^4}$ decays monotonically to zero as $m \to \infty$, there exists $M \geq 2$, such that the lower bound above is strictly positive for $m \geq M$. If $\delta = \delta_*(\varepsilon)$, the uniform bound (3.16) follows from the bounds (3.7) and (3.10) under the conditions (3.17) since $\delta(W_{1,1} - W_{m+1,m+1})$ has the same sign as $\varepsilon^2\sigma(K_{1,1,1,1} - 2K_{m+1,1,1,m+1})$. □

Theorem 3.6. *There exists $\varepsilon_0 > 0$, such that the periodic solution (1.6)–(1.7) of the GP equation (1.2) with $\varepsilon \in (0, \varepsilon_0)$ and $\delta = 0$ is spectrally stable for any values of (s, φ, θ).*

Proof. By explicit transformation, the stability of the periodic solution (1.6)–(1.7) with period $T_0 = 2\pi$ and parameters (s, φ, θ) is equivalent to the stability of the stationary solution $e^{-\frac{i}{2}t - i\mu t}\phi(x)$ in the 2π-period Poincare map. The Floquet multipliers of the Poincare map are given by the set $\{e^{i2\pi\Omega_m}\}_{m\in\mathbb{N}}$ and they are different from 1 and from each other for $\varepsilon \neq 0$ due to the explicit values (3.7) and (3.8). Therefore, the Floquet multipliers are simple and remain on the imaginary axis for sufficiently small $\varepsilon > 0$. □

4. Existence of periodic solutions

We shall use the discrete dynamical system of Section 2 and the asymptotic expansions for eigenvalues and eigenvectors of Section 3 to study periodic solutions $\mathbf{a}(t+T) = \mathbf{a}(t)$ with some period T near $T_0 = 2\pi$. The main part of the section is devoted to the proof of Theorem 1.1. Related remarks are formulated at the end of the section.

Proof of Theorem 1.1: Let $\mathbf{A} \in l^2_{1/2}(\mathbb{N})$ be a real-valued root of $\mathbf{F}(\mathbf{A};\mu,\delta) = \mathbf{0}$ defined by Proposition 1. We use a decomposition $\mathbf{a}(t) = e^{-i\mu t}\left[\mathbf{A} + \mathbf{B}(t) + i\mathbf{C}(t)\right]$ with real-valued vectors $\mathbf{B}$ and $\mathbf{C}$ to rewrite the discrete dynamical system (2.5) in the form

$$\dot{\mathbf{B}} = L_-\mathbf{C} + \sigma\mathbf{N}_-(\mathbf{B},\mathbf{C}), \qquad -\dot{\mathbf{C}} = L_+\mathbf{B} + \sigma\mathbf{N}_+(\mathbf{B},\mathbf{C}), \tag{4.1}$$

where the operators $L_\pm$ are defined below (3.4) and the vector fields $\mathbf{N}_\pm(\mathbf{B},\mathbf{C})$ contains quadratic and cubic terms with respect to $(\mathbf{B},\mathbf{C})$. By Theorem 2.4, the initial-value problem for system (4.1) is globally well-posed with a solution $\mathbf{B}(t),\mathbf{C}(t) \in C^1(\mathbb{R}, l^1_{1/2}(\mathbb{N}))$. By Proposition 2, there exists a curve $\delta = \delta_*(\varepsilon)$ such that all eigenvalues are distinct under the conditions (3.17) and (3.18). By Lemma 3.4, a solution $(\mathbf{B},\mathbf{C})$ of the discrete system (4.1) with $\delta = \delta_*(\varepsilon)$ can be uniquely represented by the series of eigenvectors (3.14) associated with the linear problem (3.4):

$$\left\{ \begin{array}{lcl} \mathbf{B}(t) & = & \sum_{m=0}^\infty b_m(t)\mathbf{B}_m + \sum_{m=0}^\infty \bar{b}_m(t)\mathbf{B}_m + \alpha(t)\partial_\mu\mathbf{A}, \\ \mathbf{C}(t) & = & i\sum_{m=0}^\infty b_m(t)\mathbf{C}_m - i\sum_{m=0}^\infty \bar{b}_m(t)\mathbf{C}_m + \beta(t)\mathbf{A}, \end{array}\right. \tag{4.2}$$

where $b_0(t)$ and $\mathbf{b}(t) = (b_1, b_2, ...)$ are complex-valued, while $\alpha(t)$ and $\beta(t)$ are real-valued. Because of the asymptotic distribution (3.15) and the equivalence of norms

$$\begin{aligned} \sum_{n=0}^\infty (1+n)|B_n| \quad &\sim \quad \langle \mathbf{B}, L_+\mathbf{B}\rangle = 2\sum_{m=0}^\infty \Omega_m \langle \mathbf{C}_m, \mathbf{B}_m\rangle |b_m|^2 + |\alpha|^2 \langle \mathbf{A}, \partial_\mu \mathbf{A}\rangle \\ &\sim \quad \sum_{n\in\mathbb{N}} (1+n)|b_n|^2, \end{aligned}$$

$\mathbf{B} \in l^2_{1/2}(\mathbb{N})$ holds if and only if $\mathbf{b} \in l^2_{1/2}(\mathbb{N})$. The linear part of system (4.1) becomes block-diagonal in the representation (4.2), yielding the evolution equations

$$\dot{b}_m - i\Omega_m b_m = \sigma N_m(b_0, \mathbf{b}, \alpha, \beta), \quad \forall m = 0,1,2,3... \tag{4.3}$$

and

$$\dot{\alpha} = \sigma S_0(b_0, \mathbf{b}, \alpha, \beta), \quad \dot{\beta} + \alpha = \sigma S_1(b_0, \mathbf{b}, \alpha, \beta), \tag{4.4}$$

where

$$N_m(b_0, \mathbf{b}, \alpha, \beta) = \frac{\langle \mathbf{C}_m, \mathbf{N}_-(\mathbf{B},\mathbf{C})\rangle + i\langle \mathbf{B}_m, \mathbf{N}_+(\mathbf{B},\mathbf{C})\rangle}{2\langle \mathbf{C}_m, \mathbf{B}_m\rangle}$$

and

$$S_0(b_0, \mathbf{b}, \alpha, \beta) = \frac{\langle \mathbf{A}, \mathbf{N}_-(\mathbf{B},\mathbf{C})\rangle}{\langle \mathbf{A}, \partial_\mu\mathbf{A}\rangle}, \qquad S_1(b_0, \mathbf{b}, \alpha, \beta) = -\frac{\langle \partial_\mu\mathbf{A}, \mathbf{N}_+(\mathbf{B},\mathbf{C})\rangle}{\langle \mathbf{A}, \partial_\mu\mathbf{A}\rangle}.$$

Using the conserved quantity Q in (2.7) and the decomposition (4.2), one can integrate the first equation of system (4.4) in the form

$$\alpha = \frac{Q_A - \|\mathbf{B}\|^2_{l^2} - \|\mathbf{C}\|^2_{l^2}}{2\langle \mathbf{A}, \partial_\mu \mathbf{A}\rangle}, \tag{4.5}$$

where $Q_A = Q - \|\mathbf{A}\|^2_{l^2}$ is constant in time $t \in \mathbb{R}$. As a result, the first equation of system (4.4) is redundant, while the second equation can be rewritten explicitly in the form

$$\dot{\beta} = \frac{\|\mathbf{B}\|^2_{l^2} + \|\mathbf{C}\|^2_{l^2} - 2\sigma \langle \partial_\mu \mathbf{A}, \mathbf{N}_+(\mathbf{B}, \mathbf{C})\rangle - Q_A}{2\langle \mathbf{A}, \partial_\mu \mathbf{A}\rangle}. \tag{4.6}$$

We are now ready to apply the method of Lyapunov–Schmidt reductions. We should work in the space $C^1_{\rm per}(\mathbb{R})$ of T-periodic functions $b_0(t)$, $\mathbf{b}(t) \in l^2_{1/2}(\mathbb{N})$, $\alpha(t)$ and $\beta(t)$, where T is close to $T_0 = 2\pi$. The period T_0 corresponds to the eigenvalue $\Omega_0 = 1$ which persists for any $\varepsilon \in \mathbb{R}$ if $\delta = \delta_*(\varepsilon)$, by Proposition 2. Let ε and s be sufficiently small. We shall prove that there exist solutions of system (4.3), (4.5) and (4.6) which are T-periodic on $t \in \mathbb{R}$ satisfying the bounds

$$|b_0(t)| \leq \varepsilon s C_0, \;\; \|\mathbf{b}(t)\|_{l^2_{1/2}} \leq \varepsilon s^2 C_b, \;\; |\alpha(t)| \leq \varepsilon^2 s^2 C_\alpha, \;\; |\beta(t)| \leq \varepsilon^2 s^2 C_\beta, \tag{4.7}$$

for all $t \in \mathbb{R}$ and some (ε, s)-independent constants $C_0, C_b, C_\alpha, C_\beta > 0$. If $b_0(t)$, $\mathbf{b}(t) \in l^2_{1/2}(\mathbb{N})$, $\alpha(t)$ and $\beta(t)$ are T-periodic functions on $t \in \mathbb{R}$ satisfying the bounds (4.7), then $\mathbf{B}(t), \mathbf{C}(t) \in l^2_{1/2}(\mathbb{N})$ are T-periodic functions of $t \in \mathbb{R}$ satisfying the bound

$$\|\mathbf{B}(t)\|_{l^2_{1/2}} + \|\mathbf{C}(t)\|_{l^2_{1/2}} \leq C \varepsilon s, \qquad \forall t \in \mathbb{R}, \tag{4.8}$$

for some (ε, s)-independent constant $C > 0$. Here we recall the expansion (3.6) for $\mathbf{A}$, $\partial_\mu \mathbf{A}$ and the fact that the eigenvectors $\mathbf{B}_m$ and $\mathbf{C}_m$ are uniformly close to the unit vectors $\mathbf{e}_m$ for sufficiently small ε. Since $\mathbf{N}_\pm(\mathbf{B}, \mathbf{C})$ is cubic with respect $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, contains quadratic terms in $(\mathbf{B}, \mathbf{C})$, and maps $l^2_{1/2}(\mathbb{N}, \mathbb{R}^2)$ to $l^2_{-1/2}(\mathbb{N}, \mathbb{R}^2)$, we obtain the bound

$$\|\mathbf{N}_\pm(\mathbf{B}(t), \mathbf{C}(t))\|_{l^2_{-1/2}} \leq C_\pm \varepsilon^3 s^2, \qquad \forall t \in \mathbb{R}, \tag{4.9}$$

for some (ε, s)-independent constants $C_\pm > 0$.

If $\beta(t)$ is T-periodic, then Q_A is found from the condition

$$Q_A = \frac{1}{T} \int_0^T \left(\|\mathbf{B}\|^2_{l^2} + \|\mathbf{C}\|^2_{l^2} - 2\sigma \langle \partial_\mu \mathbf{A}, \mathbf{N}_+(\mathbf{B}, \mathbf{C})\rangle \right) dt. \tag{4.10}$$

Equation (4.10) is a scalar equation for Q_A, where the right-hand-side is small because of the bounds (4.8) and (4.9). Therefore, there exists a unique solution Q_A of equation (4.10), such that $|Q_A| \leq C_Q \varepsilon^2 s^2$ for some $C_Q > 0$. Under the condition (4.10), there exists a unique T-periodic mean-zero solution $\beta(t)$ of equation (4.6), such that $|\beta(t)| \leq \varepsilon^2 s^2 C_\beta$ for some $C_\beta > 0$, which is the last bound in (4.7). The function $\alpha(t)$ is uniquely defined by the explicit representation (4.5) and it hence satisfies the bound $|\alpha(t)| \leq \varepsilon^2 s^2 C_\alpha$ for some $C_\alpha > 0$, which is the third bound in (4.7).

Consider now system (4.3) for $m \in \mathbb{N}$. By Lemma 3.5, $\Omega_m - m = {\rm O}(\varepsilon^2)$ uniformly in $m \in \mathbb{N}$ for sufficiently small ε. By the Implicit Function Theorem in space $C^1_{\rm per}(\mathbb{R}, l^2_{1/2}(\mathbb{N})) \times C^1_{\rm per}(\mathbb{R})$, there exists a unique T-periodic solution $\mathbf{b}(t) \in l^2_{1/2}(\mathbb{N})$ for any T-periodic function $b_0(t)$. Because of the uniform bound (4.9) and the fact that $\Omega \to 1$ as $s \to 0$, we obtain $\|\mathbf{b}(t)\|_{l^2_{1/2}} \leq \varepsilon s^2 C_b$ for some $C_b > 0$, which is the second bound in (4.7).

Eliminating the components $\mathbf{b}$, α and β from equation (4.3) for $n = 0$, we obtain a reduced evolution problem for $b_0(t)$ in the form

$$\dot{b}_0 = ib_0 + R(b_0), \tag{4.11}$$

where $R(b_0)$ is a remainder term. Explicit computations of $N_0(b_0, \mathbf{b}, \alpha, \beta)$ show that

$$R(b_0) = \varepsilon \left[iK_1(\varepsilon) b_0^2 + iK_2(\varepsilon) \bar{b}_0^2 + iK_3(\varepsilon)|b_0|^2 \right] + \mathrm{O}\left(|b_0|^3, \varepsilon |b_0| \|\mathbf{b}\| \right), \tag{4.12}$$

where $K_{1,2,3}$ are real-valued constants which are bounded for sufficiently small ε. We are looking for a T-periodic function $b_0(t)$ which satisfies the evolution problem (4.11), has the leading order $b_0 \sim \varepsilon s e^{it+i\varphi}$, where $\varphi \in \mathbb{R}$ is arbitrary, and satisfies the bound $|b_0(t)| \leq \varepsilon s C_0$ for some $C_0 > 0$. By the normal form analysis of the ODE (4.11) (see [**14**, Chapter 3]), the quadratic terms in the remainder (4.12) do not change the frequency Ω of oscillations of the periodic function $b_0(t)$ at the leading order and therefore, $|\Omega - 1| \leq C_\Omega \varepsilon^2 s^2$ for some $C_\Omega > 0$. Since the Hamiltonian function of the discrete system (2.5) is constant in time, it remains constant when the function $b_0(t)$ solves the reduced evolution problem (4.11) and all other functions are expressed through $\beta_0(t)$. By the normal form analysis of Hamiltonian systems, there exists a two-dimensional invariant manifold of system (4.11) filled with periodic solutions of frequencies close to $\Omega = 1$ and parameterized by (s, φ). □

Remark 4.1. Theorem 1.1 is reminiscent of an infinite-dimensional analogue of the Lyapunov Center Theorem for persistence of periodic orbits in a neighborhood of an elliptic stationary point (see [**15**, Chapter II]). However, due to the symmetries, a double zero eigenvalue occurs in the linear problem (3.4), and the proof of Theorem 1.1 is complicated by the analysis of the associated two-dimensional subspace. Similar theorems on persistence of k-dimensional tori in n-dimensional Hamiltonian system with $k-1$ additional conserved quantities were studied in the theorem of Nekhoroshev (see Theorem 2.3 in [**3**]).

Remark 4.2. If $\delta = 0$, the periodic solution (1.6)–(1.7) has the smallest frequency in the focusing case $\sigma = -1$, since $\Omega_1 > 1$ in the bound (3.8) for sufficiently small ε. However, the frequency $\Omega_0 = 1$ is not the smallest one in the defocusing case $\sigma = 1$ since $\Omega_1 < 1$ in the bound (3.8). Persistence of the periodic solution for the smallest frequency Ω_1 can not be proved by a simple application of the Lyapunov Center Theorem since the bounds (3.10) and (3.8) do not guarantee that the non-resonance conditions $n\Omega_1 \neq \Omega_m$ are satisfied for all $n \in \mathbb{N}$ and $m = 2, 3, ...$ By the same reason, persistence of quasi-periodic oscillations with two and more frequencies $\{\Omega_0, \Omega_1, \Omega_2, ...\}$ can not be proved for small ε with the same method.

Remark 4.3. Persistence of quasi-periodic oscillations on the tori along the Cantor set of parameter values was proved in [**13**, Section 2.5] for the GP equation (1.2) with the Hartree nonlinear function and $\delta \neq 0$. Our main result is sharper since the periodic orbit constructed in Theorem 1.1 is continuous with respect to the parameter ε. The periodic orbit corresponds to 2-periodic solutions of the GP equation (1.1).

5. Numerical Results

We illustrate the results of our manuscript through some relevant numerical computations. We start by considering the case $\delta = 0$ in the GP equation (1.2). First, we identify the relevant branch of stationary solutions of the ODE (1.3). To do so, we use a fixed point method (Newton-Raphson iteration) to solve a discretized boundary-value problem. A central difference approximation is applied to the second-order derivatives with typical spacings $\Delta x = 0.025$ and $\Delta x = 0.1$. We are using a sufficiently large computational domain $[-L, L]$ such that the boundary conditions do not affect the approximation within the considered numerical precision. The solutions $\phi(x)$ are obtained, using continuation, as a function of parameter μ. The continuation of solution branches is performed from the linear limit $\mu = 1$, both for the cases $\sigma = 1$ and $\sigma = -1$. The results are shown in Figure 1, illustrating the quantity $Q = \|\phi\|^2_{L^2}$ as a function of μ. The numerical findings are also compared to the asymptotic result (3.2) of Proposition 1 indicating the good agreement between the two results for a fairly wide parametric window.

Once the corresponding numerical solution is identified (for a given σ and μ), the linear eigenvalue problem (1.9) is approximated numerically. We use again a discretization of differential operators on a finite grid, such that the spectral problem (1.9) becomes a matrix eigenvalue problem that is solved through standard numerical linear algebra routines. The relevant lowest eigenvalues are presented in Figure 2 and are also compared with the corresponding asymptotic results (3.7) and (3.8) of Proposition 2. The dashed lines show the limiting values

$$(5.1) \quad \sigma = 1: \quad \Omega_0 = 1, \quad \lim_{\mu\to\infty} \Omega_1 = \frac{1}{\sqrt{2}}, \quad \lim_{\mu\to\infty} \Omega_m = \frac{\sqrt{m(m+1)}}{\sqrt{2}}, \qquad \forall m \geq 2.$$

Once again, the good agreement offers us a quantitative handle on the relevant eigenvalues.

Since the limit $\mu \to \infty$ of the normalized equation (1.2) corresponds to the limit $\gamma \to 0$ in the original GP equation (1.1), we notice that the eigenvalue $\Omega_1 = \frac{1}{\sqrt{2}}$ corresponds to the frequency studied in [**12, 17**]. Because the non-resonance condition $n \neq \sqrt{m(m+1)}$ for all $n, m \in \mathbb{N}$ is violated in the limit $n, m \to \infty$, the linear eigenmode corresponding to the smallest eigenvalue $\Omega_0 = \frac{1}{\sqrt{2}}$ for $\sigma = 1$ may not result in the periodic solution of the GP equation (1.2) with $\delta = 0$.

We have also examined periodic oscillations of dark solitons in the numerical simulations of the GP equation (1.2) with $\delta = 0$. A typical example is shown in Figure 3 for $\sigma = 1$ and $\mu = 1.1$ for the initial condition $u(x,0) = \phi(x) + s\phi'(x)$ with $s = 10^{-3}$. The top left panel shows the space-time contour plot of $|u(x,t)|^2$, clearly highlighting that this is a small (imperceptible, at the scale of this panel) perturbation of a stable stationary solution $\phi(x)$. The bottom left panel shows the space-time contour plot of $|u(x,t)|^2 - \phi^2(x)$, emphasizing the time-periodic oscillations of the perturbation to the stationary solution. The periodic oscillations are also visible on the top right panel where $|u(x_0,t)|^2$ is plotted versus t for $x_0 = 2$. Finally, the bottom right panel illustrates the Fourier transform of the time series of $|u(x_0,t)|^2$ (normalized to its maximum). It shows a high peak of the frequency spectrum near the value $\Omega = 1$. These results agree with Theorem 3.6 on stability of the exact periodic solutions (1.6)–(1.7) in the GP equation (1.2) with $\delta = 0$.

Finally, we have considered oscillations of the dark soliton in the parabolic potential perturbed by the decaying potential $W(x) = \text{sech}^2(x)$. Since $W_{0,0} \approx 0.726$,

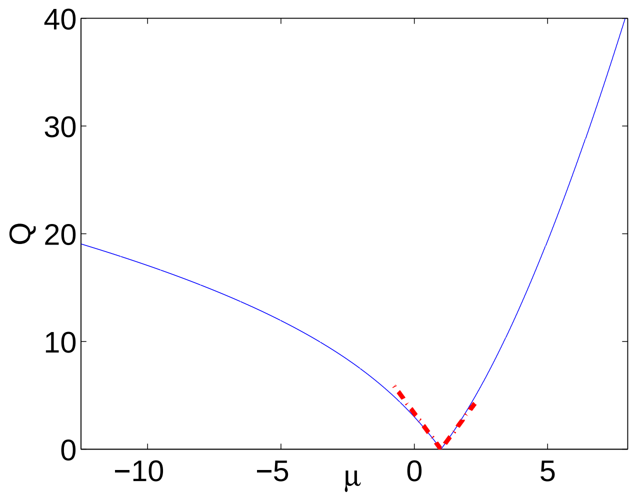

FIGURE 1. Branches of dark solitons versus μ both for the case of $\sigma = -1$ (when $\mu < 1$) and $\sigma = 1$ (when $\mu > 1$) for $\delta = 0$. The numerically obtained solution is shown by solid line and the asymptotic solution (3.2) is shown by dash-dotted line.

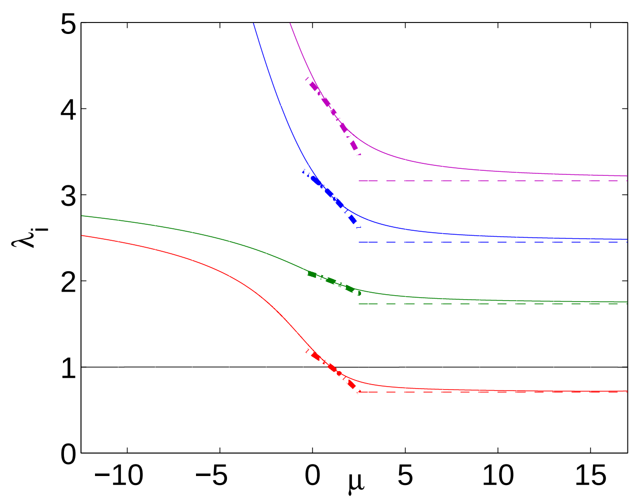

FIGURE 2. Smallest purely imaginary eigenvalues of the linear eigenvalue problem (1.9) versus μ for $\delta = 0$. The numerically obtained eigenvalues are shown by solid lines, the asymptotic results (3.7) and (3.8) are shown by dash-dotted lines, and the asymptotic results (5.1) are shown by dashed lines.

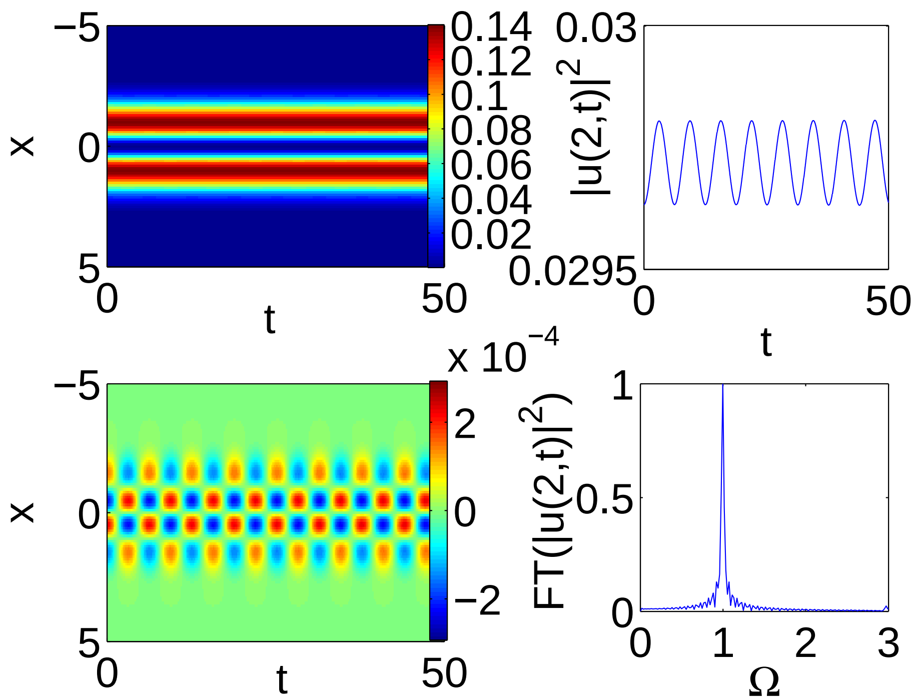

FIGURE 3. An example of the time-periodic solution of the Gross-Pitaevskii equation (1.2) for $\sigma = 1$, $\delta = 0$, $\mu = 1.1$ and $u(x,0) = \phi(x) + s\phi'(x)$ with $s = 10^{-3}$. The top left panel shows the space-time contour plot of $|u(x,t)|^2$, the bottom left panel shows the space-time contour of $|u(x,t)|^2 - \phi^2(x)$. The top right panel shows the time evolution of $|u(x_0,t)|^2$ with $x_0 = 2$, while the bottom right panel shows the Fourier transform of the time series of $|u(x_0,t)|^2$, featuring a peak at $\Omega \approx 1$.

$W_{1,1} \approx 0.402$, and $W_{2,2} \approx 0.314$, the constraints (3.17) and (3.18) are satisfied. According to the solution of the linear system for (α, β) in the proof of Proposition 2, we set the initial condition in the form

$$u(x,0) = \varepsilon\phi_1(x) + \varepsilon s\left(\phi_0(x) + r\phi_2(x)\right), \qquad r \approx \frac{W_{1,1} - W_{0,0}}{\sqrt{2}(W_{1,1} - W_{2,2})} \approx -2.603$$

with $\varepsilon = 0.5$ and $s = 0.1$. Since $\delta_*(\varepsilon) \approx 0.258\varepsilon^2 \approx 0.065$, we performed two computations with $\delta = 0.05$ and $\delta = 0.15$ shown on Figure 4. We can see from the figure that evolution of the initial data leads slowly to periodic steady-state oscillations. The Fourier spectrum of these oscillations suggest that the main frequency of the oscillations is $\Omega = 1$, according to Theorem 1.1. We recall from the theorem that the true periodic solution occurs only along the one-parameter curve $\delta = \delta_*(\varepsilon)$ on the two-parameter plane, while it remains open to prove existence of true periodic solutions for other values of (ε, δ) near the curve $\delta = \delta_*(\varepsilon)$.

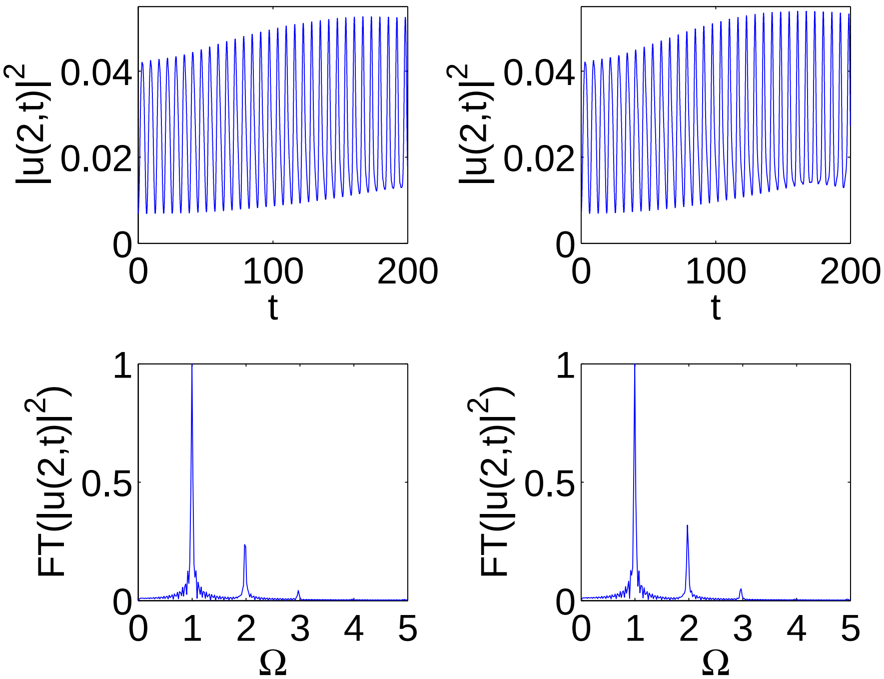

FIGURE 4. Same as the right panel of Fig. 3, but for the potential $W(x) = \mathrm{sech}^2(x)$ with $\delta = 0.05$ (left panels) and $\delta = 0.15$ (right panels). The central peak of the Fourier transform occurs at $\Omega \approx 1$.

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