J Nonlinear Sci (2013) 23:689–730
DOI 10.1007/s00332-013-9165-6

Journal of Nonlinear Science

Periodic Traveling Waves in Diatomic Granular Chains

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Received: 12 April 2012 / Accepted: 4 January 2013 / Published online: 8 February 2013

Abstract We study bifurcations of periodic traveling waves in diatomic granular chains from the anti-continuum limit, when the mass ratio between the light and heavy beads is zero. We show that every limiting periodic wave is uniquely continued with respect to the mass ratio parameter, and the periodic waves with a wavelength larger than a certain critical value are spectrally stable. Numerical computations are developed to study how this solution family is continued to the limit of equal mass ratio between the beads, where periodic traveling waves of homogeneous granular chains exist.

Keywords Diatomic granular chains · Periodic traveling waves · FPU lattice · Anti-continuum limit

Mathematics Subject Classification 34K13 · 34K20 · 34K31 · 37L60

1 Introduction

Wave propagation in granular crystals has been studied quite intensively in the past ten years. Granular crystals are thought to be closely packed chains of elastically in-

Communicated by Robert S. MacKay.

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teracting particles, which obey the Fermi–Pasta–Ulam (FPU) lattice equations with Hertzian interaction forces. Experimental work with granular crystals and their numerous applications (Daraio et al. 2006; Sen et al. 2008) stimulated theoretical and mathematical research on the granular chains of particles.

The existence of solitary waves in granular chains was considered with a number of analytical and numerical techniques. In his two-page note, MacKay (1999) showed how to adapt the technique of Friesecke and Wattis (1994) to the proof of the existence of solitary waves. English and Pego (2005) used these results to prove the double-exponential decay of spatial tails of solitary waves. Numerical convergence to the solitary wave solutions was studied by Ahnert and Pikovsky (2009). Stefanov and Kevrekidis (2012) reviewed the variational technique of Friesecke and Wattis (1994) and proved that the solitary waves are bell-shaped (single-humped).

Recently, the interest to granular crystals has shifted towards periodic traveling waves as well as traveling waves in heterogeneous (diatomic) chains, as more relevant for physical experiments (Boechler et al. 2010; Harbola et al. 2009; Ponson et al. 2010; Porter et al. 2008, 2009; Theocharis et al. 2009, 2010). Unlike solitary waves, periodic traveling waves were believed not to occur in uncompressed granular chains until their recent numerical observations (Jayaprakash et al. 2011, 2012; Starosvetsky and Vakakis 2010) (they have not been experimentally observed up to now).

Periodic wave solutions of the differential advance–delay equation were considered by James in the context of Newton's cradle (James 2011) and homogeneous granular crystals (James 2012). In particular in James (2012), the existence proof was given for wavenumbers close to π and numerical approximations suggested that periodic waves with wavelength larger than a critical value are spectrally unstable. Convergence to solitary waves in the limit of infinite wavelengths and occurrence of compactons were also illustrated numerically and asymptotically in James (2012). In more recent work James et al. (2013) showed the non-existence of time-periodic breathers in homogeneous granular crystals and the existence of these breathers in Newton's cradle, where a discrete p-Schrödinger equation provides a robust approximation.

Periodic waves in a chain of finitely many beads closed in a periodic loop were approximated by Starosvetsky et al. in homogeneous (Starosvetsky and Vakakis 2010) and heterogeneous (Jayaprakash et al. 2011, 2012) granular chains by using numerical techniques based on Poincaré maps. Interesting enough, solitary waves were found in the limit of zero mass ratio between lighter and heavy beads in Jayaprakash et al. (2011). It is explained in Jayaprakash et al. (2011) that these solitary waves are in resonance with linear waves and hence they do not persist with respect to the mass ratio parameter. Numerical results of Jayaprakash et al. (2011) indicate the existence of a countable set of the mass ratio parameter values, for which solitary waves should exist, but no rigorous studies of this problem have been developed so far. Recent work Jayaprakash et al. (2012) contains numerical results on the existence of periodic traveling waves in diatomic granular chains.

Inspired by these recent development in existence and stability of periodic traveling waves in homogeneous and heterogeneous granular crystals, we address these problems from an analytical point of view. To obtain rigorous analytical results, we rely on the anti-continuum limit of the FPU lattice, which was recently explored in the context of existence and stability of discrete multi-site breathers by Yoshimura (2011). An earlier study of the anti-continuum limit in diatomic FPU lattices was developed by Livi et al. (1997).

By using a variant of the Implicit Function Theorem, we prove that every limiting periodic wave is uniquely continued with respect to the mass ratio parameter. By the perturbation theory arguments (which are similar to the recent work (Pelinovsky and Sakovich 2012) in the context of the Klein–Gordon lattices), we also show that the periodic waves with the wavelength larger than a certain critical value are spectrally stable. Our results are different from the asymptotic calculations in Jayaprakash et al. (2011), where a different limiting solution is considered in the anti-continuum limit.

The family of periodic nonlinear waves bifurcating from the anti-continuum limit are shown numerically to extend all way to the limit of equal masses for the granular beads. The periodic traveling waves of the homogeneous granular chains considered in James (2012) are different from the periodic waves extended here from the anti-continuum limit. In other words, the periodic waves in diatomic chains do not satisfy the reductions to the periodic waves in homogeneous chains even if the mass ratio is 1. Similar traveling waves consisting of binary oscillations in homogeneous chains were considered a while ago with center manifold reduction methods (Iooss and James 2005).

The paper is organized as follows. Section 2 introduces the model and sets up the scene for the search of periodic traveling waves. Continuation from the anti-continuum limit is developed in Sect. 3. Section 4 gives perturbative results that characterize Floquet multipliers in the spectral stability problem associated with the periodic waves near the anti-continuum limit. Numerical results are collected together in Sect. 5. Section 6 concludes the paper.

2 Formalism

2.1 The Model

We consider an infinite granular chain of spherical beads of alternating masses (a diatomic granular chain). The physical configuration of the diatomic chain is shown on Fig. 1. Dynamics of the granular beads of alternating masses obey the classical Newton equations of motion,

$$\begin{cases} m\ddot{x}_n = V'(y_n - x_n) - V'(x_n - y_{n-1}), \\ M\ddot{y}_n = V'(x_{n+1} - y_n) - V'(y_n - x_n), \end{cases} \quad n \in \mathbb{Z}, \tag{1}$$

where m and M are masses of light and heavy beads, whereas $\{x_n\}_{n\in\mathbb{Z}}$ and $\{y_n\}_{n\in\mathbb{Z}}$ are deviations of the beads coordinates from their reference positions. The interaction potential V represents the Hertzian contact forces for perfect spheres and is given by

$$V(x) = \frac{1}{1+\alpha}|x|^{1+\alpha}H(-x), \tag{2}$$

where $\alpha = \frac{3}{2}$ and H is the Heaviside step function with $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x \le 0$. See review Sen et al. (2008) for a derivation of the Hertzian

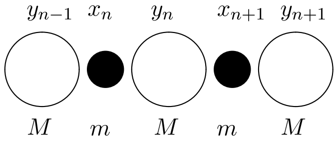

Fig. 1 Schematic representation of a diatomic granular chain

potential with $\alpha = \frac{3}{2}$ in the context of perfectly spherical granular crystals. Note that our main results can be extended to arbitrary $\alpha > 1$. The results definitely break for $\alpha = 1$ (the oscillators become harmonic) and for $0 < \alpha < 1$ (the potential V is not C^2).

The mass ratio is modeled by the parameter $\varepsilon^2 := \frac{m}{M}$. Using the substitution,

$$n \in \mathbb{Z}: \quad x_n(t) = u_{2n-1}(\tau), \quad y_n(t) = w_{2n}(\tau), \quad t = \sqrt{m}\tau, \tag{3}$$

we rewrite the system of Newton's equations (1) in the equivalent form

$$\begin{cases} \ddot{u}_{2n-1} = V'(w_{2n} - u_{2n-1}) - V'(u_{2n-1} - w_{2n-2}), \\ \ddot{w}_{2n} = \varepsilon^2 V'(u_{2n+1} - w_{2n}) - \varepsilon^2 V'(w_{2n} - u_{2n-1}), \end{cases} \quad n \in \mathbb{Z}. \tag{4}$$

The value $\varepsilon = 0$ correspond to the anti-continuum limit, when the heavy particles do not move.

At the limit of equal mass ratio $\varepsilon = 1$, we note the reduction,

$$n \in \mathbb{Z}: \quad u_{2n-1}(\tau) = U_{2n-1}(\tau), \quad w_{2n}(\tau) = U_{2n}(\tau), \tag{5}$$

for which the system of two granular chains (4) reduces to the homogeneous granular chain:

$$\ddot{U}_n = V'(U_{n+1} - U_n) - V'(U_n - U_{n-1}), \quad n \in \mathbb{Z}. \tag{6}$$

The system of Newton's equations (4) has two symmetries. One symmetry is the translational invariance of solutions with respect to τ, that is, if $\{u_{2n-1}(\tau), w_{2n}(\tau)\}_{n\in\mathbb{Z}}$ is a solution of (4), then

$$\big\{u_{2n-1}(\tau + b), w_{2n}(\tau + b)\big\}_{n\in\mathbb{Z}} \tag{7}$$

is also a solution of (4) for any $b \in \mathbb{R}$. The other symmetry is a uniform shift of coordinates $\{u_{2n-1}, w_{2n}\}_{n\in\mathbb{Z}}$, that is, if $\{u_{2n-1}(\tau), w_{2n}(\tau)\}_{n\in\mathbb{Z}}$ is a solution of (4), then

$$\big\{u_{2n-1}(\tau) + a, w_{2n}(\tau) + a\big\}_{n\in\mathbb{Z}} \tag{8}$$

is also a solution of (4) for any $a \in \mathbb{R}$.

The system of Newton's equations (4) can be cast as a Hamiltonian dynamical system with the symplectic structure:

$$\begin{aligned} &\frac{\mathrm{d}u_{2n-1}}{\mathrm{d}t} = \frac{\partial H}{\partial p_{2n-1}}, \qquad \frac{\mathrm{d}p_{2n-1}}{\mathrm{d}t} = -\frac{\partial H}{\partial u_{2n-1}}, \\ &\frac{\mathrm{d}w_{2n}}{\mathrm{d}t} = \frac{\partial H}{\partial q_{2n}}, \qquad \frac{\mathrm{d}q_{2n}}{\mathrm{d}t} = -\frac{\partial H}{\partial w_{2n}}, \quad n \in \mathbb{Z} \end{aligned} \tag{9}$$

and the Hamiltonian function

$$H = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(p_{2n-1}^2 + \varepsilon^2 q_{2n}^2\right) + \sum_{n\in\mathbb{Z}} V(w_{2n} - u_{2n-1}) + \sum_{n\in\mathbb{Z}} V(u_{2n-1} - w_{2n-2}), \quad (10)$$

written in canonical variables $\{u_{2n-1}, p_{2n-1} = \dot{u}_{2n-1}, w_{2n}, q_{2n} = \dot{w}_{2n}/\varepsilon^2\}_{n\in\mathbb{Z}}$.

2.2 Periodic Traveling Waves

We shall consider 2π-periodic solutions of the diatomic granular chain (4) satisfying

$$u_{2n-1}(\tau) = u_{2n-1}(\tau + 2\pi), \qquad w_{2n}(\tau) = w_{2n}(\tau + 2\pi), \quad \tau \in \mathbb{R},\ n \in \mathbb{Z}. \quad (11)$$

Traveling waves correspond to the special solution to the system of Newton's equations (4), which satisfies the following reduction:

$$u_{2n+1}(\tau) = u_{2n-1}(\tau + 2q), \qquad w_{2n+2}(\tau) = w_{2n}(\tau + 2q), \quad \tau \in \mathbb{R},\ n \in \mathbb{Z}, \quad (12)$$

where $q \in [0, \pi]$ is a free parameter. We note that the constraints (11) and (12) imply that there exist 2π-periodic functions u_* and w_* such that

$$u_{2n-1}(\tau) = u_*(\tau + 2qn), \qquad w_{2n}(\tau) = w_*(\tau + 2qn), \quad \tau \in \mathbb{R},\ n \in \mathbb{Z}. \quad (13)$$

In this context, q is inverse proportional to the wavelength of the periodic traveling wave over the chain $n \in \mathbb{Z}$. The functions u_* and w_* satisfy the following system of differential advance–delay equations:

$$\begin{cases} \ddot{u}_*(\tau) = V'(w_*(\tau) - u_*(\tau)) - V'(u_*(\tau) - w_*(\tau - 2q)), \\ \ddot{w}_*(\tau) = \varepsilon^2 V'(u_*(\tau + 2q) - w_*(\tau)) - \varepsilon^2 V'(w_*(\tau) - u_*(\tau)), \end{cases} \quad \tau \in \mathbb{R}. \quad (14)$$

Remark 1 A more general traveling periodic wave can be sought in the form

$$u_{2n-1}(\tau) = u_*(c\tau + 2qn), \qquad w_{2n}(\tau) = w_*(c\tau + 2qn), \quad \tau \in \mathbb{R},\ n \in \mathbb{Z},$$

where $c > 0$ is an arbitrary parameter. However, the parameter c can be normalized to one thanks to invariance of the system of Newton's equations (4) with the Hertzian potential (2) with respect to a scaling transformation.

Remark 2 For particular values $q = \frac{\pi m}{N}$, where m and N are positive integers such that $1 \le m \le N$, periodic traveling waves satisfy a system of $2mN$ second-order differential equations that follow from the system of lattice differential equations (4) subject to the periodic conditions:

$$u_{-1} = u_{2mN-1}, \qquad u_{2mN+1} = u_1, \qquad w_0 = w_{2mN}, \qquad w_{2mN+2} = w_2. \quad (15)$$

This reduction is useful for analysis of stability of periodic traveling waves and for numerical approximations.

2.3 Anti-continuum Limit

Let φ be a solution of the nonlinear oscillator equation,

$$\ddot{\varphi} = V'(-\varphi) - V'(\varphi) \quad \Rightarrow \quad \ddot{\varphi} + |\varphi|^{\alpha-1}\varphi = 0. \tag{16}$$

Because $\alpha = \frac{3}{2}$, bootstrapping arguments show that if there exists a classical 2π-periodic solution of the differential equation (16), then $\varphi \in C^3_{\mathrm{per}}(0, 2\pi)$.

The nonlinear oscillator equation (16) has the first integral

$$E = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{1+\alpha}|\varphi|^{\alpha+1}. \tag{17}$$

The phase portrait of the nonlinear oscillator (16) on the $(\varphi, \dot{\varphi})$-plane consists of a family of closed orbits around the only equilibrium point $(0, 0)$. Each orbit corresponds to the T-periodic solution for φ, where T is determined uniquely by energy E. It is well-known (James 2012; Yoshimura 2011) that, for $\alpha > 1$, the period T is a monotonically decreasing function of E such that $T \to \infty$ as $E \to 0$ and $T \to 0$ as $E \to \infty$. Therefore, there exists a unique $E_0 \in \mathbb{R}_+$ such that $T = 2\pi$ for this $E = E_0$. We also know that the nonlinear oscillator (16) is non-degenerate in the sense that $T'(E_0) \neq 0$ (to be more precise, $T'(E_0) < 0$).

In what follows, we only consider 2π-periodic functions φ which are defined by (17) for $E = E_0$. For uniqueness arguments, we shall consider initial conditions $\varphi(0) = 0$ and $\dot{\varphi}(0) > 0$, which determine uniquely one of the two odd 2π-periodic functions φ.

The limiting 2π-periodic traveling wave solution at $\varepsilon = 0$ should satisfy the constraints (12), which we do by choosing, for any fixed $q \in [0, \pi]$,

$$\varepsilon = 0: \quad u_{2n-1}(\tau) = \varphi(\tau + 2qn), \quad w_{2n}(\tau) = 0, \quad \tau \in \mathbb{R},\ n \in \mathbb{Z}. \tag{18}$$

To prove the persistence of this limiting solution with respect to the mass ratio parameter ε^2, we shall work in the Sobolev spaces of odd 2π-periodic functions for $\{u_{2n-1}\}_{n\in\mathbb{Z}}$,

$$H^k_u = \left\{u \in H^k_{\mathrm{per}}(0, 2\pi) : u(-\tau) = -u(\tau),\ \tau \in \mathbb{R}\right\}, \quad k \in \mathbb{N}_0, \tag{19}$$

and in the Sobolev spaces of 2π-periodic functions with zero mean for $\{w_{2n}\}_{n\in\mathbb{Z}}$,

$$H^k_w = \left\{w \in H^k_{\mathrm{per}}(0, 2\pi) : \int_0^{2\pi} w(\tau)\,\mathrm{d}\tau = 0\right\}, \quad k \in \mathbb{N}_0. \tag{20}$$

The constraints in (19) and (20) reflects the presence of the two symmetries (7) and (8). The two symmetries generate a two-dimensional kernel of the linearized operators. Under the constraints in (19) and (20), the kernel of the linearized operators is trivial, zero-dimensional.

It will be clear from analysis that the vector space H^k_w defined by (20) is not precise enough to prove the persistence of traveling wave solutions satisfying the constraints

(12). Instead of this space, for any fixed $q \in [0, \pi]$, we introduce the vector space $\tilde{H}^k_w$ by

$$\tilde{H}^k_w = \left\{ w \in H^k_{\text{per}}(0, 2\pi) : w(\tau) = -w(-\tau - 2q) \right\}, \quad k \in \mathbb{N}_0. \tag{21}$$

We note that $\tilde{H}^k_w \subset H^k_w$, because if the constraint $w(\tau) = -w(-\tau - 2q)$ is satisfied, then the 2π-periodic function w has zero mean. We also note that symmetry constraint in $\tilde{H}^k_w$ can be written as a shifted version of the symmetry constraint in H^k_u: $w(\tau - q) = -w(-\tau - q)$. Although this constraint is not induced directly by the symmetries (7) and (8), we find that it provides a sufficient frame for application of the Implicit Function Theorem.

2.4 Special Periodic Traveling Waves

Before developing persistence analysis, we shall point out three remarkable explicit periodic traveling solutions of the diatomic granular chain (4) for $q = 0$, $q = \frac{\pi}{2}$ and $q = \pi$. For $q = \frac{\pi}{2}$, we have an exact solution

$$q = \frac{\pi}{2}: \quad u_{2n-1}(\tau) = \varphi(\tau + \pi n), \quad w_{2n}(\tau) = 0. \tag{22}$$

This solution preserves the constraint $V'(u_{2n+1}) = V'(-u_{2n-1})$ in the system of Newton's equations (4) thanks to the symmetry $\varphi(\tau - \pi) = \varphi(\tau + \pi) = -\varphi(\tau)$ of the 2π-periodic solution of the nonlinear oscillator equation (16).

For either $q = 0$ or $q = \pi$, we obtain another exact solution,

$$q = \{0, \pi\}: \quad u_{2n-1}(\tau) = \frac{\varphi(\tau)}{(1 + \varepsilon^2)^3}, \quad w_{2n}(\tau) = \frac{-\varepsilon^2 \varphi(\tau)}{(1 + \varepsilon^2)^3}. \tag{23}$$

By construction, these solutions (22) and (23) persist for any $\varepsilon \geq 0$. We shall investigate if the continuations are unique near $\varepsilon = 0$ for these special values of q and if there exists a unique continuation of the general limiting solution (18) in ε for any other fixed value of $q \in [0, \pi]$.

Furthermore, we note that the exact solution (23) for $q = \pi$ at $\varepsilon = 1$ satisfies the constraint (5) with $U_{2n-1}(\tau) = -U_{2n}(\tau) = U_{2n}(\tau - \pi)$. This reduction indicates that the periodic traveling wave solution (23) for $q = \pi$ and $\varepsilon = 1$ satisfies the homogeneous granular chain (6) and coincides with the solution considered by James (2012). On the other hand, the exact solutions (22) for $q = \frac{\pi}{2}$ and (23) for $q = 0$ do not produce any solutions of the homogeneous granular chain at $\varepsilon = 1$. This fact implies that there exist generally two distinct solutions at $\varepsilon = 1$, one is continued from $\varepsilon = 0$ and the other one is constructed from the solution of the homogeneous granular chain (6) in James (2012).

3 Persistence of Periodic Traveling Waves Near $\varepsilon = 0$

3.1 Main Result

We consider the system of differential advance–delay equations (14). The limiting solution (18) becomes now

$$\varepsilon = 0: \quad u_*(\tau) = \varphi(\tau), \quad w_*(\tau) = 0, \quad \tau \in \mathbb{R}, \tag{24}$$

where φ is a unique odd 2π-periodic solution of the nonlinear oscillator equation (16) with $\dot{\varphi}(0) > 0$. We now formulate the main result of this section.

Theorem 1 *Fix $q \in [0, \pi]$. There is a unique C^1 continuation of 2π-periodic traveling wave (24) in ε^2, that is, there is a $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, there are a positive constant C and a unique 2π-periodic solution $(u_*, w_*) \in H^2_u \times \tilde{H}^2_w$ of the system of differential advance–delay equations (14) such that*

$$\|u_* - \varphi\|_{H^2_{\text{per}}} + \|w_*\|_{H^2_{\text{per}}} \le C\varepsilon^2. \tag{25}$$

Remark 3 By Theorem 1, the limiting solution (24) for $q \in \{0, \frac{\pi}{2}, \pi\}$ is uniquely continued in ε. These continuations coincide with the exact solutions (22) and (23).

3.2 Formal Expansions in Powers of ε

Let us first consider formal expansions in powers of ε to understand the persistence analysis from $\varepsilon = 0$. Expanding the solution of the differential advance–delay equations (14), we write

$$u_*(\tau) = \varphi(\tau) + \varepsilon^2 u_*^{(2)}(\tau) + \mathrm{o}\big(\varepsilon^2\big), \qquad w_*(\tau) = \varepsilon^2 w_*^{(2)}(\tau) + \mathrm{o}\big(\varepsilon^2\big), \tag{26}$$

and obtain the linear inhomogeneous equations

$$\ddot{w}_*^{(2)}(\tau) = F_w^{(2)}(\tau) := V'\big(\varphi(\tau + 2q)\big) - V'\big(-\varphi(\tau)\big) \tag{27}$$

and

$$\begin{aligned} &\ddot{u}_*^{(2)}(\tau) + \alpha\big|\varphi(\tau)\big|^{\alpha-1} u_*^{(2)}(\tau) \\ &\quad = F_u^{(2)}(\tau) := V''\big(-\varphi(\tau)\big) w_*^{(2)}(\tau) + V''\big(\varphi(\tau)\big) w_*^{(2)}(\tau - 2q). \end{aligned} \tag{28}$$

Because V is C^2 but not C^3, we have to truncate the formal expansion (26) at $\mathrm{o}(\varepsilon^2)$ to indicate that there are obstacles to continue the power series beyond terms of the $\mathcal{O}(\varepsilon^2)$ order.

Let us consider two differential operators

$$L_0 = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}: \quad H^2_{\text{per}}(0, 2\pi) \to L^2_{\text{per}}(0, 2\pi), \tag{29}$$

$$L = \frac{d^2}{d\tau^2} + \alpha\left|\varphi(\tau)\right|^{\alpha-1}: \quad H^2_{\mathrm{per}}(0,2\pi) \to L^2_{\mathrm{per}}(0,2\pi). \tag{30}$$

As a consequence of two symmetries, these operators are not invertible because they admit one-dimensional kernels,

$$\mathrm{Ker}(L_0) = \mathrm{span}\{1\}, \qquad \mathrm{Ker}(L) = \mathrm{span}\{\dot{\varphi}\}. \tag{31}$$

Note that the kernel of L is one-dimensional under the constraint $T'(E_0) \neq 0$ (see Lemma 3 in James (2012) for a review of this standard result).

To find uniquely solutions of the inhomogeneous equations (27) and (28) in function spaces H^2_w and H^2_u, respectively, see (19) and (20) for definition of function spaces, the source terms must satisfy the Fredholm conditions

$$\langle 1, F^{(2)}_w\rangle_{L^2_{\mathrm{per}}} = 0 \quad \text{and} \quad \langle \dot{\varphi}, F^{(2)}_u\rangle_{L^2_{\mathrm{per}}} = 0.$$

The first Fredholm condition is satisfied,

$$\begin{aligned}
&\int_0^{2\pi} \left[V'\big(\varphi(\tau+2q)\big) - V'\big(-\varphi(\tau)\big)\right] d\tau \\
&\quad = \int_0^{2\pi} V'\big(\varphi(\tau+2q)\big)\, d\tau - \int_0^{2\pi} V'\big(-\varphi(\tau)\big)\, d\tau = 0,
\end{aligned}$$

because the mean value of a periodic function is independent on the limits of integration and the function φ is odd in τ. Since $F^{(2)}_w \in L^2_w$, there is a unique solution $w^{(2)} \in H^2_w$ of the linear inhomogeneous equation (27).

The second Fredholm condition is satisfied,

$$\int_0^{2\pi} \dot{\varphi}(\tau)\left[V''\big(-\varphi(\tau)\big) w^{(2)}_*(\tau) + V''\big(\varphi(\tau)\big) w^{(2)}_*(\tau - 2q)\right] d\tau = 0,$$

if the function $F^{(2)}_u$ is odd in τ. If this is the case, then $F^{(2)}_u \in L^2_u$ and there is a unique solution $u^{(2)} \in H^2_u$ of the linear inhomogeneous equation (28). To show that $F^{(2)}_u$ is odd in τ, we will prove that $w^{(2)}_*$ satisfies the reduction

$$w^{(2)}_*(\tau) = -w^{(2)}_*(-\tau - 2q) \quad \Rightarrow \quad F^{(2)}_u(-\tau) = -F^{(2)}_u(\tau), \quad \tau \in \mathbb{R}. \tag{32}$$

It follows from the linear inhomogeneous equation (27) that

$$\begin{aligned}
\ddot{w}^{(2)}_*(\tau) + \ddot{w}^{(2)}_*(-\tau - 2q) &= V'\big(\varphi(\tau+2q)\big) - V'\big(-\varphi(\tau)\big) + V'\big(\varphi(-\tau)\big) \\
&\quad - V'\big(-\varphi(-\tau-2q)\big) = 0,
\end{aligned}$$

where the last equality appears because φ is odd in τ. Integrating this equation twice and using the fact that $w^{(2)}_* \in H^2_w$, we obtain reduction (32). Note that the reduction (32) implies that $w^{(2)}_* \in \tilde{H}^2_w$, where $\tilde{H}^2_w \subset H^2_w$ is given by (21).

3.3 Proof of Theorem 1

To prove Theorem 1, we shall consider the vector fields of the differential advance–delay equations (14),

$$\begin{cases} F_u(u(\tau), w(\tau)) := V'(w(\tau) - u(\tau)) - V'(u(\tau) - w(\tau - 2q)), \\ F_w(u(\tau), w(\tau), \varepsilon) := \varepsilon^2 V'(u(\tau + 2q) - w(\tau)) - \varepsilon^2 V'(w(\tau) - u(\tau)). \end{cases} \tag{33}$$

We are looking for a strong solution (u_*, w_*) of the system (14) satisfying the reduction,

$$u_*(-\tau) = -u_*(\tau), \qquad w_*(\tau) = -w_*(-\tau - 2q), \quad \tau \in \mathbb{R}, \tag{34}$$

that is, $u_* \in H_u^2(\mathbb{R})$ and $w_* \in \tilde{H}_w^2(\mathbb{R})$.

If $(u, w) \in H_u^2 \times \tilde{H}_w^2$, then F_u is odd in τ. Furthermore, since V is C^2, then F_u is a C^1 map from $H_u^2 \times \tilde{H}_w^2$ to L_u^2 and its Jacobian at $w = 0$ is given by

$$D_u F_u(u, 0) = -V''(-u) - V''(u) = -\alpha |u|^{\alpha - 1},$$
$$D_w F_u(u, 0) = V''(-u) - V''(u) = \alpha |u|^{\alpha - 1} \operatorname{sign}(u).$$

On the other hand, under the constraints (34), we have $F_w \in L_w^2$, because

$$\int_0^{2\pi} F_w\big(u(\tau), w(\tau), \varepsilon\big)\, d\tau = \varepsilon^2 \int_0^{2\pi} V'\big(u(\tau + 2q) + w(-\tau - 2q)\big)\, d\tau - \varepsilon^2 \int_0^{2\pi} V'\big(w(\tau) + u(-\tau)\big)\, d\tau = 0.$$

Moreover, under the constraints (34), we actually have $F_w \in \tilde{L}_w^2$ because

$$\begin{aligned} &F_w\big(u(\tau), w(\tau), \varepsilon\big) + F_w\big(u(-\tau - 2q), w(-\tau - 2q), \varepsilon\big) \\ &\quad = \varepsilon^2 V'\big(u(\tau + 2q) - w(\tau)\big) - \varepsilon^2 V'\big(w(\tau) - u(\tau)\big) \\ &\qquad + \varepsilon^2 V'\big(u(-\tau) - w(-\tau - 2q)\big) - \varepsilon^2 V'\big(w(-\tau - 2q) - u(-\tau - 2q)\big) \\ &\quad = 0. \end{aligned}$$

Since V is C^2, then F_w is a C^1 map from $H_u^2 \times \tilde{H}_w^2$ to $\tilde{L}_w^2$. We also note that $F_w(u, w, \varepsilon) = \varepsilon^2 \tilde{F}_w(u, w)$, where $\tilde{F}_w(u, w)$ is ε-independent.

Let us now define the nonlinear operator

$$\begin{cases} f_u(u, w, \varepsilon) := \frac{d^2 u}{d\tau^2} - F_u(u, w), \\ f_w(u, w, \varepsilon) := \frac{d^2 w}{d\tau^2} - \varepsilon^2 \tilde{F}_w(u, w). \end{cases} \tag{35}$$

The nonlinear operator $(f_u, f_w) : H_u^2 \times \tilde{H}_w^2 \times \mathbb{R} \to L_u^2 \times \tilde{L}_w^2$ is C^1 near the point $(\varphi, 0, 0) \in H_u^2 \times \tilde{H}_w^2 \times \mathbb{R}$. To apply the Implicit Function Theorem near this point, we need $(f_u, f_w) = (0, 0)$ at $(u, w, \varepsilon) = (\varphi, 0, 0)$ and the invertibility of the Jacobian operator (f_u, f_w) with respect to (u, w) near $(\varphi, 0, 0)$.

The Jacobian operator of (f_u, f_w) at $(\varphi, 0, 0)$ is given by the matrix

$$\begin{bmatrix} L & \alpha|\varphi|^{\alpha-1}\mathrm{sign}(\varphi) \\ 0 & L_0 \end{bmatrix},$$

where operators L and L_0 are defined by (29) and (30). The kernels of these operators in (31) are zero-dimensional in the constrained vector spaces (19) and (20) (we actually use space (21) in place of space (20)).

By the Implicit Function Theorem, there exists a C^1 continuation of the limiting solution (24) with respect to ε^2 as the 2π-periodic solutions $(u_*, w_*) \in H^2_u \times \tilde{H}^2_w$ of the system of differential advance–delay equations (14) near $\varepsilon = 0$. The proof of Theorem 1 is complete.

4 Spectral Stability of Periodic Traveling Waves Near $\varepsilon = 0$

4.1 Linearization at the Periodic Traveling Waves

We shall consider the diatomic granular chain (4), which admits for small $\varepsilon > 0$ the periodic traveling waves in the form (13), where (u_*, w_*) is defined by Theorem 1. Linearizing the system of nonlinear equations (4) at the periodic traveling waves (13), we obtain the system of linearized equations for small perturbations,

$$\begin{cases} \ddot{u}_{2n-1} = V''(w_*(\tau+2qn) - u_*(\tau+2qn))(w_{2n} - u_{2n-1}) \\ \qquad\quad - V''(u_*(\tau+2qn) - w_*(\tau+2qn-2q))(u_{2n-1} - w_{2n-2}), \\ \ddot{w}_{2n} = \varepsilon^2 V''(u_*(\tau+2qn+2q) - w_*(\tau+2qn))(u_{2n+1} - w_{2n}) \\ \qquad\quad - \varepsilon^2 V''(w_*(\tau+2qn) - u_*(\tau+2qn))(w_{2n} - u_{2n-1}), \end{cases} \tag{36}$$

where $n \in \mathbb{Z}$. A technical complication is that V'' is continuous but not continuous differentiable. This will complicate our analysis of perturbation expansions for small $\varepsilon > 0$. Note that the technical complications do not occur for the exact solutions (22) and (23). Indeed, for the exact solution (22) with $q = \frac{\pi}{2}$, the linearized system (36) is rewritten explicitly as

$$\begin{cases} \ddot{u}_{2n-1} + \alpha|\varphi|^{\alpha-1}u_{2n-1} = V''(-\varphi)w_{2n} + V''(\varphi)w_{2n-2}, \\ \ddot{w}_{2n} + 2\varepsilon^2 V''(-\varphi)w_{2n} = \varepsilon^2 V''(-\varphi)(u_{2n+1} + u_{2n-1}). \end{cases} \tag{37}$$

For the exact solution (23) with $q = 0$ or $q = \pi$, the linearized system (36) is rewritten explicitly as

$$\begin{cases} \ddot{u}_{2n-1} + \frac{\alpha}{1+\varepsilon^2}|\varphi|^{\alpha-1}u_{2n-1} = \frac{1}{1+\varepsilon^2}(V''(-\varphi)w_{2n} + V''(\varphi)w_{2n-2}), \\ \ddot{w}_{2n} + \frac{\alpha\varepsilon^2}{1+\varepsilon^2}|\varphi|^{\alpha-1}w_{2n} = \frac{\varepsilon^2}{1+\varepsilon^2}(V''(\varphi)u_{2n+1} + V''(-\varphi)u_{2n-1}). \end{cases} \tag{38}$$

In both cases, the linearized systems (37) and (38) are analytic in ε near $\varepsilon = 0$.

The system of linearized equations (36) has the same symplectic structure (9), but the Hamiltonian function is now given by

$$\begin{aligned} H = {} & \frac{1}{2}\sum_{n\in\mathbb{Z}}\bigl(p_{2n-1}^2+\varepsilon^2 q_{2n}^2\bigr) \\ & + \frac{1}{2}\sum_{n\in\mathbb{Z}} V''\bigl(w_*(\tau+2qn)-u_*(\tau+2qn)\bigr)(w_{2n}-u_{2n-1})^2 \\ & + \frac{1}{2}\sum_{n\in\mathbb{Z}} V''\bigl(u_*(\tau+2qn)-w_*(\tau+2qn-2q)\bigr)(u_{2n-1}-w_{2n-2})^2. \end{aligned} \tag{39}$$

The Hamiltonian H is quadratic in canonical variables $\{u_{2n-1}, p_{2n-1}=\dot{u}_{2n-1}, w_{2n}, q_{2n}=\dot{w}_{2n}/\varepsilon^2\}_{n\in\mathbb{Z}}$.

4.2 Main Result

Because coefficients of the linearized equations (36) are 2π-periodic in τ, we shall look for an infinite-dimensional analog of the Floquet theorem that states that all solutions of the linear system with 2π-periodic coefficients satisfies the reduction

$$\mathbf{u}(\tau+2\pi)=\mathcal{M}\mathbf{u}(\tau), \quad \tau\in\mathbb{R}, \tag{40}$$

where $\mathbf{u} := [\dots, w_{2n-2}, u_{2n-1}, w_{2n}, u_{2n+1}, \dots]$ and $\mathcal{M}$ is the monodromy operator. Eigenvalues of the monodromy operator $\mathcal{M}$ denoted by μ are called the *Floquet multipliers*.

Remark 4 Let $q=\frac{\pi m}{N}$ for some positive integers m and N such that $1\le m\le N$. In this case, the system of nonlinear equations (4) can be closed into a chain of $2mN$ second-order differential equations subject to the periodic boundary conditions (15). Similarly, the system of linearized equations (36) can also be closed as a system of $2mN$ second-order equations and the monodromy operator $\mathcal{M}$ becomes an infinite diagonal composition of an identical $4mN$-by-$4mN$ Floquet matrix.

We can find eigenvalues of the monodromy operator $\mathcal{M}$ by looking for the set of eigenvectors in the form,

$$u_{2n-1}(\tau)=U_{2n-1}(\tau)e^{\lambda\tau}, \qquad w_{2n}(\tau)=\varepsilon W_{2n}(\tau)e^{\lambda\tau}, \quad \tau\in\mathbb{R}, \tag{41}$$

where (U_{2n-1}, W_{2n}) are 2π-periodic functions and the admissible values of λ are found from the existence of these 2π-periodic functions. The admissible values of λ are called the *characteristic exponents* and they define the Floquet multipliers μ by the standard formula $\mu=e^{2\pi\lambda}$.

Eigenvectors (41) are defined as 2π-periodic solutions of the linear eigenvalue problem,

$$\begin{cases} \ddot{U}_{2n-1} + 2\lambda \dot{U}_{2n-1} + \lambda^2 U_{2n-1} \\ \quad = V''(w_*(\tau+2qn) - u_*(\tau+2qn))(\varepsilon W_{2n} - U_{2n-1}) \\ \qquad - V''(u_*(\tau+2qn) - w_*(\tau+2qn-2q))(U_{2n-1} - \varepsilon W_{2n-2}), \\ \ddot{W}_{2n} + 2\lambda \dot{W}_{2n} + \lambda^2 W_{2n} \\ \quad = \varepsilon V''(u_*(\tau+2qn+2q) - w_*(\tau+2qn))(U_{2n+1} - \varepsilon W_{2n}) \\ \qquad - \varepsilon V''(w_*(\tau+2qn) - u_*(\tau+2qn))(\varepsilon W_{2n} - U_{2n-1}). \end{cases} \tag{42}$$

Associated with simple imaginary characteristic exponents $\lambda \in \mathrm{i}\mathbb{R}$, we define the Krein signature of such eigenvalues as the sign of the 2-form associated with the symplectic structure (9):

$$\sigma = \mathrm{i} \sum_{n \in \mathbb{Z}} [u_{2n-1} \bar{p}_{2n-1} - \bar{u}_{2n-1} p_{2n-1} + w_{2n} \bar{q}_{2n} - \bar{w}_{2n} q_{2n}], \tag{43}$$

where $\{u_{2n-1}, p_{2n-1} = \dot{u}_{2n-1}, w_{2n}, q_{2n} = \dot{w}_{2n}/\varepsilon^2\}_{n\in\mathbb{Z}}$ is obtained from the corresponding eigenvector (41). By the symmetry of the linear eigenvalue problem (42), it follows that if λ is an eigenvalue, then $\bar{\lambda}$ is also an eigenvalue, whereas the 2-form σ is constant with respect to $\tau \in \mathbb{R}$.

Remark 5 The Krein signature plays an important role in the studies of spectral stability of periodic solutions (see Sect. 4 in Aubry 1997). In particular, instabilities associated with complex characteristic exponents with $\mathrm{Re}(\lambda) > 0$ typically arise when two imaginary characteristic exponents λ with opposite Krein signatures coalesce (Bridges 1990; MacKay 1986; Vougalter and Pelinovsky 2006). The count of eigenvalues of different Krein signatures is also important for the control of the total number of unstable eigenvalues in Hamiltonian dynamical systems (Chugunova and Pelinovsky 2010; Kapitula et al. 2004; Pelinovsky 2005).

If $\varepsilon = 0$, the monodromy operator $\mathcal{M}$ in (40) is block-diagonal and consists of an infinite set of 2-by-2 Jordan blocks, because the diatomic granular chain (4) is decoupled into a countable set of uncoupled second-order differential equations. As a result, the linear eigenvalue problem (42) with the limiting solution (24) admits an infinite set of 2π-periodic solutions for $\lambda = 0$,

$$\varepsilon = 0: \quad U_{2n-1}^{(0)} = c_{2n-1} \dot{\varphi}(\tau + 2qn), \quad W_{2n}^{(0)} = a_{2n}, \quad n \in \mathbb{Z}, \tag{44}$$

where $\{c_{2n-1}, a_{2n}\}_{n\in\mathbb{Z}}$ are arbitrary coefficients. Besides eigenvectors (44), there exists another countable set of generalized eigenvectors for each of the uncoupled second-order differential equations, which contribute to 2-by-2 Jordan blocks. Each block corresponds to the double Floquet multiplier $\mu = 1$ or the double characteristic exponent $\lambda = 0$. When $\varepsilon \neq 0$ but $\varepsilon \ll 1$, the characteristic exponent $\lambda = 0$ of a high algebraic multiplicity splits. We shall study the splitting of the characteristic exponents λ by the perturbation arguments.

We now formulate the main result of this section.

Theorem 2 *Fix $q = \frac{\pi m}{N}$ for some positive integers m and N such that $1 \leq m \leq N$. Let $(u_*, w_*) \in H^2_u \times \tilde{H}^2_w$ be defined by Theorem 1 for sufficiently small positive ε. Consider the linear eigenvalue problem (42) subject to $2mN$-periodic boundary conditions (15). There is a $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, there exists $q_0(\varepsilon) \in (0, \frac{\pi}{2})$ such that for every $q \in (0, q_0(\varepsilon))$ or $q \in (\pi - q_0(\varepsilon), \pi]$, no values of λ with $\mathrm{Re}(\lambda) \neq 0$ exist, whereas for every $q \in (q_0(\varepsilon), \pi - q_0(\varepsilon))$, there exist characteristic values of λ with $\mathrm{Re}(\lambda) > 0$.*

Remark 6 By Theorem 2, periodic traveling waves are spectrally stable for $q \in (0, q_0(\varepsilon))$ and $q \in (\pi - q_0(\varepsilon), \pi]$ and unstable for $q \in (q_0(\varepsilon), \pi - q_0(\varepsilon))$. The numerical value of $q_0(\varepsilon)$ as $\varepsilon \to 0$ is $q_0(0) \approx 0.915$. Therefore, the linearized system (37) for the exact solution (22) with $q = \frac{\pi}{2}$ subject to 4-periodic boundary conditions ($m = 1$ and $N = 2$) is unstable for small $\varepsilon > 0$, whereas the linearized system (38) for the exact solution (23) with $q = \pi$ subject to 2-periodic boundary conditions ($m = 1$ and $N = 1$) is stable for small $\varepsilon > 0$.

Remark 7 The result of Theorem 2 is expected to hold for all values of q in $[0, \pi]$ but the spectrum of the linear eigenvalue problem (42) for the characteristic exponent λ becomes continuous and the spectral band is connected to zero. An infinite-dimensional analog of perturbation theory is required to study the eigenvalues of the monodromy operator $\mathcal{M}$ in this case.

Remark 8 The case $q = 0$ is degenerate for an application of the perturbation theory. Nevertheless, we show numerically that the linearized system (38) for the exact solution (23) with $q = 0$ ($m = 1$ and $N \to \infty$) is stable for small $\varepsilon > 0$ and all characteristic exponents are at least double for any $\varepsilon > 0$.

4.3 Formal Perturbation Expansions

We would normally expect splitting $\lambda = \mathcal{O}(\varepsilon^{1/2})$ if the limiting linear eigenvalue problem at $\varepsilon = 0$ is diagonally decomposed into 2-by-2 Jordan blocks (Pelinovsky and Sakovich 2012). However, in the linearized diatomic granular chain (42), this splitting occurs in a higher order, that is, $\lambda = \mathcal{O}(\varepsilon)$, because the coupling between the particles of equal masses shows up at the $\mathcal{O}(\varepsilon^2)$ order of the perturbation theory. Regular perturbation computations in $\mathcal{O}(\varepsilon^2)$ would require V'' to be at least C^1, which we do not have. In the computations below, we neglect this obstacle, which is possible for at least $q = \frac{\pi}{2}$ and $q = \pi$. For other values of q, the formal perturbation expansion of this section will be justified in Sect. 4.7 with a kind of renormalization technique.

Expanding 2π-periodic solutions of the linear eigenvalue problem (42), we write

$$\lambda = \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)} + \mathrm{o}(\varepsilon^2) \tag{45}$$

and

$$\begin{cases} U_{2n-1} = U^{(0)}_{2n-1} + \varepsilon U^{(1)}_{2n-1} + \varepsilon^2 U^{(2)}_{2n-1} + \mathrm{o}(\varepsilon^2), \\ W_{2n} = W^{(0)}_{2n} + \varepsilon W^{(1)}_{2n} + \varepsilon^2 W^{(2)}_{2n} + \mathrm{o}(\varepsilon^2), \end{cases} \tag{46}$$

where the zeroth-order terms are given by (44). To determine corrections of these expansions uniquely, we shall require that

$$\langle \dot{\varphi}, U_{2n-1}^{(j)} \rangle_{L^2_{\text{per}}} = \langle 1, W_{2n}^{(j)} \rangle_{L^2_{\text{per}}} = 0, \quad n \in \mathbb{Z}, \ j = 1, 2. \tag{47}$$

Indeed, if $U_{2n-1}^{(j)}$ contains a component, which is parallel to $\dot{\varphi}$, then the corresponding term only changes the value of c_{2n-1} in the eigenvector (44), which is yet to be determined. Similarly, if a 2π-periodic function $W_{2n}^{(j)}$ has a nonzero mean value, then the mean value of $W_{2n}^{(j)}$ only changes the value of a_{2n} in the eigenvector (44), which is yet to be determined.

The linearized equations (42) are satisfied at the $\mathcal{O}(\varepsilon^0)$ order. Collecting terms at the $\mathcal{O}(\varepsilon)$ order, we obtain

$$\begin{cases} \ddot{U}_{2n-1}^{(1)} + \alpha|\varphi(\tau+2qn)|^{\alpha-1} U_{2n-1}^{(1)} \\ \quad = -2\lambda^{(1)} \dot{U}_{2n-1}^{(0)} + V''(-\varphi(\tau+2qn)) W_{2n}^{(0)} + V''(\varphi(\tau+2qn)) W_{2n-2}^{(0)}, \\ \ddot{W}_{2n}^{(1)} = -2\lambda^{(1)} \dot{W}_{2n}^{(0)} + V''(\varphi(\tau+2qn+2q)) U_{2n+1}^{(0)} \\ \qquad + V''(-\varphi(\tau+2qn)) U_{2n-1}^{(0)}. \end{cases} \tag{48}$$

Let us define solutions of the following linear inhomogeneous equations:

$$\ddot{v} + \alpha|\varphi|^{\alpha-1} v = -2\ddot{\varphi}, \tag{49}$$

$$\ddot{y}_{\pm} + \alpha|\varphi|^{\alpha-1} y_{\pm} = V''(\pm\varphi), \tag{50}$$

$$\ddot{z}_{\pm} = V''(\pm\varphi)\dot{\varphi}. \tag{51}$$

If we can find uniquely 2π-periodic solutions of these equations such that

$$\langle \dot{\varphi}, v \rangle_{L^2_{\text{per}}} = \langle \dot{\varphi}, y_{\pm} \rangle_{L^2_{\text{per}}} = \langle 1, z_{\pm} \rangle_{L^2_{\text{per}}} = 0,$$

then the perturbation equations (48) at the $\mathcal{O}(\varepsilon)$ order are satisfied with the solution:

$$\begin{cases} U_{2n-1}^{(1)} = c_{2n-1}\lambda^{(1)} v(\tau+2qn) + a_{2n} y_-(\tau+2qn) + a_{2n-2} y_+(\tau+2qn), \\ W_{2n}^{(1)} = c_{2n+1} z_+(\tau+2qn+2q) + c_{2n-1} z_-(\tau+2qn). \end{cases} \tag{52}$$

The linearized equations (42) are now satisfied up to the $\mathcal{O}(\varepsilon)$ order. Collecting terms at the $\mathcal{O}(\varepsilon^2)$ order, we obtain

$$\begin{cases} \ddot{U}_{2n-1}^{(2)} + \alpha|\varphi(\tau+2qn)|^{\alpha-1} U_{2n-1}^{(2)} \\ \quad = -2\lambda^{(1)} \dot{U}_{2n-1}^{(1)} - 2\lambda^{(2)} \dot{U}_{2n-1}^{(0)} - (\lambda^{(1)})^2 U_{2n-1}^{(0)} \\ \qquad + V''(-\varphi(\tau+2qn)) W_{2n}^{(1)} + V''(\varphi(\tau+2qn)) W_{2n-2}^{(1)} \\ \qquad - V'''(-\varphi(\tau+2qn))(w_*^{(2)}(\tau+2qn) - u_*^{(2)}(\tau+2qn)) U_{2n-1}^{(0)} \\ \qquad - V'''(\varphi(\tau+2qn))(u_*^{(2)}(\tau+2qn) - w_*^{(2)}(\tau+2qn-2q)) U_{2n-1}^{(0)}, \\ \ddot{W}_{2n}^{(2)} = -2\lambda^{(1)} \dot{W}_{2n}^{(1)} - 2\lambda^{(2)} \dot{W}_{2n}^{(0)} - (\lambda^{(1)})^2 W_{2n}^{(0)} \\ \qquad + V''(\varphi(\tau+2qn+2q))(U_{2n+1}^{(1)} - W_{2n}^{(0)}) \\ \qquad + V''(-\varphi(\tau+2qn))(U_{2n-1}^{(1)} - W_{2n}^{(0)}), \end{cases} \tag{53}$$

where corrections $u_*^{(2)}$ and $w_*^{(2)}$ are defined by expansion (26).

To solve the linear inhomogeneous equations (53), the source terms have to satisfy the Fredholm conditions because both operators L and L_0 defined by (29) and (30) have one-dimensional kernels. Therefore, we require the first equation of system (53) to be orthogonal to $\dot{\varphi}$ and the second equation of system (53) to be orthogonal to 1 on $[-\pi, \pi]$. Substituting (44) and (52) to the orthogonality conditions and taking into account the symmetry between couplings of lattice sites on $\mathbb{Z}$, we obtain the difference equations for $\{c_{2n-1}, a_{2n}\}_{n\in\mathbb{Z}}$:

$$\begin{cases} K\Lambda^2 c_{2n-1} = M_1(c_{2n+1} + c_{2n-3} - 2c_{2n-1}) + L_1\Lambda(a_{2n} - a_{2n-2}), \\ \Lambda^2 a_{2n} = M_2(a_{2n+2} + a_{2n-2} - 2a_{2n}) + L_2\Lambda(c_{2n+1} - c_{2n-1}), \end{cases} \tag{54}$$

where $\Lambda \equiv \lambda^{(1)}$, and (K, M_1, M_2, L_1, L_2) are numerical coefficients to be computed from the projections. In particular, we obtain

$$\begin{aligned} K &= \int_{-\pi}^{\pi} \big(2\dot{v}(\tau) + \dot{\varphi}(\tau)\big)\dot{\varphi}(\tau)\,\mathrm{d}\tau, \\ M_1 &= \int_{-\pi}^{\pi} V''\big(-\varphi(\tau)\big)\dot{\varphi}(\tau)z_+(\tau + 2q)\,\mathrm{d}\tau = \int_{-\pi}^{\pi} V''\big(\varphi(\tau)\big)\dot{\varphi}(\tau)z_-(\tau - 2q)\,\mathrm{d}\tau, \\ M_2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} V''\big(\varphi(\tau + 2q)\big)y_-(\tau + 2q)\,\mathrm{d}\tau = \frac{1}{2\pi}\int_{-\pi}^{\pi} V''\big(-\varphi(\tau)\big)y_+(\tau)\,\mathrm{d}\tau, \\ L_1 &= -2\int_{-\pi}^{\pi} \dot{y}_-(\tau)\dot{\varphi}(\tau)\,\mathrm{d}\tau = 2\int_{-\pi}^{\pi} \dot{y}_+(\tau)\dot{\varphi}(\tau)\,\mathrm{d}\tau, \\ L_2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} V''\big(\varphi(\tau + 2q)\big)v(\tau + 2q)\,\mathrm{d}\tau = -\frac{1}{2\pi}\int_{-\pi}^{\pi} V''\big(-\varphi(\tau)\big)v(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Note that the coefficients M_1 and M_2 need not to be computed at the diagonal terms c_{2n-1} and a_{2n} thanks to the fact that the difference equations (54) with $\Lambda = 0$ must have eigenvectors with equal values of $\{c_{2n-1}\}_{n\in\mathbb{Z}}$ and $\{a_{2n}\}_{n\in\mathbb{Z}}$, which correspond to the two symmetries of the system of linearized equations (36) related to the symmetries (7) and (8). This fact shows that the problem of limited smoothness of V'', which is C but not C^1 near zero, is not a serious obstacle in the derivation of the reduced system (54). Indeed, we show in Sect. 4.7 how to fix the perturbation expansion and to avoid this obstacle.

Difference equations (54) give a necessary and sufficient condition to solve the linear inhomogeneous equations (53) at the $\mathcal{O}(\varepsilon^2)$ order and to continue the perturbation expansions beyond this order. Before justifying this formal perturbation theory, we shall explicitly compute the coefficients (K, M_1, M_2, L_1, L_2) of the difference equations (54).

Note that the system of difference equations (54) presents a quadratic eigenvalue problem with respect to the spectral parameter Λ. Such quadratic eigenvalue problem appear often in the context of spectral stability of nonlinear waves (Chugunova and Pelinovsky 2009; Kollar 2011).

4.4 Explicit Computations of the Coefficients

We shall prove the following technical result.

Lemma 1 *Coefficients K, M_2, L_1, and L_2 are independent of q and are given by*

$$K = -\frac{4\pi^2}{T'(E_0)}, \qquad M_2 = \frac{2}{\pi T'(E_0)(\dot{\varphi}(0))^2},$$
$$L_1 = 2\pi L_2 = \frac{2(2\pi - T'(E_0)(\dot{\varphi}(0))^2)}{T'(E_0)\dot{\varphi}(0)}.$$

Consequently, $K > 0$, whereas $M_2, L_1, L_2 < 0$. On the other hand, coefficient M_1 depends on q and is given by

$$M_1 = -\frac{2}{\pi}\big(\dot{\varphi}(0)\big)^2 + I(q),$$

where

$$I(q) = I(\pi - q) := -\int_{\pi-2q}^{\pi} \ddot{\varphi}(\tau)\ddot{\varphi}(\tau + 2q)\,\mathrm{d}\tau \quad \text{for } q \in \left[0, \frac{\pi}{2}\right].$$

To prove Lemma 1, we first uniquely solve the linear inhomogeneous equations (49), (50), and (51). For Eq. (49), we note that a general solution is

$$v(\tau) = -\tau\dot{\varphi}(\tau) + b_1\dot{\varphi}(\tau) + b_2\partial_E\varphi_{E_0}(\tau), \quad \tau \in [-\pi, \pi],$$

where (b_1, b_2) are arbitrary constants and $\partial_E\varphi_{E_0}$ is the derivative of the $T(E)$-periodic solution φ_E of the nonlinear oscillator equation (16) with the first integral (17) satisfying initial conditions $\varphi_E(0) = 0$ and $\dot{\varphi}_E(0) = \sqrt{2E}$ at the value of energy $E = E_0$, for which $T(E_0) = 2\pi$. We note the equation

$$\partial_E\varphi_{E_0}(\pm\pi) = \mp\frac{1}{2}T'(E_0)\dot{\varphi}(\pm\pi), \tag{55}$$

that follows from the differentiation of equation $\varphi_E(\pm T(E)/2) = 0$ with respect to E at $E = E_0$.

To define v uniquely, we require that $\langle\dot{\varphi}, v\rangle_{L^2_{\text{per}}} = 0$. Because $\dot{\varphi}$ is even in τ, whereas $\tau\dot{\varphi}$ and $\partial_E\varphi_{E_0}$ are odd, we hence have $b_1 = 0$ and $v(0) = 0$. Hence v is odd in τ and, in order to satisfy the 2π-periodicity, we shall only require $v(\pi) = 0$, which uniquely specifies the value of b_2 by virtue of (55),

$$b_2 = \frac{\pi\dot{\varphi}(\pi)}{\partial_E\varphi_{E_0}(\pi)} = -\frac{2\pi}{T'(E_0)}.$$

As a result, we obtain

$$v(\tau) = -\tau\dot{\varphi}(\tau) - \frac{2\pi}{T'(E_0)}\partial_E\varphi_{E_0}(\tau), \quad \tau \in [-\pi, \pi]. \tag{56}$$

For Eq. (50), we can use $\varphi(\tau) \geq 0$ for $\tau \in [0, \pi]$ and $\varphi(\tau) \leq 0$ for $\tau \in [-\pi, 0]$. We can also use the symmetry $\dot{\varphi}(\pi) = -\dot{\varphi}(0)$. Integrating equations for $y_\pm$ separately, we obtain

$$y_+(\tau) = \begin{cases} 1 + a_+\dot{\varphi} + b_+\partial_E\varphi_{E_0}, & \tau \in [-\pi, 0], \\ c_+\dot{\varphi} + d_+\partial_E\varphi_{E_0}, & \tau \in [0, \pi], \end{cases}$$

$$y_-(\tau) = \begin{cases} a_-\dot{\varphi} + b_-\partial_E\varphi_{E_0}, & \tau \in [-\pi, 0], \\ 1 + c_-\dot{\varphi} + d_-\partial_E\varphi_{E_0}, & \tau \in [0, \pi]. \end{cases}$$

Continuity of $y_\pm$ and $\dot{y}_\pm$ across $\tau = 0$ defines uniquely $d_\pm = b_\pm$ and $c_\pm = a_\pm \pm \frac{1}{\dot{\varphi}(0)}$. With this definition, $\dot{y}_\pm(-\pi) = \dot{y}_\pm(\pi)$, whereas condition $y_\pm(-\pi) = y_\pm(\pi)$ sets up uniquely

$$b_\pm = \pm\frac{2}{T'(E_0)\dot{\varphi}(0)},$$

whereas constants $a_\pm$ are not specified.

To define $y_\pm$ uniquely, we again require that $\langle\dot{\varphi}, y_\pm\rangle_{L^2_{\text{per}}} = 0$. This yields the constraint on $a_\pm$,

$$a_\pm = \mp\frac{1}{2\dot{\varphi}(0)} \mp \frac{2\langle\dot{\varphi}, \partial_E\varphi_{E_0}\rangle_{L^2_{\text{per}}}}{T'(E_0)\dot{\varphi}(0)\langle\dot{\varphi}, \dot{\varphi}\rangle_{L^2_{\text{per}}}}.$$

As a result, we obtain

$$y_+(\tau) = a_+\dot{\varphi}(\tau) + b_+\partial_E\varphi_{E_0}(\tau) + \begin{cases} 1, & \tau \in [-\pi, 0], \\ \frac{\dot{\varphi}(\tau)}{\dot{\varphi}(0)}, & \tau \in [0, \pi], \end{cases} \tag{57}$$

and

$$y_-(\tau) = a_-\dot{\varphi}(\tau) + b_-\partial_E\varphi_{E_0}(\tau) + \begin{cases} 0, & \tau \in [-\pi, 0], \\ 1 - \frac{\dot{\varphi}(\tau)}{\dot{\varphi}(0)}, & \tau \in [0, \pi], \end{cases} \tag{58}$$

where $(a_\pm, b_\pm)$ are uniquely defined above.

For Eq. (51), we integrate separately on $[-\pi, 0]$ and $[0, \pi]$ to obtain

$$\dot{z}_+(\tau) = \begin{cases} c_+ - |\varphi(\tau)|^\alpha, & \tau \in [-\pi, 0], \\ c_+, & \tau \in [0, \pi], \end{cases}$$

and

$$\dot{z}_-(\tau) = \begin{cases} c_-, & \tau \in [-\pi, 0], \\ c_- + |\varphi(\tau)|^\alpha, & \tau \in [0, \pi], \end{cases}$$

where (c_+, c_-) are constants of integration and continuity of $\dot{z}_\pm$ across $\tau = 0$ have been used. To define $z_\pm$ uniquely, we require that $\langle 1, z_\pm\rangle_{L^2_{\text{per}}} = 0$. Integrating the equations above under this condition, we obtain

$$z_+(\tau) = \begin{cases} c_+\tau + d_+ - \dot{\varphi}(\tau), & \tau \in [-\pi, 0], \\ c_+\tau - d_+, & \tau \in [0, \pi], \end{cases}$$

and

$$z_-(\tau) = \begin{cases} c_-\tau + d_-, & \tau \in [-\pi, 0], \\ c_-\tau - d_- - \dot{\varphi}(\tau), & \tau \in [0, \pi], \end{cases}$$

where (d_+, d_-) are constants of integration. Continuity of $z_\pm$ across $\tau = 0$ uniquely sets coefficients $d_\pm = \pm\frac{1}{2}\dot{\varphi}(0)$. Periodicity of $\dot{z}_\pm(-\pi) = \dot{z}_\pm(\pi)$ is satisfied. Periodicity of $z_\pm(-\pi) = z_\pm(\pi)$ uniquely defines coefficients $c_\pm = \pm\frac{1}{\pi}\dot{\varphi}(0)$. As a result, we obtain

$$z_+(\tau) = \frac{1}{2\pi} \begin{cases} \dot{\varphi}(0)(2\tau + \pi) - 2\pi\dot{\varphi}(\tau), & \tau \in [-\pi, 0], \\ \dot{\varphi}(0)(2\tau - \pi), & \tau \in [0, \pi], \end{cases} \tag{59}$$

and

$$z_-(\tau) = \frac{1}{2\pi} \begin{cases} -\dot{\varphi}(0)(2\tau + \pi), & \tau \in [-\pi, 0], \\ -\dot{\varphi}(0)(2\tau - \pi) - 2\pi\dot{\varphi}(\tau), & \tau \in [0, \pi]. \end{cases} \tag{60}$$

We can now compute the coefficients (K, M_1, M_2, L_1, L_2) of the difference equations (54). For coefficients K, we integrate by parts, use Eqs. (16), (17), (56), and obtain

$$\begin{aligned} K &= \int_{-\pi}^{\pi} \dot{\varphi}(\dot{\varphi} + 2\dot{v})\,\mathrm{d}\tau \\ &= \int_{-\pi}^{\pi} \left(\dot{\varphi}^2 - 2v\ddot{\varphi}\right)\mathrm{d}\tau \\ &= \left[\tau\dot{\varphi}^2 + \frac{2\pi}{T'(E_0)}\partial_E\varphi_{E_0}\dot{\varphi}\right]\Bigg|_{\tau=-\pi}^{\tau=\pi} + \frac{2\pi}{T'(E_0)}\int_{-\pi}^{\pi}(\partial_E\varphi_{E_0}\ddot{\varphi} - \partial_E\dot{\varphi}_{E_0}\dot{\varphi})\,\mathrm{d}\tau \\ &= -\frac{4\pi}{T'(E_0)}\int_0^{\pi}\partial_E\left(\frac{1}{2}\dot{\varphi}^2 + \frac{1}{1+\alpha}\varphi^{1+\alpha}\right)_{E_0}\mathrm{d}\tau \\ &= -\frac{4\pi^2}{T'(E_0)}. \end{aligned}$$

Because $T'(E_0) < 0$, we find that $K > 0$.

For M_1, we use Eq. (51), solution (60), and obtain

$$\begin{aligned} M_1 &= \int_{-\pi}^{\pi} V''\big(-\varphi(\tau)\big)\dot{\varphi}(\tau)z_+(\tau + 2q)\,\mathrm{d}\tau \\ &= \int_{-\pi}^{\pi} \ddot{z}_-(\tau)z_+(\tau + 2q)\,\mathrm{d}\tau \\ &= -\int_{-\pi}^{\pi} \dot{z}_-(\tau)\dot{z}_+(\tau + 2q)\,\mathrm{d}\tau \\ &= \int_0^{\pi} \ddot{\varphi}(\tau)\dot{z}_+(\tau + 2q)\,\mathrm{d}\tau, \end{aligned}$$

hence, the sign of M_1 depends on q. Using solution (59), for $q \in [0, \frac{\pi}{2}]$, we obtain

$$M_1 = \frac{1}{\pi}\dot{\varphi}(0)\int_0^{\pi} \ddot{\varphi}(\tau)\,\mathrm{d}\tau - \int_{\pi-2q}^{\pi} \ddot{\varphi}(\tau)\ddot{\varphi}(\tau+2q)\,\mathrm{d}\tau$$
$$= -\frac{2}{\pi}\big(\dot{\varphi}(0)\big)^2 + I(q),$$

where

$$I(q) := -\int_{\pi-2q}^{\pi} \ddot{\varphi}(\tau)\ddot{\varphi}(\tau+2q)\,\mathrm{d}\tau.$$

On the other hand, for $q \in [\frac{\pi}{2}, \pi]$, we obtain

$$M_1 = -\frac{2}{\pi}\big(\dot{\varphi}(0)\big)^2 + \tilde{I}(q),$$

where

$$\tilde{I}(q) := -\int_0^{2\pi-2q} \ddot{\varphi}(\tau)\ddot{\varphi}(\tau+2q)\,\mathrm{d}\tau,$$

so that

$$\tilde{I}(\pi-q) = -\int_0^{2q} \ddot{\varphi}(\tau)\ddot{\varphi}(\tau-2q)\,\mathrm{d}\tau = -\int_{-2q}^{0} \ddot{\varphi}(\tau)\ddot{\varphi}(\tau+2q)\,\mathrm{d}\tau = I(q),$$

because the mean value of a periodic function does not depend on the limits of integration.

For M_2, we use Eq. (50) and obtain

$$M_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} V''(-\varphi)y_+\,\mathrm{d}\tau$$
$$= \frac{\alpha}{2\pi}\int_0^{\pi} \varphi^{\alpha-1}y_+\,\mathrm{d}\tau$$
$$= -\frac{1}{2\pi}\int_0^{\pi} \ddot{y}_+\,\mathrm{d}\tau$$
$$= \frac{1}{\pi}b_+\partial_E\dot{\phi}_{E_0}(0)$$
$$= \frac{2}{\pi T'(E_0)(\dot{\varphi}(0))^2},$$

hence, $M_2 < 0$.

For L_1, we use Eqs. (16), (17), (58), and obtain

$$L_1 = -2\int_{-\pi}^{\pi} \dot{y}_-\dot{\varphi}\,\mathrm{d}\tau$$
$$= -2b_-\int_{-\pi}^{\pi} \partial_E\dot{\varphi}_{E_0}\dot{\varphi}\,\mathrm{d}\tau$$

$$= \frac{4}{T'(E_0)\dot{\varphi}(0)}\left[\int_0^{\pi} (\partial_E \dot{\varphi}_{E_0}\dot{\varphi} - \partial_E \varphi_{E_0}\ddot{\varphi})\,\mathrm{d}\tau + \dot{\varphi}\partial_E \varphi_{E_0}\Big|_{\tau=0}^{\tau=\pi}\right]$$

$$= \frac{2(2\pi - T'(E_0)(\dot{\varphi}(0))^2)}{T'(E_0)\dot{\varphi}(0)}.$$

Because $\dot{\varphi}(0) > 0$ and $T'(E_0) < 0$, we find that $L_1 < 0$.

For L_2, we use Eqs. (16), (17), (56), and obtain

$$\begin{aligned} L_2 &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} V''(-\varphi) v\,\mathrm{d}\tau \\ &= -\frac{\alpha}{2\pi}\int_0^{\pi} \varphi^{\alpha-1} v\,\mathrm{d}\tau \\ &= \frac{1}{T'(E_0)}\int_0^{\pi} \partial_E(\varphi_{E_0})^{\alpha}\,\mathrm{d}\tau - \frac{1}{2\pi}\int_0^{\pi} \varphi^{\alpha}\,\mathrm{d}\tau \\ &= \left[\frac{1}{2\pi}\dot{\varphi} - \frac{1}{T'(E_0)}\partial_E \dot{\varphi}_{E_0}\right]\Big|_{\tau=0}^{\tau=\pi} \\ &= \frac{2\pi - T'(E_0)(\dot{\varphi}(0))^2}{\pi T'(E_0)\dot{\varphi}(0)} \\ &= \frac{1}{2\pi} L_1, \end{aligned}$$

hence, $L_2 < 0$.

The proof of Lemma 1 is complete.

4.5 Eigenvalues of the Difference Equations

Because the coefficients (K, M_1, M_2, L_1, L_2) of the difference equations (54) are independent of n, we can solve these equations by the discrete Fourier transform. Substituting

$$c_{2n-1} = C\mathrm{e}^{\mathrm{i}\theta(2n-1)}, \qquad a_{2n} = A\mathrm{e}^{\mathrm{i}2\theta n}, \tag{61}$$

where $\theta \in [0, \pi]$ is the Fourier spectral parameter, we obtain the system of linear homogeneous equations,

$$\begin{cases} K\Lambda^2 C = 2M_1(\cos(2\theta) - 1)C + 2\mathrm{i}L_1\Lambda\sin(\theta)A, \\ \Lambda^2 A = 2M_2(\cos(2\theta) - 1)A + 2\mathrm{i}L_2\Lambda\sin(\theta)C. \end{cases} \tag{62}$$

A nonzero solution of the linear system (62) exists if and only if Λ is a root of the characteristic polynomial,

$$D(\Lambda;\theta) = K\Lambda^4 + 4\Lambda^2(M_1 + KM_2 + L_1L_2)\sin^2(\theta) + 16M_1M_2\sin^4(\theta) = 0. \tag{63}$$

Since this equation is bi-quadratic, it has two pairs of roots for each $\theta \in [0, \pi]$. For $\theta = 0$, both pairs are zero, which recovers the characteristic exponent $\lambda = 0$ of algebraic multiplicity of (at least) 4 in the linear eigenvalue problem (42). For a fixed

Table 1 Squared roots of the characteristic equation (63)

Coefficients	Roots
$\Delta < 0$	$\Lambda_1^2 < 0 < \Lambda_2^2$
$0 < \Delta \le \Gamma^2,\ \Gamma > 0$	$\Lambda_1^2 \le \Lambda_2^2 < 0$
$0 < \Delta \le \Gamma^2,\ \Gamma < 0$	$\Lambda_2^2 \ge \Lambda_1^2 > 0$
$\Delta > \Gamma^2$	$\mathrm{Im}(\Lambda_1^2) > 0,\ \mathrm{Im}(\Lambda_2^2) < 0$

$\theta \in (0, \pi)$, the two pairs of roots are generally nonzero, say Λ_1^2 and Λ_2^2. The following result specifies their location.

Lemma 2 *There exists a* $q_0 \in (0, \frac{\pi}{2})$ *such that* $\Lambda_1^2 \le \Lambda_2^2 < 0$ *for* $q \in [0, q_0) \cup (\pi - q_0, \pi]$ *and* $\Lambda_1^2 < 0 < \Lambda_2^2$ *for* $q \in (q_0, \pi - q_0)$.

To classify the nonzero roots of the characteristic polynomial (63), we define

$$\Gamma := M_1 + K M_2 + L_1 L_2, \quad \Delta := 4 K M_1 M_2. \tag{64}$$

The two pairs of roots are determined in Table 1.

Using the explicit computations of the coefficients (K, M_1, M_2, L_1, L_2), we obtain

$$\Gamma = -\frac{8}{T'(E_0)} + I(q), \qquad \Delta = \frac{64}{(T'(E_0))^2}\left(1 - \frac{\pi I(q)}{2(\dot{\varphi}(0))^2}\right).$$

Because $I(q)$ is symmetric about $q = \frac{\pi}{2}$, we can restrict our consideration to the values $q \in [0, \frac{\pi}{2}]$ and use the explicit definition from Lemma 1:

$$I(q) = -\int_{\pi-2q}^{\pi} \ddot{\varphi}(\tau)\ddot{\varphi}(\tau + 2q)\,\mathrm{d}\tau, \quad q \in \left[0, \frac{\pi}{2}\right].$$

We claim that $I(q)$ is a positive, monotonically increasing function in $[0, \frac{\pi}{2}]$ starting with $I(0) = 0$.

Because $\ddot{\varphi}(\tau) = -|\varphi(\tau)|^{\alpha-1}\varphi(\tau)$, we realize that $\ddot{\varphi}(\tau) \le 0$ for $\tau \in [0, \pi]$, whereas $\ddot{\varphi}(\tau + 2q) \ge 0$ for $\tau \in [\pi - 2q, \pi]$. Hence, $I(q) \ge 0$ for any $2q \in [0, \pi]$. Moreover, I is a continuously differentiable function of q, because the first derivative,

$$\begin{aligned} I'(q) &= -2\int_{\pi-2q}^{\pi} \ddot{\varphi}(\tau)\dddot{\varphi}(\tau + 2q)\,\mathrm{d}\tau \\ &= 2\int_{\pi-2q}^{\pi} \dddot{\varphi}(\tau)\ddot{\varphi}(\tau + 2q)\,\mathrm{d}\tau \\ &= -2\alpha\int_{\pi-2q}^{\pi} \left|\varphi(\tau)\right|^{\alpha-1}\dot{\varphi}(\tau)\ddot{\varphi}(\tau + 2q)\,\mathrm{d}\tau, \end{aligned}$$

is continuous for all $2q \in [0, \pi]$. Because $\dot{\varphi}(\tau)$ and $\ddot{\varphi}(\tau)$ are odd and even with respect to $\tau = \frac{\pi}{2}$, respectively, and $\dot{\varphi}(\tau) \ge 0$ for $\tau \in [0, \frac{\pi}{2}]$, we have $I'(q) \ge 0$ for all

$2q \in [0, \pi]$. Therefore, $I(q)$ is monotonically increasing from $I(0) = 0$ to

$$I\left(\frac{\pi}{2}\right) = -\int_0^{\pi} \ddot{\varphi}(\tau)\ddot{\varphi}(\tau+\pi)\,d\tau = \int_0^{\pi} \left(\ddot{\varphi}(\tau)\right)^2 d\tau > 0.$$

Hence, for all $q \in [0, \frac{\pi}{2}]$, we have $\Gamma > 0$ and

$$\Gamma^2 - \Delta = I(q)\left(I(q) - \frac{16}{T'(E_0)} + \frac{32\pi}{(T'(E_0)\dot{\varphi}(0))^2}\right) \geq 0,$$

where $\Delta = \Gamma^2$ if and only if $q = 0$. Therefore, only the first two lines of Table 1 can occur.

For $q = 0$, $I(0) = 0$, hence $M_1 < 0$, $\Delta > 0$ and $\Delta = \Gamma^2$. The second line of Table 1 gives $\Lambda_1^2 = \Lambda_2^2 < 0$. All characteristic exponents are purely imaginary and degenerate, thanks to the explicit computations:

$$\Lambda_1^2 = \Lambda_2^2 = -\frac{4}{\pi^2}\sin^2(\theta). \tag{65}$$

The proof of Lemma 2 is achieved if there is $q_0 \in (0, \frac{\pi}{2})$ such that the first line of Table 1 yields $\Lambda_1^2 < 0 < \Lambda_2^2$ for $q \in (q_0, \frac{\pi}{2}]$ and the second line of Table 2 yields $\Lambda_1^2 < \Lambda_2^2 < 0$ for $q \in (0, q_0)$. Because I is a monotonically increasing function of q and $\Delta > 0$ for $q = 0$, the existence of $q_0 \in (0, \frac{\pi}{2})$ follows by continuity if $\Delta < 0$ for $q = \frac{\pi}{2}$. Since $K > 0$ and $M_2 < 0$, we need to prove that $M_1 > 0$ for $q = \frac{\pi}{2}$ or, equivalently,

$$I\left(\frac{\pi}{2}\right) > \frac{2}{\pi}\left(\dot{\varphi}(0)\right)^2.$$

Because $\dot{\varphi}$ is a 2π-periodic function with zero mean, the Poincaré inequality yields

$$I\left(\frac{\pi}{2}\right) = \frac{1}{2}\int_{-\pi}^{\pi} \left(\ddot{\varphi}(\tau)\right)^2 d\tau \geq \frac{1}{2}\int_{-\pi}^{\pi} \left(\dot{\varphi}(\tau)\right)^2 d\tau.$$

On the other hand, using Eqs. (16), (17), and integration by parts, we obtain

$$\frac{1}{2}\int_{-\pi}^{\pi} \left(\dot{\varphi}(\tau)\right)^2 d\tau = -\frac{1}{2}\int_{-\pi}^{\pi} \varphi(\tau)\ddot{\varphi}(\tau)\,d\tau = \frac{1}{2}\int_{-\pi}^{\pi} \left|\varphi(\tau)\right|^{\alpha+1} d\tau = \frac{2\pi(\alpha+1)}{(\alpha+3)}E,$$

where the last equality is obtained by integrating the first invariant (17) on $[-\pi, \pi]$. Therefore, we obtain

$$I\left(\frac{\pi}{2}\right) \geq \frac{2\pi(\alpha+1)}{(\alpha+3)}E = \frac{\pi(\alpha+1)}{(\alpha+3)}\left(\dot{\varphi}(0)\right)^2 > \frac{2}{\pi}\left(\dot{\varphi}(0)\right)^2,$$

where the last inequality is obtained for $\alpha = \frac{3}{2}$ based on the fact that $\frac{5\pi^2}{18} \approx 2.74 > 1$. Therefore, $M_1 > 0$ and hence, $\Delta < 0$ for $q = \frac{\pi}{2}$. The proof of Lemma 2 is complete.

Numerical approximations of coefficients Γ and Δ versus q is shown on Fig. 2. We can see from the figure that the sign change of Δ occurs at $q_0 \approx 0.915$.

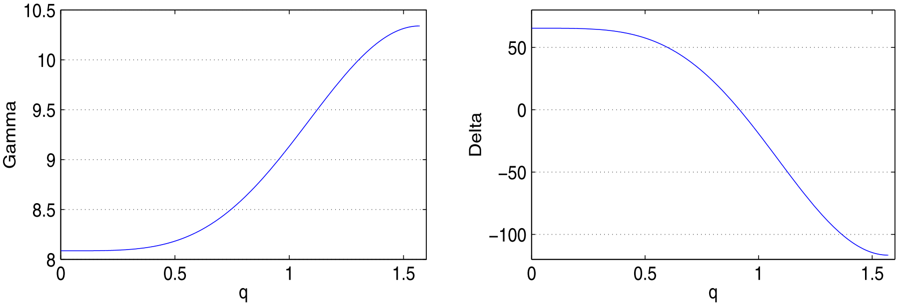

Fig. 2 Coefficients Γ (*left*) and Δ (*right*) versus q

4.6 Krein Signature of Eigenvalues

Because the eigenvalue problem (62) is symmetric with respect to reflection of θ about $\frac{\pi}{2}$, that is, $\sin(\theta) = \sin(\pi - \theta)$, some roots $\Lambda \in \mathbb{C}$ of the characteristic polynomial (63) produce multiple characteristic exponents λ in the linear eigenvalue problem (42) at the $\mathcal{O}(\varepsilon)$ order of the asymptotic expansion (45). To control splitting and persistence of the imaginary characteristic values of λ with respect to perturbations, we shall look at the Krein signature of the 2-form σ defined by (43). The following result allows us to compute σ asymptotically as $\varepsilon \to 0$.

Lemma 3 *For every $q \in (0, q_0)$, the 2-form σ for every eigenvector of the linear eigenvalue problem* (42) *generated by the perturbation expansion* (46) *associated with the root $\Lambda \in \mathrm{i}\mathbb{R}_+$ of the characteristic equation* (63) *is nonzero.*

Using the representation (41) for $\lambda = \mathrm{i}\omega$ with $\omega \in \mathbb{R}_+$, we rewrite σ in the form

$$\begin{aligned}\sigma = 2\omega \sum_{n\in\mathbb{Z}} \big[|U_{2n-1}|^2 + |W_{2n}|^2\big] \\ + \mathrm{i} \sum_{n\in\mathbb{Z}} [U_{2n-1}\dot{\bar{U}}_{2n-1} - \bar{U}_{2n-1}\dot{U}_{2n-1} + W_{2n}\dot{\bar{W}}_{2n} - \bar{W}_{2n}\dot{W}_{2n}].\end{aligned}$$

Now using perturbation expansion $\omega = \varepsilon\Omega + \mathcal{O}(\varepsilon^2)$, where $\Lambda = \mathrm{i}\Omega \in \mathrm{i}\mathbb{R}_+$ is a root of the characteristic equation (63), and the perturbation expansions (46) for the eigenvector, we compute

$$\sigma = \varepsilon \sum_{n\in\mathbb{Z}} \sigma_n^{(1)} + \mathcal{O}(\varepsilon^2),$$

where

$$\begin{aligned}\sigma_n^{(1)} = 2\Omega\big(|c_{2n-1}|^2\dot{\varphi}^2(\tau + 2qn) + |a_{2n}|^2\big) \\ + \mathrm{i}\big(c_{2n-1}\dot{\bar{U}}_{2n-1}^{(1)} - \bar{c}_{2n-1}\dot{U}_{2n-1}^{(1)}\big)\dot{\varphi}(\tau + 2qn) \\ - \mathrm{i}\big(c_{2n-1}\bar{U}_{2n-1}^{(1)} - \bar{c}_{2n-1}U_{2n-1}^{(1)}\big)\ddot{\varphi}(\tau + 2qn) + \mathrm{i}\big(a_{2n}\dot{\bar{W}}_{2n}^{(1)} - \bar{a}_{2n}\dot{W}_{2n}^{(1)}\big).\end{aligned}$$

Using representation (52), this becomes

$$\sigma_n^{(1)} = 2\Omega\big(|c_{2n-1}|^2 E_0 + |a_{2n}|^2\big) + \mathrm{i}(c_{2n-1}\bar{a}_{2n} - \bar{c}_{2n-1}a_{2n})E_- \\ + \mathrm{i}(c_{2n-1}\bar{a}_{2n-2} - \bar{c}_{2n-1}a_{2n-2})E_+,$$

where E_0 and $E_\pm$ are τ-independent constants from the first integrals:

$$E_0 = \dot{\varphi}^2 + \dot{\varphi}\dot{v} - \ddot{\varphi}v,$$
$$E_\pm = \dot{\varphi}\dot{y}_\pm - \ddot{\varphi}y_\pm - \dot{z}_\pm.$$

Using explicit computations (56)–(60) of functions v, $y_\pm$, and $z_\pm$, we obtain

$$E_0 = -\frac{2\pi}{T'(E_0)}, \qquad E_\pm = \pm\frac{2\pi - T'(E_0)(\dot{\varphi}(0))^2}{\pi T'(E_0)\dot{\varphi}(0)},$$

and hence we have

$$\sigma_n^{(1)} = 2\Omega\left(\frac{K}{2\pi}|c_{2n-1}|^2 + |a_{2n}|^2\right) \\ - \mathrm{i}L_2(c_{2n-1}\bar{a}_{2n} - \bar{c}_{2n-1}a_{2n} - c_{2n-1}\bar{a}_{2n-2} + \bar{c}_{2n-1}a_{2n-2}).$$

Substituting the discrete Fourier transform representation (61) of the eigenvector of the reduced eigenvalue problem (54), we obtain

$$\sigma_n^{(1)} = 2\Omega\left(\frac{K}{2\pi}C^2 + A^2\right) - 4L_2\sin(\theta)CA \\ = \frac{1}{\pi\Omega}\big(\Omega^2 KC^2 + 8\pi M_2\sin^2(\theta)A^2\big),$$

where the second equation of the linear system (62) has been used. Using now the first equation of the linear system (62), we obtain

$$\sigma_n^{(1)} = \frac{C^2}{\pi L_1 L_2\Omega^3}\big[KL_1L_2\Omega^4 + M_2\big(K\Omega^2 - 4M_1\sin^2(\theta)\big)^2\big]. \tag{66}$$

Note that $\sigma_n^{(1)}$ is independent of n, hence periodic boundary conditions are used to obtain a finite expression for the 2-form σ.

We consider $q \in (0, q_0)$ and $\theta \in (0, \pi)$, so that $\Omega \neq 0$ and $C \neq 0$. Then, $\sigma_n^{(1)} = 0$ if and only if

$$KL_1L_2\Omega^4 + M_2\big(K\Omega^2 - 4M_1\sin^2(\theta)\big)^2 = 0.$$

Using the explicit expressions for coefficients (K, M_1, M_2, L_1, L_2) in Lemma 1, we factorize the left hand side as follows:

$$KL_1L_2\Omega^4 + M_2\big(K\Omega^2 - 4M_1\sin^2(\theta)\big)^2 \\ = \big(\Omega^2 + T'(E_0)M_1M_2\sin^2(\theta)\big)$$

$$\times \left(\frac{32\pi^2}{(T'(E_0))^2} \left(1 - \frac{T'(E_0)(\dot{\varphi}(0))^2}{4\pi} \right) \Omega^2 + \frac{16}{T'(E_0)} M_1 \sin^2(\theta) \right). \quad (67)$$

For every $q \in (0, q_0)$, $M_1 < 0$, so that the second bracket is strictly positive (recall that $T'(E_0) < 0$). Now the first bracket vanishes at

$$\Omega^2 = \frac{-2M_1}{\pi(\dot{\varphi}(0))^2} \sin^2(\theta).$$

Substituting this constraint to the characteristic equation (63) yields after straightforward computations:

$$D(\mathrm{i}\Omega; \theta) = \frac{8M_1 \sin^4(\theta)}{\pi \dot{\varphi}^2(0)} \left(1 - \frac{2\pi}{T'(E_0)\dot{\varphi}^2(0)} \right) I(q),$$

which is nonzero for all $q \in (0, q_0)$ and $\theta \in (0, \pi)$. Therefore, $\sigma_n^{(1)}$ does not vanish if $q \in (0, q_0)$ and $\theta \in (0, \pi)$. By continuity of the perturbation expansions in ε, σ does not vanish too. The proof of Lemma 3 is complete.

Remark 9 For every $q \in (0, q_0)$, all roots $\Lambda \in \mathrm{i}\mathbb{R}_+$ of the characteristic equation (63) are divided into two equal sets, one has $\sigma_n^{(1)} > 0$ and the other one has $\sigma_n^{(1)} < 0$. This follows from the factorization

$$D(\mathrm{i}\Omega; \theta) = -\frac{4\pi^2}{T'(E_0)} \left(\Omega^2 - \frac{4}{\pi^2} \sin^2(\theta) \right)^2 - 4I(q) \left(\Omega^2 - \frac{8}{\pi T'(E_0)(\dot{\varphi}(0))^2} \sin^2(\theta) \right) \sin^2(\theta).$$

As $q \to 0$, $I(q) \to 0$ and perturbation theory for double roots (65) for $q = 0$ yields

$$\Omega^2 = \frac{4}{\pi^2} \sin^2(\theta) \pm \frac{2}{\pi^2} \sin^2(\theta) \sqrt{\left|T'(E_0)\right| I(q) \left(1 - \frac{2\pi}{T'(E_0)(\dot{\varphi}(0))^2} \right)} + \mathcal{O}\big(I(q)\big).$$

Using the factorization formula (67), the sign of $\sigma_n^{(1)}$ is determined by the expression

$$\Omega^2 + T'(E_0) M_1 M_2 \sin^2(\theta) = \pm \frac{2}{\pi^2} \sin^2(\theta) \sqrt{\left|T'(E_0)\right| I(q) \left(1 - \frac{2\pi}{T'(E_0)(\dot{\varphi}(0))^2} \right)} + \mathcal{O}\big(I(q)\big),$$

which justifies the claim for small positive q. By Lemma 3, $\sigma_n^{(1)}$ does not vanish for all $q \in (0, q_0)$ and $\theta \in (0, \pi)$, therefore the splitting of all roots $\Lambda \in \mathrm{i}\mathbb{R}_+$ into two equal sets persists for all values of $q \in (0, q_0)$.

4.7 Proof of Theorem 2

To conclude the proof of Theorem 2, we develop rigorous perturbation theory in the case when $q = \frac{\pi m}{N}$ for some positive integers m and N such that $1 \le m \le N$. In this case, the linear eigenvalue problem (42) can be closed at $2mN$ second-order differential equations subject to $2mN$-periodic boundary conditions (15) and we are looking for $4mN$ eigenvalues λ, which are characteristic exponents for a $4mN \times 4mN$ Floquet matrix.

At $\varepsilon = 0$, we have $2mN$ double Jordan blocks for $\lambda = 0$. The $2mN$ eigenvectors are given by (44). The $2mN$-periodic boundary conditions are incorporated in the discrete Fourier transform (61) if

$$\theta = \frac{\pi k}{mN} \equiv \theta_k(m, N), \quad k = 0, 1, \ldots, mN - 1.$$

Because the characteristic equation (63) for each $\theta_k(m, N)$ returns 4 roots, we count $4mN$ roots of the characteristic equation (63), as many as there are eigenvalues λ in the linear eigenvalue problem (42) closed at $2mN$ second-order differential equations. As long as the roots are non-degenerate (if $\Delta \neq \Gamma^2$) and different from zero (if $\Delta \neq 0$), the first-order perturbation theory predicts splitting of $\lambda = 0$ into symmetric pairs of nonzero eigenvalues. The zero eigenvalue of multiplicity 4 persists and corresponds to the value $\theta_0(m, N) = 0$. It is associated with the symmetries (7) and (8) of the diatomic granular chain.

The nonzero eigenvalues are located hierarchically with respect to the values of $\sin^2(\theta)$ for $\theta = \theta_k(m, N)$ with $1 \le k \le mN - 1$. Because $\sin(\theta) = \sin(\pi - \theta)$, every nonzero eigenvalue corresponding to $\theta_k(m, N) \neq \frac{\pi}{2}$ is double. Because all eigenvalues $\lambda \in \mathrm{i}\mathbb{R}_+$ have a definite Krein signature by Lemma 3 and the sign of $\sigma_n^{(1)}$ in (66) is same for both eigenvalues with θ and $\pi - \theta$, the double eigenvalues $\lambda \in \mathrm{i}\mathbb{R}$ are structurally stable with respect to parameter continuations (Chugunova and Pelinovsky 2010) in the sense that they split along the imaginary axis beyond the leading-order perturbation theory.

Remark 10 The argument based on the Krein signature cannot be applied to the case of double real eigenvalues $\Lambda \in \mathbb{R}_+$, which may split off the real axis to the complex domain. However, both real and complex eigenvalues contribute to the count of unstable eigenvalues with the account of their multiplicities.

It remains to address the issue that the perturbation theory in Sect. 4.3 uses computations of V''', which is not a continuous function of its argument. To deal with this issue, we use a renormalization technique. We note that if (u_*, w_*) is a solution of the differential advance–delay equations (14) given by Theorem 1, then differentiation of the first equation of the system yields

$$\begin{aligned} \dddot{u}_*(\tau) &= V''\big(w_*(\tau) - u_*(\tau)\big)\big(\dot{w}_*(\tau) - \dot{u}_*(\tau)\big) \\ &\quad - V''\big(u_*(\tau) - w_*(\tau - 2q)\big)\big(\dot{u}_*(\tau) - \dot{w}_*(\tau - 2q)\big), \end{aligned} \tag{68}$$

where the right-hand side is a continuous function of τ.

Using (68), we substitute

$$U_{2n-1} = c_{2n-1}\dot{u}_*(\tau + 2qn) + \mathcal{U}_{2n-1}, \qquad W_{2n} = \mathcal{W}_{2n},$$

for an arbitrary choice of $\{c_{2n-1}\}_{n\in\mathbb{Z}}$, into the linear eigenvalue problem (42) and obtain

$$\begin{cases} \ddot{\mathcal{U}}_{2n-1} + 2\lambda\dot{\mathcal{U}}_{2n-1} + \lambda^2\mathcal{U}_{2n-1} \\ \quad = V''(w_*(\tau+2qn) - u_*(\tau+2qn))(\varepsilon\mathcal{W}_{2n} - \mathcal{U}_{2n-1}) \\ \qquad - V''(u_*(\tau+2qn) - w_*(\tau+2qn-2q))(\mathcal{U}_{2n-1} - \varepsilon\mathcal{W}_{2n-2}), \\ \qquad - (2\lambda\ddot{u}_*(\tau+2qn) + \lambda^2\dot{u}_*(\tau+2qn))c_{2n-1} \\ \qquad - V''(w_*(\tau+2qn) - u_*(\tau+2qn))\dot{w}_*(\tau+2qn)c_{2n-1} \\ \qquad - V''(u_*(\tau+2qn) - w_*(\tau+2qn-2q))\dot{w}_*(\tau+2qn-2q)c_{2n-1}, \\ \ddot{\mathcal{W}}_{2n} + 2\lambda\dot{\mathcal{W}}_{2n} + \lambda^2\mathcal{W}_{2n} \\ \quad = \varepsilon V''(u_*(\tau+2qn+2q) - w_*(\tau+2qn))(\mathcal{U}_{2n+1} - \varepsilon\mathcal{W}_{2n}) \\ \qquad - \varepsilon V''(w_*(\tau+2qn) - u_*(\tau+2qn))(\varepsilon\mathcal{W}_{2n} - \mathcal{U}_{2n-1}) \\ \qquad + \varepsilon V''(u_*(\tau+2qn+2q) - w_*(\tau+2qn))\dot{u}_*(\tau+2qn+2q)c_{2n-1} \\ \qquad + \varepsilon V''(w_*(\tau+2qn) - u_*(\tau+2qn))\dot{u}_*(\tau+2qn)c_{2n-1}. \end{cases} \tag{69}$$

When we repeat decompositions of the perturbation theory in Sect. 4.3, we write

$$\lambda = \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)} + \mathrm{o}(\varepsilon^2),$$
$$\mathcal{U}_{2n-1} = \varepsilon\mathcal{U}_{2n-1}^{(1)} + \varepsilon^2\mathcal{U}_{2n-1}^{(2)} + \mathrm{o}(\varepsilon^2),$$
$$\mathcal{W}_{2n} = a_{2n} + \varepsilon\mathcal{W}_{2n}^{(1)} + \varepsilon^2\mathcal{W}_{2n}^{(2)} + \mathrm{o}(\varepsilon^2),$$

for an arbitrary choice of $\{a_{2n}\}_{n\in\mathbb{Z}}$. Substituting this decomposition to system (69), we obtain equations at the $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\varepsilon^2)$ orders, which do not require computations of V'''. Hence, the system of difference equations (54) is justified at the $\mathcal{O}(\varepsilon^2)$ order and the splitting of the eigenvalues λ obeys roots of the characteristic equation (63). Persistence of roots beyond the $\mathcal{O}(\varepsilon^2)$ order holds by the standard perturbation theory for isolated eigenvalues of the Floquet matrix. The proof of Theorem 2 is complete.

5 Numerical Results

We obtain numerical approximations of the periodic traveling waves (12) in the case $q = \frac{\pi}{N}$, where N is an integer, when the diatomic granular chain (4) can be closed as the following system of $2N$ differential equations:

$$\begin{cases} \ddot{u}_{2n-1}(t) = (w_{2n}(t) - u_{2n-1}(t))_+^\alpha - (u_{2n-1}(t) - w_{2n-2}(t))_+^\alpha, \\ \ddot{w}_{2n}(t) = \varepsilon^2(u_{2n-1}(t) - w_{2n}(t))_+^\alpha - \varepsilon^2(w_{2n}(t) - u_{2n+1}(t))_+^\alpha, \end{cases} \tag{70}$$

subject to the periodic boundary conditions

$$u_{-1} = u_{2N-1}, \qquad u_{2N+1} = u_1, \qquad w_0 = w_{2N}, \qquad w_{2N+2} = w_2. \tag{71}$$

The periodic traveling waves (12) corresponds to 2π-periodic solutions of system (70) satisfying the reduction

$$u_{2n+1}(t) = u_{2n-1}\left(t + \frac{2\pi}{N}\right), \qquad w_{2n+2}(t) = w_{2n}\left(t + \frac{2\pi}{N}\right), \tag{72}$$

For convenience and uniqueness, we look for an odd function $u_1(t) = -u_1(-t)$ with

$$u_1(0) = 0 \quad \text{and} \quad \dot{u}_1(0) > 0. \tag{73}$$

By Theorem 1, the traveling wave solutions satisfying (72) and (73) are continued uniquely for small values of ε from the limiting solution with $u_1 = \varphi$ and $w_2 = 0$. We continue numerically this branch of solutions with respect to parameter ε in the interval $[0, 1]$.

5.1 Existence of Periodic Traveling Waves

In order to obtain 2π-periodic traveling wave solutions to the nonlinear system (70), we use the shooting method. Our shooting parameters are given by the initial conditions

$$\{u_{2n-1}(0), \dot{u}_{2n-1}(0), w_{2n}(0), \dot{w}_{2n}(0)\}_{1 \le n \le N}.$$

Since $u_1(0) = 0$, this gives a set of $2N - 1$ shooting parameters. However, for solutions satisfying the traveling wave reduction (72), we can use symmetries of the nonlinear system of differential equations (70) to reduce the number of shooting parameters to N parameters.

For two particles ($N = 1$ or $q = \pi$), the existence and stability problems are trivial. The exact solution (23) is uniquely continued for all $\varepsilon \in [0, 1]$ and matches at $\varepsilon = 1$ with the exact solution of the homogeneous granular chain considered in James (2012). This solution is spectrally stable with respect to 2-periodic perturbations for all $\varepsilon \in [0, 1]$ because the characteristic exponent $\lambda = 0$ has algebraic multiplicity four, which coincides with the total number of admissible characteristic values of λ in the linearized system (38) closed at two second-order differential equations.

For four particles ($N = 2$ or $q = \frac{\pi}{2}$), the nonlinear system (70) is written explicitly as

$$\begin{cases} \ddot{u}_1(t) = (w_4(t) - u_1(t))_+^\alpha - (u_1(t) - w_2(t))_+^\alpha, \\ \ddot{w}_2(t) = \varepsilon^2 (u_1(t) - w_2(t))_+^\alpha - \varepsilon^2 (w_2(t) - u_3(t))_+^\alpha, \\ \ddot{u}_3(t) = (w_2(t) - u_3(t))_+^\alpha - (u_3(t) - w_4(t))_+^\alpha, \\ \ddot{w}_4(t) = \varepsilon^2 (u_3(t) - w_4(t))_+^\alpha - \varepsilon^2 (w_4(t) - u_1(t))_+^\alpha. \end{cases} \tag{74}$$

We are looking for 2π-periodic functions satisfying the traveling wave reduction:

$$u_3(t) = u_1(t + \pi), \qquad w_4(t) = w_2(t + \pi). \tag{75}$$

We note that the system (74) is invariant with respect to the following transformation:

$$\begin{aligned} u_1(-t) &= -u_1(t), & w_2(-t) &= -w_4(t), \\ u_3(-t) &= -u_3(t), & w_4(-t) &= -w_2(t). \end{aligned} \tag{76}$$

Table 2 Error between numerical and exact solutions for branch 1

AbsTol of Shooting Method	AbsTol of ODE solver	L^∞ error
$\mathcal{O}(10^{-12})$	$\mathcal{O}(10^{-15})$	4.5×10^{-14}
	$\mathcal{O}(10^{-10})$	3.0×10^{-11}
$\mathcal{O}(10^{-8})$	$\mathcal{O}(10^{-15})$	4.5×10^{-14}
	$\mathcal{O}(10^{-10})$	3.0×10^{-11}

A 2π-periodic solution of this system satisfying (76) must also satisfy $u_1(\pi) = u_3(\pi) = 0$ and $w_2(\pi) = -w_4(\pi)$. Then, the constraints of the traveling wave reduction (75) yields the additional condition $w_4(\pi) = w_2(0)$.

To approximate a solution of the initial-value problem for the nonlinear system (74) satisfying (76), we only need four shooting parameters (a_1, a_2, a_3, a_4) in the initial condition:

$$\begin{aligned} u_1(0) &= 0, \quad \dot{u}_1(0) = a_1, \quad w_2(0) = a_2, \quad \dot{w}_2(0) = a_3, \\ u_3(0) &= 0, \quad \dot{u}_3(0) = a_4, \quad w_4(0) = -a_2, \quad \dot{w}_4(0) = a_3. \end{aligned}$$

The solution of the initial-value problem corresponds to a 2π-periodic traveling wave solution only if the following four conditions are satisfied:

$$u_1(\pi) = 0, \quad w_2(\pi) + w_4(\pi) = 0, \quad w_2(0) - w_4(\pi) = 0, \quad u_3(\pi) = 0. \tag{77}$$

These four conditions fully specify the shooting method for the four parameters (a_1, a_2, a_3, a_4). Additionally, the solution of the initial-value problem must satisfy two more conditions:

$$\dot{w}_2(\pi) - \dot{w}_4(\pi) = 0, \quad \dot{w}_2(0) - \dot{w}_4(\pi) = 0, \tag{78}$$

but these additional conditions are redundant for the shooting method. We have been checked conditions (78) a posteriori, after the shooting method has converged to a solution.

We are now able to run the shooting method based on conditions (77). The error of this numerical method is composed from the error of an ODE solver and the error in finding zeros for the functions above. We use the built-in MATLAB function `ode113` on the interval $[0, \pi]$ as an ODE solver and then use the transformation (76) to extend the solutions to the interval $[-\pi, \pi]$ or $[0, 2\pi]$.

Figure 3 (top left) plots $w_2(0)$ versus ε for the three solution branches obtained by the shooting method. The first solution branch (labeled branch 1) exists for all $\varepsilon \in [0, 1]$ and is shown on the top right panel for $\varepsilon = 1$. This branch coincides with the exact solution (22). The error in the supremum norm between the numerical and exact solutions $\|u_1 - \varphi\|_{L^\infty}$ can be found in Table 2.

We can see from the top left panel of Fig. 3 that a pitchfork bifurcation occurs at $\varepsilon = \varepsilon_0 \approx 0.72$ and results in the appearance of two symmetrically reflected branches (labeled branches 2 and 2′). These branches with $w_2(0) \neq 0$ extend to $\varepsilon = 1$ (bottom panels) to recover two traveling wave solutions of the homogeneous granular

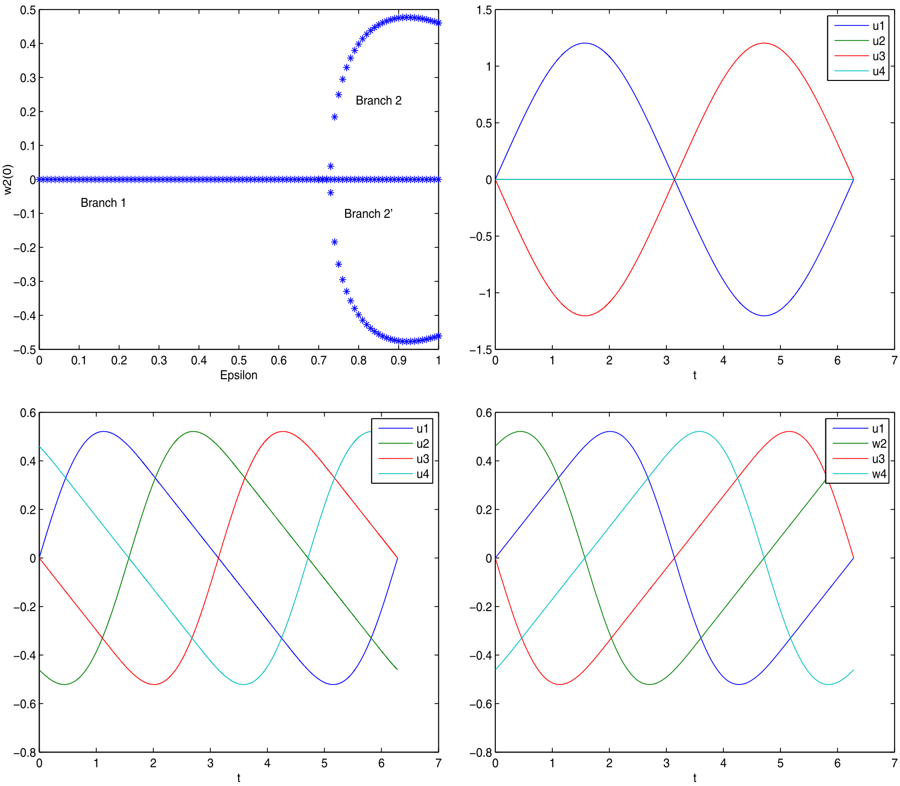

Fig. 3 Traveling wave solutions for $N = 2$: the solution of the diatomic granular chain continued from $\varepsilon = 0$ to $\varepsilon = 1$ (*top right*) and two solutions of the homogeneous granular chain at $\varepsilon = 1$ (*bottom left and right*). The *top left panel* shows the value of $w_2(0)/\varepsilon$ for all three solutions branches versus ε

chain (6). The solution of branch 2 satisfies the traveling wave reduction $U_{n+1}(t) = U_n(t + \frac{\pi}{2})$ and was previously approximated numerically by James (2012). The other solution of branch 2′ satisfies the traveling wave reduction $U_{n+1}(t) = U_n(t - \frac{\pi}{2})$ and was previously obtained numerically by Starosvetsky and Vakakis (2010).

For $N = 2$ ($q = \frac{\pi}{2}$), the solution of branch 2′ given by $\{\tilde{u}_{2n-1}, \tilde{w}_{2n}\}_{n\in\{1,2\}}$ is obtained from the solution of branch 2 given by $\{u_{2n-1}, w_{2n}\}_{n\in\{1,2\}}$, by means of the symmetry

$$\tilde{u}_1(t) = -u_3(t), \quad \tilde{w}_2(t) = -w_2(t), \quad \tilde{u}_3(t) = -u_1(t), \quad \tilde{w}_4(t) = -w_4(t), \tag{79}$$

which holds for any $\varepsilon > 0$. (Of course, both solutions 2 and 2′ exist only for $\varepsilon \in (\varepsilon_0, 1]$ because of the pitchfork bifurcation at $\varepsilon = \varepsilon_0 \approx 0.72$.) The solution of branch 1 is the invariant reduction $\tilde{u}_{2n-1} = u_{2n-1}$, $\tilde{w}_{2n} = w_{2n}$ with respect to the symmetry (79) so that it satisfies $w_2(t) = w_4(t) = 0$ for all t.

For six particles ($N=3$ or $q=\frac{\pi}{3}$), the nonlinear system (70) is written explicitly as

$$\begin{cases} \ddot{u}_1(t) = (w_6(t)-u_1(t))_+^\alpha - (u_1(t)-w_2(t))_+^\alpha, \\ \ddot{w}_2(t) = \varepsilon^2(u_1(t)-w_2(t))_+^\alpha - \varepsilon^2(w_2(t)-u_3(t))_+^\alpha, \\ \ddot{u}_3(t) = (w_2(t)-u_3(t))_+^\alpha - (u_3(t)-w_4(t))_+^\alpha, \\ \ddot{w}_4(t) = \varepsilon^2(u_3(t)-w_4(t))_+^\alpha - \varepsilon^2(w_4(t)-u_5(t))_+^\alpha, \\ \ddot{u}_5(t) = (w_4(t)-u_5(t))_+^\alpha - (u_5(t)-w_6(t))_+^\alpha, \\ \ddot{w}_6(t) = \varepsilon^2(u_5(t)-w_6(t))_+^\alpha - \varepsilon^2(w_6(t)-u_1(t))_+^\alpha. \end{cases} \tag{80}$$

We are looking for 2π-periodic functions satisfying the traveling wave reduction:

$$\begin{aligned} u_5(t) &= u_3\left(t+\frac{2\pi}{3}\right) = u_1\left(t+\frac{4\pi}{3}\right), \\ w_6(t) &= w_4\left(t+\frac{2\pi}{3}\right) = w_2\left(t+\frac{4\pi}{3}\right). \end{aligned} \tag{81}$$

We note that the system (80) is invariant with respect to the following transformation:

$$\begin{aligned} u_1(-t) &= -u_1(t), \qquad & w_2(-t) &= -w_6(t), \\ u_3(-t) &= -u_5(t), \qquad & w_4(-t) &= -w_4(t). \end{aligned} \tag{82}$$

A 2π-periodic solution of this system satisfying (82) must also satisfy $u_1(\pi) = w_4(\pi) = 0$, $w_2(\pi) = -w_6(\pi)$, and $u_3(\pi) = -u_5(\pi)$. Then, the constraints of the traveling wave reduction (81) yield the conditions $u_3(\pi) = -u_1(\frac{\pi}{3})$ and $w_4(\pi) = -w_2(\frac{\pi}{3})$.

To approximate a solution of the initial-value problem for the nonlinear system (80) satisfying (82), we only need six shooting parameters $(a_1, a_2, a_3, a_4, a_5, a_6)$ in the initial condition:

$$\begin{aligned} &u_1(0) = 0, \quad &&\dot{u}_1(0) = a_1, \quad &&w_2(0) = a_2, \quad &&\dot{w}_2(0) = a_3, \\ &u_3(0) = a_4, \quad &&\dot{u}_3(0) = a_5, \quad &&w_4(0) = 0, \quad &&\dot{w}_4(0) = a_6, \\ &u_5(0) = -a_4, \quad &&\dot{u}_5(0) = a_5, \quad &&w_6(0) = -a_2, \quad &&\dot{w}_6(0) = a_3. \end{aligned}$$

This solution corresponds to a 2π-periodic traveling wave solution only if it satisfies the following six conditions:

$$\begin{aligned} &u_1(\pi) = 0, \qquad &&w_2(\pi) + w_6(\pi) = 0, \qquad &&u_3(\pi) + u_5(\pi) = 0, \\ &u_1\left(\frac{\pi}{3}\right) + u_3(\pi) = 0, \qquad &&w_2\left(\frac{\pi}{3}\right) + w_4(\pi) = 0, \qquad &&w_4(\pi) = 0. \end{aligned}$$

The six conditions determines the shooting method for the six parameters $(a_1, a_2, a_3, a_4, a_5, a_6)$. Additional conditions,

$$\begin{aligned} &\dot{w}_2(\pi) - \dot{w}_6(\pi) = 0, \qquad &&\dot{u}_3(\pi) - \dot{u}_5(\pi) = 0, \\ &\dot{u}_1\left(\frac{\pi}{3}\right) - \dot{u}_3(\pi) = 0, \qquad &&\dot{w}_2\left(\frac{\pi}{3}\right) - \dot{w}_4(\pi) = 0, \end{aligned}$$

are to be checked a posteriori, after the shooting method has converged to a solution.

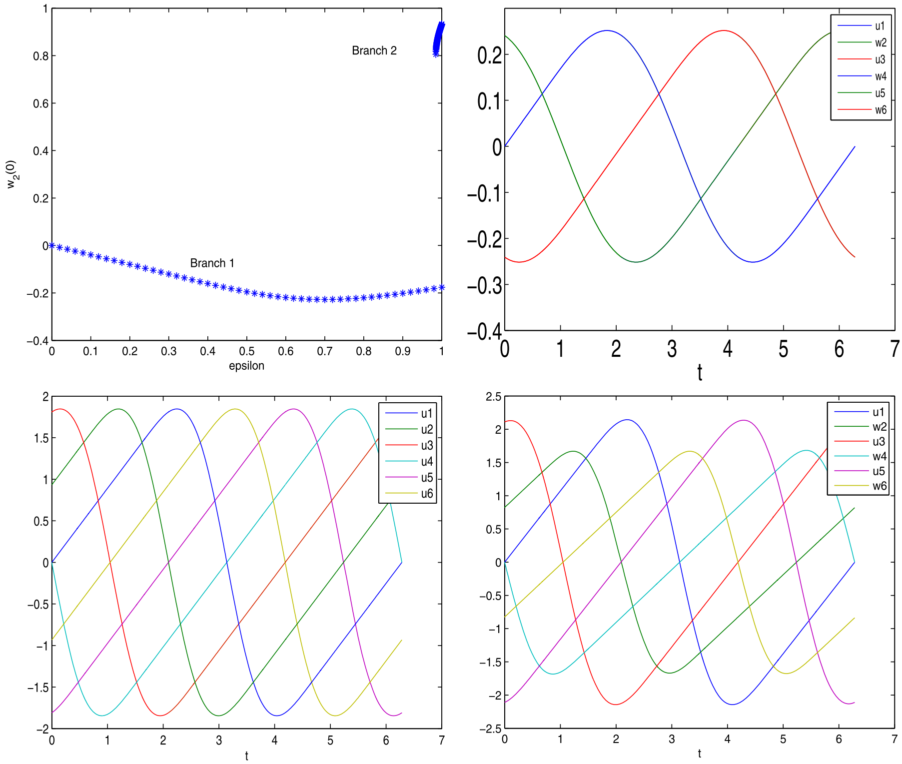

Fig. 4 Traveling wave solutions for $N = 3$: the solution of branch 1 is continued from $\varepsilon = 0$ to $\varepsilon = 1$ (*top right*) and the solution of branch 2 is continued from $\varepsilon = 1$ (*bottom left*) to $\varepsilon = 0.985$ (*bottom right*). The *top left panel* shows the value of $w_2(0)/\varepsilon$ for solution branches 1 and 2 versus ε

Figure 4 (top left) shows two solution branches obtained by the shooting method. Again, $w_2(0)$ is plotted versus ε. Branch 1 is continued from $\varepsilon = 0$ to $\varepsilon = 1$ (top right) without any pitchfork bifurcation in $\varepsilon \in (0, 1)$. Branch 2 is continued from $\varepsilon = 1$ (bottom left) starting with a numerical solution of the homogeneous granular chain (6) satisfying the reduction $U_{n+1}(t) = U_n(t + \frac{\pi}{3})$ to $\varepsilon = 0.985$ (bottom right), where the branch terminates according to our shooting method. We have not been able so far to detect numerically any other branch of traveling wave solutions near branch 2 for $\varepsilon = 0.985$. This unusual bifurcation may be induced by discontinuity in the nonsmooth dynamical system (80). Detailed analysis of this bifurcation will remain opened for further studies. See di Bernardo and Hogan (2010) for a review of discontinuity-induced bifurcations.

We use the same technique for $N = 4$ and show similar results on Fig. 5. Branch 1 is uniquely continued from $\varepsilon = 0$ to $\varepsilon = 1$ (top right), whereas branch 2 is continued from $\varepsilon = 1$ (bottom left) starting with a numerical solution of the homogeneous granular chain (6) satisfying the reduction $U_{n+1}(t) = U_n(t + \frac{\pi}{4})$ to $\varepsilon = 0.9$ (bottom right), where the branch terminates.

Note the occurrence of the free flight near $t = 0$ and the shock near $t = \pi$ in the component u_1 of the solution of branch 2 at $\epsilon = 1$ on Figs. 4 and 5, which were

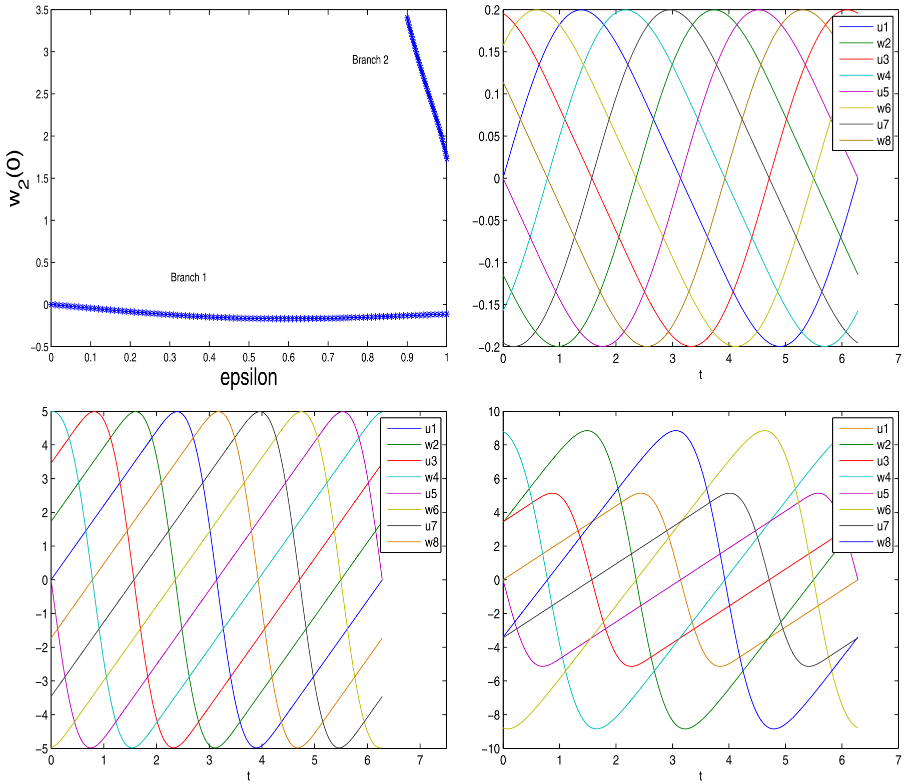

Fig. 5 Traveling wave solutions for $N = 4$: the solution of branch 1 continued from $\varepsilon = 0$ to $\varepsilon = 1$ (*top right*) and the solution of branch 2 continued from $\varepsilon = 1$ (*bottom left*) to $\varepsilon = 0.9$ (*bottom right*). The *top left panel* shows the value of $w_2(0)/\varepsilon$ for solution branches 1 and 2 versus ε

studied in James (2012). For the solution of branch 1 at $\epsilon = 1$, the dynamics of beads is compensated so that no free flights or shocks are observed. Also note that similar traveling wave solutions in diatomic granular chains with $N = 2, 3, 4$ were observed numerically in Jayaprakash et al. (2012).

5.2 Stability of Periodic Traveling Waves

To determine stability of the different branches of periodic traveling wave solutions of the diatomic granular chain (4), we compute Floquet multipliers of the monodromy matrix for the linearized system (36). To do this, we use the traveling wave solution obtained with the shooting method and the MATLAB function `ode113` to compute the fundamental matrix solution of the linearized system (36) on the interval $[0, 2\pi]$.

By Theorem 2, the traveling waves of branch 1 for $N = 2$ ($q = \frac{\pi}{2}$) are unstable for small values of ε. Figure 6 (top) shows real and imaginary parts of the characteristic exponents associated with branch 1 for all values of ε in $[0, 1]$. Only positive values of $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ are shown, moreover, $\mathrm{Im}(\lambda) \in [0, \frac{1}{2}]$ because of 1-periodicity of the characteristic exponents along the imaginary axis.

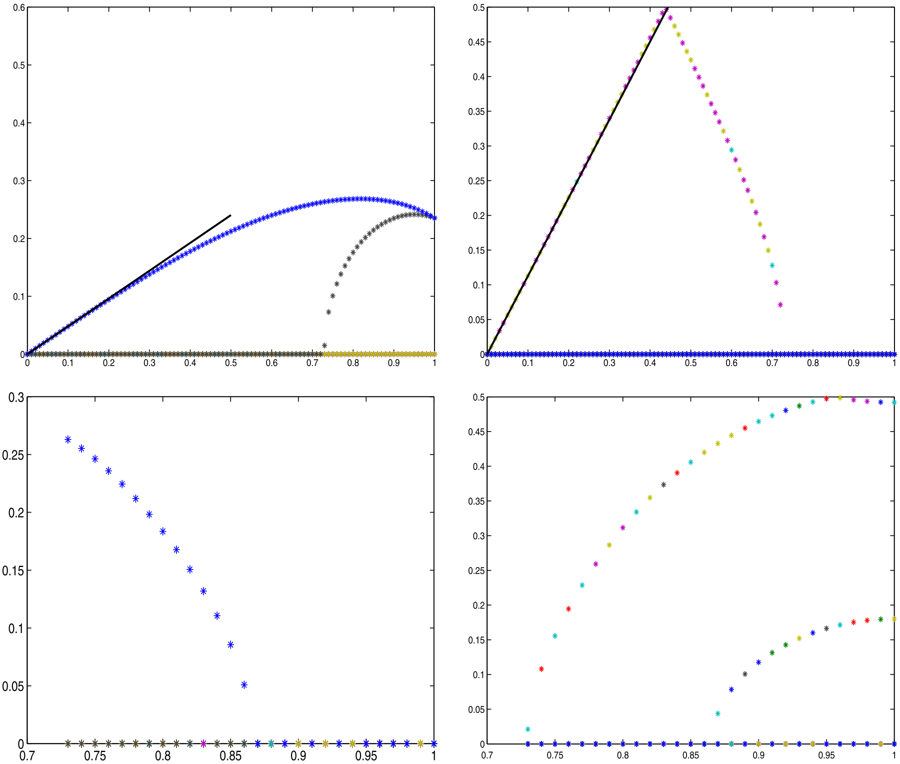

Fig. 6 Real (*left*) and imaginary (*right*) parts of the characteristic exponents λ versus ε for $N = 2$ for branch 1 (*top*) and branch 2 (*bottom*). *Solid lines on the top panels* display the asymptotic results from the roots of the characteristic equation (63), whereas *dots* show the numerical approximations of characteristic exponents

Thanks to the periodic boundary conditions, the system of linearized equations (37) for $N = 2$ is closed at four second-order linearized equations, which have 8 characteristic exponents as follows. The exponent $\lambda = 0$ has multiplicity 4 for small positive ε, and two pairs of nonzero exponents (one is real, the other one is purely imaginary) bifurcate according to the roots of the characteristic equation (63) for $\theta = \frac{\pi}{2}$. These asymptotic approximations are shown on the top panels of Fig. 6 by solid lines, in excellent agreement with the numerical data (dots). We see that the unstable real λ persist for all values of ε in $[0, 1]$. The pitchfork bifurcation at $\varepsilon = \varepsilon_0 \approx 0.72$ in Fig. 3 (top left) corresponds to the coalescence of the pair of purely imaginary characteristic exponents on Fig. 6 (top right) and appearance of a new pair of real characteristic exponents for $\varepsilon > \varepsilon_0$ on Fig. 6 (top left). Therefore, the branch continued from $\varepsilon = 0$ is unstable for all $\varepsilon \in [0, 1]$.

Bottom panels on Fig. 6 shows real and imaginary parts of the characteristic exponents associated with branch 2 (same for $2'$ by symmetry) for all values of ε in $[\varepsilon_0, 1]$. We see that these traveling waves are spectrally stable near $\varepsilon = 1$, in agreement with the numerical results of James (2012). When ε is decreased, these traveling waves lose spectral stability near $\varepsilon = \varepsilon_1 \approx 0.86$ because of coalescence of a pair of

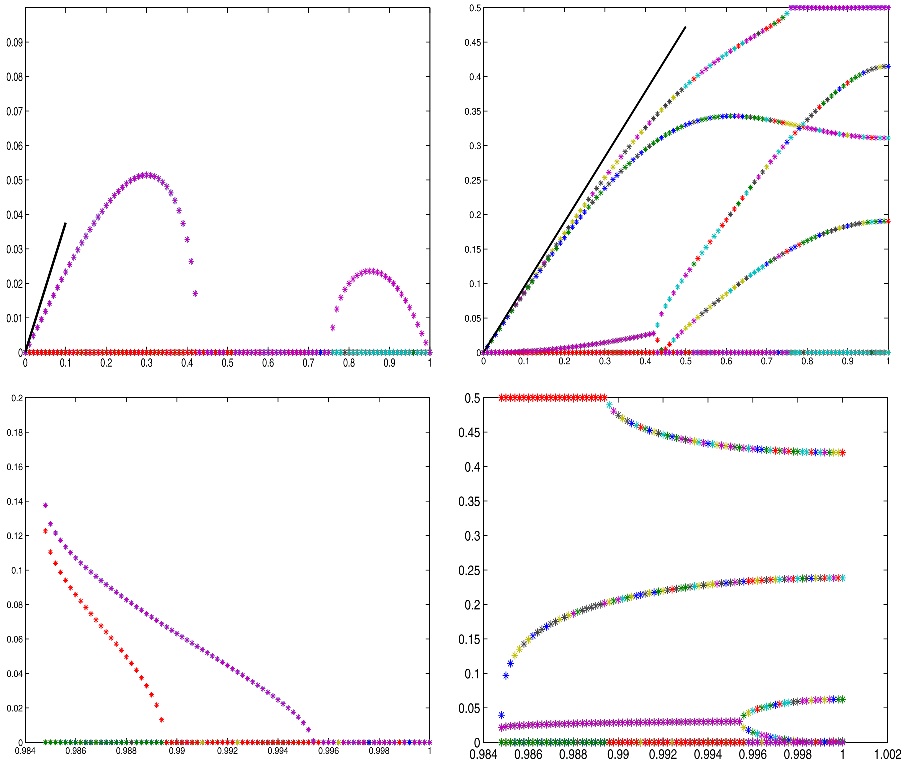

Fig. 7 Real (*left*) and imaginary (*right*) parts of the characteristic exponents λ versus ε for $N = 3$ for branch 1 (*top*) and branch 2 (*bottom*). *Solid lines* and *dots* have the same meaning as in Fig. 6

purely imaginary characteristic exponents, which creates a pair of real characteristic exponents for $\varepsilon < \varepsilon_1$. The two solution branches disappear as a result of the pitchfork bifurcation at $\varepsilon = \varepsilon_0 \approx 0.72$, which is also induced by the coalescence of the second pair of purely imaginary characteristic exponents.

For $N = 3$ ($q = \frac{\pi}{3}$), the system of linearized equations (36) is closed at six second-order linearized equations. Besides the characteristic exponent $\lambda = 0$ of multiplicity four, we have 8 nonzero characteristic exponents λ. The characteristic equation (63) with $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$ predicts a double pair of real λ and a double pair of purely imaginary λ. Figure 7 (top) shows Re(λ) (left) and Im(λ) (right) for solutions of branch 1. The double pair of purely imaginary λ split along the imaginary axis for small $\varepsilon > 0$, as explained in Sects. 4.6 and 4.7. On the other hand, the double pair of real λ splits along the transverse direction and results in occurrence of a quartet of complex-valued λ for small $\varepsilon > 0$. These complex characteristic exponents approach the imaginary axis at $\varepsilon = \varepsilon_1 \approx 0.43$ (Neimark–Sacker bifurcation) and then split along the imaginary axis as two pairs of purely imaginary λ for $\varepsilon > \varepsilon_1$. We also have one pair of purely imaginary λ continued from $\varepsilon = 0$ that approaches the line $\pm\frac{i}{2}$ (corresponding to the Floquet multiplier at -1) at $\varepsilon = \varepsilon_2 \approx 0.72$ (period-doubling bifurcation). That pair splits in a complex plane for $\varepsilon > \varepsilon_2$ (the corresponding Floquet multipliers are real and negative). In summary, the periodic traveling wave of branch 1 for $N = 3$ is stable for $\varepsilon \in (\varepsilon_1, \varepsilon_2)$ but unstable near $\varepsilon = 0$ and $\varepsilon = 1$.

Figure 7 (bottom) shows $\mathrm{Re}(\lambda)$ (left) and $\mathrm{Im}(\lambda)$ (right) for solutions of branch 2 that exist only for $\varepsilon \in [\varepsilon_*, 1]$, where $\varepsilon_* \approx 0.985$. All four pairs of the characteristic exponents λ are purely imaginary near $\varepsilon = 1$. This corresponds to the numerical results for stability of traveling waves in the homogeneous granular chain in James (2012). Two pairs coalesce at $\varepsilon \approx 0.995$ and split in a complex quartet of characteristic exponents (Neimark–Sacker bifurcation). Another pair approaches the line $\pm\frac{i}{2}$ at $\varepsilon \approx 0.989$ and then splits in a complex plane to yield real and negative Floquet multipliers (period-doubling bifurcation). The final remaining pair of purely imaginary λ crosses zero near $\varepsilon = \varepsilon_* \approx 0.985$ that indicates that termination of branch 2 is related to a local (discontinuity-induced ?) bifurcation.

Recall that the coefficient M_1 changes sign at $q \approx 0.915$, as explained in Sect. 4.5. Therefore, for $N \geq 4$, the characteristic equation (63) for any values of θ predicts pairs of purely imaginary λ only. This is illustrated on the top panels of Fig. 8 for $N = 4$ ($q = \frac{\pi}{4}$). We can see that all double pairs of purely imaginary λ split along the imaginary axis for small $\varepsilon > 0$ and that the periodic traveling waves of branch 1 remain stable for all $\varepsilon \in [0, 1]$. The figure also illustrate the validity of asymptotic approximations obtained from roots of the characteristic equation (63).

It is interesting that Fig. 8 (top right) shows safe coalescence of characteristic exponents for larger values of ε. Recall from Remark 9 that the characteristic exponents have opposite Krein signature for small values of ε in such a way that larger exponents on Fig. 8 (top right) have negative Krein signature and smaller exponents have positive Krein signature. It is typical to observe instabilities arise after the coalescence of two purely imaginary eigenvalues of the opposite Krein signature (MacKay 1986) but this only happens when the double eigenvalue at the coalescence point is not semi-simple. When the double eigenvalue is semi-simple or if some perturbation terms to the Jordan blocks are identically zero, the coalescence does not introduce any instabilities (Bridges 1990; Vougalter and Pelinovsky 2006). This is precisely what we observe in Fig. 8 (top right). After coalescence, for larger values of ε, the purely imaginary characteristic exponents λ reappear as simple exponents with opposite Krein signature, that is, the exponents with positive Krein signature are now above the ones with negative Krein signature.

Figure 8 (bottom) shows $\mathrm{Re}(\lambda)$ (left) and $\mathrm{Im}(\lambda)$ (right) for solutions of branch 2 that exist only for $\varepsilon \in [\varepsilon_*, 1]$, where $\varepsilon_* \approx 0.90$. Besides three pairs of purely imaginary characteristic exponents λ, there is one pair of real λ and a complex quartet near $\varepsilon = 1$. The pair of real λ corresponds to the numerical results for instability of traveling waves in the homogeneous granular chain (James 2012), for which instability occurs for $q \lessapprox 0.9$. The quartet of complex λ gives additional instability, which is not captured by the reductions to the homogeneous granular chain. Several more instabilities arise as ε is decreased from $\varepsilon = 1$ for solutions of branch 2 because of bifurcations of pairs of purely imaginary exponents λ. Branch 2 is unstable in the entire existence interval $[\varepsilon_*, 1]$.

Figure 9 shows the stability of solutions of branch 1 for $N = 5$ (left) and $N = 6$ (right). These figures are included to illustrate the safe splitting of purely imaginary

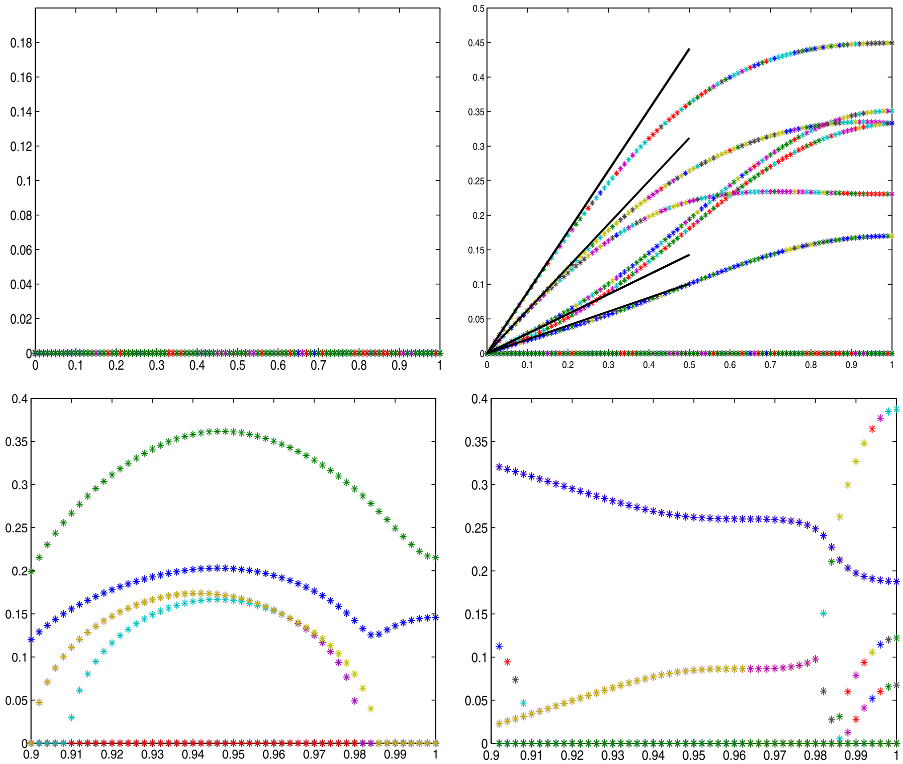

Fig. 8 Real (*left*) and imaginary (*right*) parts of the characteristic exponents λ versus ε for $N = 4$ for branch 1 (*top*) and branch 2 (*bottom*). *Solid lines* and *dots* have the same meaning as in Fig. 6

exponents along the imaginary axis near $\varepsilon = 0$, as well as safe coalescence of purely imaginary exponents of opposite Krein signature that never results in the occurrence of complex exponents. The solutions of branch 1 for $N = 5, 6$ remain stable for all values of $\varepsilon \in [0, 1]$.

5.3 Stability of the Uniform Periodic Oscillations

The periodic solution with $q = 0$ (which is no longer a traveling wave but a uniform oscillation of all sites of the diatomic granular chain) is given by the exact solution (23). Spectral stability of this solution is obtained from the linearized system (38). Using the boundary conditions

$$u_{2n+1} = e^{2i\theta} u_{2n-1}, \qquad w_{2n+2} = e^{2i\theta} w_{2n}, \quad n \in \mathbb{Z},$$

where $\theta \in [0, \pi]$ is a continuous parameter, we close the system at two second-order equations,

$$\begin{cases} \ddot{u} + \frac{\alpha}{1+\varepsilon^2} |\varphi|^{\alpha-1} u = \frac{1}{1+\varepsilon^2} (V''(-\varphi) + V''(\varphi) e^{-2i\theta}) w, \\ \ddot{w} + \frac{\alpha\varepsilon^2}{1+\varepsilon^2} |\varphi|^{\alpha-1} w = \frac{\varepsilon^2}{1+\varepsilon^2} (V''(-\varphi) + V''(\varphi) e^{2i\theta}) u. \end{cases} \tag{83}$$

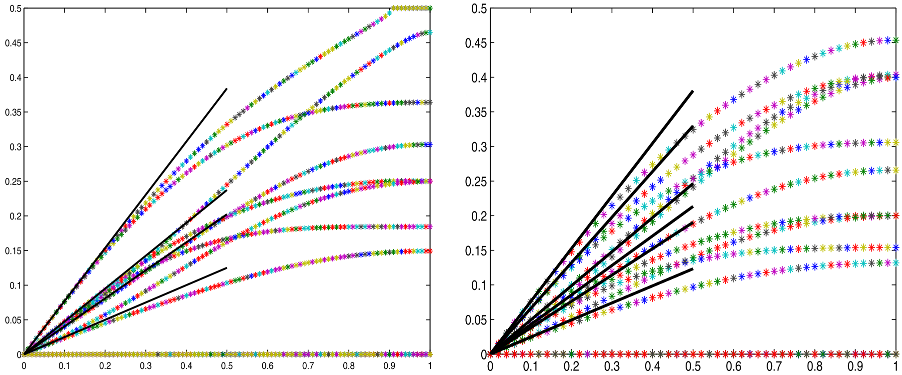

Fig. 9 Imaginary parts of the characteristic exponents λ versus ε for $N = 5$ (*left*) and $N = 6$ (*right*). The real part of all the exponents is zero. *Solid lines* and *dots* have the same meaning as in Fig. 6

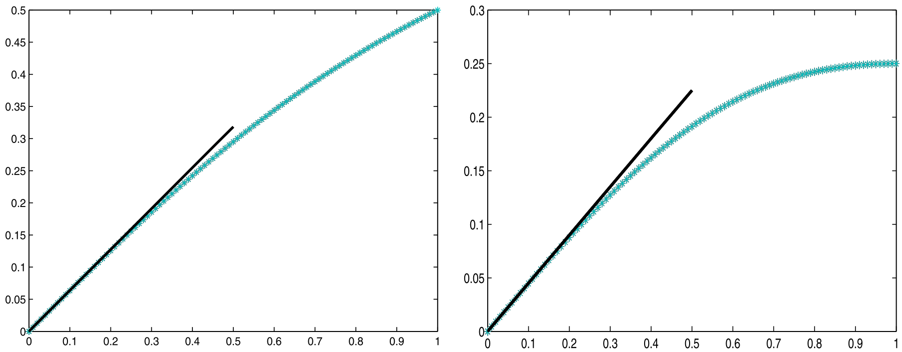

Fig. 10 Imaginary parts of the characteristic exponents λ versus ε for $\theta = \frac{\pi}{2}$ (*left*) and $\theta = \frac{\pi}{4}$ (*right*). The real part of all the exponents is zero. *Solid lines* and *dots* have the same meaning as in Fig. 6

The characteristic equation (63) for $q = 0$ predicts a double pair (65) of purely imaginary Λ for any $\theta \in (0, \pi)$. We confirm numerically that the double pair is preserved for all $\varepsilon \in [0, 1]$.

Figure 10 shows the imaginary part of the characteristic exponents λ of the linearized system (83) for $\theta = \frac{\pi}{2}$ (left) and $\theta = \frac{\pi}{4}$ (right). Similar results are obtained for other values of θ. Therefore, the periodic solution with $q = 0$ remains stable for all values of $\varepsilon \in [0, 1]$.

The pattern on Fig. 10 suggests a hidden symmetry in this case. Suppose λ_θ is a characteristic exponent of the system (83) for the eigenvector

$$\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} U_\theta(t) \\ W_\theta(t) \end{bmatrix} e^{\lambda_\theta t}, \tag{84}$$

where $U_\theta(t)$ and $W_\theta(t)$ are 2π-periodic and the subscript θ indicates that the system (83) depends explicitly on θ. Recall that the unperturbed solution satisfies the symmetry $\varphi(t + \pi) = -\varphi(t)$ for all t. Using this symmetry and the trivial identity

$e^{2\pi i} = 1$, we can verify that there is another solution of the system (83) with the same θ for the characteristic exponent $\lambda_{\pi-\theta}$:

$$\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} U_{\pi-\theta}(t+\pi) \\ e^{2i\theta} W_{\pi-\theta}(t+\pi) \end{bmatrix} e^{\lambda_{\pi-\theta} t}. \tag{85}$$

From the symmetry of roots (65) and the corresponding characteristic exponents, we have $\lambda_\theta = \lambda_{\pi-\theta}$. The eigenvectors (84) and (85) are generally linearly independent and coexist for the same value of $\lambda = \lambda_\theta = \lambda_{\pi-\theta}$. This argument explains the double degeneracy of characteristic exponents λ for the case $q = 0$ for all values of $\varepsilon \in [0, 1]$.

6 Conclusion

We have studied periodic traveling waves in a diatomic granular chain by continuing these solutions from the anti-continuum limit, when the mass ratio between the light and heavy beads is zero. We have shown that every limiting periodic wave is uniquely continued for small mass ratios. Although the vector fields of the granular dimer chain equations are not C^2 at the origin, we can still use the implicit function theorem to guarantee that the continuation is C^1 with respect to the mass ratio parameter ε^2. We have also used rigorous perturbation theory to compute characteristic exponents in the linearized stability problem. From this theory, we have seen that the periodic waves with a wavelength larger than a certain critical value are spectrally stable for small mass ratios.

Numerical computations are developed to show that the stability of these periodic waves with larger wavelengths extends all way to the limit of equal mass ratio. On the other hand, we have also computed periodic traveling waves that are continued from solutions of the homogeneous granular chain at the equal mass ratio, their spectral stability, and their terminations for smaller mass ratios.

Among open problems, we have not clarified the nature of bifurcation, where the solutions of branch 2 terminate at a $\varepsilon_* \in (0, 1)$ for $N = 3, 4$. We have not been able to find another solution for ε near ε_*. Safe coalescence of purely imaginary characteristic exponents λ of opposite Krein signatures is also amazing and we have not been able to find the hidden symmetry that would explain why the eigenvalues of opposite Krein signatures remain stable past the coalescence point. These problems as well as analysis of the periodic traveling wave solutions for other values of q will wait for further studies.

Finally, our paper may inspire further experimental work with diatomic granular crystals. We hope that some stable configurations of periodic traveling waves in a periodic granular chain of finitely many beads produced in our studies might be found experimentally to play important role in the dynamics of uncompressed granular crystals.

Acknowledgements The research of the authors was supported by funding from the NSERC Discovery grant and the Ministry of Education and Science of Russian Federation (project 14.B37.21.0868). The authors thank G. James, P. Kevrekidis, and Y. Starosvetsky for useful discussions.

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