DYNAMICS OF THE BLACK SOLITON IN A REGULARIZED NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. We consider a family of regularized defocusing nonlinear Schrödinger (NLS) equations proposed in the context of the cubic NLS equation with a bounded dispersion relation. The time evolution is well-posed if the black soliton is perturbed by a small perturbation in the Sobolev space $H^s(\mathbb{R})$ with $s > \frac{1}{2}$. We prove that the black soliton is spectrally stable (unstable) if the regularization parameter is below (above) some explicitly specified threshold. We illustrate the stable and unstable dynamics of the perturbed black solitons by using the numerical finite-difference method. The question of orbital stability of the black soliton is left open due to the mismatch of the function spaces for the energy and momentum conservation.

1. INTRODUCTION

Dark solitons are the depression waves propagating steadily along the continuous wave background. The name is drawn from the realms of nonlinear optics, where the continuous wave background supports the light of constant intensity and the dark solitons reduce the light intensity during transmission [17, 23]. The most extreme case in the family of dark solitons is the black soliton, for which the light intensity drops to zero and the wave is spatially localized with zero speed. Such standing waves are common in many models of nonlinear optics and Bose–Einstein condensation in one, two, and three spatial dimensions [16].

The canonical model for the dark and black solitons is the defocusing nonlinear Schrödinger (NLS) equation

$$(1.1) \qquad i\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0,$$

where $\psi(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$. The exact traveling wave solutions for the dark solitons are given by

$$(1.2) \qquad \psi(t,x) = [\gamma \tanh(\gamma(x - 2ct)) + ic]\, e^{-2it}, \quad \gamma := \sqrt{1 - c^2},$$

where $c \in (-1, 1)$ is a free parameter for the half wave speed. For $c = 0$, the dark soliton $\psi(t,x) = \tanh(x)e^{-2it}$ is referred as the black soliton.

Compared to the canonical NLS equation (1.1), we address the following family of regularized defocusing NLS equations

$$(1.3) \qquad i(1 - \mu^2\partial_x^2)\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0,$$

where $\psi(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ and $\mu > 0$ is a small parameter. The model was proposed in the context of nonlinear optics for ultra-short pulses [7,18] (see [9, Section 4.1.4, eq. (72)]) as an example of the NLS equation with a bounded dispersion relation. The linear part of the model has the dispersion relation

$$\omega(k) = \frac{k^2}{1 + \mu^2 k^2}, \quad k \in \mathbb{R},$$

obtained for the Fourier modes $u(t,x) \sim e^{ikx - i\omega(k)t}$. The additional term $-i\mu^2 \psi_{txx}$ in (1.3) compared to the canonical model (1.1) leads to the bounded dispersion relation in the interval $[0, \mu^{-2}]$ for the frequencies ω. This kind of regularization of the dispersion relation is popular in models of fluid dynamics where the canonical Korteweg–de Vries equation is replaced by a regularized Benjamin–Bona–Mahony equation [4]. A focusing version of the regularized NLS equation was studied in [3] in the context of global well-posedness of the initial-value problem.

We are mainly interested in the stability of the black soliton which is the standing wave solution of the defocusing NLS equation (1.3) with nonzero boundary conditions. If the black soliton is stable in the time evolution, as is known for the canonical NLS equation (1.1) [5, 6, 10, 13, 14, 20], then it plays an important role in nonlinear optics as a carrier of information inside the spatially modulated periodic waves [15, 19, 22] (see review in [1]).

In order to study the black soliton of the model (1.3), we normalize the boundary conditions to unity without loss of generality and consider solutions satisfying

$$|\psi(t,x)| \to 1 \quad \text{as } |x| \to \infty.$$

With the transformation

$$(1.4) \qquad \psi(t,x) = e^{-2it} u(t,\xi), \quad \xi = \frac{x}{\sqrt{1 - 2\mu^2}},$$

the NLS equation (1.3) can be rewritten in the equivalent form as

$$(1.5) \qquad i(1 - \epsilon^2 \partial_\xi^2) u_t + u_{\xi\xi} + 2(1 - |u|^2) u = 0,$$

where

$$\epsilon := \frac{\mu}{\sqrt{1 - 2\mu^2}}.$$

The mapping $\mu \to \epsilon$ is monotonically increasing for $\mu \in (0, \frac{1}{\sqrt{2}})$ with $\epsilon \to \infty$ as $\mu \to \frac{1}{\sqrt{2}}$.

The black soliton is the steady-state (time-independent) solution of the transformed NLS equation (1.5). It is available in the explicit form $u(t,\xi) = \varphi(\xi) := \tanh(\xi)$, which coincides with the black soliton of the defocusing NLS equation (1.1) given by (1.2) for $c = 0$.

Local well-posedness of the regularized model (1.5) can be studied in the space of bounded smooth functions with nonzero boundary conditions at infinity. One general method is to consider a decomposition $u(t,\xi) = \varphi(\xi) + v(t,\xi)$, where the perturbation $v(t,\cdot)$ is a continuous function of t in Sobolev spaces with respect to the spatial coordinate [11, 12, 24]. This is our first result presented in Theorem 1.

Theorem 1. *For every $v_0 \in H^s(\mathbb{R})$ with $s > \frac{1}{2}$, there exist the maximal existence time $\tau_0 \in (0, \infty]$ and a unique solution of the NLS equation (1.5) in the form $u = \varphi + v$, where $\varphi(\xi) = \tanh(\xi)$ and $v \in C^1([0, \tau_0), H^s(\mathbb{R}))$ such that $v(0, \cdot) = v_0$.*

Moreover, for any $\tau \in (0, \tau_0)$, the solution $v \in C^1([0, \tau], H^s(\mathbb{R}))$ depends Lipschitz continuously on the initial data in some neighborhood of $v_0 \in H^s(\mathbb{R})$.

Replacing the solution u of Theorem 1 with $\varphi + v$, where $\varphi(\xi) = \tanh(\xi)$ is the black soliton and $v := U + iV$ is a small perturbation allows us to reformulate the stability of the black soliton at the level of linearized approximation. Separation of the real part U and the imaginary part V gives the following linearized equations:

$$(1.6) \qquad (1 - \epsilon^2 \partial_\xi^2)U_t = L_- V, \qquad (1 - \epsilon^2 \partial_\xi^2)V_t = -L_+ U,$$

where the linear operators $L_\pm$ in $L^2(\mathbb{R})$ with $\mathrm{Dom}(L_\pm) \subset L^2(\mathbb{R})$ are defined by the differential expressions

$$L_+ = -\partial_\xi^2 + 6\varphi^2 - 2,$$
$$L_- = -\partial_\xi^2 + 2\varphi^2 - 2.$$

Separation of variables in system (1.6) gives the spectral stability problem in the form

$$(1.7) \qquad \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \lambda(1 - \epsilon^2 \partial_\xi^2) \begin{bmatrix} U \\ V \end{bmatrix} \Leftrightarrow \begin{array}{l} L_- V = \lambda(1 - \epsilon^2 \partial_\xi^2)U, \\ -L_+ U = \lambda(1 - \epsilon^2 \partial_\xi^2)V. \end{array}$$

It is natural to consider the spectral problem (1.7) for fixed $\epsilon \neq 0$ in $H_\epsilon^1(\mathbb{R}) \times H_\epsilon^1(\mathbb{R})$, where $H_\epsilon^1(\mathbb{R})$ is the Hilbert space equipped with the associated inner product

$$(1.8) \qquad (f, g)_\epsilon := \int_{\mathbb{R}} \left[\bar{f}g + \epsilon^2 \bar{f}'g' \right] d\xi,$$

and the induced norm $\| \cdot \|_\epsilon := \sqrt{(\cdot, \cdot)_\epsilon}$. The Hilbert space $H_\epsilon^1(\mathbb{R})$ for fixed $\epsilon \neq 0$ is equivalent to $H^1(\mathbb{R})$, which is the form domain of L_+ in $L^2(\mathbb{R})$ and is a subset of the form domain of L_- in $L^2(\mathbb{R})$. The inner product and the norm in $L^2(\mathbb{R})$ correspond to $\epsilon = 0$ and we use notations $(\cdot, \cdot)$ and $\| \cdot \|$ instead of $(\cdot, \cdot)_0$ and $\| \cdot \|_0$.

The essential spectrum of the spectral stability problem (1.7) can be found in the limit $|\xi| \to \infty$ since $|\varphi(\xi)| \to 1$ exponentially fast. By using the Fourier transform, we find that the essential spectrum is located at

$$(1.9)$$

$$\sigma_c = \overline{\left\{ ik \frac{\sqrt{4 + k^2}}{1 + \epsilon^2 k^2}, \ k \in \mathbb{R} \right\}} = \begin{cases} i\left[-\dfrac{1}{\epsilon^2}, \dfrac{1}{\epsilon^2} \right], & \epsilon \in \left(0, \dfrac{1}{\sqrt{2}}\right], \\ i\left[-\dfrac{2}{\sqrt{4\epsilon^2 - 1}}, \dfrac{2}{\sqrt{4\epsilon^2 - 1}} \right], & \epsilon \in \left(\dfrac{1}{\sqrt{2}}, \infty\right). \end{cases}$$

Hence, the essential spectrum is neutrally stable and the stability or instability of the black soliton depends on isolated eigenvalues λ outside σ_c.

Remark 1. We say that the black soliton φ is spectrally unstable if the spectral problem (1.7) admits an isolated eigenvalue $\lambda_0 \in \mathbb{C}$ with the corresponding eigenfunction $(U, V) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ such that $\mathrm{Re}(\lambda_0) > 0$. The black soliton is spectrally stable if no such eigenvalue exists.

Remark 2. Isolated eigenvalues may exist on $i\mathbb{R} \backslash \sigma_c$ but such eigenvalues do not contribute to spectral instability of the black soliton.

Our second result is the following stability theorem, which is the main result of this paper.

Theorem 2. *Let $\epsilon_0 = (5/8)^{1/4}$. The black soliton is spectrally stable for $\epsilon \in (0, \epsilon_0]$ and spectrally unstable for $\epsilon \in (\epsilon_0, \infty)$.*

Remark 3. The stability threshold $\epsilon_0 = (5/8)^{1/4}$ is approximately $\epsilon_0 \approx 0.89$. In view of the transformation (1.4), the stability threshold in the original NLS model (1.3) is given by

$$\mu_0 = \sqrt{\frac{\sqrt{5}}{\sqrt{8} + 2\sqrt{5}}} \approx 0.5534.$$

The black soliton is spectrally stable for $\mu \in (0, \mu_0]$ and spectrally unstable for $\mu \in (\mu_0, \mu_1)$, where $\mu_1 = \frac{1}{\sqrt{2}} \approx 0.7071$.

It is tempting to extend the spectral stability of Theorem 2 to the orbital stability of the black soliton similar to [5, 10, 14] for the canonical NLS equation (1.1). The proof of orbital stability of the black soliton was also developed for other NLS models such as the quintic NLS equation [2], the coupled NLS systems [8], and the NLS equation with intensity-dependent dispersion [21]. Orbital stability is usually proven with the use of conserved quantities. Our third result specifies the conserved quantities of the regularized NLS equation (1.5).

Theorem 3. *Let* $u = \varphi + v$ *with* $v \in C^1([0, \tau_0), H^s(\mathbb{R}))$ *be the local solution in Theorem 1 with* $s > \frac{1}{2}$. *Then, energy*

$$(1.10) \qquad E(u) = \int_{\mathbb{R}} \left[|u_\xi|^2 + (1 - |u|^2)^2 \right] d\xi$$

and momentum

$$(1.11) \qquad P(u) = i \int_{\mathbb{R}} \left[(\bar{u}u_\xi - \bar{u}_\xi u) + \epsilon^2 (\bar{u}_\xi u_{\xi\xi} - \bar{u}_{\xi\xi} u_\xi) \right] d\xi$$

are well-defined for $s \geq 1$ *and* $s \geq 2$ *respectively, and their values are independent of* $t \in [0, \tau_0)$.

Remark 4. Besides the energy and momentum conservation, the NLS equation (1.5) admits also mass conservation,

$$(1.12) \qquad M(u) = \int_{\mathbb{R}} \left[\epsilon^2 |u_\xi|^2 + |u|^2 - 1 \right] d\xi,$$

if the solution $u = \varphi + v$ with $v \in C^1([0, \tau_0), H^s(\mathbb{R}))$ and $s \geq 1$ satisfies $v(t, \cdot) \in L^1(\mathbb{R})$ for $t \in [0, \tau_0)$. The mass conservation plays no role in the proof of orbital stability of the black soliton in the canonical NLS equation (1.1) [5, 14].

Remark 5. The proof of orbital stability of the black soliton is an open problem for the regularized NLS equation (1.5) because of the mismatch between the energy and momentum spaces. The energy arguments only provide control of the perturbation in the weighted $H^1(\mathbb{R})$ spaces with the exponential weight [14], where the weight is needed due to the lack of coercivity of the quadratic form associated with the linearized operator L_-, see also [2, 10, 21]. However, the momentum conservation is not defined in the weighted $H^1(\mathbb{R})$ space and the energy arguments do not control perturbations in the weighted $H^2(\mathbb{R})$ space.

Remark 6. Orbital stability of the black soliton with respect to spatially odd perturbations can be proven in the weighted $H^1(\mathbb{R})$ space independently on the values of ϵ in $(0, \infty)$. The proof is based on the decomposition

$$E(\varphi + U + iV) - E(\varphi) = Q_-(U) + Q_-(V) + \|\eta\|^2,$$

where

$$Q(U) = \int_{\mathbb{R}} (|U_\xi|^2 - 2(1 - \varphi^2)|U|^2)d\xi,$$

and

$$\eta := 2U\varphi + U^2 + V^2 \in L^2(\mathbb{R}).$$

The perturbation $U + iV \in \mathcal{H}$ is defined in the weighted space with odd spatial symmetry:

$$\mathcal{H} := \{f \in H_{\mathrm{loc}}^1(\mathbb{R}) : \quad f' \in L^2(\mathbb{R}), \quad \sqrt{1 - \varphi^2}f \in L^2(\mathbb{R}), \quad f(-x) = -f(x), \quad x \in \mathbb{R}\}.$$

Since the negative eigenvalue of L_- in $\mathcal{H}$ corresponds to the spatially even eigenfunction, L_- is nonnegative in $\mathcal{H}$ and the zero eigenvalue of L_- with the eigenfunction $\varphi \in \mathcal{H}$ is isolated from the strictly positive rest of its spectrum. Coercivity of the energy difference holds under a single orthogonality constraint

$$\langle \varphi, V \rangle_{\mathcal{H}} := \int_{\mathbb{R}} [\varphi'V' + (1 - \varphi^2)\varphi V]d\xi = 0$$

and it allows to control the $\mathcal{H}$-norm of $U + iV \in \mathcal{H}$ and the $L^2(\mathbb{R})$-norm of $\eta \in L^2(\mathbb{R})$, see [14, 21]. The single orthogonality constraint can be satisfied in time $t > 0$ by choosing the phase parameter $\theta(t)$ in the orbit $\{e^{-\theta}[\varphi + U + iV]\}_{\theta \in \mathbb{R}/2\pi\mathbb{Z}}$.

The remainder of this paper is organized as follows. Section 2 presents the proof of Theorems 1 and 3. Section 3 is devoted to the proof of Theorem 2. The numerical illustrations of the stable and unstable dynamics of the perturbed black soliton are contained in Section 4. Section 5 concludes the paper with a summary and the discussion of further directions.

2. Local well-posedness and conserved quantities

Let us write $u = \varphi + v$, where $\varphi(\xi) = \tanh(\xi)$, and reduce the NLS equation (1.5) to the evolutionary form:

$$(2.1) \quad v_t = i(1 - \epsilon^2\partial_\xi^2)^{-1}\left[v_{\xi\xi} + 2(1 - 2\varphi^2)v - 2\varphi^2\bar{v} - 2\varphi(v^2 + 2|v|^2) - 2|v|^2v\right],$$

where we have used $\varphi'' + 2(1 - \varphi^2)\varphi = 0$ for $\varphi(\xi) = \tanh(\xi)$. The proof of Theorem 1 follows from the contraction mapping principle according to the following arguments.

Proof of Theorem 1. We recall that $H^s(\mathbb{R})$ forms a Banach algebra with respect to pointwise multiplication if $s > \frac{1}{2}$. Hence,

$$F(v) := v_{\xi\xi} + 2(1 - 2\varphi^2)v - 2\varphi^2\bar{v} - 2\varphi(v^2 + 2|v|^2) - 2|v|^2v$$

is an operator from $H^s(\mathbb{R})$ to $H^{s-2}(\mathbb{R})$ which maps any fixed bounded ball $B(v_0) \subset H^s(\mathbb{R})$ centered at $v_0 \in H^s(\mathbb{R})$ into a bounded set in $H^{s-2}(\mathbb{R})$, and moreover is Lipschitz continuous on $B(v_0)$. Furthermore, $(1 - \epsilon^2\partial_\xi^2)^{-1}$ is a bounded operator from $H^{s-2}(\mathbb{R})$ to $H^s(\mathbb{R})$ for every $\epsilon > 0$. The integral formulation of the evolution equation (2.1) is given by

$$v(t, \cdot) = v_0 + i\int_0^t (1 - \epsilon^2\partial_\xi^2)^{-1}F(v(t', \cdot))dt'.$$

It follows from the contraction mapping principle that there exists a local solution $v \in C^0([0, t_0], H^s(\mathbb{R}))$ of the integral equation for sufficiently small $t_0 > 0$. Moreover, since the operator $i(1 - \epsilon^2\partial_\xi^2)^{-1}F : H^s(\mathbb{R}) \to H^s(\mathbb{R})$ maps the bounded

ball $B(v_0) \subset H^s(\mathbb{R})$ into a bounded set in $H^s(\mathbb{R})$, the solution v belongs to $C^1([0,t_0], H^s(\mathbb{R}))$. Continuing smoothly the local solution to the maximal time $\tau_0 > 0$ (which could be finite or infinite) yields the solution $v \in C^1([0,\tau_0), H^s(\mathbb{R}))$ stated in Theorem 1.

Lipschitz continuous dependence of $v \in C^1([0,t_0], H^s(\mathbb{R}))$ on the initial data in some neighborhood of $v_0 \in H^s(\mathbb{R})$ is obtained from the contraction principle by Gronwall's estimates. Iterating the estimates to every $\tau \in (0,\tau_0)$, we obtain Lipschitz continuous dependence of $v \in C^1([0,\tau], H^s(\mathbb{R}))$ on the initial data in some neighborhood of $v_0 \in H^s(\mathbb{R})$. $\square$

By using the transformation $u = \varphi + v$, we can rewrite (1.10), (1.11), and (1.12) in the form:

$$E(\varphi + v) = E(\varphi) + \hat{E}(v),$$
$$P(\varphi + v) = P(\varphi) + \hat{P}(v),$$
$$M(\varphi + v) = M(\varphi) + \hat{M}(v),$$

with

$$\hat{E}(v) = \int_{\mathbb{R}} [|v_\xi|^2 - 2(1 - 2\varphi^2)|v|^2 + \varphi^2(v^2 + \bar{v}^2) + 2\varphi|v|^2(v + \bar{v}) + |v|^4]d\xi,$$

$$\hat{P}(v) = i\int_{\mathbb{R}} [2\varphi'(\bar{v} - v) + (\bar{v}v_\xi - \bar{v}_\xi v) + 2\epsilon^2\varphi''(\bar{v}_\xi - v_\xi) + \epsilon^2(\bar{v}_\xi v_{\xi\xi} - \bar{v}_{\xi\xi} v_\xi)]d\xi,$$

$$\hat{M}(v) = \int_{\mathbb{R}} [(\varphi - \epsilon^2\varphi'')(v + \bar{v}) + \epsilon^2|v_\xi|^2 + |v|^2]d\xi,$$

where we have used $\varphi'' + 2(1 - \varphi^2)\varphi = 0$. The proof of Theorem 3 is obtained from specific computations for the NLS equation (2.1).

Proof of Theorem 3. The energy functional $\hat{E}(v) : H^s(\mathbb{R}) \to \mathbb{R}$ is smooth in v and $\bar{v}$ for $s \geq 1$. The evolution equation (2.1) can be cast to the Hamiltonian form

$$(2.2) \qquad \frac{dv}{dt} = -i(1 - \epsilon^2\partial_\xi^2)^{-1}\nabla_{\bar{v}}\hat{E}(v),$$

where $\nabla_{\bar{v}}\hat{E}(v) = -F(v)$ is the variational derivative of $\hat{E}(v)$ with respect to $\bar{v}$. Since the operator $(1 - \epsilon^2\partial_\xi^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint, the energy $\hat{E}(v)$ is constant in time $t \in [0,\tau_0)$ for the solution $v \in C^1([0,\tau_0), H^s(\mathbb{R}))$ with $s \geq 1$ which exists by Theorem 1.

The momentum functional $\hat{P}(v) : H^s(\mathbb{R}) \to \mathbb{R}$ is smooth in v and $\bar{v}$ for $s \geq 2$. To prove its conservation, we consider the solution $v \in C^1([0,\tau_0), H^s(\mathbb{R}))$ for $s \geq 2$ which exists by Theorem 1. Since $P(\varphi) = 0$, we can work equivalently with $P(u) = \hat{P}(v)$. Differentiating (1.5) in ξ, multiplying it by $\bar{u}$, adding the complex conjugate, and integrating in ξ over $\mathbb{R}$ yield:

$$i\int_{\mathbb{R}} \left[\bar{u}(1 - \epsilon^2\partial_\xi^2)u_{t\xi} - u(1 - \epsilon^2\partial_\xi^2)\bar{u}_{t\xi}\right]d\xi$$
$$+ \int_{\mathbb{R}} [\bar{u}u_{\xi\xi\xi} + u\bar{u}_{\xi\xi\xi}]d\xi + 2\int_{\mathbb{R}} \left[\bar{u}u_\xi + u\bar{u}_\xi - 3|u|^2(|u|^2)_\xi\right]d\xi = 0.$$

Multiplying (1.5) by $\bar{u}_\xi$, adding the complex conjugate, and integrating in ξ over $\mathbb{R}$ yield

$$i \int_{\mathbb{R}} \left[\bar{u}_\xi (1 - \epsilon^2 \partial_\xi^2) u_t - u_\xi (1 - \epsilon^2 \partial_\xi^2) \bar{u}_t \right] d\xi$$

$$+ \int_{\mathbb{R}} \left[\bar{u}_\xi u_{\xi\xi} + u_\xi \bar{u}_{\xi\xi} \right] d\xi + 2 \int_{\mathbb{R}} \left[\bar{u}_\xi u + u_\xi \bar{u} - |u|^2 (|u|^2)_\xi \right] d\xi = 0.$$

Subtracting the two equations and integrating by parts under the boundary conditions $u(t, \xi) \to \pm 1$ as $\xi \to \pm\infty$ give conservation of $P(u) = \hat{P}(v)$ in time $t \in [0, \tau_0)$. $\qquad\square$

Remark 7. The mass functional $\hat{M}(v) : H^s(\mathbb{R}) \to \mathbb{R}$ is smooth in v and $\bar{v}$ for $s \geq 1$ under the additional condition $v \in L^1(\mathbb{R})$ since $\varphi(\xi) \to \pm 1$ as $\xi \to \pm\infty$. Assuming the existence of the solution $v \in C^1([0, \tau_0), H^s(\mathbb{R}))$ for $s \geq 1$ such that $v(t, \cdot) \in L^1(\mathbb{R})$ for $t \in [0, \tau_0)$, we can prove conservation of $\hat{M}(v)$ as follows. Multiplying (1.5) by $\bar{u}$, subtracting the complex conjugate, and integrating in ξ over $\mathbb{R}$ yield

$$i \int_{\mathbb{R}} \left[\bar{u} (1 - \epsilon^2 \partial_\xi^2) u_t + u (1 - \epsilon^2 \partial_\xi^2) \bar{u}_t \right] d\xi + \int_{\mathbb{R}} \left[\bar{u} u_{\xi\xi} - u \bar{u}_{\xi\xi} \right] d\xi = 0.$$

Integrating by parts under the boundary conditions $u(t, \xi) \to \pm 1$ as $\xi \to \pm\infty$ gives conservation of $M(u)$ in time $t \in [0, \tau_0)$.

3. Spectral stability and instability of the black soliton

In order to prove Theorem 2, we first clarify the spectral properties of the Schrödinger operator $L_+ : \text{Dom}(L_+) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$, where $L_+ = -\partial_\xi^2 + 6\varphi^2 - 2 = -\partial_\xi^2 + 4 - 6 \text{ sech}^2(\xi)$. As is well-known, any Schrödinger operator with bounded potential can be extended to its form domain, hence we can write $L_+ : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$. Similarly, we can consider bounded operators $(1 - \epsilon^2 \partial_\xi^2)^{-1/2} : L^2(\mathbb{R}) \to H^1(\mathbb{R})$ and $(1 - \epsilon^2 \partial_\xi^2)^{-1/2} : H^{-1}(\mathbb{R}) \to L^2(\mathbb{R})$. By compositing the three operators above, we obtain a bounded operator

$$\mathcal{L}_+ = (1 - \epsilon^2 \partial_\xi^2)^{-1/2} L_+ (1 - \epsilon^2 \partial_\xi^2)^{-1/2} : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Coercivity of $L_+ : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$ and $\mathcal{L}_+ : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is obtained in Lemma 3.1.

Lemma 3.1. *There exists $C > 0$ such that*

$$(3.1) \qquad \langle L_+ U, U \rangle \geq C \|U\|^2, \quad \text{for every } U \in H^1(\mathbb{R}) : \ (U, \varphi') = 0$$

and

$$(3.2) \qquad (\mathcal{L}_+ W, W) \geq C \|W\|^2, \quad \text{for every } W \in L^2(\mathbb{R}) : \ (W, W_0) = 0,$$

where $\langle L_+ U, U \rangle$ is the dual action of $L_+ U \in H^{-1}(\mathbb{R})$ on $U \in H^1(\mathbb{R})$ and $W_0 := (1 - \epsilon^2 \partial_\xi^2)^{1/2} \varphi'$.

Proof. We have $L_+ \varphi' = 0$ and $\varphi' \in H^1(\mathbb{R})$ due to the translational invariance and the exponential decay of $\varphi'(\xi) \to 0$ as $|\xi| \to \infty$. Since $\varphi'(\xi) > 0$ for all $\xi \in \mathbb{R}$, the zero eigenvalue of L_+ in $L^2(\mathbb{R})$ is the lowest eigenvalue separated from the rest of the spectrum in $L^2(\mathbb{R})$ by a gap. The spectral theorem implies the coercivity bound (3.1).

By using the transformation $U = (1 - \epsilon^2 \partial_\xi^2)^{-1/2} W$ between $U \in H^1(\mathbb{R})$ and $W \in L^2(\mathbb{R})$, we get $\mathcal{L}_+ W_0 = 0$ with $W_0 := (1 - \epsilon \partial_\xi^2)^{1/2} \varphi' \in L^2(\mathbb{R})$. The zero eigenvalue of $\mathcal{L}_+$ in $L^2(\mathbb{R})$ is separated from the continuous spectrum of $\mathcal{L}_+$ by a gap:

$$\sigma_c(\mathcal{L}_+) = \overline{\left\{ \frac{4 + k^2}{1 + \epsilon^2 k^2}, \quad k \in \mathbb{R} \right\}} = \left\{ \begin{array}{ll} [4, \epsilon^{-2}], & \epsilon \in (0, \frac{1}{2}), \\ [\epsilon^{-2}, 4], & \epsilon \in (\frac{1}{2}, \infty). \end{array} \right.$$

The coercivity bound (3.1) in $L^2(\mathbb{R})$ implies a coercivity bound in $H^1(\mathbb{R})$, that is,

$$(L_+ U, U) \geq C \|U\|_\epsilon^2, \quad \text{for every } U \in H^1(\mathbb{R}): \ (U, \varphi') = 0.$$

This bound is equivalent to (3.2) since if $U = (1 - \epsilon^2 \partial_\xi^2)^{-1/2} W$, then $(L_+ U, U) = (\mathcal{L}_+ W, W)$, $\|U\|_\epsilon^2 = \|W\|^2$, and $(U, \varphi') = (W, W_0)$. $\qquad\square$

By Lemma 3.1, we can define the constrained operator

$$\mathcal{T}_+ := \mathcal{L}_+|_{\{W_0\}^\perp} \ : \ L^2(\mathbb{R})|_{\{W_0\}^\perp} \mapsto L^2(\mathbb{R})|_{\{W_0\}^\perp},$$

Thanks to the bound (3.2), $\mathcal{T}_+$ is invertible and strictly positive with a bounded and strictly positive inverse $\mathcal{T}_+^{-1}$.

In addition to Lemma 3.1, we also need a technical computation related to the Schrödinger operator $L_- : \mathrm{Dom}(L_-) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$, where $L_- = -\partial_\xi^2 + 2\varphi^2 - 2 = -\partial_\xi^2 - 2\,\mathrm{sech}^2(\xi)$. The technical computation is given by Lemma 3.2.

Lemma 3.2. *The linear inhomogeneous equation*

$$(3.3) \qquad\qquad L_- V_\varphi = (1 - \epsilon^2 \partial_\xi^2) \varphi'$$

admits a unique even and bounded solution V_φ satisfying

$$(\varphi', V_\varphi)_\epsilon = -1 + \frac{8}{5} \epsilon^4.$$

Hence $(\varphi', V_\varphi)_\epsilon \leq 0$ if $\epsilon \in [0, \epsilon_0]$ and $(\varphi', V_\varphi)_\epsilon > 0$ if $\epsilon \in (\epsilon_0, \infty)$, where $\epsilon_0 := (5/8)^{1/4}$.

Proof. A general solution V_φ of the linear equation (3.3) can be found in the explicit form by substitutions as

$$V_\varphi(\xi) = -\frac{1}{2}(1 + 2\epsilon^2) + \frac{3}{2}\epsilon^2 \mathrm{sech}^2(\xi) + c_1 \tanh(\xi) + c_2 \left[\xi \tanh(\xi) - 1 \right],$$

where c_1 and c_2 are arbitrary constants. If V_φ is required to be even and bounded, then $c_1 = 0$ and $c_2 = 0$ respectively. Although $V_\varphi \notin L^2(\mathbb{R})$, the inner product $(\varphi', V_\varphi)_\epsilon$ makes sense due to the exponential decay of $\varphi'(\xi) \to 0$ as $|\xi| \to \infty$ and can be computed explicitly:

$$(\varphi', V_\varphi)_\epsilon = \int_\mathbb{R} \left[(1 - 4\epsilon^2)\mathrm{sech}^2(\xi) + 6\epsilon^2 \mathrm{sech}^4(\xi) \right] \left[-\frac{1}{2}(1 + 2\epsilon^2) + \frac{3}{2}\epsilon^2 \mathrm{sech}^2(\xi) \right] d\xi$$

$$= -(1 + 2\epsilon^2)(1 - 4\epsilon^2) + 2\epsilon^2(1 - 4\epsilon^2) - 4\epsilon^2(1 + 2\epsilon^2) + \frac{48}{5}\epsilon^4$$

$$= -1 + \frac{8}{5}\epsilon^4,$$

which yields the result. $\qquad\square$

We are now ready to prove Theorem 2.

Proof of Theorem 2. By Fredholm's theory, if there exists a solution of the spectral problem (1.7) in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ for $\lambda \neq 0$, then the component V satisfies the orthogonality condition $(\varphi', V)_\epsilon = 0$. We will show that an eigenvalue $\lambda_0 \in \mathbb{C}\backslash i\mathbb{R}$ exists if and only if $\epsilon \in (\epsilon_0, \infty)$. Since, for every eigenvalue λ_0, $-\lambda_0$ is also an eigenvalue, this will prove the theorem.

For any eigenvalue $\lambda_0 \neq 0$, the corresponding eigenvector $(U, V) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ must satisfy $(\varphi', V)_\epsilon = 0$. The second equation of the system (1.7) can be written in the form

$$(3.4) \qquad \mathcal{L}_+(1 - \epsilon^2 \partial_\xi^2)^{1/2} U = -\lambda_0 (1 - \epsilon^2 \partial_\xi^2)^{1/2} V,$$

where the right-hand side is orthogonal to $W_0 = (1 - \epsilon^2 \partial_\xi^2)^{1/2} \varphi'$ since

$$((1 - \epsilon^2 \partial_\xi^2)^{1/2} V, W_0) = ((1 - \epsilon^2 \partial_\xi^2)^{1/2} V, (1 - \epsilon^2 \partial_\xi^2)^{1/2} \varphi') = (V, \varphi')_\epsilon = 0.$$

Furthermore, we have

$$((1 - \epsilon^2 \partial_\xi^2)^{1/2} U, W_0) = ((1 - \epsilon^2 \partial_\xi^2)^{1/2} U, (1 - \epsilon^2 \partial_\xi^2)^{1/2} \varphi') = (U, \varphi')_\epsilon.$$

One can select the eigenfunction uniquely by requiring that $(U, \varphi')_\epsilon = 0$ after adding a multiple of φ' to U. With this convention, $\mathcal{L}_+$ in (3.4) can be replaced by $\mathcal{T}_+$ so that equation (3.4) can be solved in the form

$$(1 - \epsilon^2 \partial_\xi^2)^{1/2} U = -\lambda_0 \mathcal{T}_+^{-1} (1 - \epsilon^2 \partial_\xi^2)^{1/2} V.$$

Substituting this formula into the first equation of the system (1.7) yields

$$L_- V = -\lambda_0^2 (1 - \epsilon^2 \partial_\xi^2)^{1/2} \mathcal{T}_+^{-1} (1 - \epsilon^2 \partial_\xi^2)^{1/2} V.$$

Since this eigenvalue problem is self-adjoint and $(1 - \epsilon^2 \partial_\xi^2)^{1/2} \mathcal{T}_+^{-1} (1 - \epsilon^2 \partial_\xi^2)^{1/2} :$ $H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$ is a strictly positive operator by Lemma 3.1, we obtain $\lambda_0^2 \in \mathbb{R}$, i.e. $\lambda_0 \in \mathbb{R} \cup i\mathbb{R}$. Therefore, λ_0 is an eigenvalue in $\mathbb{C}\backslash i\mathbb{R}$ if and only if $-\lambda_0^2 < 0$. Consequently, such an eigenvalue exists if and only if

$$(3.5) \qquad \inf_{\substack{V \in H_\epsilon^1(\mathbb{R})\backslash\{0\} \\ (\varphi', V)_\epsilon = 0}} \frac{\langle L_- V, V \rangle}{\langle (1 - \epsilon^2 \partial_\xi^2)^{1/2} \mathcal{T}_+^{-1} (1 - \epsilon^2 \partial_\xi^2)^{1/2} V, V \rangle} < 0.$$

Since the denominator is strictly positive by Lemma 3.1, the sign of the left-hand side of (3.5) is determined by the sign of $-\mu_0^2$ in a simpler variational problem

$$(3.6) \qquad -\mu_0^2 = \inf_{\substack{V \in H_\epsilon^1(\mathbb{R})\backslash\{0\} \\ (\varphi', V)_\epsilon = 0}} \frac{\langle L_- V, V \rangle}{\|V\|^2}.$$

As is shown in [20], the sign of $-\mu_0^2$ in (3.6) depends on the sign of

$$\lim_{\lambda \to 0^-} ((L_- - \lambda I)^{-1} (1 - \epsilon^2 \partial_\xi^2) \varphi', (1 - \epsilon^2 \partial_\xi^2) \varphi') = \lim_{\lambda \to 0^-} ((L_- - \lambda I)^{-1} L_- V_\varphi, (1 - \epsilon^2 \partial_\xi^2) \varphi')$$
$$= (V_\varphi, \varphi')_\epsilon,$$

which changes the sign if $\epsilon = \epsilon_0$. The following dichotomy exists (see [20, Theorem 1.1]):

(i) $-\mu_0^2 \geq 0$ if $(\varphi', V_\varphi)_\epsilon \leq 0$,
(ii) $-\mu_0^2 < 0$ if $(\varphi', V_\varphi)_\epsilon > 0$.

In case (i), the condition (3.5) is not satisfied so that no isolated eigenvalue λ_0 with $\mathrm{Re}(\lambda_0) > 0$ of the spectral problem (1.7) exists. This is the spectral stability case which corresponds to $\epsilon \in (0, \epsilon_0]$. In case (ii), the condition (3.5) is satisfied so that there exists an isolated positive eigenvalue $\lambda_0 \in \mathbb{R}$. This is the spectral instability case which corresponds to $\epsilon \in (\epsilon_0, \infty)$. $\square$

4. Numerical illustrations

We approximate solutions of the NLS equation (1.5) numerically by using a finite-difference method. The line $\mathbb{R}$ is truncated on the symmetric interval $[-L, L]$ for sufficiently large $L > 0$ subject to the Neumann boundary conditions $u_\xi(t, -L) = u_\xi(t, L) = 0$ for every $t > 0$. One can also use the inhomogeneous Dirichlet conditions $u(t, \pm L) = \pm 1$ as an alternative truncation, which we do not report here.

Representing the solution $u(t, \xi)$ as a column vector $\mathbf{u}(t)$ on an equally spaced grid of $2K + 1$ grid points on the interval $[-L, L]$ yields the evolutionary problem in the form

$$(4.1) \qquad i(1 - \epsilon^2 A_N)\frac{d\mathbf{u}}{dt} + A_N \mathbf{u} + 2(1 - |\mathbf{u}|^2)\mathbf{u} = 0,$$

where $\mathbf{u}(t) : \mathbb{R} \to \mathbb{C}^{2K+1}$ and $A_N \in \mathbb{M}^{(2K+1)\times(2K+1)}$ is the matrix approximation of the central difference for the second spatial derivative which incorporates the Neumann boundary conditions at the end points,

$$(A_N \mathbf{u})_k = \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2}, \qquad 2 \leq k \leq 2K,$$

and

$$(A_N \mathbf{u})_1 = \frac{2(u_2 - u_1)}{h^2}, \quad (A_N \mathbf{u})_{2K+1} = \frac{2(u_{2K} - u_{2K+1})}{h^2}.$$

As is well-known, the discretization error of the central difference has the order of $\mathcal{O}(h^2)$.

Iterations in time are performed with the Crank–Nicholson method over an equally spaced temporal grid with the time step τ. The numerical approximation of $\mathbf{u}(t)$ at the time level $t_m = m\tau$ is denoted by $\mathbf{u}^{(m)}$. The Crank–Nicolson method is given by the iterative rule:

$$\left(1 - \epsilon^2 A_N - \frac{i\tau}{2}A_N - i\tau(1 - |\mathbf{u}^{(m+1)}|^2)\right)\mathbf{u}^{(m+1)}$$

$$(4.2) \qquad = \left(1 - \epsilon^2 A_N + \frac{i\tau}{2}A_N + i\tau(1 - |\mathbf{u}^{(m)}|^2)\right)\mathbf{u}^{(m)},$$

for integer $m \geq 0$. As is also well-known, the discretization error of the Crank–Nicolson method has the global error of $\mathcal{O}(\tau^2)$ and the stability of iterations is unconditional with respect to the time step τ relative to h.

Iterations of the nonlinear system (4.2) are fully implicit. In order to make them explicit, we invert the matrix in the left side at each iteration m by using the previous value of $|\mathbf{u}^{(m)}|^2$,

$$\tilde{\mathbf{u}}^{(m+1)}$$

$$= \left(1 - \epsilon^2 A_N - \frac{i\tau}{2}A_N - i\tau(1 - |\mathbf{u}^{(m)}|^2)\right)^{-1}\left(1 - \epsilon^2 A_N + \frac{i\tau}{2}A_N + i\tau(1 - |\mathbf{u}^{(m)}|^2)\right)\mathbf{u}^{(m)},$$
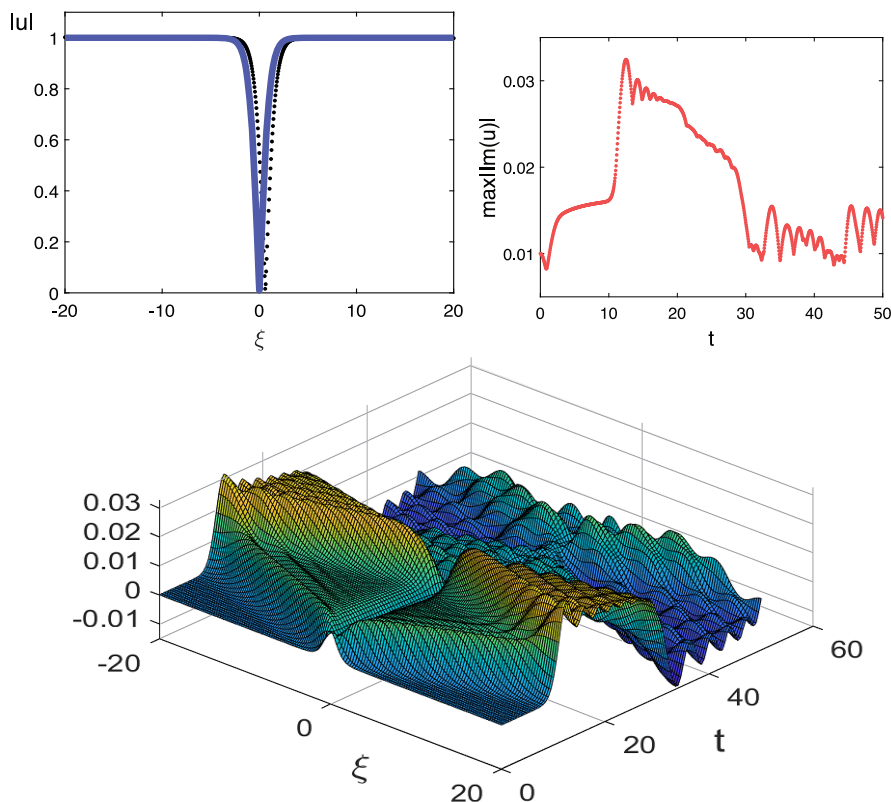
FIGURE 1. Numerical approximation of the dynamics of the perturbed black soliton for $\epsilon = 0.5$. Top left: The profile of $|u|$ versus ξ for time $t = 0$ (blue line) and time $t = 50$ (black dots). Top right: The maximal value of $|\mathrm{Im}(u)|$ versus t. Bottom: the solution surface for $\mathrm{Im}(u)$ on the (t, ξ) plane.

and use Heun's predictor–corrector method for $\mathbf{u}^{(m+1)}$ to restore the second-order accuracy of the time iterations. The initial data was chosen as

$$\mathbf{u}^{(0)} = \tanh(\xi_k) + ia\,\mathrm{sech}^2(\xi_k),$$

where $a > 0$ is the amplitude factor for the perturbation to the black soliton. The perturbation is needed to observe the stable versus unstable dynamics of the perturbed black soliton in the time evolution since spatially odd perturbations to the black soliton are orbitally stable, see Remark 6. We have chosen $L = 20$, $K = 400$, and $a = 0.01$.

Figure 1 shows the outcomes of the numerical simulations for $\epsilon = 0.5$. According to Theorem 2, the black soliton is spectrally stable for this value of ϵ since $\epsilon_0 = (5/8)^{1/4} \approx 0.89$. Indeed, we observe that the initial perturbation pushes the black soliton to the right for a small distance (top left panel) but the fluctuation of the imaginary part of the solution is bounded in the time evolution (top right panel). The solution surface for the imaginary part (bottom panel) shows that the perturbations to the black soliton are pushed towards the boundaries during the
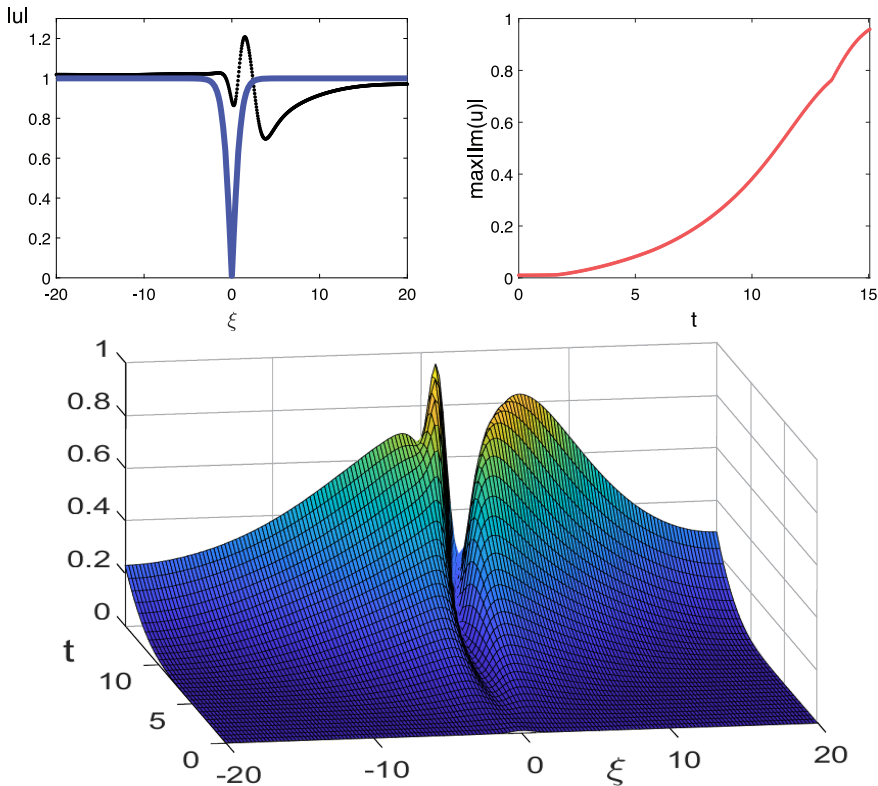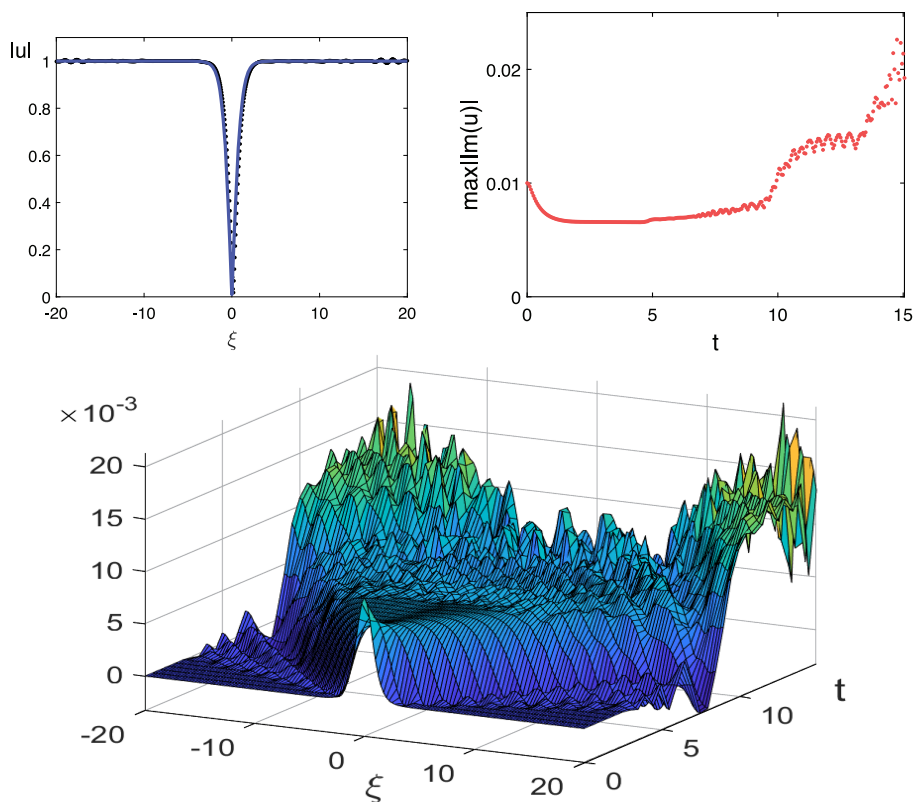
FIGURE 2. The same as on Figure 1 but for $\epsilon = 1$

time evolution where they are reflected due to the Neumann boundary conditions. The perturbed black soliton preserves its shape in the case $\epsilon = 0.5$.

In comparison with the stable dynamics of the perturbed black soliton for $\epsilon = 0.5 < \epsilon_0$, Figure 2 shows the unstable dynamics for $\epsilon = 1 > \epsilon_0$. The perturbations to the black soliton in $\mathrm{Im}(u)$ grow from the initial value of $a = 0.01$ towards the unit magnitude. As a result, the black soliton is completely destroyed in the time evolution and the final profile of $|u|$ versus ξ for $t = 10$ (black dots on the top left panel) shows non-solitonic solutions.

We have performed computations for ϵ closer to ϵ_0 and observed the same stable and unstable dynamics of the perturbed black soliton similar to Figures 1 and 2. The actual value of the instability threshold depends generally on the half-length L of the truncated interval $[-L, L]$.

Finally, we can also inspect the time evolution of the perturbed black soliton in the canonical defocusing NLS equation (1.1) which corresponds to the NLS model (1.5) with $\epsilon = 0$. Figure 3 shows that the perturbed black soliton is stable in the time evolution but the perturbations become noisy in the time evolution due to multiple reflections from the boundaries. This dynamics agrees well with the property of the NLS equation (1.1) that the imaginary part of the perturbation is not controlled in the Sobolev space of $H^1(\mathbb{R})$ since the energy and momentum are not coercive in $H^1(\mathbb{R})$. In comparison, perturbations of the black soliton in

FIGURE 3. The same as on Figure 1 but for $\epsilon = 0$

the regularized NLS equation (1.5) with $\epsilon > 0$ are well-defined in the space $H^1(\mathbb{R})$ as a continuously differentiable function of time and the imaginary part of the perturbations shown in Figures 1 and 2 is much smoother in the spatial coordinate compared to the one shown in Figure 3.

5. DISCUSSION

We have shown analytically and illustrated numerically that the regularized NLS equation (1.5) admits smooth time-dependent solutions near the black soliton with perturbations defined in $C^1([0, \tau_0), H^s(\mathbb{R}))$ with $s > \frac{1}{2}$. We have shown the spectral stability of the black soliton for $\epsilon \leq \epsilon_0 := (5/8)^{1/4}$ and spectral instability for $\epsilon > \epsilon_0$. The question of orbital stability of the black soliton is left open since the energy conservation can be used to control a weighted $H^1(\mathbb{R})$ norm of the perturbation, whereas the momentum conservation is only defined for perturbations in $H^s(\mathbb{R})$ with $s \geq 2$, see Remark 5.

Among further directions of the research, one can consider generalizations of the regularized NLS equation with cubic nonlinearity in the form

$$(5.1) \qquad i(1 - \epsilon^2 \partial_x^2 - \delta^2 |u|^2)u_t + u_{xx} + 2(1 - |u|^2)u = 0,$$

with two parameters $\epsilon \geq 0$ and $\delta \in [0, 1]$. The case $\epsilon = 0$ and $\delta = 1$ was considered in [21] as the NLS model with intensity-dependent dispersion,

$$(5.2) \qquad i(1 - |u|^2)u_t + u_{xx} + 2(1 - |u|^2)u = 0.$$

We proved in [21] that the perturbations to the black soliton were controlled in a weighted $H^1(\mathbb{R})$ space from the energy and momentum conservation of the NLS model (5.2). Spectral stability of the black soliton was also proven by characterizing the purely discrete spectrum of the spectral stability problem. The question of local well-posedness for the perturbations of the black soliton was left open since the black soliton satisfies the boundary conditions $u(t, x) \to \pm 1$ as $x \to \pm\infty$, where the evolution equation (5.2) is singular.

In the combined NLS model (5.1) with sufficiently small $\epsilon > 0$ and $\delta \in (0, 1)$, one can achieve both the local well-posedness and the spectral stability of the black soliton. Although the orbital stability problem might still be out of reach for $\epsilon > 0$ due to the mismatch between the function spaces for the energy and momentum conservation, the combined model (5.1) can be used to study the limit $\delta \to 1$ and to shed more light on the well-posedness theory for the NLS model (5.2).

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