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Dark solitons in external potentials

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Abstract. We consider the persistence and stability of dark solitons in the Gross–Pitaevskii (GP) equation with a small decaying potential. We show that families of black solitons with zero speed originate from extremal points of an appropriately defined *effective potential* and persist for sufficiently small strength of the potential. We prove that families at the maximum points are generally unstable with exactly one real positive eigenvalue, while families at the minimum points are generally unstable with exactly two complex-conjugated eigenvalues with positive real part. This mechanism of destabilization of the black soliton is confirmed in numerical approximations of eigenvalues of the linearized GP equation and full numerical simulations of the nonlinear GP equation. We illustrate the monotonic instability associated with the real eigenvalues and the oscillatory instability associated with the complex eigenvalues and compare the numerical results of evolution of a dark soliton with the predictions of Newton's particle law for its position.

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1. Introduction

Dark solitons are solutions of nonlinear PDEs in the space of one dimension with non-zero boundary conditions and a non-zero phase shift. They are represented by a family of traveling waves extending from the limit of zero speed (so-called *black* solitons) to the limit of sound speed (so-called *grey* solitons). From a physical point of view, dark solitons are waves in defocusing nonlinear systems which move along the modulationally stable continuous-wave background.

The original interest in studies of dark solitons emerged, roughly, two decades ago in the context of nonlinear optics, where dark solitons provide modulations of the light intensity of an optical beam traveling in a planar waveguide [22]. The main model for dark solitons in nonlinear optics is the generalized nonlinear Schrödinger (NLS) equation

$$iu_t = -\frac{1}{2}u_{xx} + f(|u|^2)u, \tag{1.1}$$

where $u(x,t) : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{C}$ and $f(q) : \mathbb{R}_+ \to \mathbb{R}$. We assume that $f(q)$ is a smooth monotonically increasing function on $q \in \mathcal{I} \subset \mathbb{R}_+$. Particular examples of $f(q)$ include the cubic NLS with $f(q) = q$, the cubic-quintic NLS with $f(q) = \alpha q + q^2$, $\alpha \in \mathbb{R}$ and the saturable NLS with $f = -1/(1 + \beta q)$, $\beta \in \mathbb{R}_+$.

Among others, several analytical results were important in the development of dark solitons in recent years: perturbation theory based on renormalized power [24] and momentum [23], orbital stability of dark solitons [1, 2], completeness of eigenfunctions in the cubic NLS [8, 15], inverse scattering for the vector cubic NLS equation [41], construction of the Evans function for dark solitons in the perturbed cubic NLS [19], asymptotic analysis of the radiation and dynamics of dark solitons [26, 27, 35], and spectral analysis of transverse instabilities of one-dimensional dark solitons [25, 36].

Subsequently, rigorous analysis of the existence and stability of dark solitons was developed in the last decade based on the earlier physical literature. In particular, Zhidkov [45] proved local existence of solutions of the Cauchy problem, de Bouard [4] proved spectral and nonlinear instability of stationary bubbles (black solitons with zero phase shift), Lin [28] proved the criterion for orbital stability and instability of dark solitons (for non-zero velocities), Maris [29] studied bifurcation of dark solitons (for non-zero velocities) in the presence of a delta-function potential and its generalizations, and Di Menza and Gallo [31] proved recently the stability criterion for kinks (black solitons with a non-zero phase shift). Extensions of the existence and stability theory in two and higher dimensions were also developed by J.C. Saut and his co-workers.

While many mathematical results are now available for solutions of the generalized NLS equation (1.1) with non-zero boundary conditions and a non-zero phase shift, dark solitons have suffered a decreasing popularity in the context of nonlinear optics. This is not only because they possess infinite energy due to non-zero boundary conditions but also because it is difficult from the experimental point of view to separate the effects of the dark soliton dynamics and the background dynamics.

Nevertheless, the interest in these nonlinear waveforms has recently been rejuvenated by the rapid development of a new area of physics, namely the field of Bose–Einstein condensates (BECs) [39, 40]. In the latter setting, dark solitons typically move along the nonlinear ground state trapped by the external potentials [42]. The main model for BECs is a modification of the NLS equation (1.1) with an external potential, which is called the Gross–Pitaevskii (GP) equation,

$$iu_t = -\frac{1}{2} u_{xx} + f(|u|^2)u + \epsilon V(x) u, \tag{1.2}$$

where $\epsilon \in \mathbb{R}$ is the strength of the potential $V(x)$ and $V(x) : \mathbb{R} \to \mathbb{R}$ is assumed to be a smooth function satisfying one of the three properties:

(i) $V(x)$ is bounded and decaying, e.g.

$$\exists C > 0,\ \kappa > 0 : \quad |V(x)| \le C e^{-\kappa |x|}, \quad \forall x \in \mathbb{R} \tag{1.3}$$

(ii) $V(x)$ is bounded but non-decaying, e.g. $V(x+d) = V(x)$, $x \in \mathbb{R}$ with period

$d > 0$

(iii) $V(x)$ is unbounded, e.g. $V(x) = x^2 + \tilde{V}(x)$, where $\tilde{V}(x)$ is bounded on $x \in \mathbb{R}$.

The last case is of particular interest in the context of Bose–Einstein condensates, where dynamics of localized dips in the nonlinear ground state trapped by the unbounded potential was studied in many recent papers, see [5, 34] for surveys of results. Although cases (ii) and (iii) have been initially our primary motivations, this paper covers only the case (i) when $V(x)$ is bounded and exponentially decaying as in (1.3). In particular, we consider the class of symmetric potentials $V(-x) = V(x)$ with two examples

$$V_1(x) = -\mathrm{sech}^2\left(\frac{\kappa x}{2}\right), \qquad V_2(x) = x^2 e^{-\kappa|x|}, \qquad \forall x \in \mathbb{R}. \tag{1.4}$$

While the potentials (1.4) are, perhaps, less customary than the standard magnetic (parabolic) and optical lattice (periodic) potentials [21], they are nonetheless still quite physically relevant. In particular, the potential $V_1(x)$ corresponds to a red-detuned laser beam potential, analogous to the one used in [32]. The potential $V_2(x)$ represents an all-optically trapped BEC, as modeled in [7] and experimentally implemented in [3].

The strategy of our work is to exploit solutions of the GP equation (1.2) in the limit of small strength ϵ. Starting with the limit $\epsilon = 0$, where both existence and stability of dark solitons are known from the analysis of the NLS equation (1.1), we shall use the method of Lyapunov–Schmidt reductions for small ϵ. From the technical point of view, we use local bifurcation analysis of solutions of the ODEs with non-zero boundary conditions similarly to [30], persistence analysis of isolated eigenvalues in the problems with small potentials similarly to [18], the count of eigenvalues in the generalized eigenvalue problem for self-adjoint operators with no spectral gaps similarly to [9], the Evans function construction similar to [19], and the construction of L^2 eigenfunctions of the stability problem with fast and slow exponential decay similarly to [33]. Since our starting point is the case of $\epsilon = 0$, we will also give alternative proofs to the existence and stability of black solitons in the NLS equation (1.1), which complement the recent work of [31].

Our main results are listed as follows.

(i) Let $u_0 = \phi_0(x - s)e^{-i\omega t + i\theta}$ be a solution of the NLS equation (1.1) with $s \in \mathbb{R}$, $\theta \in \mathbb{R}$, $\omega = f(q_0)$, $q_0 \in \mathcal{I} \subset \mathbb{R}_+$ and $\phi_0(x) : \mathbb{R} \mapsto \mathbb{R}$ converges to $\pm\sqrt{q_0}$ as $x \to \pm\infty$ exponentially fast. Let s_0 be a simple root of the function

$$M'(s) = \int_{\mathbb{R}} V'(x) \left[q_0 - \phi_0^2(x - s)\right] dx. \tag{1.5}$$

Then, there exists a solution $u_\epsilon = \phi_\epsilon(x - s_\epsilon)e^{-i\omega t + i\theta}$ of the GP equation (1.2) with $V(x)$ in (1.3) and ϵ sufficiently small, where $\phi_\epsilon : \mathbb{R} \mapsto \mathbb{R}$ converges to $\pm\sqrt{q_0}$ as $x \to \pm\infty$ exponentially fast, while $\phi_\epsilon(x)$ and (s_ϵ) is ϵ-close to $\phi_0(x)$ and s_0 in L^∞-norm.

(ii) Let the solution u_0 be spectrally stable in the time evolution of the NLS equation (1.1). Then, the solution u_ϵ is spectrally unstable in the time evolution of the GP equation (1.2) for sufficiently small ϵ with exactly one real positive eigenvalue if $M''(s_0) < 0$ and exactly two complex-conjugate eigenvalues with positive real part if $M''(s_0) > 0$.

(iii) Let the function $u(x,0) : \mathbb{R} \mapsto \mathbb{R}$ be close to the solution $u_\epsilon(x - s(0), 0)$ in L^∞-norm for some $s(0)$ close to s_ϵ. Then, the solution $u(x,t) : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{C}$ of the Cauchy problem to the GP equation (1.2) remains close to the solution $u_\epsilon(x - s(t), t)$, where the varying coordinate $s(t)$ solves the Newton's particle equation

$$\mu_0 \ddot{s} - \epsilon \lambda_0 M''(s) \dot{s} = -\epsilon M'(s), \qquad 0 \le t < T. \tag{1.6}$$

Here $M(s)$ is the effective potential in (1.5), (μ_0, λ_0) are constants representing the soliton's mass and gain terms, and the time $T > 0$ is of the order of $\mathrm{O}(1/\epsilon)$.

Statement (i) is formulated and proved in Section 2 (see Theorem 2.12). Statement (ii) is formulated and proved in Section 3 (see Theorems 3.11, 3.14, and 3.15) under some non-degeneracy assumptions (see Corollary 3.16 and Remark 3.17). The two complex-conjugate eigenvalues with positive real part in the statement (ii) for $M''(s_0) > 0$ follow from the linearized version of the Newton's particle equation (1.6) with $\mu_0 > 0$ and $\lambda_0 > 0$, which is rigorously derived in Section 4 (see Theorem 4.11). Constants (μ_0, λ_0) are identified in Remark 4.13. Numerical studies of the linearization around a black soliton are reported in Section 5. Statement (iii) is a conjecture which is tested against appropriately crafted numerical experiments in Section 6. The summary and open problems are discussed in Section 7.

2. Existence analysis of dark and black solitons

We first consider the family of traveling solutions of the NLS equation (1.1). After we formulate the conditions for existence of dark and black solitons, we address persistence of stationary solutions in the GP equation (1.2) for small ϵ and the potential $V(x)$ in (1.3). We conclude this section with applications of the persistence analysis to the two potentials (1.4), which indicate that the families of black solitons bifurcate from the extremal points of the effective potential $M(s)$ defined in (1.5).

Definition 2.1. *Any traveling solution of the NLS equation (1.1) of the form*

$$u(x,t) = U(x - vt) e^{-i\omega t}, \quad U(z) = \Phi(z) e^{i\Theta(z)}, \quad z = x - vt, \tag{2.1}$$

is called a dark soliton if $U : \mathbb{R} \mapsto \mathbb{C}$, $\Phi : \mathbb{R} \mapsto \mathbb{R}_+$, and $\Theta : \mathbb{R} \mapsto [-\pi, \pi]$ are smooth functions of their arguments, which converge exponentially fast to the boundary conditions

$$\lim_{z \to \pm\infty} \Phi(z) = \sqrt{q_0}, \qquad \lim_{z \to \pm\infty} \Theta(z) = \Theta_\pm. \tag{2.2}$$

Here $(\omega, v) \in \mathcal{D} \subset \mathbb{R}^2$, $q_0 \in \mathcal{I} \subset \mathbb{R}_+$ *and* $\Theta_\pm \in [-\pi, \pi]$ *are parameters of the solution. Moreover, the functions* $\Phi(z)$ *and* $\Theta(z)$ *can be chosen to satisfy the normalization conditions* $\Phi'(0) = 0$ *and* $\Theta_+ = 0$. *Additionally, we require that* $\Phi(z) < \sqrt{q_0}$ *on* $z \in \mathbb{R}$.

Remark 2.2. The normalization conditions for $\Phi(z)$ and $\Theta(z)$ use the gauge $[u(x,t) \to u(x,t)e^{i\theta}, \forall \theta \in \mathbb{R}]$ and translational $[u(x,t) \to u(x-s,t), \forall s \in \mathbb{R}]$ invariance of the NLS equation (1.1), while the linear growth of $\Theta(z)$ in z is excluded by the Galileo invariance $[u(x,t) \to u(x-kt,t)e^{ikx-ik^2t/2}, \forall k \in \mathbb{R}]$.

Theorem 2.3. *Let* $f(q)$ *be* $C^1(\mathbb{R}_+)$ *and fix* $q_0 \in \mathbb{R}_+$ *such that* $f'(q_0) > 0$ *and* $c = \sqrt{q_0 f'(q_0)} > 0$. *A dark soliton* $U(z)$ *of Definition 2.1 exists if* $\omega = f(q_0)$, $v \in (-c, c)$, *and there exists a simple largest root* q_1 *in* $(0, q_0)$ *of the function*

$$\tilde{W}(q) = \int_q^{q_0} (f(q_0) - f(q))\, dq - \frac{v^2(q-q_0)^2}{2q} \tag{2.3}$$

with $\tilde{W}'(q_1) > 0$. *Moreover, for these solutions,* $\Phi(z)$ *has a global minimum at* $z = 0$ *with* $0 < \Phi(0) < \sqrt{q_0}$ *and* $\Theta(z)$ *is monotonically decreasing for* $v > 0$ *and increasing for* $v < 0$.

Proof. It follows immediately that the function $U(z)$ satisfies the second-order ODE:

$$-ivU' + \frac{1}{2}U'' + (\omega - f(|U|^2))U = 0, \tag{2.4}$$

while the functions $\Phi(z)$ and $\Theta(z)$ satisfy the ODE system in the hydrodynamic form:

$$\Phi'' - (\Theta')^2\Phi + 2v\Theta'\Phi + 2(\omega - f(\Phi^2))\Phi = 0 \tag{2.5}$$

$$(\Theta'\Phi^2 - v\Phi^2)' = 0 \tag{2.6}$$

Integrating the ODE (2.6) under the boundary conditions (2.2), we obtain

$$\Theta' = v\frac{\Phi^2 - q_0}{\Phi^2}. \tag{2.7}$$

As a result, the ODE (2.5) reads as follows

$$\Phi'' + 2(\omega - f(\Phi^2))\Phi + v^2\frac{\Phi^4 - q_0^2}{\Phi^3} = 0. \tag{2.8}$$

The equilibrium point $\Phi = \sqrt{q_0}$ exists if and only if $\omega = f(q_0)$ and it is a non-degenerate hyperbolic point if and only if $v^2 < c^2 = q_0 f'(q_0)$. Integrating the second-order ODE (2.8) subject to the boundary conditions (2.2), we obtain the quadrature

$$(\Phi')^2 - 2W(\Phi^2) + v^2\frac{(\Phi^2 - q_0)^2}{\Phi^2} = 0, \tag{2.9}$$

where

$$W(q) = \int_q^{q_0} (f(q_0) - f(q))\, dq. \tag{2.10}$$

If there exists a simple largest root q_1 in $(0, q_0)$ of the function $\tilde{W}(q)$ in (2.3) with $\tilde{W}'(q_1) > 0$, then the trajectory from the hyperbolic point $\Phi = \sqrt{q_0}$ turns at $\Phi = \sqrt{q_1}$ and returns back to the hyperbolic point $\Phi = \sqrt{q_0}$ forming a homoclinic orbit, thus proving the statement. □

Remark 2.4. It is not difficult to prove from the ODE analysis that if $v \in (-c, 0) \cup (0, c)$ and $f \in C^\infty(\mathbb{R}_+)$, then $U(z)$ is in fact $C^\infty(\mathbb{R})$ and $U(z)$ converges to $\sqrt{q_0} e^{i\Theta_\pm}$ as $z \to \pm\infty$ at the exponential rate $\mathrm{O}(e^{-\sqrt{c^2-v^2}z})$.

Definition 2.5. *The limiting solution* $\phi_0(x) = \lim_{v \downarrow 0} U(x)$ *in a family of dark solitons of Definition 2.1 is said to be a black soliton if* $\phi_0(x)$ *is a real-valued smooth function on* $x \in \mathbb{R}$. *The black soliton is called a bubble if* $\phi_0(-x) = \phi_0(x)$ *with* $0 < \phi_0(0) < \sqrt{q_0}$ *and* $\lim_{x\to\pm\infty} \phi_0(x) = \sqrt{q_0}$, *while it is called a kink if* $\phi_0(-x) = -\phi_0(x)$ *and* $\lim_{x\to\pm\infty} \phi_0(x) = \pm\sqrt{q_0}$.

Theorem 2.6. *Let* $f(q)$ *satisfy the same conditions as in Theorem 2.3 and* $W(q)$ *be defined by (2.10). A kink solution of Definition 2.5 exists if and only if* $\omega = f(q_0)$, $v = 0$, *and* $W(q) > 0$ *for all* $0 \le q < q_0$. *A bubble solution of Definition 2.5 exists if and only if* $\omega = f(q_0)$, $v = 0$, *and there exists a simple largest root* q_1 *in* $(0, q_0)$ *of the function* $W(q)$ *with* $W'(q_1) > 0$.

Proof. It follows from the equation (2.7) that $\Theta(z) \neq 0$ on $z \in \mathbb{R}$ for $v \neq 0$ and hence no black solitons may exist for $v \neq 0$. Let $v = 0$ and consider the real-valued solution $\phi_0(x)$ of the second-order ODE

$$\frac{1}{2}\phi_0'' + (\omega - f(\phi_0^2))\phi_0 = 0. \tag{2.11}$$

The equilibrium points $\phi_0 = \pm\sqrt{q_0}$ exist if and only if $\omega = f(q_0)$ and they are non-degenerate hyperbolic points. Integrating the second-order ODE (2.11) under the boundary conditions in Definition 2.5, we obtain

$$(\phi_0')^2 - 2W(\phi_0^2) = 0,$$

where $W(q)$ is defined by (2.10). If $W(q) > 0$ for any $q \in [0, q_0)$, then the outgoing trajectory from the hyperbolic point $\phi_0 = \sqrt{q_0}$ connects the incoming trajectory to the hyperbolic point $\phi_0 = -\sqrt{q_0}$ forming a heteroclinic orbit (a kink). If there exists a simple largest root q_1 in $(0, q_0)$ of the function $W(q)$ with $W'(q_1) > 0$, then the trajectory from the hyperbolic point $\phi_0 = \sqrt{q_0}$ turns at $\phi_0 = \sqrt{q_1}$ and returns back to the point $\phi_0 = \sqrt{q_0}$ forming a homoclinic orbit (a bubble). If the root q_1 is multiple or if $q_1 = 0$, there exists a front solution $\phi_0(x)$ from $\lim_{x\to-\infty} \phi_0(x) = \sqrt{q_1}$ to $\lim_{x\to\infty} \phi_0(x) = \sqrt{q_0}$ which is neither kink nor black soliton.

It remains to prove that the family of dark solitons $U(x)$ for $0 < v < c$ converges to the black soliton $\phi_0(x)$ as $v \downarrow 0$ in L^∞-norm. The proof follows from the quadrature (2.9). There exist unique classical solutions $\phi_\pm(z)$ on $z \in \mathbb{R}_\pm$ for any $v > 0$ with $\Phi_\pm(0) = \sqrt{q_*}$ found from the largest root of the function $\tilde{W}(q)$ on $(0, q_0)$. In the limit $v \downarrow 0$, the root q_* converges to 0 if $W(0) > 0$ and no other

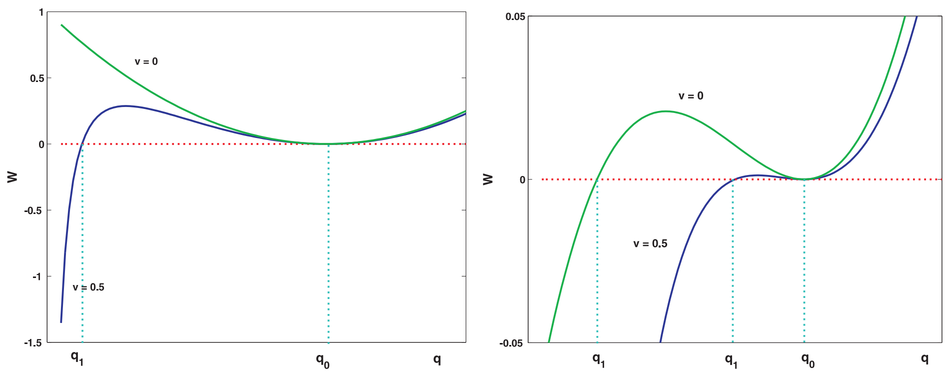

Figure 1. Functions $\tilde{W}(q)$ and $W(q)$ for the cubic NLS with $f(q) = q$ (left) and the cubic–quintic NLS with $f(q) = -\frac{3}{2}q + q^2$ (right) with $q_0 = 1$.

roots of $W(q)$ exists on $q \in [0, q_0)$. Otherwise, the root q_* converges to q_1, where q_1 is the largest root of $W(q)$ on $q \in [0, q_0)$. It follows from the ODE (2.7) in the limit $v \downarrow 0$ that $\Theta(z)$ is piecewise constant function on $z \in \mathbb{R}$ with a possible jump discontinuity at $z = 0$.

Let us write $x = z$ for $v = 0$. In the case $q_* \downarrow 0$ as $v \downarrow 0$, the two smooth solutions $\Phi_\pm(x)$ are glued into one smooth real-valued solution $\phi_0(x)$ if and only if $\Theta(x) = 0$ on $x \in \mathbb{R}_+$ and $\Theta(x) = \pi$ on $x \in \mathbb{R}_-$ (under the normalization $\Theta_+ = 0$). This limiting solution becomes a kink. In the case $q_* \downarrow q_1$ as $v \downarrow 0$ and q_1 is a simple root of $W(q)$, the two smooth solutions $\Phi_\pm(x)$ are glued into a smooth real-valued solution $\phi_0(x)$ if and only if $\Theta(x) = 0$ on $x \in \mathbb{R}$ (under the same normalization). This limiting solution becomes a bubble. □

Remark 2.7. Due to their potential stability in the time evolution of the NLS equation (1.1), only kinks are considered in the GP equation (1.2) for sufficiently small ϵ. Bubbles are always unstable in the time evolution of the NLS equation (1.1) [4]. Figure 1 illustrates functions $\tilde{W}(q)$ and $W(q)$ in Theorems 2.3 and 2.6 for kinks (left) and bubbles (right).

Definition 2.8. *Any stationary solution of the GP equation (1.2) of the form*

$$u(x,t) = \phi_\epsilon(x) e^{-if(q_0)t + i\theta}, \qquad \theta \in \mathbb{R}$$

is called a kink mode if $\phi_\epsilon(x)$ is a real-valued smooth function on $x \in \mathbb{R}$, which converges to $\pm\sqrt{q_0}$ exponentially fast for any $\epsilon \in \mathbb{R}$.

Lemma 2.9. *Let $\phi_\epsilon(x)$ be a kink mode of Definition 2.8 and let $V(x)$ satisfy (1.3). Then, for any $\epsilon \neq 0$,*

$$\int_{\mathbb{R}} V'(x) \left[q_0 - \phi_\epsilon^2(x) \right] dx = 0. \tag{2.12}$$

Proof. Stationary solutions of Definition 2.8 satisfy the second-order ODE:

$$\frac{1}{2}\phi_\epsilon'' + \left(f(q_0) - f(\phi_\epsilon^2)\right)\phi_\epsilon = \epsilon V(x)\phi_\epsilon, \tag{2.13}$$

which is generated by the Hamiltonian function

$$E(\phi_\epsilon, \phi_\epsilon', x) = \frac{1}{2}(\phi_\epsilon')^2 - W(\phi_\epsilon^2) + \epsilon V(x)\left[q_0 - \phi_\epsilon^2\right],$$

where $W(q)$ is given by (2.10). Therefore, the change of $E(\phi_\epsilon(x), \phi_\epsilon'(x), x)$ at the classical solution $\phi_\epsilon(x)$ of the second-order ODE (2.13) is given by

$$\frac{dE}{dx} = \epsilon V'(x)\left[q_0 - \phi_\epsilon^2(x)\right].$$

Integrating this equation on $x \in \mathbb{R}$ and using the boundary conditions $\lim_{x\to\pm\infty} E(\phi_\epsilon(x), \phi_\epsilon'(x), x) = 0$, we derive the condition (2.12). □

Remark 2.10. If $V(-x) = V(x)$ and $\phi_\epsilon(-x) = -\phi_\epsilon(x)$ on $x \in \mathbb{R}$, the necessary condition (2.12) is always satisfied. The center of the kink is located at $x = 0$, which is the minimal point of $V(x)$ if $V''(0) > 0$ and maximal point if $V''(0) < 0$.

Remark 2.11. The necessary condition (2.12) specifies restrictions on the shape of the kink mode $\phi_\epsilon(x)$ but does not give us any information about its existence. By using the smallness of ϵ, we will show that this condition is equivalent to the bifurcation equation in the Lyapunov–Schmidt reduction technique. A similar result for bright solitons was obtained in [18]. A different role of the condition (2.12) was exploited in [37] in the context of local bifurcations of small gap solitons in finite periodic potentials $V(x)$.

Theorem 2.12. *Let $\phi_0(x)$ be a kink of Definition 2.5. Let s_0 be a simple root of the function*

$$M'(s) = \int_{\mathbb{R}} V'(x)\left[q_0 - \phi_0^2(x-s)\right]dx, \qquad s \in \mathbb{R}, \tag{2.14}$$

such that $M'(s_0) = 0$ and $M''(s_0) \neq 0$. Let $f(q)$ be $C^2(\mathbb{R}_+)$ and $V(x)$ be $C^2(\mathbb{R})$ satisfying (1.3). Then, there exists a unique continuation of $\phi_0(x - s_0)$ to a kink mode $\phi_\epsilon(x - s_\epsilon)$ of Definition 2.8 for sufficiently small ϵ, such that $\phi_\epsilon(x)$ and s_ϵ are ϵ-close to $\phi_0(x)$ and s_0 in the L^∞-norm.

*Proof.*We use the decomposition $\phi_\epsilon(x) = \phi_0(x - s) + \varphi(x, \epsilon, s)$ and rewrite the second-order ODE (2.13) for $\phi_\epsilon(x)$ as the root of the nonlinear operator-valued function

$$F(\varphi, \epsilon, s) = L_+\varphi + N(\varphi, s, \epsilon) + \epsilon V(x)\left[\phi_0(x-s) + \varphi\right] = 0, \tag{2.15}$$

where $L_+ : H^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is the self-adjoint operator parameterized by s

$$L_+ = -\frac{1}{2}\partial_x^2 + f(\phi_0^2) - f(q_0) + 2\phi_0^2 f'(\phi_0^2),$$

and $N(\varphi, \epsilon, s) : H^1(\mathbb{R}) \mapsto H^1(\mathbb{R})$ is the nonlinear vector field

$$N = \phi_0 \left[f((\phi_0 + \varphi)^2) - f(\phi_0^2) - 2\phi_0 \varphi f'(\phi_0^2) \right] + \varphi \left[f((\phi_0 + \varphi)^2) - f(\phi_0^2) \right],$$

such that $N(\varphi, \epsilon, s) = \mathrm{O}(\|\varphi\|_{H^1}^2)$ as $\|\varphi\|_{H^1} \to 0$ (since $f \in C^2(\mathbb{R}_+)$). Because $\phi_0(x)$ converges to $\pm\sqrt{q_0}$ as $x \to \pm\infty$ exponentially fast, the essential spectrum of L_+ is bounded from below by $2c^2 > 0$. The operator L_+ may have isolated positive eigenvalues and no negative eigenvalues since the kernel $L_+ \phi'(x - s) = 0$ is a positive definite ground state. Therefore, the method of Lyapunov–Schmidt reductions can be applied. Projection of $F(\varphi, \epsilon, s)$ onto $\mathrm{Ker}(L_+)^\perp$ defines a unique smooth map $(x, \epsilon, s) \mapsto \varphi \in \mathrm{Ker}(L_+)^\perp$ such that $\|\varphi\|_{H^1} = \mathrm{O}(\epsilon)$ as $\epsilon \to 0$. By the Sobolev Embedded Theorem, $\|\varphi\|_{L^\infty} = \mathrm{O}(\epsilon)$ and $\varphi(x)$ can be decomposed as follows:

$$\varphi = \epsilon \varphi_1(x) + \tilde{\varphi}(x, \epsilon, s),$$

where $\varphi_1(x)$ is specified below and $\|\tilde{\varphi}\|_{L^\infty} = \mathrm{O}(\epsilon^2)$. Projection of $F(\varphi, \epsilon, s)$ onto $\mathrm{Ker}(L_+)$ defines the bifurcation equation:

$$G(\epsilon, s) = \epsilon \left(\phi_0', V(x)(\phi_0 + \varphi) \right) + \left(\phi_0', N(\varphi, \epsilon, s) \right) = \frac{\epsilon}{2} M'(s) + \tilde{G}(\epsilon, s), \tag{2.16}$$

where $(\cdot, \cdot)$ is the standard inner product in $L^2(\mathbb{R})$ and $\tilde{G}(\epsilon, s) = \mathrm{O}(\epsilon^2)$ as $\epsilon \to 0$. The equation $\frac{1}{\epsilon} G(\epsilon, s) = 0$ is solved by using the Implicit Function Theorem with respect to the variable s. If $M'(s_0) = 0$ and $M''(s_0) \neq 0$, then the simple root is $s = s_0 + \tilde{s}(\epsilon)$ and $\tilde{s} = \mathrm{O}(\epsilon)$ as $\epsilon \to 0$. By the Lyapunov–Schmidt Reduction Theorem for the root of (2.15), a unique continuation of $\phi_0(x - s_0)$ into $\phi_\epsilon(x - s_\epsilon)$ exists. In particular, the correction term $\varphi_1(x)$ satisfies the inhomogeneous problem

$$L_+ \varphi_1 = -V(x) \phi_0(x - s_0), \tag{2.17}$$

which has a unique solution $\varphi_1 \in \mathrm{Ker}(L_+)^\perp$ by the Fredholm Alternative (since $M'(s_0) = 0$). □

Remark 2.13. The renormalization of $\phi^2 \mapsto (\phi^2 - q_0)$ is not needed if the potential $V(x)$ satisfies the condition (1.3). We expect that the same quantity $M(s)$ with the renormalization above can be useful to treat the other cases (ii) and (iii) of the potential term $V(x)$. However, the method of Lyapunov–Schmidt reductions does not work for these cases since the perturbation term $\epsilon V(x) \phi_0(x)$ does not belong to $H^1(\mathbb{R})$.

Example 2.14. When the cubic NLS is considered with $f(s) = s$, the value $q_0 \in \mathbb{R}_+$ can be normalized by $q_0 = 1$. In this case, the second-order ODE (2.4) admits an exact solution

$$U(z) = k \tanh(kz) + iv, \qquad k = \sqrt{1 - v^2}, \tag{2.18}$$

where $v \in (-1, 1)$. The black soliton corresponds to the kink $\phi_0(x) = \tanh x$. When the potential $V(x)$ is even $V(-x) = V(x)$, the function $M'(s)$ in (2.14) can be split into two parts:

$$M'(s) = \int_{\mathbb{R}_+} V'(x) \left[\phi_0^2(x+s) - \phi_0^2(x-s) \right] dx = L'(s) - L'(-s),$$

where

$$L(s) = \int_{\mathbb{R}_+} V(x) \left[q_0 - \phi_0^2(x-s) \right] dx. \tag{2.19}$$

If $V(x)$ is $C^2(\mathbb{R})$ and satisfies the decay condition (1.3), the function $L(s)$ is $C^2(\mathbb{R})$ and $L(s) \to 0$ exponentially fast as $|s| \to \infty$. Therefore, $M'(0) = L'(0) - L'(0) = 0$ and one family of kink modes bifurcates from $s_0 = 0$. Additional families of kink modes of the GP equation (1.2) may bifurcate if $L(0)$ and $L''(0)$ are of the same sign. In this case, two global extrema of $L(s) + L(-s)$ exist at $s_0 = \pm s_*$ with $s_* > 0$, such that two other families of kink modes bifurcate from $s_0 = \pm s_*$.
When $V = V_1(x)$, $\phi_0 = \tanh x$, and $q_0 = 1$, the function $L(s)$ is computed in the implicit form

$$L(s) = -\int_{\mathbb{R}_+} \mathrm{sech}^2\left(\frac{\kappa x}{2}\right) \mathrm{sech}^2(x-s) dx.$$

Clearly $L(0) < 0$ and, as can be seen from Fig. 2 (top left panel), $L''(0) > 0$ for any $\kappa \neq 0$. Additionally, Fig. 2 (middle and bottom left panels) suggests that $M(s) < 0$ and $M(s) \to 0$ as $s \to \infty$ for any κ. Therefore, there is only one kink mode that bifurcates from $s_0 = 0$, where $V_1(x)$ has a minimum.
When $V = V_2(x)$, the function $L(s)$ is computed in the implicit form:

$$L(s) = \int_{\mathbb{R}_+} x^2 e^{-\kappa x} \mathrm{sech}^2(x-s) dx,$$

where $\kappa \in \mathbb{R}_+$. The above $L(s)$ can be expressed as a generalized hypergeometric function, however, we will not reproduce the resulting expression here. Instead, we note that $L(0) > 0$ and

$$L''(0) = \int_{\mathbb{R}_+} x^2 e^{-\kappa x} \left[4\mathrm{sech}^2 x - 6\mathrm{sech}^4 x \right] dx.$$

When $\kappa = 0$, $L''(0) = 2$. By using the Laplace method for computations of the integrals, one can find that $L''(0) = -4\kappa^{-3} + \mathrm{O}(\kappa^{-5})$ as $\kappa \to \infty$. Therefore, there exists $\kappa_0^{\pm} \in \mathbb{R}_+$ with $\kappa_0^- \leq \kappa_0^+$, such that $L''(0) > 0$ for $0 < \kappa < \kappa_0^-$ and $L''(0) < 0$ for $\kappa > \kappa_0^+$. As can be seen from Fig. 2 (top right panel), $\kappa_0^- = \kappa_0^+ \approx 3.21$. Additionally, Fig. 2 (middle and bottom right panels) suggest that there exist three kink modes for $0 < \kappa < \kappa_0$: two modes with $s_0 = \pm s_*$ are associated with the global maxima of the effective potential $M(s)$, while the mode with $s_0 = 0$ is associated with the local minimum of $M(s)$. When $\kappa > \kappa_0$, no kink modes with $s_0 = \pm s_*$ exist but the mode at $s_0 = 0$ corresponds to the global maximum of the effective potential $M(s)$. Hence, the structure of kink modes corresponds to a *subcritical pitchfork bifurcation* in the parameter κ, such that three solutions exist (at $s_0 = 0$ and $s_0 = \pm s_*$) for $0 < \kappa < \kappa_0$ and only one solution persists for

$\kappa > \kappa_0$. We point out that the effective potential $M(s)$ gives a different prediction in comparison with the true potential $V_2(x)$ which possesses a minimum at $x = 0$ and two maxima at $x = \pm 2/\kappa$, *for all* κ.

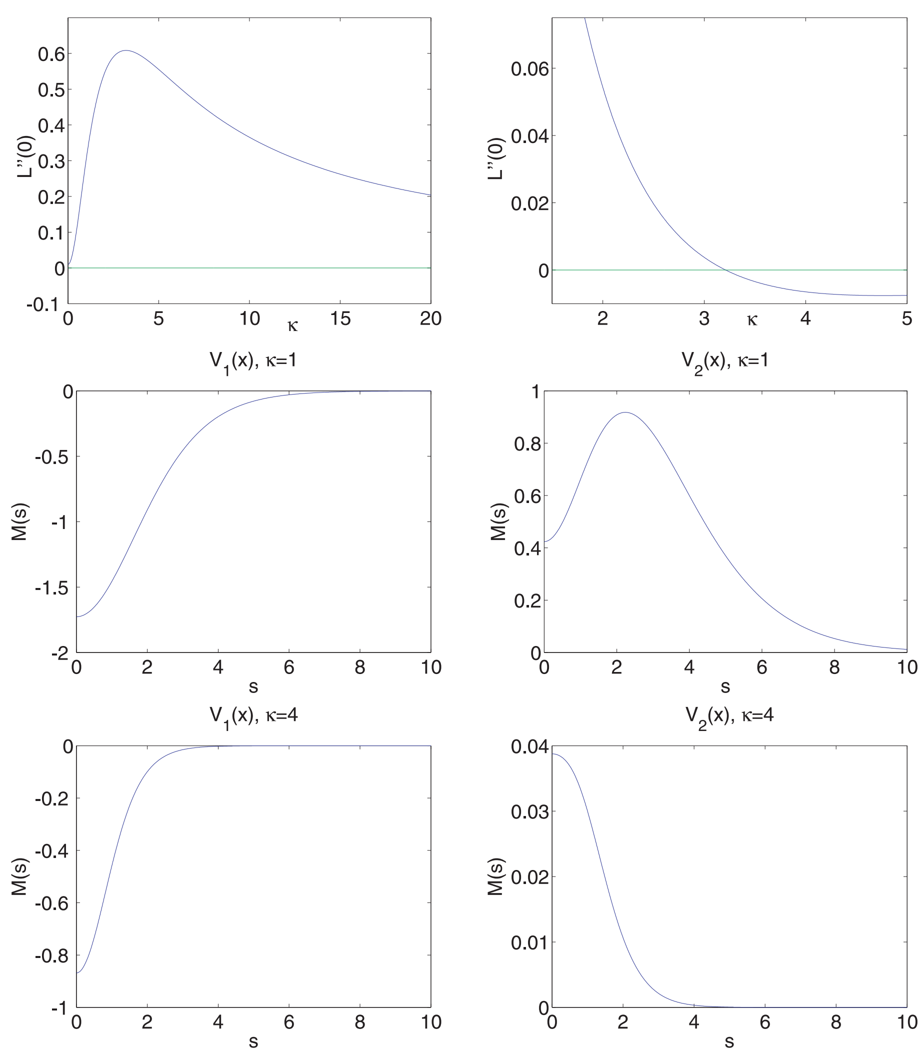

Figure 2. The effective potential evaluated numerically for $V_1(x)$ (left) and $V_2(x)$ (right): the quantity $L''(0)$ versus κ (top panels) and the function $M(s)$ for $\kappa = 1$ (middle panels) and $\kappa = 4$ (bottom panels).

3. Stability analysis of black solitons

We first consider the spectral stability of black solitons in the NLS equation (1.1). We will obtain conditions for spectral stability and instability of kinks and bubbles and then extend these conditions to kink modes of the GP equation (1.2). In the end of this section, we will apply these conditions to the kink modes related to the two potentials (1.4).

Definition 3.1. *Let $\phi_0(x)$ be a black soliton of Definition 2.5. A black soliton is said to be spectrally unstable in the time evolution of the NLS equation (1.1) if there exists an eigenvector $(u,w) \in L^2(\mathbb{R},\mathbb{C}^2)$ of the spectral problem*

$$L_+ u = -\lambda w, \qquad L_- w = \lambda u, \tag{3.1}$$

for an eigenvalue λ with $\mathrm{Re}(\lambda) > 0$, where

$$L_+ = -\frac{1}{2}\partial_x^2 + f(\phi_0^2) - f(q_0) + 2\phi_0^2 f'(\phi_0^2), \qquad L_- = -\frac{1}{2}\partial_x^2 + f(\phi_0^2) - f(q_0). \tag{3.2}$$

Otherwise, a black soliton is said to be spectrally stable.

Remark 3.2. The spectral problem (3.1) arises in the linearization of the NLS equation (1.1) by using the expansion

$$u(x,t) = e^{-if(q_0)t} \times \left[\phi_0(x) + e^{\lambda t}\left[u(x) + iw(x)\right] + e^{\bar{\lambda} t}\left[\bar{u}(x) + i\bar{w}(x)\right] + \mathrm{O}(\|u\|^2 + \|w\|^2)\right].$$

It will be clear from analysis of the system (3.1) that the spectral instability of black solitons is always associated with a real positive eigenvalue λ, while the spectral stability of black solitons (under a non-degeneracy constraint) corresponds to the case when a black soliton is a ground state of an equivalent variational principle. It is relatively straightforward to develop the nonlinear analysis for these two cases and to show that the spectral instability and stability of black solitons (under a non-degeneracy constraint) correspond to their orbital instability and stability. See [4, 28, 31] for nonlinear analysis.

Proposition 3.3. *Let $u(x)$ satisfy $u_x \in L^2(\mathbb{R})$, $|u|^2 - q_0 \in L^2(\mathbb{R})$ and $u(x) \neq 0$ $\forall x \in \mathbb{R}$. Let the renormalized energy $E_r[u]$ and momentum $P_r[u]$ of the NLS equation (1.1) be defined by*

$$E_r[u] = \frac{1}{2}\int_{\mathbb{R}} \left[|u_x|^2 + 2\int_{|u|^2}^{q_0} \left(f(q_0) - f(q)\right) dq \right] dx, \tag{3.3}$$

$$P_r[u] = \frac{i}{2}\int_{\mathbb{R}} (\bar{u}u_x - u\bar{u}_x)\left(1 - \frac{q_0}{|u|^2}\right) dx. \tag{3.4}$$

A family of dark solitons of Definition 2.1 is a critical point of the Lyapunov functional $\Lambda[u] = E_r[u] + vP_r[u]$.

Proof. By direct differentiation, if $u = U(x)$ satisfies the second-order ODE (2.4) with $\omega = f(q_0)$, then the variational derivative $E_r'[u]|_{u=U} + vP_r'[u]|_{u=U}$ is zero. □

Remark 3.4. The Lyapunov functional $\Lambda[u]$ can be expanded by an extra term

$$\Lambda[u] = E_r[u] + vP_r[u] + CS[u], \tag{3.5}$$

where C is arbitrary constant and $S[u]$ is the Casimir functional with zero variational derivative. It is given by

$$S[u] = \frac{i}{2}\int_{\mathbb{R}} \left(\frac{\bar{u}_x}{\bar{u}} - \frac{u_x}{u}\right) dx = [\arg(u)]\,|_{x\to-\infty}^{x\to\infty} \tag{3.6}$$

and it represents the total phase shift of $u(x)$ on $x \in \mathbb{R}$ subject to $\lim_{|x|\to\infty} |u(x)| = \sqrt{q_0} \neq 0$. In order to define the constant C uniquely, let us add a constraint on the variational problem by requiring that if $u = \phi e^{i\theta}$, $\theta' = v(1 - q_0/\phi^2)$, and $\phi = \Phi(x)$ satisfies the second-order ODE (2.8), then the first variation $\tilde{\Lambda}'[\phi]|_{\phi=\Phi}$ is zero, where $\tilde{\Lambda}[\phi] = \Lambda[u]|_{u=\phi e^{i\theta}}$, with $\theta' = v(1 - q_0/\phi^2)$. By direct differentiation, this constraint immediately results in $C = 0$.

Remark 3.5. The renormalized momentum (3.4) was constructed in [23] as a difference between the standard momentum $P[u]$ associated with a solution $u(x)$ and the value $P[u_0]$ evaluated at the background solution $u_0 = \sqrt{q_0}e^{i\,\mathrm{sign}(x)S_0/2}$, where the value $S_0 = S[u]$ is related to the total phase shift of the solution $u(x)$. It was shown in [1, 28] that the renormalized momentum $P_r[u]$ computed at the family of dark solitons $U(x)$ of Definition 2.1 defines the spectral stability and instability of dark solitons in the sense that the dark soliton is spectrally stable if $P_r'(v) \geq 0$ and unstable if $P_r'(v) < 0$ where $P_r(v) = P_r[U]$. (The degenerate case $P_r'(v) = 0$ corresponds to the dark solitons which are spectrally stable and orbitally unstable. Under the non-degeneracy constraint $P_r'(v) \neq 0$, the spectral stability and instability corresponds to the orbital stability and instability, see [28]. In what follows, we will consider dark solitons under the non-degeneracy constraint $P_r'(v) \neq 0\ \forall v \in (-c, c)$.) We will show that the limit $v \downarrow 0$ is well defined and the quantity $P_r'|_{v\downarrow 0}$ determines spectral stability and instability of kinks of Definition 2.5.

Lemma 3.6. *Let $U(x)$ be a family of dark solitons of Definition 2.1 and $f(q)$ be $C^2(\mathbb{R}_+)$. Then, (i) the function $P_r(v) = P_r[U]$ is C^1 on $v \in (-c, 0) \cup (0, c)$ and (ii) the limiting quantity $P_r'|_{v\downarrow 0}$ is well-defined.*

Proof. (i) By construction of dark solitons in Theorem 2.3, the function $P_r(v) = P_r[U]$ is represented by

$$\begin{aligned} P_r(v) &= \frac{i}{2}\int_{\mathbb{R}} \left(\bar{U}U' - U\bar{U}'\right)\left(1 - \frac{q_0}{|U|^2}\right) dx \\ &= -v\int_{\mathbb{R}} \Phi^2(x)\left(1 - \frac{q_0}{\Phi^2(x)}\right)^2 dx = vN(v) + q_0 S(v), \end{aligned} \tag{3.7}$$

where $N(v)$ and $S(v)$ is the total power and phase shift of the dark solitons:

$$N(v) = \int_{\mathbb{R}} \left(q_0 - \Phi^2(x)\right) dx, \qquad S(v) = \int_{\mathbb{R}} \Theta'(x) dx = \Theta_+ - \Theta_-. \tag{3.8}$$

By the ODE theory for the system (2.7)–(2.8) with $\Phi(x) > 0$ on $x \in \mathbb{R}$, the map $v \mapsto (\Phi, \Theta)$ is C^1 on $v \in (-c, 0) \cup (0, c)$, such that $N(v)$ and $S(v)$ are smooth functions and so is $P_r(v)$.

(ii) We will show that the functions $N(v)$, $S(v)$ and $P_r(v)$ remain smooth in the limit $v \downarrow 0$. Let $U(x)$ be a dark soliton for $v \in (0, c)$ according to Definition 2.1 and $\phi_0(x)$ be a black soliton according to Definition 2.5. Let us consider

$$\tilde{U}(x) = \frac{U(x) - \phi_0(x)}{v} = \tilde{U}_r(x) + i\tilde{U}_i(x), \qquad v \in (0, c),$$

where $\tilde{U}_r(x)$ and $\tilde{U}_i(x)$ are real-valued functions on $x \in \mathbb{R}$. By the construction of $U(x)$ and $\phi_0(x)$ in Theorems 2.3 and 2.6, it is clear that $\tilde{U}_r, \tilde{U}_i \in L^\infty(\mathbb{R})$ are continuous in v for $v \in (0, c)$. We need to prove that these functions remain continuous in v as $v \downarrow 0$. By separating the real and imaginary parts in the ODEs (2.4) and (2.11), we obtain an equivalent ODE system for $\tilde{U}_r(x)$ and $\tilde{U}_i(x)$:

$$v\tilde{U}_i' + \frac{1}{v}\phi_0 \left(f(\phi_0^2) - f(|U|^2)\right) + \frac{1}{2}\tilde{U}_r'' + \tilde{U}_r \left[f(q_0) - f(|U|^2)\right] = 0, \tag{3.9}$$

$$-\phi_0' - v\tilde{U}_r' + \frac{1}{2}\tilde{U}_i'' + \tilde{U}_i \left[f(q_0) - f(|U|^2)\right] = 0, \tag{3.10}$$

where $|U|^2 = (\phi_0 + v\tilde{U}_r)^2 + v^2\tilde{U}_i^2$. Let us rewrite the ODE (3.10) as an inhomogeneous problem:

$$(L_- + \tilde{L}_-)\tilde{U}_i = F_-, \tag{3.11}$$

where operator L_- is defined by (3.2) and

$$\tilde{L}_- = f(|U|^2) - f(\phi_0^2), \qquad F_- = -\phi_0' - v\tilde{U}_r'.$$

Since $|U(x)|^2$, $\phi_0^2(x)$ converge to q_0 and $\tilde{U}_r(x)$, $\tilde{U}_i(x)$, $\phi_0(x)$ converge to some constants exponentially fast as $|x| \to \infty$, it is clear that $\tilde{L}_-$ is a relatively compact perturbation to L_- and $F_- \in L^2(\mathbb{R})$ for $v \in (0, c)$. By continuity of the solution $\phi_0(x) = \lim_{v \downarrow 0} U(x)$ in Theorem 2.6, we know that $\|\tilde{L}_-\|_{L^\infty} = \mathrm{o}(1)$ and $v\|\tilde{U}_r'\|_{L^\infty} = \mathrm{o}(1)$ as $v \downarrow 0$.

Since $L_-\phi_0 = 0$ and $\phi_0 \in L^\infty(\mathbb{R})$, then $\forall f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ there exists $L_-^{-1} f \in L^\infty(\mathbb{R})$ if and only if $(\phi_0, f) = 0$. The following computation shows that this condition for $f = F_- - \tilde{L}_-\tilde{U}_i$ is equivalent to the ODE (3.10) and is thus satisfied on $v \in (0, c)$:

$$\begin{aligned} (\phi_0, f) &= -(\phi_0, \phi_0') - v(\phi_0, \tilde{U}_r') + \left(\phi_0 \left[f(\phi_0^2) - f(q_0)\right], \tilde{U}_i\right) \\ &\quad + \left(\phi_0, \tilde{U}_i \left[f(q_0) - f(|U|^2)\right]\right) \\ &= -(\phi_0, \phi_0') - v(\phi_0, \tilde{U}_r') + \frac{1}{2}(\phi_0'', \tilde{U}_i) + \left(\phi_0, \tilde{U}_i \left[f(q_0) - f(|U|^2)\right]\right) \end{aligned}$$

$$= \left(\phi_0, -\phi_0' - v\tilde{U}_r' + \frac{1}{2}\tilde{U}_i'' + \tilde{U}_i \left[f(q_0) - f(|U|^2) \right] \right) = 0.$$

Therefore, a solution of the inhomogeneous problem (3.11) on $v \in (0, c)$ can be written in the form

$$\tilde{U}_i = -\left(L_- + \tilde{L}_- \right)^{-1} \left(\phi_0' + v\tilde{U}_r' \right).$$

Since $\|\tilde{L}_-\|_{L^\infty} = \mathrm{o}(1)$ and $v\|\tilde{U}_r'\|_{L^\infty} = \mathrm{o}(1)$ as $v \downarrow 0$, there exists a solution $\tilde{U}_i \in L^\infty(\mathbb{R})$ uniformly in $v \in [0, c)$, such that $\tilde{U}_i|_{v\downarrow 0} = -L_-^{-1}\phi_0' \in L^\infty(\mathbb{R})$. Therefore, the function $\mathrm{Im}U(x)$ is smooth as $v \downarrow 0$ and $\mathrm{Im}\partial_v U(x)|_{v\downarrow 0} = \tilde{U}_i|_{v\downarrow 0} = -L_-^{-1}\phi_0'$.

We can now use the fact that $v\|\tilde{U}_i\|_{L^\infty} = \mathrm{O}(v)$ as $v \downarrow 0$. Since $f(q)$ is $C^2(\mathbb{R}_+)$ and $v\|\tilde{U}_r\|_{L^\infty} = \mathrm{o}(v)$ as $v \downarrow 0$, there exists a function $g(\phi_0, v\tilde{U}_r)$ for small $v\tilde{U}_r$ such that

$$f((\phi_0 + v\tilde{U}_r)^2) - f(\phi_0^2) - 2v\phi_0\tilde{U}_r f'(\phi_0^2) = v\tilde{U}_r g(\phi_0, v\tilde{U}_r),$$

where $\|g(\phi_0, v\tilde{U}_r)\|_{L^\infty} = \mathrm{o}(v)$ as $v \downarrow 0$. Using these facts, we rewrite the ODE (3.9) as an inhomogeneous problem:

$$(L_+ + \tilde{L}_+)\tilde{U}_r = F_+, \tag{3.12}$$

where operator L_+ is defined by (3.2) and

$$\tilde{L}_+ = f(|U|^2) - f(\phi_0^2) + \phi_0 g(\phi_0, v\tilde{U}_r), \quad F_+ = v\tilde{U}_i' + \frac{1}{v}\phi_0 \left[f((\phi_0 + v\tilde{U}_r)^2) - f(|U|^2) \right].$$

It is clear that $F_+ \in L^\infty(\mathbb{R})$ for $v \in (0, c)$. Since $L_+\phi_0' = 0$ and $\phi_0'(x) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, then $\forall f \in L^\infty(\mathbb{R})$ there exists $L_+^{-1} f \in L^\infty(\mathbb{R})$ if and only if $(\phi_0', f) = 0$. The following computation shows that this condition for $f = F_+ - \tilde{L}_+\tilde{U}_r$ is equivalent to the ODE (3.9) and is thus satisfied on $v \in (0, c)$:

$$v(\phi_0', \tilde{U}_i') + \frac{1}{v}\left(\phi_0'\phi_0, \left[f(\phi_0^2) - f(|U|^2) \right] \right) + \left(\phi_0' \left[f(\phi_0^2) + 2\phi_0^2 f'(\phi_0^2) - f(|U|^2) \right], \tilde{U}_r \right)$$

$$= v(\phi_0', \tilde{U}_i') + \frac{1}{v}\left(\phi_0'\phi_0, \left[f(\phi_0^2) - f(|U|^2) \right] \right) + \frac{1}{2}(\phi_0''', \tilde{U}_r) + \left(\phi_0', \tilde{U}_r \left[f(q_0) - f(|U|^2) \right] \right)$$

$$= \left(\phi_0', v\tilde{U}_i' + \frac{1}{v}\phi_0 \left[f(\phi_0^2) - f(|U|^2) \right] + \frac{1}{2}\tilde{U}_r'' + \tilde{U}_r \left[f(q_0) - f(|U|^2) \right] \right) = 0.$$

Therefore, a solution of the inhomogeneous problem (3.12) on $v \in (0, c)$ can be written in the form

$$\tilde{U}_r = -\left(L_+ + \tilde{L}_+ \right)^{-1} \left(v\tilde{U}_i' + \frac{1}{v}\phi_0 \left[f((\phi_0 + v\tilde{U}_r)^2) - f(|U|^2) \right] \right).$$

Since $\|\tilde{L}_+\|_{L^\infty} = \mathrm{o}(v)$ and $v\|\tilde{U}_i\|_{L^\infty} = \mathrm{O}(v)$ as $v \downarrow 0$, there exists a solution $\tilde{U}_r \in L^\infty(\mathbb{R})$ uniformly in $v \in [0, c)$, such that $\tilde{U}_r|_{v\downarrow 0} = 0$ (the homogeneous solution $\phi_0'(x)$ is removed from $\tilde{U}_r(x)$ due to the symmetry in $U(x)$). Therefore, the function $\mathrm{Re}U(x)$ is smooth as $v \downarrow 0$ and $\mathrm{Re}\partial_v U(x)|_{v\downarrow 0} = \tilde{U}_r|_{v\downarrow 0} = 0$. As a result, the map $v \mapsto U$ is C^1 on $v \in [0, c)$, such that $N(v)$ and $S(v)$ are smooth functions as $v \downarrow 0$ and so is $P_r(v)$. □

Corollary 3.7. *The following identities are true for* $v \in (-c,0) \cup (0,c)$

$$P_r'(v) = i \int_{\mathbb{R}} \left(U' \partial_v \bar{U} - \bar{U}' \partial_v U \right) dx = 2 \int_{\mathbb{R}} \left(\mathrm{Re} U' \, \mathrm{Im} \partial_v U - \mathrm{Im} U' \, \mathrm{Re} \partial_v U \right) dx \quad (3.13)$$

and, as $v \downarrow 0$,

$$P_r'|_{v \downarrow 0} = 2 \left(\phi_0', \mathrm{Im} \partial_v U|_{v \downarrow 0} \right) = N|_{v \downarrow 0} + q_0 S'|_{v \downarrow 0}. \quad (3.14)$$

Proof. By Lemma 3.6, the quantity $P_r'(v)$ is continuous on $v \in [0,c)$. The first identity in (3.13) follows by direct differentiation:

$$\begin{aligned} P_r'(v) &= \frac{i}{2} \int_{\mathbb{R}} \left(U' \partial_v \bar{U} + \bar{U} \partial_v U' - \bar{U}' \partial_v U - U \partial_v \bar{U}' \right) dx + \frac{i q_0}{2} \int_{\mathbb{R}} \partial_v \left(\frac{\bar{U}'}{\bar{U}} - \frac{U'}{U} \right) dx \\ &= i \int_{\mathbb{R}} \left(U' \partial_v \bar{U} - \bar{U}' \partial_v U \right) dx + \frac{i}{2} \left(\bar{U} \partial_v U - U \partial_v \bar{U} \right) |_{x \to -\infty}^{x \to \infty} + \frac{i q_0}{2} \partial_v \log \left(\frac{\bar{U}}{U} \right) \Big|_{x \to -\infty}^{x \to \infty} \\ &= i \int_{\mathbb{R}} \left(U' \partial_v \bar{U} - \bar{U}' \partial_v U \right) dx. \end{aligned}$$

Other identities follow by the substitution $U(x) = \mathrm{Re} U(x) + i \mathrm{Im} U(x)$, by the smoothness of $U(x)$ with respect to $v \in [0,c)$ and by the relation (3.7). □

Example 3.8. Following Example 2.14, we consider the cubic NLS with $f(s) = s$ and $q_0 = 1$. By using the exact solution (2.18), we find for $v \in [0,1)$

$$N(v) = 2\sqrt{1 - v^2}, \quad S(v) = -2 \mathrm{arctan} \frac{\sqrt{1 - v^2}}{v},$$

such that

$$P_r'(v) = 4\sqrt{1 - v^2}, \qquad S'(v) = \frac{2}{\sqrt{1 - v^2}},$$

and $P_r'|_{v \downarrow 0} = 4$, $S'|_{v \downarrow 0} = 2$.

Lemma 3.9. *(i) Let* $\phi_0(x)$ *be a kink of Definition 2.5. Then, the spectrum of* L_+ *in* $L^2(\mathbb{R})$ *consists of the positive continuous spectrum bounded away from zero by* $2c^2$*, the kernel with the eigenfunction* $\phi_0'(x)$ *and, possibly, a finite number of positive eigenvalues in* $(0, 2c^2)$*. The spectrum of* L_- *in* $L^2(\mathbb{R})$ *consists of the non-negative continuous spectrum and a single negative eigenvalue.*

(ii) Let $\phi_0(x)$ *be a bubble of Definition 2.5. Then, the spectrum of* L_+ *in* $L^2(\mathbb{R})$ *consists of the positive continuous spectrum bounded away from zero by* $2c^2$*, the kernel with the eigenfunction* $\phi_0'(x)$*, a single negative eigenvalue and, possibly, a finite number of positive eigenvalues in* $(0, 2c^2)$*. The spectrum of* L_- *in* $L^2(\mathbb{R})$ *consists of the non-negative continuous spectrum.*

Proof. Since $\phi_0^2(x)$ converges to q_0 exponentially fast as $|x| \to \infty$, operators $L_\pm$ in (3.2) are self-adjoint Schrödinger operators on the domain $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$, which have absolutely continuous spectrum $\sigma_c(L_\pm)$, a finite number of isolated eigenvalues of finite multiplicities $\sigma_p(L_\pm)$, and no embedded eigenvalues or residual spectrum [14]. By the Weyl's Essential Spectrum Lemma, $\sigma_c(L_+) \geq 2q_0 f'(q_0) = 2c^2 > 0$ and $\sigma_c(L_-) \geq 0$, such that the continuous spectrum of L_+ is bounded away from zero and the continuous spectrum of L_- touches zero. Moreover, $L_+\phi_0'(x) = 0$ and $L_-\phi_0(x) = 0$ due to the translational and gauge symmetries of the NLS equation (1.1), such that L_+ has a simple kernel in $L^2(\mathbb{R})$ while L_- has no kernel in $L^2(\mathbb{R})$.

(i) In the case of kinks, $\phi_0(x)$ has a single zero on $x \in \mathbb{R}$. By the Sturm Nodal Theorem, $\sigma_p(L_+)$ contains no negative eigenvalues and $\sigma_p(L_-)$ contains exactly one negative eigenvalue.

(ii) In the case of bubbles, $\phi_0(x)$ has no zeros on $x \in \mathbb{R}$. By the Sturm Nodal Theorem, $\sigma_p(L_+)$ contains exactly one negative eigenvalue and $\sigma_p(L_-)$ contains no negative eigenvalues. □

Lemma 3.10. *Consider the constrained space*

$$X_c = \left\{ w \in L^2(\mathbb{R}) : \quad (w, \phi_0') = 0 \right\}, \tag{3.15}$$

where $\phi_0(x)$ is a black soliton of Definition 2.5. In the case of kinks, the operator L_- has exactly one negative eigenvalue in X_c if $P_r'|_{v\downarrow 0} < 0$ and no negative eigenvalues if $P_r'|_{v\downarrow 0} > 0$, where $P_r'|_{v\downarrow 0}$ is defined by Lemma 3.6. In the case of bubbles, the operator L_- is non-negative in X_c.

*Proof.*We consider a constrained variational problem:

$$(L_- - \mu)w = -\nu\phi_0', \qquad w \in H^2(\mathbb{R}), \ \ \mu \notin \sigma(L_-), \tag{3.16}$$

where μ is the spectral parameter and ν is the Lagrange multiplier. By Lemma 3.9, there exists a unique solution $w \in H^2(\mathbb{R})$ for any $\mu \in (\mu_0, 0)$, where μ_0 is the only negative eigenvalue of L_- in the case of kinks and $\mu_0 = -\infty$ in the case of bubbles. By the standard variational theory (e.g. see [1, 31]), the smooth function $g(\mu) = -(\phi_0', (L_- - \mu)^{-1}\phi_0')$ is decreasing on $\mu \in (\mu_0, 0)$ from $\lim_{\mu \downarrow \mu_0} g(\mu) = +\infty$ in the case of kinks or $\lim_{\mu \to -\infty} g(\mu) = 0$ in the case of bubbles. Therefore, operator L_- is non-negative in X_c in the case of bubbles. It is non-negative in X_c if $\lim_{\mu \uparrow 0} g(\mu) > 0$ and has a single eigenvalue in X_c on $\mu \in (\mu_0, 0)$ if $\lim_{\mu \uparrow 0} g(\mu) < 0$ in the case of kinks. We need to show that $\lim_{\mu \uparrow 0} g(\mu) = P_r'|_{v\downarrow 0}$.

Since $H^2(\mathbb{R}) \subset L^\infty(\mathbb{R})$ and there exists a solution of the inhomogeneous problem $L_- w_0 = -\phi_0'$ in $w_0 \in L^\infty(\mathbb{R})$, then a solution of the variational problem (3.16) in $L^\infty(\mathbb{R})$ is uniform in $\mu \in (\mu_0, 0]$. Moreover, it follows from smoothness by Lemma 3.6 that

$$w_0 = \tilde{U}_i|_{v\downarrow 0} = \mathrm{Im}\partial_v U(x)|_{v\downarrow 0} + c\phi_0,$$

where c is arbitrary. By the relation (3.14), we obtain $\lim_{\mu \uparrow 0} g(\mu) = (\phi_0', \mathrm{Im}\partial_v U|_{v\downarrow 0}) = \frac{1}{2} P_r'|_{v\downarrow 0}$. □

Theorem 3.11. *(i) Let $\phi_0(x)$ be a kink of Definition 2.5. Then, it is spectrally stable if $P_r'|_{v\downarrow 0} > 0$ and unstable if $P_r'|_{v\downarrow 0} < 0$ with exactly one real positive eigenvalue λ in the spectral problem (3.1).*

(ii) Let $\phi_0(x)$ be a bubble of Definition 2.5. Then, it is spectrally unstable with exactly one real positive eigenvalue λ in the spectral problem (3.1).

Proof. Let λ be a non-zero eigenvalue of the spectral problem (3.1) corresponding to an eigenvector $(u, w) \in H^2(\mathbb{R}, \mathbb{C}^2)$. Then, L_+ is invertible in X_c defined by (3.15) and the component $w(x) \in X_c$ can be found from the generalized eigenvalue problem

$$L_- w = \gamma L_+^{-1} w, \qquad \gamma = -\lambda^2, \qquad w \in X_c. \tag{3.17}$$

Due to the equivalence above, all *non-zero* eigenvalues λ of the spectral problem (3.1) can be recovered from the *non-zero* eigenvalues γ of the generalized eigenvalue problem (3.17). The operators $L_\pm$ satisfy properties P1–P2 of the recent paper [9]. Even though L_- has no spectral gap near the origin, one can shift the generalized eigenvalue problem to the equivalent form,

$$(L_- + \delta L_+^{-1}) w = (\gamma + \delta) L_+^{-1} w, \qquad 0 < \delta < \delta_0, \tag{3.18}$$

where δ_0 is the distance from $\gamma = 0$ to the first negative eigenvalue γ if it exists or $\delta_0 = \infty$ if not. Now

$$\sigma_c(L_- + \delta L_+^{-1}) \geq \frac{\delta}{2c^2} > 0,$$

and the operator $\tilde{L}_- = L_- + \delta L_+^{-1}$ has the spectral gap near the origin.

(i) In the case of kinks, the operator L_+^{-1} is positive in X_c. By Theorem 3 of [9], the problem (3.18) has no eigenvalues $\gamma \in \mathbb{C}$ with $\mathrm{Im}(\gamma) \neq 0$, has no eigenvalues $\gamma \in \mathbb{R}_+$ such that $(w, L_+^{-1} w) \leq 0$ and has exactly $N = \dim(H^-_{L_- + \delta L_+^{-1}})$ eigenvalues $\gamma \in \mathbb{R}_-$, where $H^-_{L_- + \delta L_+^{-1}} \subset X_c$ is the invariant negative subspace of X_c with respect to $L_- + \delta L_+^{-1}$. We will prove that $N = \dim(H^-_{L_- + \delta L_+^{-1}}) = \dim(H^-_{L_-})$ for sufficiently small $\delta > 0$. Indeed, continuity of isolated negative eigenvalues of L_- in δ follows by the perturbation theory since δL_+^{-1} is a relatively compact perturbation to L_- in X_c. Therefore, $\dim(H^-_{L_- + \delta L_+^{-1}}) \geq \dim(H^-_{L_-})$ for sufficiently small $\delta > 0$. Consider a splitting $X_c = H^-_{L_-} \oplus H^+_{L_-}$, where $H^-_{L_-}$ ($H^+_{L_-}$) is negative (non-negative) invariant subspace of X_c with respect to L_-, such that $\dim(H^-_{L_-}) < \infty$ and $\dim(H^+_{L_-}) = \infty$. Since L_+^{-1} is a strictly positive operator in X_c, we have

$$\forall \delta > 0, \ \ \forall w \in H^+_{L_-} : \quad (w, (L_- + \delta L_+^{-1}) w) \geq \delta (w, L_+^{-1} w) > 0.$$

Therefore, the operator $(L_- + \delta L_+^{-1})$ is strictly positive on $H^+_{L_-}$ and $\dim(H^-_{L_- + \delta L_+^{-1}}) \leq \dim(H^-_{L_-})$. We have thus proved that $\dim(H^-_{L_- + \delta L_+^{-1}}) = \dim(H^-_{L_-})$. By Lemma 3.10, $\dim(H^-_{L_-}) = 1$ if $P_r'|_{v\downarrow 0} < 0$ and $\dim(H^-_{L_-}) = 0$ if $P_r'|_{v\downarrow 0} > 0$. In the former case, $N = 1$ and a kink is spectrally unstable. In the latter case, $N = 0$ and a kink is spectrally stable.

(ii) In the case of bubbles, the operator L_- has no negative eigenvalues in X_c by Lemma 3.10. Therefore, Theorem 3 of [9] guarantees that the generalized eigenvalue problem (3.18) has no eigenvalues $\gamma \in \mathbb{C}$ with $\mathrm{Im}(\gamma) \neq 0$, has no eigenvalues $\gamma \in \mathbb{R}_+$ such that $(w, L_+^{-1} w) \leq 0$ and has exactly $N = \dim(H^-_{L_+^{-1}})$ eigenvalues $\gamma \in \mathbb{R}_-$, where $H^-_{L_+^{-1}} \subset X_c$ is the invariant negative subspace of X_c with respect to L_+^{-1}. Since all eigenvectors of L_+ for non-zero eigenvalues are orthogonal to ϕ_0' and belong to X_c, it follows immediately that $N = \dim(H^-_{L_+^{-1}}) = 1$, and a bubble is spectrally unstable. □

Remark 3.12. The statement (i) of Theorem 3.11 extends the stability–instability theorem in [28] from dark solitons with $v \neq 0$ to kinks with $v = 0$. The same result with $\lim_{\mu \uparrow 0} g(\mu)$ instead of $P_r'|_{v\downarrow 0}$ was obtained in [31] by the Vakhitov–Kolokolov method (similar to [1]) and the variational principle (similar to [4]). However, the relation $\lim_{\mu\uparrow 0} g(\mu) = \frac{1}{2} P_r'|_{v\downarrow 0}$ was not proved in [31].

The statement (ii) of Theorem 3.11 was proved differently in [4] by using a variational technique. We note that $P_r(v) < 0$ for $v \in (0, c)$ and $\lim_{v\downarrow 0} P_r(v) = 0$ in the case of bubbles of Definition 2.5. Therefore, if $P_r'|_{v\downarrow 0} \neq 0$, then $P_r'|_{v\downarrow 0} < 0$, such that the statement (i) for kinks extends formally to the statement (ii) for bubbles.

Definition 3.13. *Let $\phi_\epsilon(x)$ be a kink mode of Definition 2.8. A kink mode is said to be spectrally unstable in the time evolution of the GP equation (1.2) if there exists an eigenvector $(u, w) \in L^2(\mathbb{R}, \mathbb{C}^2)$ of the spectral problem*

$$\mathcal{L}_+ u = -\lambda w, \qquad \mathcal{L}_- w = \lambda u, \tag{3.19}$$

for an eigenvalue λ with $\mathrm{Re}(\lambda) > 0$, where

$$\mathcal{L}_+ = -\frac{1}{2}\partial_x^2 + f(\phi_\epsilon^2) - f(q_0) + 2\phi_\epsilon^2 f'(\phi_\epsilon^2) + \epsilon V(x),\ \mathcal{L}_- = -\frac{1}{2}\partial_x^2 + f(\phi_\epsilon^2) - f(q_0) + \epsilon V(x). \tag{3.20}$$

Otherwise, a kink mode is said to be spectrally stable.

Theorem 3.14. *Let $\phi_\epsilon(x)$ be a kink mode of Definition 2.8. Assume that the operators $\mathcal{L}_\pm$ have $n_\pm$ negative eigenvalues and empty kernels in $L^2(\mathbb{R})$. Assume that all embedded (purely imaginary) eigenvalues of the spectral problem (3.19) are algebraically simple. Then, the spectral problem (3.19) has exactly N_c complex eigenvalues λ in the first quadrant, N_i^- purely imaginary eigenvalues λ with $\mathrm{Im}(\lambda) > 0$ and $(w, \mathcal{L}_+^{-1} w) \leq 0$, and $N_r = N_r^+ + N_r^-$ real positive eigenvalues λ, where N_r^+ corresponds to eigenvalues with $(w, \mathcal{L}_+^{-1} w) \geq 0$ and N_r^- corresponds to eigenvalues with $(w, \mathcal{L}_+^{-1} w) \leq 0$, such that*

$$N_r^- + N_i^- + N_c = n_+, \qquad N_r^+ + N_i^- + N_c = n_-, \tag{3.21}$$

where multiple eigenvalues are accounted up to their algebraic multiplicities.

Proof. Since $V(x) \to 0$ and $\phi_\epsilon^2(x) \to q_0$ exponentially fast as $|x| \to \infty$, operators $\mathcal{L}_\pm$ have the absolutely continuous spectrum such that $\sigma_c(\mathcal{L}_+) \geq 2c^2 > 0$ and $\sigma_c(\mathcal{L}_-) \geq 0$. Since the kernel of $\mathcal{L}_+$ is empty in $L^2(\mathbb{R})$ by assumption, the generalized eigenvalue problem (3.17) for operators $\mathcal{L}_\pm$ is rewritten in the unconstrained space:

$$\mathcal{L}_- w = \gamma \mathcal{L}_+^{-1} w, \qquad \gamma = -\lambda^2, \qquad w \in L^2(\mathbb{R}). \tag{3.22}$$

The kernel of $\mathcal{L}_-$ is empty in $L^2(\mathbb{R})$ since $\mathcal{L}_- \phi_\epsilon = 0$ and $\phi_\epsilon \in L^\infty(\mathbb{R})$, $\phi_\epsilon \notin L^2(\mathbb{R})$. The generalized eigenvalue problem (3.22) can be rewritten in the equivalent form,

$$(\mathcal{L}_- + \delta \mathcal{L}_+^{-1}) w = (\gamma + \delta) \mathcal{L}_+^{-1} w, \tag{3.23}$$

where $\delta > 0$ is sufficiently small. Properties P1–P2 of [9] are satisfied and Theorem 3 of [9] gives the relations

$$N_r^- + N_i^- + N_c = \dim(H^-_{\mathcal{L}_+^{-1}}), \qquad N_r^+ + N_i^- + N_c = \dim(H^-_{\mathcal{L}_- + \delta \mathcal{L}_+^{-1}})$$

for sufficiently small $\delta > 0$, where $H^-_{\mathcal{L}_+^{-1}}$ and $H^-_{\mathcal{L}_- + \delta\mathcal{L}_+^{-1}}$ are invariant negative subspaces of $L^2(\mathbb{R})$ with respect to $\mathcal{L}_+^{-1}$ and $\mathcal{L}_- + \delta \mathcal{L}_+^{-1}$ respectively. It follows immediately that $\dim(H^-_{\mathcal{L}_+^{-1}}) = n_+$. By continuity of eigenvalues and the relative compactness of $\mathcal{L}_+^{-1}$ with respect to $\mathcal{L}_-$, it follows that $\dim(H^-_{\mathcal{L}_-}) \leq \dim(H^-_{\mathcal{L}_- + \delta\mathcal{L}_+^{-1}})$. We shall prove that $\dim(H^-_{\mathcal{L}_- + \delta\mathcal{L}_+^{-1}}) = \dim(H^-_{\mathcal{L}_-})$. The operator $\mathcal{L}_- + \delta \mathcal{L}_+^{-1}$ may have additional negative eigenvalues compared to operator $\mathcal{L}_-$ if and only if some eigenvalues bifurcate as $\delta \neq 0$ from the end point of the continuous spectrum of $\mathcal{L}_-$ by means of the edge bifurcation [11, 20, 43]. In order to analyze the edge bifurcation, we rewrite the eigenvalue problem $(\mathcal{L}_- + \delta \mathcal{L}_+^{-1}) w = \mu w$ in the equivalent form:

$$(\mathcal{L} + \delta \mathcal{M})\, w = \mu w, \qquad w \in L^2(\mathbb{R}),$$

where

$$\mathcal{L} = \mathcal{L}_- + \delta \left(2c^2 - \frac{1}{2}\partial_x^2 \right)^{-1},$$

$$\mathcal{M} = \mathcal{L}_+^{-1} \left[f(\phi_\epsilon^2) - f(q_0) + 2\phi_\epsilon^2 f'(\phi_\epsilon^2) + \epsilon V(x) \right] \left(2c^2 - \frac{1}{2}\partial_x^2 \right)^{-1}$$

where $\mathcal{M}$ is a relatively compact perturbation to the unbounded operator $\mathcal{L}$. The continuous spectrum of $\mathcal{L}$ is bounded from below by $\sigma_c(\mathcal{L}) \geq \frac{\delta}{2c^2}$. By the theory of edge bifurcations (see review in [20]), the new eigenvalue $\mu = \mu_\delta$, if it bifurcates from the end point of $\sigma_c(\mathcal{L})$, has the expansion $\mu_\delta = \frac{\delta}{2c^2} - \alpha\delta^2 + \mathrm{O}(\delta^3)$, where α is positive constant. Therefore, there exists sufficiently small $\delta > 0$, such that $\mu_\delta > 0$. As a result, the edge bifurcation does not change the number of negative eigenvalues of $\mathcal{L}_- + \delta \mathcal{L}_+^{-1}$ compared to $\mathcal{L}_-$ and $\dim(H^-_{\mathcal{L}_- + \delta\mathcal{L}_+^{-1}}) = \dim(H^-_{\mathcal{L}_-}) = n_-$. □

Theorem 3.15. *Let $\phi_0(x)$ be a kink of Definition 2.5 and $M'(s)$ be defined by (2.14), such that $M'(s_0) = 0$ and $M''(s_0) \neq 0$. Then, the operators $\mathcal{L}_\pm$ have $n_\pm$ negative eigenvalues and empty kernels in $L^2(\mathbb{R})$ for sufficiently small ϵ with $n_+ = 1$, $n_- = 1$ for $M''(s_0) > 0$ and $n_+ = 0$, $n_- = 1$ for $M''(s_0) < 0$.*

Proof. It follows from the proof of Theorem 2.12 that $\phi_\epsilon = \phi_0(x-s) + \epsilon\varphi_1(x) + \tilde{\varphi}(x,\epsilon,s)$ and $s = s_0 + \tilde{s}(\epsilon)$, where $\|\tilde{\varphi}\|_{L^\infty} = \mathrm{O}(\epsilon^2)$ and $|\tilde{s}| = \mathrm{O}(\epsilon)$ as $\epsilon \to 0$. Therefore, operators $\mathcal{L}_\pm$ in (3.20) are represented by

$$\mathcal{L}_\pm = L_\pm + \epsilon M_\pm + \tilde{M}_\pm,$$

where $L_\pm$ are given by (3.2), $M_\pm$ are given by

$$M_+ = V(x) + 6\phi_0\varphi_1 f(\phi_0^2) + 4\phi_0^3\varphi_1 f''(\phi_0^2), \qquad M_- = V(x) + 2\phi_0\varphi_1 f(\phi_0^2),$$

and $\|\tilde{M}_\pm\|_{L^\infty} = \mathrm{O}(\epsilon^2)$ as $\epsilon \to 0$. We note that $\epsilon M_+ + \tilde{M}_+ = \epsilon V(x) + D_\varphi N(\varphi, s, \epsilon)$, where $D_\varphi N$ is the Jacobian of the nonlinear function in (2.15). By using the inhomogeneous equation (2.17) for the correction term $\varphi_1(x)$, we compute

$$(\phi_0'(x-s_0), M_+\phi_0'(x-s_0)) = -(\phi_0', V'\phi_0) - (\phi_0', L_+\varphi_1') = -\frac{1}{2}M''(s_0). \quad (3.24)$$

By the regular perturbation theory, the zero eigenvalue of L_+ becomes a non-zero eigenvalue λ_ϵ of $\mathcal{L}_+$ for small ϵ, such that

$$\lambda_\epsilon = \epsilon\lambda_1 + \tilde{\lambda}, \qquad \lambda_1 = -\frac{M''(s_0)}{2\|\phi_0'\|_{L^2}^2},$$

where $\tilde{\lambda} = \mathrm{O}(\epsilon^2)$ as $\epsilon \to 0$. Since the zero eigenvalue of L_+ is simple and positive eigenvalues of L_+ are bounded away from zero, the kernel of $\mathcal{L}_+$ is empty, such that $n_+ = 1$ for $M''(s_0) > 0$ and $n_+ = 0$ for $M''(s_0) < 0$ for sufficiently small ϵ. By the Implicit Function Theorem applied to $\phi_\epsilon(x)$, the function $\phi_\epsilon(x)$ has only one node on $x \in \mathbb{R}$ for sufficiently small ϵ if $\phi_0(x)$ has only one simple zero at $x = 0$. We recall that $\mathcal{L}_-\phi_\epsilon = 0$ and $\phi_\epsilon \in L^\infty(\mathbb{R})$, $\phi_\epsilon \notin L^2(\mathbb{R})$. By the Sturm Nodal Theorem, the kernel of $\mathcal{L}_-$ is empty and $n_- = 1$ for sufficiently small ϵ. □

Corollary 3.16. *A kink mode with $M''(s_0) < 0$ is spectrally unstable with exactly one real positive eigenvalue λ in the spectral problem (3.19) for sufficiently small ϵ. A kink mode with $M''(s_0) > 0$ may have up to two unstable eigenvalues λ in the spectral problem (3.19).*

Proof. If $M''(s_0) < 0$, then $n_+ = 0$, $n_- = 1$ and the count of eigenvalues (3.21) gives $N_r^+ = N_i^- = N_c = 0$ and $N_r^- = 1$. If $M''(s_0) > 0$, then $n_+ = n_- = 1$ and the count of eigenvalues may give either $N_i^- + N_c = 1$, $N_r^+ = N_r^- = 0$ or $N_i^- = N_c = 0$, $N_r^+ = N_r^- = 1$. In the cases $N_c = 1$ or $N_r^+ = N_r^- = 1$, there are two unstable and no embedded eigenvalues in the spectral problem (3.19). In the case $N_i^- = 1$, the pair of embedded eigenvalues is simple, such that the last assumption of Theorem 3.14 is satisfied. □

Remark 3.17. Asymptotic approximations of eigenvalues λ and precise statements on unstable eigenvalues in the case $M''(s_0) > 0$ are obtained in Section 4 under non-degeneracy assumptions $P_r'|_{v\downarrow 0} \neq 0$ and $S'|_{v\downarrow 0} \neq 0$. By Corollary 4.12, the case $N_i^- = N_c = 0$, $N_r^+ = N_r^- = 1$ occurs for $P_r'|_{v\downarrow 0} < 0$ and the case $N_r^+ = N_r^- = N_i^- = 0$, $N_c = 1$ occurs for $P_r'|_{v\downarrow 0} > 0$.

Example 3.18. Continuing Examples 2.14 and 3.8, we consider the cubic NLS equation with $f(s) = s$, $q_0 = 1$ and $P_r'|_{v\downarrow 0} = 4 > 0$. When the potential $V(x)$ is even with $V(-x) = V(x)$, one family of kink modes with $s_0 = 0$ always bifurcates for $\epsilon \neq 0$.

When $V = V_1(x)$, it follows from Fig. 2 (left panel) that the kink mode with $s_0 = 0$ corresponds to the minimum of $M(s)$ and it is unstable with two complex conjugate eigenvalues, according to Remark 3.17.

When $V = V_2(x)$, it follows from Fig. 2 (right panel) that there exists $0 < \kappa_0 < \infty$ such that a pair of kink modes bifurcates from $s_0 = \pm s_*$ for $0 < \kappa < \kappa_0$. These modes correspond to the maxima of the effective potential $M(s)$ and they are unstable with one real eigenvalue, according to Corollary 3.16. In this case, the kink mode with $s_0 = 0$ corresponds to a minimum of the effective potential $M(s)$ and it is unstable with two complex eigenvalues. On the other hand, there is only one extremum (maximum) of $M(s)$ for $\kappa > \kappa_0$, such that only one kink mode with $s_0 = 0$ exists in this case and is unstable with a simple real positive eigenvalue. This scenario indicates the subcritical pitchfork bifurcation at $\kappa = \kappa_0$. We will illustrate this bifurcation in Section 5.

4. Eigenfunctions and eigenvalues of kinks

We develop asymptotic analysis of the spectral problem (3.19) in the limit of small λ and ϵ. This asymptotic analysis is needed to complete the stability analysis of kink modes with $M''(s_0) > 0$ which is not conclusive in Corollary 3.16. We will show that if a kink is stable in the linear problem (3.1) for $\epsilon = 0$, then the pair of zero eigenvalues of the spectral problem (3.19) at $\epsilon = 0$ splits into a pair of purely imaginary eigenvalues at $\mathrm{O}(\epsilon)$ and bifurcates into a quartet of four complex eigenvalues (two of which are unstable) at $\mathrm{O}(\epsilon^{3/2})$. These eigenvalues λ for small ϵ correspond to the eigenvectors $(u, w) \in L^2(\mathbb{R}, \mathbb{C}^2)$ in the spectral problem (3.19), persistence of which in ϵ follows by Theorem 3.14.

From a technical point of view, our analysis is complicated by the fact that the eigenvalues $\lambda \in i\mathbb{R}$ are embedded into the continuous spectrum of the non-self-adjoint problem (3.19). One way to deal with this problem is to introduce exponential weights which move branches of the continuous spectrum from the imaginary axis (see [38] and references therein). However, there are two branches of the continuous spectrum, and, independently of the weight parameter, one branch moves to the left and the other branch moves to the right of the imaginary axis. Any eigenvalues that bifurcate off the imaginary axis may become resonant poles when the weight parameter is sent to zero unless specific information about the decay rate of eigenfunctions is available. However, if this information were available, one could avoid the technique of exponential weights and perform a direct analysis of the eigenfunctions and eigenvalues of the problem (3.19).

Another way to deal with this problem is to consider the Evans function with analysis of fast and slow decaying solutions (see [19] and reference therein). By using the Gap Lemma, the Evans function can be appropriately extended across the continuous spectrum with a full account of the branch points on the imaginary axis. Information about small eigenvalues is drawn from the derivatives of the Evans function with respect to λ and ϵ near $\lambda = 0$ and $\epsilon = 0$. However, computational formulas become more and more involved when higher-order derivatives of the Evans function are needed.

Our treatment of the problem brings together the analysis of fast and slow decaying solutions in the two approaches above. To avoid complications, it is based on direct analysis of eigenfunctions and eigenvalues of the spectral problem (3.19) expanded in powers of $\epsilon^{1/2}$. We will obtain a characteristic equation for small eigenvalue λ versus small parameter ϵ.

Lemma 4.1. *Let $\phi_0(x)$ be a kink of Definition 2.5 and $U(x)$ be a dark soliton of Definition 2.1 for $v > 0$. Let operators $L_\pm$ be defined by (3.2). The uncoupled homogeneous problems*

$$L_+ u_0 = 0, \qquad L_- w_0 = 0$$

admit four linearly independent solutions:

(i) *exponentially decaying eigenfunction $u_0 = \phi_0'(x)$ in $L^2(\mathbb{R})$*
(ii) *bounded eigenfunction $w_0 = \phi_0(x)$ in $L^\infty(\mathbb{R})$*
(iii) *unbounded linearly growing solution $w_0 = x\phi_0(x) - \mathrm{Im}\partial_v U(x)|_{v\downarrow 0}$*
(iv) *and an unbounded exponentially growing solution u_0*

The uncoupled inhomogeneous problems

$$L_+ u_1 = -w_0, \qquad L_- w_1 = u_0$$

admit solutions in the same order:

(i) *bounded eigenfunction $w_1 = -\mathrm{Im}\partial_v U(x)|_{v\downarrow 0}$ in $L^\infty(\mathbb{R})$*
(ii) *a bounded eigenfunction u_1 in $L^\infty(\mathbb{R})$*
(iii) *an unbounded exponentially growing solution u_1 if $S'|_{v\downarrow 0} \neq 0$*
(iv) *and an unbounded exponentially growing solution w_1*

The uncoupled inhomogeneous problems

$$L_+ u_2 = -w_1, \qquad L_- w_2 = u_1$$

admit no solutions in $L^\infty(\mathbb{R})$ if $P_r'|_{v\downarrow 0} \neq 0$.

Proof. It follows from the proofs of Lemmas 3.9 and 3.10 that

$$L_+\phi_0' = 0, \quad L_-\phi_0 = 0, \quad L_- x\phi_0 = -\phi_0', \quad L_-\mathrm{Im}\partial_v U(x)|_{v\downarrow 0} = -\phi_0', \qquad (4.1)$$

which proves (i)–(iii) for u_0 and w_0 and (i) for w_1. Existence of an exponentially growing solution u_0 in (iv) follows from the fact that the Wronskian determinant of two linearly independent solutions of $L_+ u_0 = 0$ is constant in x. Existence of bounded solution u_1 in (ii) follows from the fact that $(\phi_0', \phi_0) = 0$, such that $L_+^{-1} \phi_0 \in L^\infty(\mathbb{R})$. The solution u_1 in (iii) grows exponentially since the Fredholm Alternative is not satisfied for w_0 in (iii):

$$(\phi_0', x\phi_0(x) - \mathrm{Im}\partial_v U(x)|_{v\downarrow 0}) = \frac{1}{2} N|_{v\downarrow 0} - \frac{1}{2} P_r'|_{v\downarrow 0} = -\frac{q_0}{2} S'|_{v\downarrow 0} \neq 0, \tag{4.2}$$

where the relation (3.14) has been used. Existence of an exponentially growing solution w_1 in (iv) follows from the fact that $L_-^{-1} u_0$ has the same exponential growth in x as u_0 in (iv). The solution u_2 of $L_+ u_2 = \mathrm{Im}\partial_v U(x)|_{v\downarrow 0}$ grows exponentially since the Fredholm Alternative is not satisfied:

$$(\phi_0', \mathrm{Im}\partial_v U(x)|_{v\downarrow 0}) = \frac{1}{2} P_r'|_{v\downarrow 0} \neq 0$$

The solution w_2 of $L_- w_2 = u_1$ in (ii) grows linearly due to the same reason since

$$(\phi_0, u_1) = -(L_+ u_1, u_1) \neq 0.$$

The last inequality is due to the non-negativity of L_+ for kinks and the orthogonality of the odd function u_1 to the even function ϕ_0' of the kernel of L_+. □

Definition 4.2. *Let λ be fixed in the strip $\{\lambda \in \mathbb{C} : 0 < \mathrm{Re}\lambda < c^2\}$ and define $\kappa_\pm(\lambda)$ from the roots of the characteristic equations*

$$\mathrm{Re}\kappa_\pm > 0 : \quad \kappa_\pm^2 = 2c^2 \left(1 \pm \sqrt{1 - \frac{\lambda^2}{c^4}}\right), \tag{4.3}$$

such that $\kappa_+ \kappa_- = 2\lambda$ and $\kappa_+ = \sqrt{4c^2 - \kappa_-^2}$.

Remark 4.3. The roots $\kappa_\pm$ can be expanded in the Taylor series near $\lambda = 0$, such that

$$\kappa_+(\lambda) = 2c \left(1 - \frac{\lambda^2}{8c^4} + \mathrm{O}(\lambda^4)\right), \qquad \kappa_-(\lambda) = \frac{\lambda}{c} \left(1 + \frac{\lambda^2}{8c^4} + \mathrm{O}(\lambda^4)\right). \tag{4.4}$$

Lemma 4.4. *Let $\phi_\epsilon(x)$ be a kink mode of Definition 2.8 for $|\epsilon| < \epsilon_0$, where $\epsilon_0 > 0$. There exist four fundamental solutions (u, w) of the spectral problem (3.19) for any λ in the strip $\{\lambda \in \mathbb{C} : 0 < \mathrm{Re}\lambda < c^2\}$, such that*

$$\begin{pmatrix} u_\pm \\ w_\pm \end{pmatrix} \to \begin{pmatrix} \kappa_\pm \\ -\kappa_\mp \end{pmatrix} e^{\kappa_\pm x} \qquad as \qquad x \to -\infty \tag{4.5}$$

and

$$\begin{pmatrix} \tilde{u}_\pm \\ \tilde{w}_\pm \end{pmatrix} \to \begin{pmatrix} \kappa_\pm \\ -\kappa_\mp \end{pmatrix} e^{-\kappa_\pm x} \qquad as \qquad x \to +\infty \tag{4.6}$$

Proof. For any λ in the strip $\{\lambda \in \mathbb{C} : 0 < \text{Re}\lambda < c^2\}$, the two roots $\kappa_+(\lambda)$ and $\kappa_-(\lambda)$ of the characteristic equations (4.3) are distinct and $\text{Re}\kappa_\pm > 0$. Existence of four linearly independent solutions with the exponential tails in (4.5) and (4.6) follows by the Coddington–Levinson's Theorem for ODEs [10] under the condition that $V(x) \to 0$ and $\phi_\epsilon^2(x) \to q_0$ exponentially fast as $|x| \to \infty$. □

Definition 4.5. *The determinant of the four fundamental solutions in Lemma 4.4 for any $x \in \mathbb{R}$ is called the Evans function $E(\lambda, \epsilon)$ of the spectral problem (3.19), namely*

$$E(\lambda,\epsilon) = \det \begin{vmatrix} u_+ & \tilde{u}_+ & u_- & \tilde{u}_- \\ u'_+ & \tilde{u}'_+ & u'_- & \tilde{u}'_- \\ w_+ & \tilde{w}_+ & w_- & \tilde{w}_- \\ w'_+ & \tilde{w}'_+ & w'_- & \tilde{w}'_- \end{vmatrix}. \tag{4.7}$$

Remark 4.6. Because the Wronskian determinant of any four particular solutions of the ODE (3.19) is independent of x, the values of $E(\lambda, \epsilon)$ are independent of x.

Lemma 4.7. *Let $\phi_\epsilon(x)$ be a kink mode of Definition 2.8, while $\lambda = \frac{1}{2}\kappa_-\kappa_+$ and $\kappa_+ = \sqrt{4c^2 - \kappa_-^2}$. The four fundamental solutions and the Evans function $E(\lambda, \epsilon)$ of the spectral problem (3.19) are analytically continued in variable κ_- near $\kappa_- = 0$ for any $\epsilon \in \mathbb{R}$.*

Proof. Let us unfold the branch point $\lambda = 0$ with the transformation

$$\lambda = \frac{\kappa_+\kappa_-}{2}, \qquad c^2 = \frac{\kappa_+^2 + \kappa_-^2}{4}. \tag{4.8}$$

The spectral problem (3.19) is rewritten explicitly as follows:

$$\left[-\partial_x^2 + \kappa_+^2 + \kappa_-^2 + 2V_+(x)\right] u = -\kappa_+\kappa_- w, \qquad \left[-\partial_x^2 + 2V_-(x)\right] w = \kappa_+\kappa_- u, \tag{4.9}$$

where

$$V_+(x) = f(\phi_\epsilon^2) - f(q_0) + 2\phi_\epsilon^2 f'(\phi_\epsilon^2) - 2q_0 f'(q_0) + \epsilon V(x),$$
$$V_-(x) = f(\phi_\epsilon^2) - f(q_0) + \epsilon V(x).$$

Since the ODE system (4.9) depends analytically on $(\kappa_+, \kappa_-) \in \mathbb{C}^2$ and the boundary conditions (4.5)–(4.6) are also analytic in variables (κ_+, κ_-), the four fundamental solutions are analytic on $(\kappa_+, \kappa_-) \in \mathbb{C}^2$ and so is the Evans function $E(\lambda, \epsilon)$ as a determinant of analytic functions for any fixed $x \in \mathbb{R}$. The unfolding transformation (4.8) implies that the parameter $c \in \mathbb{C}$ is arbitrary. Since $c \in \mathbb{R}_+$ is fixed, this leads to the constraint $\kappa_+ = \sqrt{4c^2 - \kappa_-^2}$, which is locally analytic near $\kappa_- = 0$. □

Remark 4.8. The Evans function was constructed in [19] for the cubic NLS equation with a perturbation. It was found in [19] that the function $E(\lambda)$ is analytic in a small domain near $\lambda = 0$ with $\text{Re}\lambda > 0$, its zeros coincide with eigenvalues λ with the account of their algebraic multiplicities, and it is analytically continued in the variable κ_- near the point $\lambda = 0$ ($\kappa_- = 0$). Lemma 4.7 repeats these arguments with an alternative analysis based on the unfolding transformation (4.8) similarly to the recent work [12].

Example 4.9. Continuing Example 2.14, we compute the Evans function $E(\lambda)$ explicitly for the cubic NLS with $f(s) = s$ and $q_0 = 1$. There exist explicit solutions of the spectral problem (3.1) for the cubic NLS (see, e.g. [25]). By using these solutions, we obtain the explicit representation of the eigenvectors (u, w) satisfying the boundary conditions (4.5) and (4.6):

$$u_\pm = -\frac{2}{2+\kappa_\pm} e^{\kappa_\pm x}\left(\operatorname{sech}^2 x + \kappa_\pm \tanh x - \frac{1}{2}\kappa_\pm^2\right),$$
$$w_\pm = -\frac{\kappa_\mp}{2+\kappa_\pm} e^{\kappa_\pm x}\left(\kappa_\pm - 2\tanh x\right)$$

and

$$\tilde{u}_\pm = -\frac{2}{2+\kappa_\pm} e^{-\kappa_\pm x}\left(\operatorname{sech}^2 x - \kappa_\pm \tanh x - \frac{1}{2}\kappa_\pm^2\right),$$
$$\tilde{w}_\pm = -\frac{\kappa_\mp}{2+\kappa_\pm} e^{-\kappa_\pm x}\left(\kappa_\pm + 2\tanh x\right).$$

The Evans function $E(\lambda)$ is computed explicitly as the determinant of the four fundamental solutions in the form

$$E(\lambda) = \frac{4\kappa_+^3 \kappa_-^3 (\kappa_+^2 - \kappa_-^2)^2}{(\kappa_+ + 2)^2 (\kappa_- + 2)^2}, \tag{4.10}$$

such that $E(\lambda) = 8\lambda^3\left(1 - \lambda + \mathrm{O}(\lambda^2)\right)$ as $\lambda \to 0$ with $\mathrm{Re}\lambda > 0$. The validity of all explicit formulas has been confirmed by using Wolfram's Mathematica.

Theorem 4.10. *Let $f(q)$ be $C^\infty(\mathbb{R}_+)$ and $V(x)$ be $C^2(\mathbb{R})$ satisfying (1.3). Let $M''(s_0) > 0$ and $P_r'|_{v\downarrow 0} \neq 0$ in Theorems 2.12 and 3.11. Then, the spectral problem (3.19) for sufficiently small ϵ admits at least two small eigenvalues $\lambda = \pm\Lambda(\epsilon)$ with $(u, w) \in L^2(\mathbb{R}, \mathbb{C}^2)$, $\mathrm{Re}\Lambda\ \mathrm{Im}\Lambda \geq 0$, and $\lim_{\epsilon\to 0} \Lambda(\epsilon) = 0$, such that*

(i) *$\Lambda(\epsilon)$ is infinitely smooth with respect to $\epsilon^{1/2}$.*

(ii) *(u, w) is infinitely smooth with respect to $\epsilon^{1/2}$ and*

$$\lim_{\epsilon\to 0} u(x) = \phi_0'(x), \qquad \lim_{\epsilon\to 0} w(x) = 0, \tag{4.11}$$

up to an arbitrary multiplicative factor.

(iii) *(u, w) admits an asymptotic expansion as $\lambda \to 0$, $\mathrm{Re}\lambda > 0$ for large $\pm x \gg 1$:*

$$\begin{pmatrix} u \\ w \end{pmatrix} \to a_\pm\left(1 + \mathrm{O}(\lambda)\right)\begin{pmatrix} \lambda/c + \mathrm{O}(\lambda^3) \\ -2c + \mathrm{O}(\lambda^2)\end{pmatrix}\left(1 \mp \frac{\lambda x}{c} + \mathrm{O}(\lambda x)^2\right), \tag{4.12}$$

where $a_\pm$ are some constants and $\lambda = \Lambda(\epsilon)$.

Proof. If $M''(s_0) > 0$, there exists at least one small eigenvalue $\gamma = \Gamma(\epsilon)$ of the generalized eigenvalue problem (3.22) with the eigenvector $w \in L^2(\mathbb{R})$, such that $\mathrm{Im}\Gamma(\epsilon) \leq 0$ and $\lim_{\epsilon\to 0} \Gamma(\epsilon) = 0$. Since $\gamma = -\lambda^2$, there must exist a zero of the Evans function $E(\lambda, \epsilon)$ at $\lambda = \Lambda(\epsilon) = \sqrt{-\Gamma(\epsilon)}$ for sufficiently small ϵ, such that $\mathrm{Re}\Lambda\ \mathrm{Im}\Lambda \geq 0$ and $\lim_{\epsilon\to 0} \Lambda(\epsilon) = 0$.

(i) By Lemma 4.7, the Evans function $E(\lambda, \epsilon)$ is analytically continued in $\kappa_- = \lambda/c + \mathrm{O}(\lambda^3)$ near $\lambda = 0$. It is also infinitely smooth in ϵ near $\epsilon = 0$. (Indeed, the potential terms $V_\pm(x)$ in the representation (4.9) are infinitely smooth in ϵ and exponentially decaying as $|x| \to \infty$.) In addition, the Evans function $E(\lambda, \epsilon)$ has the following properties:

$$E(\lambda, 0) = \alpha\lambda^3 + \mathrm{O}(\lambda^4), \qquad E(0, \epsilon) = 0,$$

where α is a numerical constant. According to Lemma 4.1, the triple root of $E(\lambda, 0)$ corresponds to the solutions u_0 and w_1 in (i) due to translational invariance and the solutions w_0 and u_1 in (ii) due to the gauge invariance. The former subspace results in the double root $\lambda = 0$ of $E(\lambda, 0)$, while the latter subspace results in a single root $\kappa_- = 0$ ($\lambda = 0$) of $E(\lambda, 0)$ [20]. By Lemma 4.1, $\alpha \neq 0$ is equivalent to the condition $P_r'|_{v\downarrow 0} \neq 0$. The constraint $E(0, \epsilon) = 0$ follows from existence of $\phi_\epsilon \in L^\infty(\mathbb{R})$ such that $\mathcal{L}_-\phi_\epsilon = 0$. As a result, the Evans function is expanded near $\lambda = 0$ and $\epsilon = 0$ as follows:

$$E(\lambda, \epsilon) = \lambda\left(\alpha\lambda^2 + \beta\epsilon + \mathrm{O}(\lambda^3, \lambda\epsilon, \epsilon^2)\right), \tag{4.13}$$

where β is another numerical constant. Since the gauge invariance is preserved while the translational invariance of dark solitons is destroyed by Theorem 3.15, $\beta \neq 0$ is equivalent to the condition that $M''(s_0) \neq 0$. It follows from expansion (4.13) and the smoothness of $E(\lambda, \epsilon)$ that the root $\Lambda(\epsilon)$ is infinitely smooth in $\epsilon^{1/2}$.

(ii) By the construction of the Evans function $E(\lambda, \epsilon)$, the eigenvector (u, w) is spanned by the four fundamental solutions in Lemma 4.4. By Lemma 4.7, these solutions are analytic in κ_- near $\kappa_- = 0$. By Remark 4.3, $\kappa_- = \lambda/c + \mathrm{O}(\lambda^3)$ near $\lambda = 0$ and by (i) of Theorem 4.10, the root $\lambda = \Lambda(\epsilon)$ of $E(\lambda, \epsilon) = 0$ is infinitely smooth in $\epsilon^{1/2}$. Therefore, the eigenvector (u, w) is also infinitely smooth in $\epsilon^{1/2}$. By Lemma 4.1, the kernel of operators $\mathcal{L}_+$ and $\mathcal{L}_-$ is one-dimensional in $L^2(\mathbb{R})$, such that the limiting relation (4.11) holds (eigenvectors are defined up to an arbitrary multiplicative factor).

(iii) It follows from the decay (4.5) and (4.6) that there exist constants $A_\pm(\lambda)$ and $B_\pm(\lambda)$, such that

$$\begin{pmatrix} u \\ w \end{pmatrix} \to A_\pm \begin{pmatrix} \kappa_- \\ -\kappa_+ \end{pmatrix} e^{\mp\kappa_- x} + B_\pm \begin{pmatrix} \kappa_+ \\ -\kappa_- \end{pmatrix} e^{\mp\kappa_+ x} \quad \text{as} \quad x \to \pm\infty, \tag{4.14}$$

where $\kappa_\pm$ are defined by the characteristic equation (4.3) for $\lambda = \Lambda(\epsilon)$. By expansion (4.4), the slowly decaying solutions $e^{\mp\kappa_- x}$ are expanded to the form (4.12), while the fast decaying solutions $e^{\mp\kappa_+ x}$ are not written in (4.12). □

Theorem 4.11. *Let $f(q)$ be $C^2(\mathbb{R}_+)$ and $V(x) \in C^2(\mathbb{R})$ satisfy (1.3). Let $M''(s_0) \neq 0$, $P'_r|_{v\downarrow 0} \neq 0$, and $S'|_{v\downarrow 0} \neq 0$ for a black soliton $\phi_0(x)$ of Definition 2.5. Let $\lambda = \Lambda(\epsilon)$ be a small eigenvalue of the spectral problem (3.19) for sufficiently small ϵ with $\mathrm{Re}\Lambda > 0$. Then, the value of $\Lambda(\epsilon)$ is given by the root of the characteristic equation:*

$$\mathrm{Re}\Lambda > 0: \quad (P'_r|_{v\downarrow 0})\,\Lambda^2 - \epsilon \frac{q_0 (S'|_{v\downarrow 0})^2 M''(s_0)}{2c(P'_r|_{v\downarrow 0})} \Lambda + \epsilon M''(s_0) = \mathrm{O}(\epsilon^2). \tag{4.15}$$

Proof. By Theorem 3.15, the spectral problem (3.19) for sufficiently small ϵ can be written in the form:

$$\left[L_+ + \epsilon M_+ + \mathrm{O}(\epsilon^2)\right] u = -\lambda w, \qquad \left[L_- + \epsilon M_- + \mathrm{O}(\epsilon^2)\right] w = \lambda u.$$

By Theorem 4.10, the eigenvector (u, w) and the eigenvalue $\lambda = \Lambda(\epsilon)$ can be expanded in powers of $\epsilon^{1/2}$:

$$u = \phi'_0(x) + \epsilon^{1/2} u_1 + \epsilon u_2 + \epsilon^{3/2} u_3 + \mathrm{O}(\epsilon^2), \quad w = \epsilon^{1/2} w_1 + \epsilon w_2 + \epsilon^{3/2} w_3 + \mathrm{O}(\epsilon^2),$$

and $\lambda = \epsilon^{1/2}\lambda_1 + \epsilon\lambda_2 + \epsilon^{3/2}\lambda_3 + \mathrm{O}(\epsilon^2)$. The first-order corrections terms (u_1, w_1) satisfy the system

$$L_+ u_1 = 0, \qquad L_- w_1 = \lambda_1 \phi'_0. \tag{4.16}$$

By the expansion (4.12), we are looking for a solution with $u_1 \in L^2(\mathbb{R})$ and $w_1 \in L^\infty(\mathbb{R})$. By Lemma 4.1, the explicit solution is

$$u_1 = c_1 \phi'_0(x), \qquad w_1 = a_1 \phi_0(x) - \lambda_1 \mathrm{Im}\partial_v U(x)|_{v\downarrow 0},$$

where (a_1, c_1) are parameters. Without loss of generality, we can set $c_1 = 0$. The second-order corrections terms (u_2, w_2) satisfy the system

$$L_+ u_2 = -\lambda_1 w_1 - M_+ \phi'_0, \qquad L_- w_2 = \lambda_2 \phi'_0 \tag{4.17}$$

By the Fredholm alternative for L_+, we obtain the constraint

$$-\lambda_1 (\phi'_0, w_1) - (\phi'_0, M_+ \phi'_0) = 0.$$

By using (3.14) and (3.24), the constraint is equivalent to the characteristic equation

$$\frac{1}{2}(P'_r|_{v\downarrow 0})\lambda_1^2 + \frac{1}{2} M''(s_0) = 0. \tag{4.18}$$

Since $w_1 \in L^\infty(\mathbb{R})$ and the Fredholm constraint is satisfied, there exists a solution $u_2 \in L^\infty(\mathbb{R})$ of the first inhomogeneous equation (4.17). There exists also a solution $w_2 \in L^\infty(\mathbb{R})$ of the second inhomogeneous equation (4.17) similarly to the solution $w_1 \in L^\infty(\mathbb{R})$. By the expansion (4.12), we shall add a homogeneous linearly growing solution w_0 in (iii) of Lemma 4.1 which is not in $L^\infty(\mathbb{R})$, such that $w_2 \notin L^\infty(\mathbb{R})$. As a result, the second-order corrections terms are written in the form

$$\begin{aligned} u_2 &= c_2 \phi'_0(x) + \tilde{u}_2(x), \\ w_2 &= a_2 \phi_0(x) - \lambda_2 \mathrm{Im}\partial_v U(x)|_{v\downarrow 0} + b_2 \left[x\phi_0(x) - \mathrm{Im}\partial_v U(x)|_{v\downarrow 0}\right], \end{aligned}$$

where (a_2, b_2, c_2) are parameters ($c_2 = 0$ without loss of generality) and $\tilde{u}_2 \in L^\infty(\mathbb{R})$ is a solution of the inhomogeneous equation

$$L_+ \tilde{u}_2 = -\lambda_1 a_1 \phi_0 - \frac{M''(s_0)}{P_r'|_{v\downarrow 0}} \mathrm{Im}\partial_v U(x)|_{v\downarrow 0} - M_+ \phi_0'.$$

The third-order corrections terms (u_3, w_3) satisfy the system

$$L_+ u_3 = -\lambda_2 w_1 - \lambda_1 w_2, \qquad L_- w_3 = \lambda_3 \phi_0' + \lambda_1 u_2 - M_- w_1. \tag{4.19}$$

By the Fredholm alternative for L_+, we obtain the constraint

$$-\lambda_2(\phi_0', w_1) - \lambda_1(\phi_0', w_2) = 0$$

which is equivalent by virtue of (4.2) to the characteristic equation

$$(P_r'|_{v\downarrow 0})\lambda_1\lambda_2 + \frac{1}{2} q_0 (S'|_{v\downarrow 0})\lambda_1 b_2 = 0. \tag{4.20}$$

Combining two corrections λ_1 and λ_2 in $\Lambda(\epsilon) = \epsilon^{1/2}\lambda_1 + \epsilon\lambda_2 + \mathrm{O}(\epsilon^{3/2})$, we rewrite the characteristic equations (4.18) and (4.20) in the form

$$(P_r'|_{v\downarrow 0})\Lambda^2 + \epsilon q_0 (S'|_{v\downarrow 0}) b_2 \Lambda + \epsilon M''(s_0) = \mathrm{O}(\epsilon^2). \tag{4.21}$$

In order to find b_2 for $\mathrm{Re}\Lambda > 0$, we need to consider $w(x) = \epsilon^{1/2} w_1 + \epsilon w_2 + \mathrm{O}(\epsilon^{3/2})$ for large $\pm x \gg 1$:

$$\begin{aligned} w(x) \to \pm\sqrt{q_0}\Big[&\epsilon^{1/2}\left(a_1 - \lambda_1 \partial_v \Theta^\pm|_{v\downarrow 0}\right) + \epsilon\big(a_2 - \lambda_2\partial_v\Theta^\pm|_{v\downarrow 0} \\ &+ b_2(x - \partial_v\Theta^\pm|_{v\downarrow 0})\big) + \mathrm{O}(\epsilon^{3/2})\Big] \end{aligned} \tag{4.22}$$

where $\Theta^\pm = \lim_{x\to\pm\infty} \Theta(x)$ and $\lim_{x\to\pm\infty} \phi_0(x) = \pm\sqrt{q_0}$ have been used. Matching the asymptotic expansions (4.12) and (4.22), we find a linear system on parameters (a_1, b_2):

$$c b_2 = \mp \lambda_1 \left(a_1 - \lambda_1 \partial_v \Theta^\pm|_{v\downarrow 0}\right).$$

The linear system has the explicit solution

$$2cb_2 = \lambda_1^2 \partial_v \left(\Theta^+ - \Theta^-\right)|_{v\downarrow 0}, \qquad 2\lambda_1 a_1 = \lambda_1^2 \partial_v \left(\Theta^+ + \Theta^-\right)|_{v\downarrow 0},$$

such that

$$b_2 = \frac{1}{2c}(S'|_{v\downarrow 0})\lambda_1^2 = -\frac{(S'|_{v\downarrow 0})M''(s_0)}{2c(P_r'|_{v\downarrow 0})},$$

where $S = \Theta^+ - \Theta^-$. As a result, the characteristic equation (4.21) reduces to the form (4.15). □

Corollary 4.12. *Let $P_r'|_{v\downarrow 0} \neq 0$ and $S'|_{v\downarrow 0} \neq 0$ for a kink mode of Theorems 2.12, 3.14, 3.15 and 4.11 for sufficiently small ϵ. Then,*

- *If $P_r'|_{v\downarrow 0} > 0$, a kink mode with $M''(s_0) > 0$ has precisely one quartet of small complex eigenvalues ($N_r = N_i^- = 0$, $N_c = 1$), while a kink mode with $M''(s_0) < 0$ has precisely one pair of small real eigenvalues ($N_r = 1$, $N_i^- = N_c = 0$) in the spectral problem (3.19).*

- *If $P_r'|_{v\downarrow 0} < 0$, a kink mode with $M''(s_0) > 0$ has precisely one pair of small real eigenvalues and one pair of finite real eigenvalues ($N_r = 2$, $N_i^- = N_c = 0$), while a kink mode with $M''(s_0) < 0$ has precisely one pair of finite real eigenvalues and no small eigenvalues ($N_r = 1$, $N_i^- = N_c = 0$) in the spectral problem (3.19).*

Remark 4.13. Comparison of the characteristic equation (4.15) and the linearized version of the Newton's particle law (1.6) shows that

$$\mu_0 = P_r'|_{v\downarrow 0}, \qquad \lambda_0 = \frac{q_0 (S'|_{v\downarrow 0})^2}{2c(P_r'|_{v\downarrow 0})}.$$

Both constants are positive if $P_r'|_{v\downarrow 0} > 0$, i.e. if the kink is stable in the spectral problem (3.1).

Remark 4.14. Characteristic equation (4.15) can be derived from the power expansion of the Evans function $E(\lambda, \epsilon)$ of Definition 4.5:

$$E(\lambda, \epsilon) = \lambda \left(\alpha \lambda^2 + \tilde{\alpha} \lambda^3 + \beta \epsilon + \tilde{\beta} \lambda \epsilon + \mathrm{O}(\lambda^4, \lambda^2 \epsilon, \epsilon^2) \right), \tag{4.23}$$

where (α, β) are constants from the expansion (4.13) and $(\tilde{\alpha}, \tilde{\beta})$ are new constants. Since computations of these constants from derivatives of $E(\lambda, \epsilon)$ are technically involved, these computations are replaced in Theorem 4.11 with direct expansions of eigenvectors and eigenvalues of the spectral problem (3.19) in powers of $\epsilon^{1/2}$.

Example 4.15. Continuing Example 3.8 we consider the cubic NLS with $f(s) = s$ and $q_0 = 1$, where $P_r'|_{v\downarrow 0} = 4$ and $S'|_{v\downarrow 0} = 2$. As a result, the characteristic equation (4.15) is written explicitly by

$$\mathrm{Re}\Lambda > 0: \quad \Lambda^2 + \frac{\epsilon}{4} M''(s_0) \left(1 - \frac{\Lambda}{2} \right) = \mathrm{O}(\epsilon^2). \tag{4.24}$$

This equation has only one real-valued root $\Lambda(\epsilon) > 0$ for $M''(s_0) < 0$ and two complex-conjugate roots with $\mathrm{Re}\Lambda(\epsilon) > 0$ for $M''(s_0) > 0$ provided that $\epsilon > 0$ is sufficiently small. The validity of the expansion (4.24) will be tested in Section 5.

Remark 4.16. If the characteristic equation (4.24) is formally applied to the cubic GP equation (1.2) with $f(s) = s$, $q_0 = 1$ and $V(x) = x^2$, we obtain $M''(s) = 2 \int_{\mathbb{R}} \mathrm{sech}^2(x) dx = 4$, such that the characteristic equation (4.24) is

$$\Lambda^2 - \frac{\epsilon}{2} \Lambda + \epsilon = \mathrm{O}(\epsilon^2).$$

This characteristic equation was derived in [34] with a formal method for slow dynamics of dark solitons subject to radiative boundary conditions. The validity of radiative boundary conditions for parabolic potentials $V(x) = x^2$ can not be verified by the present analysis.

Remark 4.17. The characteristic equation (4.15) can be rewritten in the form

$$(P_r'|_{v\downarrow 0})\,\Lambda^2 + \frac{q_0(S'|_{v\downarrow 0})^2}{2c}\Lambda^3 = -\epsilon M''(s_0) + \mathrm{O}(\epsilon^2).$$

The left-hand-side of this equation was derived in Eqs. (19)-(20) of [35] and Eqs. (2.37)–(2.38) of [33] in a more general context of dark solitons with $v \in (-c, c)$. The method of [35] was based on asymptotic theory for slow dynamics of dark solitons, while the method of [33] was based on slow decay conditions for eigenfunctions of the linearized problem. Here we have replaced these formal methods with the rigorous proof of Theorems 4.10 and 4.11.

5. Numerical approximations of eigenvalues

We test the predictions of the characteristic equation (4.24) for the cubic GP equation (1.2) with $f(s) = s$ and the two potentials $V_1(x)$ and $V_2(x)$ in (1.4). In particular, we focus on examining the dependencies of small unstable eigenvalues of the spectral problem (3.19) versus parameters ϵ and κ. The numerical approximations of the kink mode $\phi_\epsilon(x)$ are obtained by means of fixed point iterations of the ODE (2.13). The iterations are applied to a finite-difference discretization of the computational domain $x \in [-L, L]$ on a grid of N nodes (typically $N = 1600$) with a spacing Δx (typically $\Delta x = 0.2$). Subsequently, the spectral problem (3.19) is discretized in a matrix eigenvalue problem that, in turn, is solved through standard numerical linear algebra routines.

In the case of the potential $V_1(x)$, as is considered in Examples 2.14 and 3.18, the positive-definite sign of $M''(0)$ leads to a sole kink mode bifurcating from $s_0 = 0$ (and staying at $s = 0$ by Remark 2.10). The kink mode is located at the minimum of the effective potential $M(s)$ and is unstable due to a complex quartet of eigenvalues according to the characteristic equation (4.24).

The numerical results on Figures 3 and 4 fully confirm the above picture. Fig. 3 shows only one solution $\phi_\epsilon(x)$ of the ODE (2.13) with the potential $V_1(x)$ and a unique quartet of complex eigenvalues $\lambda = \lambda_r + i\lambda_i$ in the spectral problem (3.19). The left panel of Fig. 4 shows the real part of this quartet as a function of κ for a given $\epsilon = 0.2$, while the right panel shows the relevant real part as a function of ϵ for a given $\kappa = 1$. The predictions of the characteristic equation (4.24) are shown by dashed-dotted lines, while the numerically obtained eigenvalues are shown by thick lines.

The non-monotonic behavior of the real part of complex eigenvalues is produced by the truncation of the computational domain $x \in [-L, L]$ and subsequent discretization on a finite grid. This numerical phenomenon is explained in [17] (see their Figure 2) as follows. The continuous spectrum of the spectral problem (3.19) becomes a finite spectral band along the imaginary axis near $\lambda = 0$ due to the truncation and the band is represented by isolated eigenvalues due to the discretization. The quartet of eigenvalues bifurcates from the point $\lambda = 0$ in the

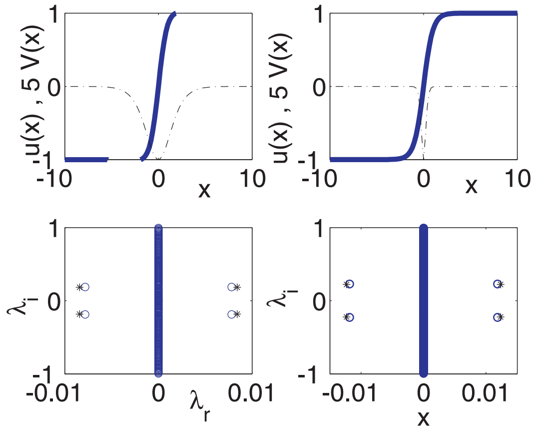

Figure 3. Bifurcation results for the potential $V_1(x)$. The top panels show the kink mode (solid line) and the potential $V_1(x)$ amplified by a factor of 5 (dash-dotted line) for $(\epsilon, \kappa) = (0.2, 1.1)$ (left) and $(\epsilon, \kappa) = (0.2, 6.4)$ (right). The bottom panels show the corresponding spectrum of the linearized problem (the numerical result is shown by circles, while the theoretical prediction is shown by stars).

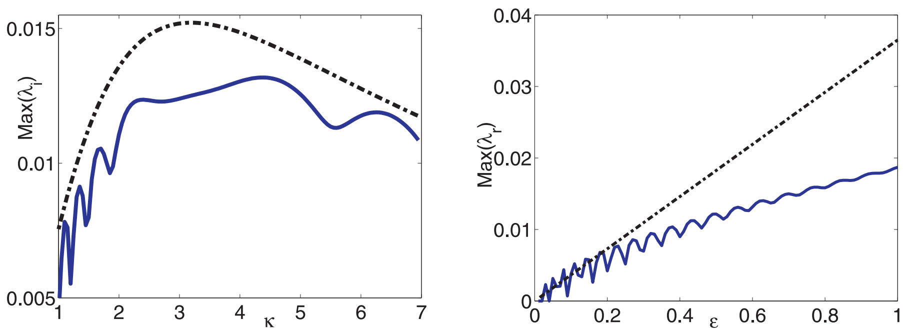

Figure 4. The left panel shows the theoretical (dash-dotted line) and numerical (solid line) dependence on κ of the real part of the unstable complex eigenvalue for fixed $\epsilon = 0.2$. The right panel shows the dependence of the same quantity on ϵ for fixed $\kappa = 1$.

direction of the imaginary axis with small real part for small ϵ and interferes with eigenvalues from the discretized continuous spectrum. This interference leads to the non-monotonic behavior of the real part of the relevant eigenvalues on Fig. 4. We note that this effect is *not* present for real eigenvalues bifurcating from the point $\lambda = 0$.

In the case of the potential $V_2(x)$, we have a more interesting phenomenology. While the potential always has two maxima at $x = \pm 2/\kappa$, the *effective potential* $M(s)$ possesses two maxima at $s_0 = \pm s_*$, $s_* \neq 0$ and a minimum at $s_0 = 0$ for

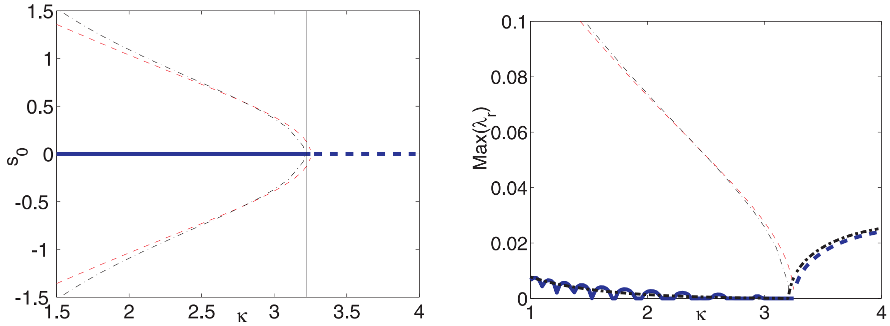

Figure 5. The subcritical pitchfork bifurcation in parameter κ for the potential $V_2(x)$ for fixed $\epsilon = 0.2$. The left panel shows the center of mass s_0 of the kink modes ($s_0 \neq 0$ by dashed line, $s_0 = 0$ by thick solid and dashed lines). The theoretical predictions of s_0 are shown by dash-dotted line. The vertical line gives the theoretical prediction for the bifurcation point $\kappa = \kappa_0$. The right panel shows the real part of the unstable eigenvalues for the relevant kink modes, using the same symbolism as the left panel. The theoretical predictions of eigenvalues are shown by thick and thin dash-dotted lines, respectively for the branches with $s_0 = 0$ and $s_0 \neq 0$.

$\kappa < \kappa_0 \approx 3.21$ and only one maximum at $s_0 = 0$ for $\kappa > \kappa_0$. Branches of solutions with $M''(s_0) < 0$ are unstable due to a pair of real eigenvalues, while the branch of solutions with $M''(s_0) > 0$ is unstable due to a quartet of complex eigenvalues, according to the characteristic equation (4.24). The transition of the kink mode $s_0 = 0$ from $\kappa < \kappa_0$ (when $M''(0) > 0$) to $\kappa > \kappa_0$ (when $M''(0) < 0$) indicates a subcritical pitchfork bifurcation at $\kappa = \kappa_0$.

The numerical results for the potential $V_2(x)$ are shown on Figures 5, 6 and 7. Branches of the solutions with $s_0 = \pm s_*$ are denoted by a dashed line, the branch of the solution $s_0 = 0$ with $M''(s_0) > 0$ is denoted by thick solid line and the same branch with $M''(s_0) < 0$ is denoted by thick dashed line. The corresponding predictions from the extremal points of the effective potential $M(s)$ and from the characteristic equation (4.24) are shown by dash-dotted lines (thick for $s_0 = 0$ and thin for $s_0 = \pm s_*$).

The bifurcation point is found numerically to be $\kappa_0 \approx 3.26$ for $\epsilon = 0.2$ in a very good agreement with the value $\kappa_0 = 3.21$ obtained from $M(s)$ with $M''(0) = 0$. The computational error is approximately 1.5%. The values of s_0 for the kink mode $\phi_\epsilon(x)$ are obtained numerically from its "center of mass" defined by

$$s_0 = \frac{\int_{-\infty}^{\infty} x(1 - |\phi_\epsilon|^2)dx}{\int_{-\infty}^{\infty}(1 - |\phi_\epsilon|^2)dx}. \tag{5.1}$$

The values of s_0 are plotted on the left panel of Fig. 5 for $\epsilon = 0.2$ in a good agreement with the value s_* obtained from $M(s)$ with $M'(s_*) = 0$.

The solution profile $\phi_\epsilon(x)$ and the corresponding linearization spectra for the different branches and for particular choices of (ϵ, κ) are shown on Fig. 6. In

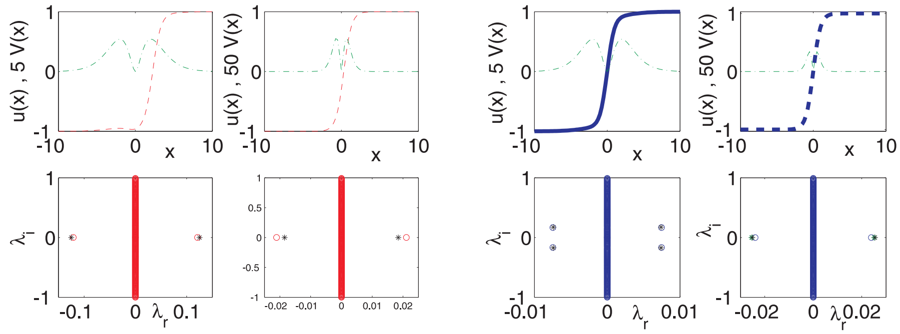

Figure 6. The left quartet of panels shows the solutions with $s_0 \neq 0$ (dashed lines) and the potential $V_2(x)$ (dash-dotted line) for $(\epsilon, \kappa) = (0.2, 1.0)$ and $(\epsilon, \kappa) = (0.2, 3.1)$. The corresponding spectrum features a pair of real eigenvalues (numerical results are shown by circles and the theoretical predictions are shown by stars). The right quartet of panels shows the similar picture for the kink mode with $s_0 = 0$ for $(\epsilon, \kappa) = (0.2, 1.025)$ and $(\epsilon, \kappa) = (0.2, 3.975)$. The corresponding spectrum features either a quartet of complex eigenvalues or a pair of real eigenvalues.

agreement with the characteristic equation (4.24), the kink modes with $s_0 = \pm s_*$ for $\kappa < \kappa_0$ and with $s_0 = 0$ for $\kappa > \kappa_0$ has a pair of small real eigenvalues, while the kink mode with $s_0 = 0$ for $\kappa < \kappa_0$ has a quartet of small complex eigenvalues. The right panel of Fig. 5 shows the real parts of unstable eigenvalues for each kink mode.

Fig. 7 shows the dependence of the relevant eigenvalues versus ϵ for a fixed $\kappa = 1 < \kappa_0$. In this case, three branches of kink modes exist and the branch with $s_0 \neq 0$ has a pair of real eigenvalues, while the branch with $s_0 = 0$ has a quartet of complex eigenvalues. We can see that the non-monotonic behavior of the real part of unstable eigenvalues is only observed for the quartet of complex eigenvalues. We can also see from all figures of this section that the agreement between numerical and theoretical results is excellent for $\epsilon < 0.3$ and deteriorates for $\epsilon > 0.3$ due to the truncation error $O(\epsilon^2)$ in the characteristic equation (4.24).

6. Numerical simulations of the GP equation

We examine the dynamics of the unstable kink modes in the full GP equation (1.2) by using direct numerical simulations. The time-evolution problem is approximated by the fourth-order Runge-Kutta method applied to the spatial discretization of the GP equation. The output of the fixed point iterations was used as an input in the time integrator with the time step Δt (typically $\Delta t = 0.001$). The results of the time evolution are compared against the effective Newton's particle equation (1.6) for the position $s(t)$ of the center of dark soliton $\phi_\epsilon(x - s(t))$.

In order to test the theoretical result, we have to use the following numerical

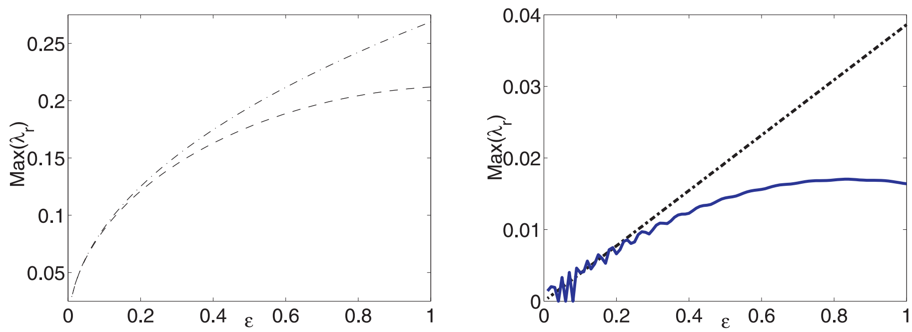

Figure 7. The real part of the unstable eigenvalues versus ϵ for $\kappa = 1$ for the solution branches with $s_0 \neq 0$ (left panel) and with $s_0 = 0$ (right panel). The numerical results are shown by dashed and thick solid lines, while the theoretical predictions are shown by dash-dotted lines.

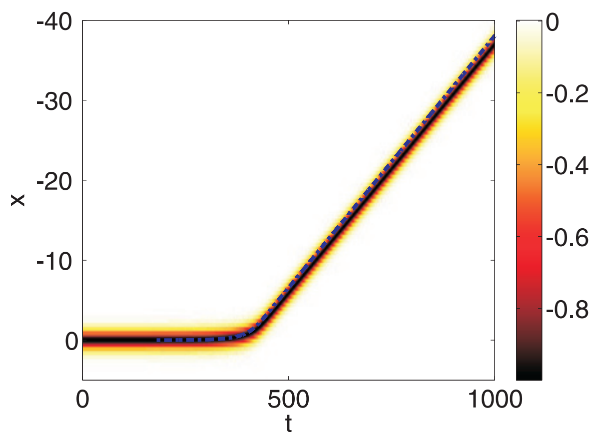

Figure 8. The unstable evolution of the kink mode with $s_0 = 0$ for the potential $V_2(x)$ with $\kappa = 4$ and $\epsilon = 0.2$. The dashed-dotted line shows the result of the Newton's law initialized around $t = 180$.

technique. The initial condition $u(x, 0)$ of the GP equation (1.2) is specified in the form of the kink mode $\phi_\epsilon(x)$ plus a small (typically 10^{-4}) perturbation multiple of its most unstable eigenmode. The time-evolution problem is integrated for an initial period $0 < t < t_0$, during which the dark soliton acquires a small speed due to instabilities, which quickly grows for $t > t_0$. At the time instance $t = t_0$, we approximate the values of $s(t_0)$ and $\dot{s}(t_0)$ by using the center of mass (5.1) at $t = t_0$ and at earlier time instances. The Newton's particle equation (1.6) is initialized at $t = t_0$ with given $s(t_0)$ and $\dot{s}(t_0)$ and then integrated with the fourth-order Runge–Kutta method.

Figures 8 and 9 illustrate the time evolution of an unstable dark soliton, which

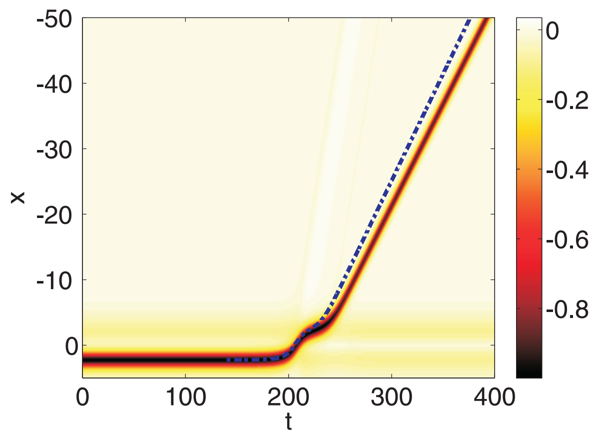

Figure 9. The unstable evolution of the kink mode with $s_0 \approx 2.23$ for the potential $V_2(x)$ with $\kappa = 1$ and $\epsilon = 0.2$. The dashed-dotted line shows the result of the Newton's law initialized around $t = 140$.

possesses a pair of real eigenvalues in the linearization spectrum. Fig. 8 corresponds to the potential $V_2(x)$ with $\kappa = 4 > \kappa_0$, when the kink mode with $s_0 = 0$ has a real eigenvalue $\lambda = 0.0241$ (the theoretical prediction of the linearized Newton's particle equation is $\lambda \approx 0.0253$). We observe from the figure that the unstable kink mode undertakes a monotonic transition to a stable dark soliton, which escapes the double-humped potential $V_2(x)$ and travels with an asymptotically

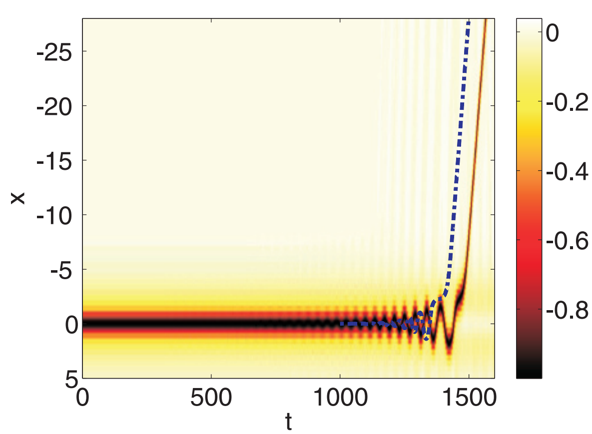

Figure 10. The unstable evolution of the kink mode with $s_0 = 0$ for the potential $V_2(x)$ with $\kappa = 1$ and $\epsilon = 0.2$. The dashed-dotted line shows the result of the Newton's law initialized at $t \approx 1000$.

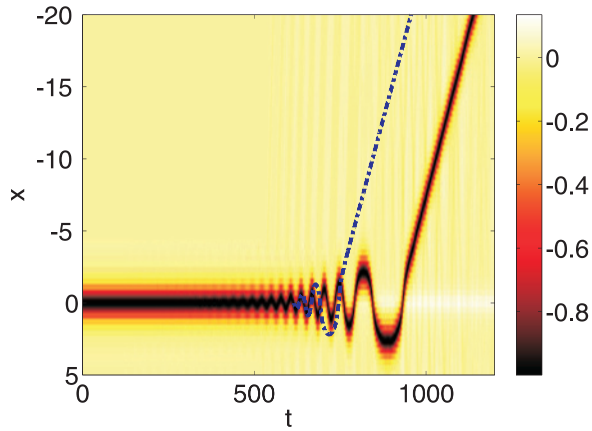

Figure 11. The unstable evolution of the kink mode with $s_0 = 0$ for the potential $V_1(x)$ with $\kappa = 6.4$ and $\epsilon = 0.2$. The dash-dotted line shows the result of the Newton's law initialized around $t \approx 600$.

constant speed. The predictions of the Newton's particle equation shown by thick dash-dotted line captures the entire process accurately but slightly precedes the full time-evolution of the GP equation. The latter discrepancy can be attributed to the larger value of λ for the unstable eigenvalue.

Fig. 9 shows a similar monotonic transition for the potential $V_2(x)$ with $\kappa = 1$, when the kink mode with $s_0 = s_* \approx 2.23$ has a pair of real unstable eigenvalues. In this case, the theoretical prediction $\lambda \approx 0.1251$ exceeds the numerically obtained value $\lambda = 0.1211$ too and the prediction of the Newton particle equation precedes its counterpart from the GP equation. It is worth to note the qualitative agreement between the two time evolutions, including the small "leg" formed in the trajectory as the dark soliton passes the unstable kink mode with $s_0 = -s_* \approx -2.23$.

Figures 10 and 11 illustrate the time evolution of an unstable dark soliton, which possesses a quartet of complex eigenvalues in the linearization spectrum. Fig. 10 corresponds to the potential $V_2(x)$ with $\kappa = 1$, when the kink mode with $s_0 = 0$ has a quartet of complex eigenvalues with the real part $\mathrm{Re}\lambda = 0.06606$ (the theoretical prediction of the linearized Newton particle equation is $\mathrm{Re}\lambda \approx 0.07724$). We observe from the figure that the unstable kink mode oscillates in a local potential well of the double-humped potential $V_2(x)$ with an increasing amplitude due to unstable complex eigenvalues. When the oscillations reach a large amplitude, the dark soliton escapes the maximum of the effective potential and transforms to a steadily moving soliton. The predictions of the Newton particle equation represent this dynamics correctly with a larger deviation from the full GP equation in comparison with the case of monotonic transitions.

Fig. 11 shows a similar oscillatory behavior of an unstable kink mode with $s_0 = 0$ for the potential $V_1(x)$ with $\kappa = 6.4$. In this case, the theoretical prediction

$\mathrm{Re}\lambda \approx 0.0123$ exceeds again the numerically obtained value $\mathrm{Re}\lambda = 0.0118$ and the prediction of the Newton particle equation precedes its counterpart from the GP equation.

In the end, we mention that the rigorous derivation of the Newton particle equation for slow dynamics of a bright soliton in an external potential has been reported recently in [6, 13, 16]. Derivation of its counterpart (1.6) for slow dynamics of a dark soliton is an open problem of analysis. Our numerical results suggest that this Newton particle equation is highly appropriate for understanding the nonlinear time-evolution of dark solitons in the GP equation with small external potentials.

7. Conclusion

We have systematically analyzed the persistence and stability of dark solitons in the presence of small decaying potentials. We have shown how the effective potential can be used to predict bifurcations of kink modes in a small potential and to approximate small unstable eigenvalues of the linearization spectrum. These theoretical results have been tested against the numerical bifurcation results indicating excellent qualitative and good quantitative agreement. We have also conjectured a dynamical evolution equation (the Newton particle law) that can be used, quite successfully, to describe the motion of the kink modes and the manifestation of their instabilities.

One of the directions of interest for future studies is to expand the present results to other types of potentials which include periodic and confining potentials. While, as argued in the text, we expect many of the qualitative features to persist, periodic or growing potentials may possess additional interesting properties due to the presence of spectral bands or purely discrete spectrum in the linearization problem.

Another open direction would involve extending the present analysis to the two-dimensional setting and, in particular, to the case of vortices in the presence of external potentials. While some of the techniques applied herein (e.g. perturbative expansions) would apply to the latter case as well, others are more geared towards the one-dimensional setting (e.g. the Evans function technique). It would be especially interesting to generalize our current results to the two-dimensional GP equation.

We add that a similar use of the Lyapunov–Schmidt reduction technique and the Evans function formalism is recently reported for the solitary waves of coupled Korteweg–de Vries equations in [44].

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