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Periodic waves in the discrete mKdV equation: Modulational instability and rogue waves

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ABSTRACT

We derive the traveling periodic waves of the discrete modified Korteweg–de Vries equation by using a nonlinearization method. Modulational stability of the traveling periodic waves is studied from the squared eigenfunction relation and the Lax spectrum. We use numerical approximations to show that, similar to the continuous counterpart, the family of dnoidal solutions is modulationally stable and the family of cnoidal solutions is modulationally unstable. Consequently, algebraic solitons propagate on the dnoidal wave background and rogue waves (spatially and temporally localized events) are dynamically generated on the cnoidal wave background.

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1. Introduction

The modified Korteweg–de Vries (mKdV) equation is an important model for dynamics of long internal waves in the cases when the quadratic nonlinearity vanishes [1,2]. This model is integrable with the inverse scattering transform method [3], which makes it attractive for many studies, e.g., the existence, stability, and dynamics of the traveling periodic waves [4,5].

Spatial discretizations of the integrable models are also integrable with the inverse scattering transform method [6]. One of such models was obtained in [7] as the spatial discretization of the modified Korteweg–de Vries (mKdV) equation. We call it the dmKdV equation and write it in the normalized form:

$$\dot{u}_n = (1 + u_n^2)(u_{n+1} - u_{n-1}), \quad n \in \mathbb{Z}, \tag{1.1}$$

where the dot represents the derivative of $\{u_n(t)\}_{n\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ with respect to the time variable $t \in \mathbb{R}$. In the continuum limit, long waves of small amplitudes can be modeled by

$$u_n(t) = \varepsilon \mathfrak{u}(\xi, \tau), \quad \xi := \varepsilon(n + 2t), \quad \tau := \frac{1}{3}\varepsilon^3 t, \tag{1.2}$$

where ε is a formal small parameter. Substituting (1.2) into (1.1) and neglecting higher-order terms beyond the formal order of $\mathcal{O}(\varepsilon^4)$ yield the mKdV equation

$$\mathfrak{u}_\tau = 6\mathfrak{u}^2 \mathfrak{u}_\xi + \mathfrak{u}_{\xi\xi\xi}. \tag{1.3}$$

Traveling periodic waves of the mKdV equation have been studied recently in many details. Two families are given by the Jacobi dnoidal and cnoidal functions [5], namely

$$\begin{aligned} \mathfrak{u}(\xi, \tau) &= \mathrm{dn}(\xi + (2 - k^2)\tau; k), \\ \mathfrak{u}(\xi, \tau) &= k\mathrm{cn}(\xi + (2k^2 - 1)\tau; k), \end{aligned} \tag{1.4}$$

where $k \in (0, 1)$ is elliptic modulus and the solutions can be generalized by using the translational and scaling invariance of the mKdV equation. The two traveling periodic wave solutions (1.4) are continued into two families expressed by the rational functions of the Jacobi elliptic functions [8].

Stability of dnoidal and cnoidal waves of the mKdV equation with respect to periodic perturbations of the same period was studied in [9–11]. Extension of the stability analysis to the other families of traveling periodic waves was obtained in [12,13]. Modulational stability of these solutions with respect to long perturbations was studied in [4,5,14], where it was shown that the dnoidal waves are modulationally stable and the cnoidal waves are modulationally unstable. These stability results are illustrated in Figs. 1 and 2 for the dnoidal and cnoidal waves respectively. The description of how the figures were generated can be found in Appendix A. Since the stability spectrum is on the imaginary axis for the dnoidal waves (Fig. 1), the dnoidal waves are spectrally (and modulationally) stable. On the other hand, the stability spectrum for the cnoidal waves (Fig. 2) contains a band outside the imaginary axis and the band intersects the origin, hence the cnoidal waves are spectrally (and modulationally) unstable.

The immediate consequence of the modulational stability results is that the rogue waves (spatially and temporally local-

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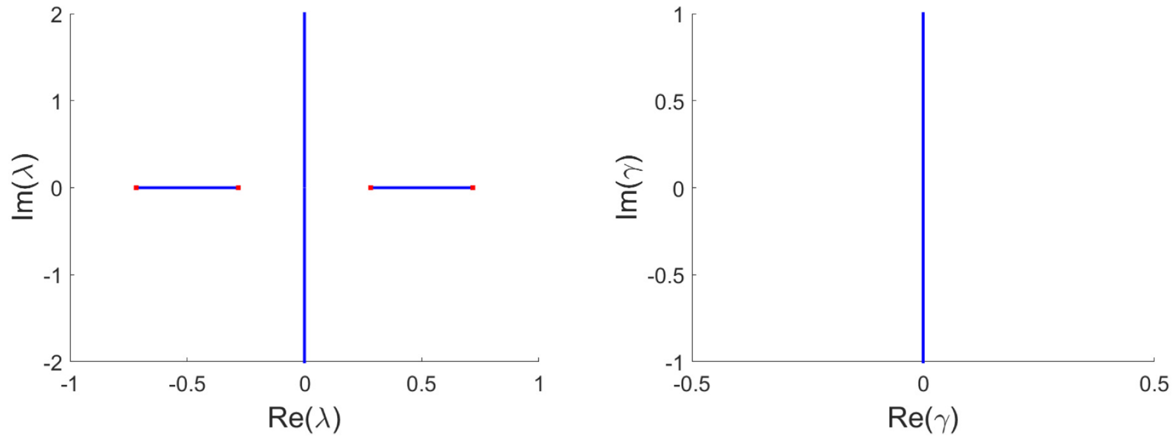

Fig. 1. Lax spectrum (left) and stability spectrum (right) for the dnoidal wave with $k = 0.9$.

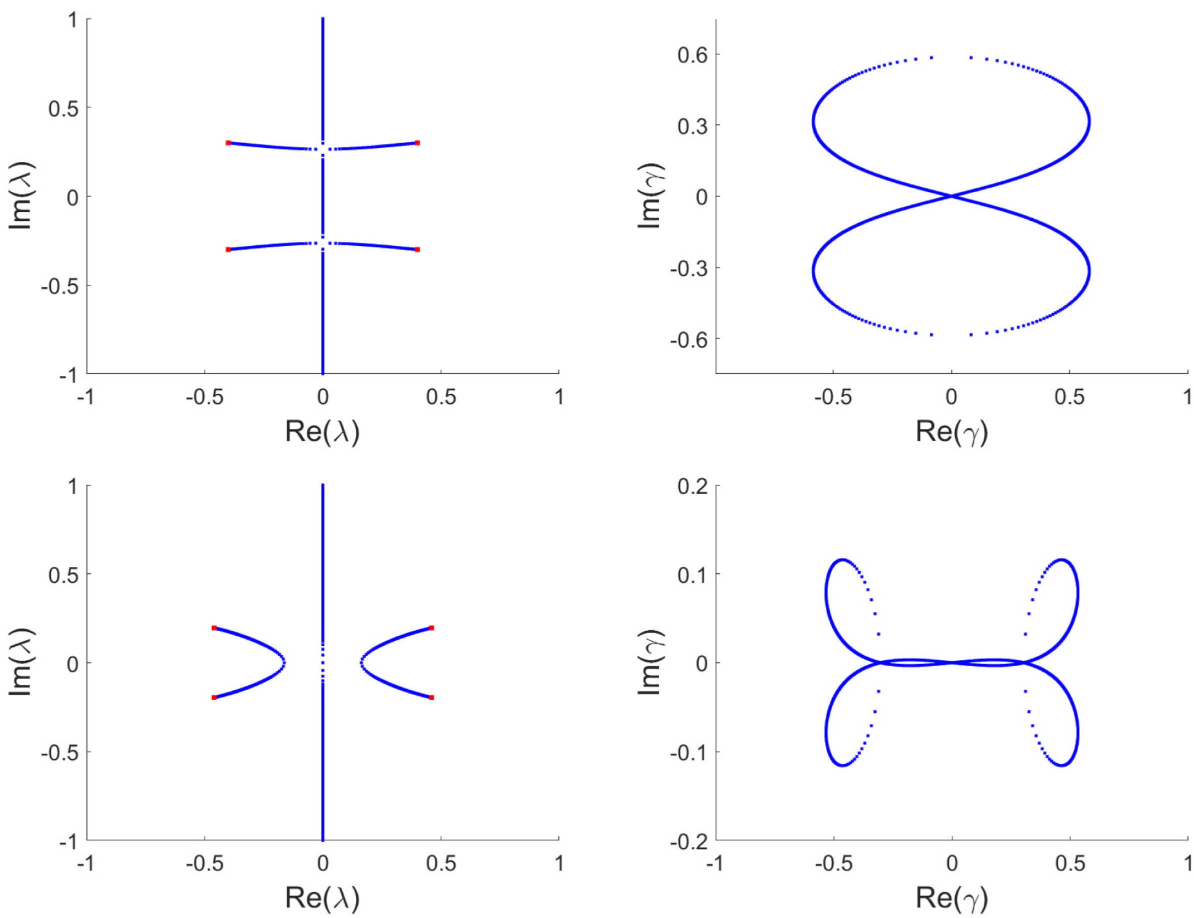

Fig. 2. Lax spectrum (left) and stability spectrum (right) for the cnoidal wave with $k = 0.8$ (top) and $k = 0.92$ (bottom).

ized events) only arise on the background of the cnoidal waves whereas a steady propagation of an algebraic soliton (a spatially decaying traveling wave) is observed on the background of the dnoidal waves [15]. Similar results were extended to the more general families of traveling periodic waves of the mKdV equation [8,16]. A general approach towards analysis of modulational stability of traveling periodic waves of the mKdV and other related equations can be found in [17].

The dmKdV equation is a real-valued flow of the integrable hierarchy of complex-valued discrete equations which starts at the Ablowitz–Ladik (AL) equation [18]. Rogue waves on the constant background have been constructed for the Ablowitz–Ladik (AL) equation [19,20] and the complex dmKdV (Hirota) equations [21,22]. Rogue waves are related to the modulation instability of the constant-amplitude waves which leads to appearance of breathers localized on the lattice [23,24].

Computations of rogue waves and breathers are usually performed with the Darboux transformations by using explicit solutions of the Lax equations considered at the constant-amplitude waves. It is much harder to analyze the Lax equations on the traveling waves, when they are either spatially periodic or spatially decaying. Constructions of such rogue waves on the periodic wave background have been elaborated for the continuous systems such as the NLS equation [25–27], the mKdV equation [8, 15], the sine–Gordon equation [28,29], and the derivative NLS equation [30–32]. No such rogue waves have been constructed for the discrete counterparts of these equations to the best of our knowledge. On the other hand, exact solutions for the traveling periodic solutions of the discrete equations have been previously constructed in the literature [33,34].

The nonlinearization method [35] has been used to characterize the traveling periodic waves in the continuous integrable systems, see [8,15,25,26,29,31] and references therein. This method is considerably more complicated when it is applied to the discrete nonlinear equations such as the hierarchy of the AL equations [36]. Another complication is that different forms of the Darboux transformations have been obtained for the AL equations based on different Lax representations [37]. To study the modulational stability and to construct rogue wave solutions on the periodic background, one needs to select the correct Darboux transformation.

As the first step towards the ultimate goal of studying traveling periodic waves of the AL equation, we consider here the case of the dmKdV equation and make a precise connection between the traveling periodic waves and the squared eigenfunctions of the Lax system of linear equations.

Among outcomes of the nonlinearization method, we give numerically-assisted solutions of the modulational stability problem for the two families of the traveling periodic waves. We confirm in a complete analogy with the case of the continuous mKdV equation that the dnoidal waves are modulationally stable and the cnoidal waves are modulationally unstable. We also construct new solutions of the dmKdV equation by using the Darboux transformations. The new solutions describe the steady propagation of an algebraic soliton on the dnoidal wave background and the appearance of a rogue wave on the cnoidal wave background.

Although the analytical methods of our work rely on integrability of the dmKdV equation, the results are important for applications of rogue waves in a variety of physical systems. For instance, periodic waves are common in nonintegrable lattices on the background of numerically approximated breathers [24]. Exact solutions constructed for the limiting integrable discrete systems can be used as seeds of numerical methods for more general nonintegrable lattices.

The paper is organized as follows. Section 2 presents the integrable symplectic map which arises from the Lax system in the nonlinearization method. Section 3 shows that the class of traveling periodic waves of the dmKdV equation is obtained from integrability of the symplectic map. Two families of traveling periodic waves are available in the closed analytical form by means of the Jacobi (dnoidal and cnoidal) elliptic functions. Section 4 gives a numerically-assisted solution of the modulational stability problem for the two families of traveling periodic waves. Sections 5 and 6 present respectively the one-fold and two-fold Darboux transformations, which are then used to construct respectively the algebraic soliton propagating on the dnoidal wave background and the rogue waves generated on the cnoidal wave background. Section 7 contains the summary and the roadmap for further studies. Appendix A gives details of the Lax and stability spectra for the dnoidal and cnoidal waves in the continuous mKdV equation. Appendix B presents the proof that the constant wave solution is modulationally stable in the dmKdV equation. Appendix C gives the proof of the two-fold Darboux transformation for the dmKdV equation assisted with symbolic computations.

2. Nonlinearization method with a single eigenvalue

Eq. (1.1) is a compatibility condition for the following Lax pair of linear equations:

$$\begin{aligned} &\varphi_{n+1} = U(u_n,\lambda)\varphi_n,\\ &U(u_n,\lambda) = \frac{1}{\sqrt{1+u_n^2}}\begin{pmatrix} \lambda & u_n \\ -u_n & \lambda^{-1}\end{pmatrix}, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &\dot{\varphi}_n = V(u_n,\lambda)\varphi_n,\\ &V(u_n,\lambda) = \begin{pmatrix} \frac{1}{2}\left(\lambda^2-\lambda^{-2}\right) & \lambda u_n + \lambda^{-1}u_{n-1} \\ -\lambda u_{n-1} - \lambda^{-1}u_n & -\frac{1}{2}\left(\lambda^2-\lambda^{-2}\right)\end{pmatrix}, \end{aligned} \tag{2.2}$$

where $\{\varphi_n(t)\}_{n\in\mathbb{Z}} \in (\mathbb{C}^2)^{\mathbb{Z}}$ depends on time t and $\lambda \in \mathbb{C}$ is a spectral parameter which is independent of n and t.

Remark 1. There exists another Lax pair of the same dmKdV Eq. (1.1) given by

$$U(u_n,\lambda) = \begin{pmatrix} \lambda & u_n \\ -u_n & \lambda^{-1}\end{pmatrix}$$

and

$$V(u_n,\lambda) = \begin{pmatrix} \frac{1}{2}\left(\lambda^2-\lambda^{-2}\right) + u_n u_{n-1} & \lambda u_n + \lambda^{-1}u_{n-1} \\ -\lambda u_{n-1} - \lambda^{-1}u_n & -\frac{1}{2}\left(\lambda^2-\lambda^{-2}\right) + u_n u_{n-1}\end{pmatrix}.$$

However, this Lax pair cannot be used to connect the squared eigenfunctions of the linear equations with the traveling periodic waves of the dmKdV equation in the nonlinearization method.

Let $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ be a nontrivial solution to the linear system (2.1) and (2.2) for a specific value $\lambda = \lambda_1 \in \mathbb{C}$. The following symmetry exists:

> *If $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ is a solution for $\lambda = \lambda_1$, then $\varphi_n = (-q_n, p_n)^{\mathrm{T}}$ is a solution for $\lambda = \lambda_1^{-1}$.*

Following the nonlinearization method [36], we assume that $\lambda_1^4 \neq 1$ and consider the following constraint between the potential $\{u_n(t)\}_{n\in\mathbb{Z}}$ and the squared eigenfunction $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ with $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ for a single eigenvalue $\lambda = \lambda_1$:

$$u_n = \lambda_1 p_n^2 + \lambda_1^{-1} q_n^2, \qquad n \in \mathbb{Z}. \tag{2.3}$$

Substituting (2.3) into the spatial part (2.1) of Lax system with $\lambda = \lambda_1$, we obtain the discrete map $\varphi_{n+1} = (\mathcal{E}\varphi)_n$, $n \in \mathbb{Z}$ given by

$$(\mathcal{E}\varphi)_n := \frac{1}{\sqrt{1+(\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2)^2}}\begin{pmatrix} \lambda_1 p_n + (\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2) q_n \\ \lambda_1^{-1} q_n - (\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2) p_n \end{pmatrix}. \tag{2.4}$$

In addition to the constraint (2.3), another constraint is given by

$$u_{n-1} = \lambda_1^{-1} p_n^2 + \lambda_1 q_n^2, \qquad n \in \mathbb{Z}. \tag{2.5}$$

This constraint is obtained by using (2.1), (2.3), and the shift operators E and E^{-1} defined by $(E\varphi)_n := \varphi_{n+1}$ and $(E^{-1}\varphi)_n := \varphi_{n-1}$ in the following computation:

$$\begin{aligned} u_{n-1} &= E^{-1}(\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2)\\ &= E^{-1}\left[\frac{\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2 + u_n^2(\lambda_1^{-1} q_n^2 + \lambda_1 p_n^2)}{1+u_n^2}\right]\\ &= E^{-1}\left[\frac{\lambda_1^{-1}(\lambda_1 p_n + u_n q_n)^2 + \lambda_1(\lambda_1^{-1} q_n - u_n p_n)^2}{1+u_n^2}\right]\\ &= E^{-1}\left[\lambda_1^{-1}(E p_n)^2 + \lambda_1 (E q_n)^2\right]\\ &= \lambda_1^{-1} p_n^2 + \lambda_1 q_n^2. \end{aligned}$$

It follows from (2.3) and (2.5) that the squared eigenfunction $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ and the potential $\{u_n(t)\}_{n\in\mathbb{Z}}$ satisfy the following inverse relations:

$$p_n^2 = \frac{\lambda_1 u_n - \lambda_1^{-1} u_{n-1}}{\lambda_1^2 - \lambda_1^{-2}}, \quad q_n^2 = \frac{\lambda_1 u_{n-1} - \lambda_1^{-1} u_n}{\lambda_1^2 - \lambda_1^{-2}}, \tag{2.6}$$

where $\lambda_1^4 \neq 1$ is assumed.

The following lemma characterizes the discrete map $\varphi_{n+1} = (\mathcal{E}\varphi)_n$ given by (2.4) as the symplectic map in $\mathbb{C}^2$ with the 2-form $dp_n \wedge dq_n$ [38].

Lemma 1. *The map $\mathcal{E}$ given by (2.4) is symplectic in $\mathbb{C}^2$ with the 2-form $dp_n \wedge dq_n$.*

Proof. The map is symplectic if $dEp_n \wedge dEq_n = dp_n \wedge dq_n$. In order to prove this relation we use (2.3) and obtain that

$$du_n = 2\lambda_1 p_n dp_n + 2\lambda_1^{-1} q_n dq_n.$$

It follows from (2.1) that

$$dEp_n = \frac{(1+u_n^2)(\lambda_1 dp_n + u_n dq_n + q_n du_n) - (\lambda_1 p_n + u_n q_n) u_n du_n}{(1+u_n^2)^{\frac{3}{2}}},$$

$$dEq_n = \frac{(1+u_n^2)(\lambda_1^{-1} dq_n - u_n dp_n - p_n du_n) - (\lambda_1^{-1} q_n - u_n p_n) u_n du_n}{(1+u_n^2)^{\frac{3}{2}}}.$$

Since $dp_n \wedge dp_n = dq_n \wedge dq_n = 0$, it follows by direct computation that

$$\begin{aligned} dEp_n \wedge dEq_n =& \frac{1}{(1+u_n^2)^2} \left[(1+u_n^2)(\lambda_1 dp_n + u_n dq_n + q_n du_n) \right. \\ & \wedge (\lambda_1^{-1} dq_n - u_n dp_n - p_n du_n) \\ & - u_n(\lambda_1^{-1} q_n - u_n p_n)(\lambda_1 dp_n + u_n dq_n) \wedge du_n \\ & \left. - u_n(\lambda_1 p_n + u_n q_n) du_n \wedge (\lambda_1^{-1} dq_n - u_n dp_n) \right] \\ =& dp_n \wedge dq_n \\ & + \frac{1}{(1+u_n^2)} (\lambda_1^{-1} q_n du_n \wedge dq_n - \lambda_1 p_n dp_n \wedge du_n) \\ =& dp_n \wedge dq_n, \end{aligned}$$

where the expression for du_n has been used to cancel the last term. This proves the relation $dEp_n \wedge dEq_n = dp_n \wedge dq_n$. □

Substituting now (2.3) and (2.5) into the temporal part (2.2) of Lax system with $\lambda = \lambda_1$, we obtain the following nonlinear Hamiltonian system

$$\frac{dp_n}{dt} = \frac{\partial H}{\partial q_n}, \quad \frac{dq_n}{dt} = -\frac{\partial H}{\partial p_n}, \tag{2.7}$$

where $n \in \mathbb{Z}$ is fixed and the Hamiltonian $H = H(p_n, q_n)$ is given by

$$\begin{aligned} H(p_n, q_n) =& \frac{1}{2}(\lambda_1^2 - \lambda_1^{-2}) p_n q_n \\ & + \frac{1}{2}(\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2)(\lambda_1^{-1} p_n^2 + \lambda_1 q_n^2). \end{aligned} \tag{2.8}$$

Since $H(p_n, q_n)$ is t-independent, the Hamiltonian system (2.7)–(2.8) is integrable in the Liouville sense.

The following lemma ensures that the Hamiltonian $H(p_n, q_n)$ is a constant of motion for the symplectic map given by (2.4).

Lemma 2. *Let $F_1 := 2H(p_n, q_n)$. Then, F_1 is independent of $n \in \mathbb{Z}$.*

Proof. It follows by differentiating of the constraint (2.3) in t that

$$\dot{u}_n = \lambda_1^3 p_n^2 + \lambda_1^{-3} q_n^2 - u_{n-1} + 2u_n(\lambda_1^2 - \lambda_1^{-2}) p_n q_n. \tag{2.9}$$

By using the dmKdV equation (1.1), the Lax system (2.1) and (2.2) with $\lambda = \lambda_1$, and the relation (2.9), we obtain that

$$\begin{aligned} &(\lambda_1^2 - \lambda_1^{-2}) p_{n+1} q_{n+1} + u_{n+1} u_n \\ &= \frac{(\lambda_1^2 - \lambda_1^{-2})}{1+u_n^2} (\lambda_1 p_n + u_n q_n)(\lambda_1^{-1} q_n - u_n p_n) + u_{n+1} u_n \\ &= (\lambda_1^2 - \lambda_1^{-2}) p_n q_n + u_n u_{n+1} \\ &\quad + \frac{u_n}{1+u_n^2} (u_{n-1} - \lambda_1^3 p_n^2 - \lambda_1^{-3} q_n - 2u_n p_n q_n (\lambda_1^2 - \lambda_1^{-2})) \\ &= (\lambda_1^2 - \lambda_1^{-2}) p_n q_n + u_n u_{n-1}, \end{aligned}$$

which implies that F_1 is independent of $n \in \mathbb{Z}$. □

Since $F_1 = 2H(p_n, q_n)$ is independent of both n and t, the squared eigenfunction $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ with $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ and the potential $\{u_n(t)\}_{n\in\mathbb{Z}}$ satisfy the following relation:

$$p_n q_n = \frac{F_1 - u_n u_{n-1}}{\lambda_1^2 - \lambda_1^{-2}}. \tag{2.10}$$

Substituting (2.6) and (2.10) into (2.9) yields

$$\dot{u}_n = (\lambda_1^2 + \lambda_1^{-2} + 2F_1) u_n - 2u_{n-1}(1+u_n^2),$$

which together with (1.1) results in the second-order difference equation

$$(1+u_n^2)(u_{n+1} + u_{n-1}) = \omega u_n, \quad n \in \mathbb{Z}, \tag{2.11}$$

where $\omega := \lambda_1^2 + \lambda_1^{-2} + 2F_1$. Expanding $(F_1 - u_n u_{n-1})^2$ with the help of (2.6) and (2.10) yields the conserved quantity for (2.11) in the form:

$$F_1^2 + u_n^2 + u_{n-1}^2 + u_n^2 u_{n-1}^2 = \omega u_n u_{n-1}. \tag{2.12}$$

Consequently, the second-order difference equation (2.11) is an integrable map with the first-order invariant (2.12).

To summarize, the integrable symplectic map $\mathcal{E}$ given by (2.4) and the integrable Hamiltonian system (2.7)–(2.8) obtained with the help of the relations (2.3) and (2.5) between the potential $\{u_n(t)\}_{n\in\mathbb{Z}}$ and the squared eigenfunction $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ with $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ is relevant for the class of solutions of the dmKdV equation given by solutions of the integrable second-order difference equation (2.11) with parameter $\omega := \lambda_1^2 + \lambda_1^{-2} + 2F_1$, where λ_1 is an eigenvalue satisfying $\lambda_1^4 \neq 1$ and $F_1 = 2H(p_n, q_n)$ is independent of $n \in \mathbb{Z}$ and $t \in \mathbb{R}$.

3. Class of traveling periodic waves

Integrability of the symplectic map $\mathcal{E}$ given by (2.4) and the Hamiltonian system (2.7)–(2.8) can be deduced by using the following Lax matrix [36]:

$$W(p_n, q_n, \lambda) = \begin{pmatrix} \frac{1}{2} - \frac{\lambda_1^2 p_n q_n}{\lambda^2 - \lambda_1^2} + \frac{\lambda_1^{-2} p_n q_n}{\lambda^2 - \lambda_1^{-2}} & \frac{\lambda \lambda_1 p_n^2}{\lambda^2 - \lambda_1^2} + \frac{\lambda \lambda_1^{-1} q_n^2}{\lambda^2 - \lambda_1^{-2}} \\ -\frac{\lambda \lambda_1 q_n^2}{\lambda^2 - \lambda_1^2} - \frac{\lambda \lambda_1^{-1} p_n^2}{\lambda^2 - \lambda_1^{-2}} & -\frac{1}{2} + \frac{\lambda_1^2 p_n q_n}{\lambda^2 - \lambda_1^2} - \frac{\lambda_1^{-2} p_n q_n}{\lambda^2 - \lambda_1^{-2}} \end{pmatrix}, \tag{3.1}$$

where $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ with $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ is a nontrivial solution of the Lax system (2.1) and (2.2) with $\lambda = \lambda_1$ used in the constraints (2.3) and (2.5) and $\lambda \in \mathbb{C}$ is an arbitrary spectral parameter. The symplectic map $\varphi_{n+1} = (\mathcal{E}\varphi)_n$ is equivalent to the discrete map equation

$$W(p_{n+1}, q_{n+1}, \lambda) U(u_n, \lambda_1) - U(u_n, \lambda_1) W(p_n, q_n, \lambda) = 0. \tag{3.2}$$

The Hamiltonian system (2.7)–(2.8) is equivalent to the time-dependent equation

$$\frac{d}{dt} W(p_n, q_n, \lambda) = V(u_n, \lambda_1) W(p_n, q_n, \lambda) - W(p_n, q_n, \lambda) V(u_n, \lambda_1). \tag{3.3}$$

In both Eqs. (3.2) and (3.3), u_n and u_{n-1} are related to p_n^2 and q_n^2 by means of the constraints (2.3) and (2.5).

The following lemma introduces the polynomial $P(\lambda)$ associated with the second-order difference equation (2.11), roots of which give admissible values for λ_1 in the constraints (2.3) and (2.5).

Lemma 3. *For every ω in the second-order difference equation (2.11) and every F_1 in the first-order invariant (2.12), $\pm\lambda_1$ and $\pm\lambda_1^{-1}$ are simple roots of the polynomial $P(\lambda)$ given by*

$$P(\lambda) := \lambda^8 - 2\omega\lambda^6 + (2+\omega^2 - 4F_1^2)\lambda^4 - 2\omega\lambda^2 + 1. \tag{3.4}$$

Proof. Since $\det W(p_n, q_n, \lambda)$ is a generating function of conserved quantities for the symplectic map $\mathcal{E}$ given by (2.4) and the Hamiltonian system (2.7)–(2.8) [39,40], it is independent of n and t. Using (2.8) and (3.1), we compute $\det W(p_n, q_n, \lambda)$ in the explicit form

$$\det W(p_n, q_n, \lambda) = -\frac{1}{4} + \frac{\lambda^2 F_1}{(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_1^{-2})}.$$

Hence $\pm\lambda_1$ and $\pm\lambda_1^{-1}$ are simple poles of $\det W(p_n, q_n, \lambda)$. On the other hand, we can rewrite $W(p_n, q_n, \lambda)$ by using (2.6) and (2.10) in the form

$$W(p_n, q_n, \lambda) = \begin{pmatrix} \frac{1}{2} - \frac{\lambda^2(F_1 - u_n u_{n-1})}{(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_1^{-2})} & \frac{\lambda(\lambda^2 u_n - u_{n-1})}{(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_1^{-2})} \\ -\frac{\lambda(\lambda^2 u_{n-1} - u_n)}{(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_1^{-2})} & -\frac{1}{2} + \frac{\lambda^2(F_1 - u_n u_{n-1})}{(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_1^{-2})} \end{pmatrix},$$

from which we compute with the use of the conserved quantity (2.12) that

$$\det W(p_n, q_n, \lambda) = -\frac{P(\lambda)}{4(\lambda^2-\lambda_1^2)^2(\lambda^2-\lambda_1^{-2})^2},$$

where $P(\lambda)$ is given by (3.4). Since $\det W(p_n, q_n, \lambda)$ has simple poles at $\pm\lambda_1$ and $\pm\lambda_1^{-1}$, then $\pm\lambda_1$ and $\pm\lambda_1^{-1}$ are roots of the polynomial $P(\lambda)$. □

Remark 2. Symmetry of coefficients of $P(\lambda)$ in (3.4) supports the quadruple symmetry of its roots at $\{\pm\lambda_1, \pm\lambda_1^{-1}\}$. The polynomial $P(\lambda)$ has eight simple roots which form two quadruplets:

$$P(\lambda) = (\lambda^2-\lambda_1^2)(\lambda^2-\lambda_1^{-2})(\lambda^2-\lambda_2^2)(\lambda^2-\lambda_2^{-2}),$$

where λ_2 is related to λ_1 and F_1 by

$$\lambda_1^2 + \lambda_1^{-2} - \lambda_2^2 - \lambda_2^{-2} + 4F_1 = 0.$$

Two families of periodic solutions are available in the literature (see, e.g., [41]). We give details of these solutions separately and show that they are traveling wave solutions of the dmKdV equation of the form $u_n(t) = \phi(\alpha n + ct)$ with appropriately chosen $\alpha > 0$ and $c > 0$.

Remark 3. We have no proof that the second-order difference equation (2.11) admits no other periodic solutions except for the two families of dnoidal and cnoidal waves. Similarly, we have no proof that a general traveling wave solution of the dmKdV equation (1.1) in the form $u_n(t) = \phi(\alpha n + ct)$ with some $\alpha > 0$ and $c \in \mathbb{R}$ must necessarily be a solution of the second-order difference equation (2.11). Nevertheless, it is remarkable that the second-order difference equation (2.11) describes the real-valued standing wave solutions of the complex-valued AL equation, which is related to the dmKdV equation in the AL hierarchy.

3.1. Dnoidal periodic wave solutions

These are solutions of the dmKdV equation (1.1) and the second-order difference equation (2.11) given by the Jacobi dnoidal elliptic function:

$$u_n(t) = A\,\mathrm{dn}(\xi; k), \quad \xi := \alpha n + ct,$$

where $k \in (0,1)$ is elliptic modulus and $A > 0$ is defined up to the reflection symmetry:

If $u_n(t)$ is a solution, then $-u_n(t)$ is also a solution.

We are looking for appropriately chosen $\alpha > 0$ and $c > 0$ in addition to $k \in (0,1)$ and $A > 0$. Another (translation) parameter can always be included due to the translational symmetry:

If $u_n(t)$ is a solution, then $u_n(t+t_0)$ is also a solution for every $t_0 \in \mathbb{R}$.

Using the formula for addition of the Jacobi elliptic functions,

$$\mathrm{dn}(\xi \pm \alpha; k) = \frac{\mathrm{dn}(\xi;k)\mathrm{dn}(\alpha;k) \mp k^2 \mathrm{sn}(\xi;k)\mathrm{cn}(\xi;k)\mathrm{sn}(\alpha;k)\mathrm{cn}(\alpha;k)}{1 - k^2\mathrm{sn}^2(\xi;k)\mathrm{sn}^2(\alpha;k)},$$

we obtain from (2.11) that

$$\omega = 2(1+A^2)\mathrm{dn}(\alpha;k), \quad \omega\,\mathrm{sn}^2(\alpha;k) = 2A^2\mathrm{dn}(\alpha;k).$$

Since $A > 0$, this gives the unique expressions for A and ω:

$$A = \frac{\mathrm{sn}(\alpha;k)}{\mathrm{cn}(\alpha;k)}, \quad \omega = \frac{2\mathrm{dn}(\alpha;k)}{\mathrm{cn}^2(\alpha;k)}, \quad \alpha \in (0, K(k)),$$

where the cutoff $K(k)$ for the range of α corresponds to the first positive zero of $\mathrm{cn}(\alpha;k) = 0$. Similarly, we obtain from (1.1) that

$$c = 2(1+A^2)\,\mathrm{sn}(\alpha;k)\mathrm{cn}(\alpha;k),$$
$$c\,\mathrm{sn}^2(\alpha;k) = 2A^2\mathrm{sn}(\alpha;k)\mathrm{cn}(\alpha;k),$$

which yields $c = 2A$. The exact dnoidal periodic wave solution can be written explicitly as

$$u_n(t) = \frac{\mathrm{sn}(\alpha;k)}{\mathrm{cn}(\alpha;k)}\mathrm{dn}(\alpha n + ct; k), \quad c = \frac{2\mathrm{sn}(\alpha;k)}{\mathrm{cn}(\alpha;k)}, \quad \omega = \frac{2\mathrm{dn}(\alpha;k)}{\mathrm{cn}^2(\alpha;k)}, \tag{3.5}$$

where $\alpha \in (0, K(k))$ and $k \in (0,1)$ are arbitrary parameters.

- As $k \to 0$, the dnoidal wave degenerates to the constant wave
$$u_n(t) = \tan(\alpha), \quad c = 2\tan(\alpha), \quad \omega = 2\sec^2(\alpha), \quad \alpha \in \left(0, \frac{\pi}{2}\right), \tag{3.6}$$
- As $k \to 1$, the dnoidal wave degenerates into the solitary wave
$$u_n(t) = \sinh(\alpha)\mathrm{sech}(\alpha n + ct), \quad c = 2\sinh(\alpha), \quad \omega = 2\cosh(\alpha), \quad \alpha \in (0, \infty). \tag{3.7}$$

Taking the conserved quantity (2.12) at $\xi = 0$ yields

$$\begin{aligned} F_1^2 &= A^2\left(\omega\mathrm{dn}(\alpha;k) - 1 - \mathrm{dn}^2(\alpha;k) - A^2\mathrm{dn}^2(\alpha;k)\right) \\ &= \frac{\mathrm{sn}^2(\alpha;k)}{\mathrm{cn}^4(\alpha;k)}\left(2\mathrm{dn}^2(\alpha;k) - \mathrm{cn}^2(\alpha;k) - \mathrm{dn}^2(\alpha;k)\mathrm{cn}^2(\alpha;k) - \mathrm{dn}^2(\alpha;k)\mathrm{sn}^2(\alpha;k)\right) \\ &= (1-k^2)\frac{\mathrm{sn}^4(\alpha;k)}{\mathrm{cn}^4(\alpha;k)}, \end{aligned}$$

where we have used the fundamental relation for the Jacobi elliptic functions. This allows us to compute

$$F_1 = \sigma_1 \sqrt{1-k^2}\frac{\mathrm{sn}^2(\alpha;k)}{\mathrm{cn}^2(\alpha;k)}, \quad \sigma_1 = \pm 1, \tag{3.8}$$

after which the bi-quadratic equation $\lambda_1^2 + \lambda_1^{-2} = \omega - 2F_1$ is solved by

$$\lambda_1^2 = \frac{(1-\sigma_1\sigma_2 \mathrm{sn}(\alpha;k))(\mathrm{dn}(\alpha;k) + \sigma_2\sqrt{1-k^2}\mathrm{sn}(\alpha;k))}{\mathrm{cn}^2(\alpha;k)},$$
$$\sigma_2 = \pm 1.$$

The product of squared eigenvalues for different values of σ_2 and same value of σ_1 is equal to one. As a result, there exist exactly four pairs of real eigenvalues $\{\pm\lambda_1, \pm\lambda_1^{-1}, \pm\lambda_2, \pm\lambda_2^{-1}\}$ ordered by

$$0 < \lambda_1 < \lambda_2 < 1 < \lambda_2^{-1} < \lambda_1^{-1},$$

where

$$\lambda_1 = \frac{\sqrt{(1-\mathrm{sn}(\alpha;k))\left(\mathrm{dn}(\alpha;k) - \sqrt{1-k^2}\mathrm{sn}(\alpha;k)\right)}}{\mathrm{cn}(\alpha;k)} \tag{3.9}$$

and

$$\lambda_2 = \frac{\sqrt{(1-\mathrm{sn}(\alpha;k))\left(\mathrm{dn}(\alpha;k) + \sqrt{1-k^2}\mathrm{sn}(\alpha;k)\right)}}{\mathrm{cn}(\alpha;k)}, \tag{3.10}$$

where we have used that $\mathrm{dn}(\alpha;k) > \sqrt{1-k^2} > \sqrt{1-k^2}\mathrm{sn}(\alpha;k)$ for $\alpha \in (0, K(k))$. In the two limiting cases, we have

- $\lambda_2 \to 1$ as $k \to 0$ and
$$\lambda_1 \to \frac{1-\sin(\alpha)}{\cos(\alpha)}, \quad \alpha \in \left(0, \frac{\pi}{2}\right). \tag{3.11}$$
- $\lambda_2 \to \lambda_1$ as $k \to 1$ with
$$\lambda_1 \to \sqrt{\cosh(\alpha) - \sinh(\alpha)}, \quad \alpha \in (0, \infty). \tag{3.12}$$

3.2. Cnoidal periodic wave solutions

These are solutions of the dmKdV equation (1.1) and the second-order difference equation (2.11) given by the Jacobi cnoidal elliptic function:

$$u_n(t) = A\mathrm{cn}(\xi;k), \quad \xi := \alpha n + ct,$$

with four parameters $A > 0$, $\alpha > 0$, $c > 0$, and $k \in (0,1)$ as before. Using the formula for addition of the Jacobi elliptic functions,

$$\mathrm{cn}(\xi \pm \alpha; k) = \frac{\mathrm{cn}(\xi;k)\mathrm{cn}(\alpha;k) \mp \mathrm{sn}(\xi;k)\mathrm{dn}(\xi;k)\mathrm{sn}(\alpha;k)\mathrm{dn}(\alpha;k)}{1-k^2\mathrm{sn}^2(\xi;k)\mathrm{sn}^2(\alpha;k)},$$

we obtain similarly to the dnoidal solutions that

$$A = k\frac{\mathrm{sn}(\alpha;k)}{\mathrm{dn}(\alpha;k)},$$
$$c = \frac{2\mathrm{sn}(\alpha;k)}{\mathrm{dn}(\alpha;k)}, \quad \omega = \frac{2\mathrm{cn}(\alpha;k)}{\mathrm{dn}^2(\alpha;k)}, \quad \alpha \in (0, K(k)),$$

where $K(k)$ corresponds to the first maximum of $\mathrm{sn}(\alpha;k)$. The exact cnoidal periodic wave solution can be written explicitly as

$$u_n(t) = k\frac{\mathrm{sn}(\alpha;k)}{\mathrm{dn}(\alpha;k)}\mathrm{cn}(\alpha n + ct; k), \qquad c = \frac{2\mathrm{sn}(\alpha;k)}{\mathrm{dn}(\alpha;k)}, \quad \omega = \frac{2\mathrm{cn}(\alpha;k)}{\mathrm{dn}^2(\alpha;k)}, \tag{3.13}$$

where $\alpha \in (0, K(k))$ and $k \in (0,1)$ are arbitrary parameters. The limit $k \to 0$ gives the trivial solution, whereas the limit $k \to 1$ gives the same solitary wave (3.7). Taking the conserved quantity (2.12) at $\xi = 0$ yields

$$\begin{aligned} F_1^2 &= A^2\left(\omega\mathrm{cn}(\alpha;k) - 1 - \mathrm{cn}^2(\alpha;k) - A^2\mathrm{cn}^2(\alpha;k)\right) \\ &= \frac{k^2\mathrm{sn}^2(\alpha;k)}{\mathrm{dn}^4(\alpha;k)}\left(2\mathrm{cn}^2(\alpha;k) - \mathrm{dn}^2(\alpha;k) - \mathrm{dn}^2(\alpha;k)\mathrm{cn}^2(\alpha;k)\right. \\ &\quad \left. - \; k^2\mathrm{sn}^2(\alpha;k)\mathrm{cn}^2(\alpha;k)\right) \\ &= -k^2(1-k^2)\frac{\mathrm{sn}^4(\alpha;k)}{\mathrm{dn}^4(\alpha;k)}, \end{aligned}$$

where we have used the fundamental relation for the Jacobi elliptic functions. This allows us to compute

$$F_1 = i\sigma_1 k\sqrt{1-k^2}\frac{\mathrm{sn}^2(\alpha;k)}{\mathrm{dn}^2(\alpha;k)}, \quad \sigma_1 = \pm 1, \tag{3.14}$$

after which the bi-quadratic equation $\lambda_1^2 + \lambda_1^{-2} = \omega - 2F_1$ is solved by

$$\lambda_1^2 = \frac{(1-\sigma_1\sigma_2 k\mathrm{sn}(\alpha;k))(\mathrm{cn}(\alpha;k) + i\sigma_2\sqrt{1-k^2}\mathrm{sn}(\alpha;k))}{\mathrm{dn}^2(\alpha;k)},$$
$$\sigma_2 = \pm 1.$$

There exist exactly two complex quadruplets of eigenvalues $\{\pm\lambda_1, \pm\lambda_1^{-1}, \pm\bar{\lambda}_1, \pm\bar{\lambda}_1^{-1}\}$, where

$$\lambda_1 = \frac{\sqrt{(1-k\mathrm{sn}(\alpha;k))(\mathrm{cn}(\alpha;k) + i\sqrt{1-k^2}\mathrm{sn}(\alpha;k))}}{\mathrm{dn}(\alpha;k)}, \tag{3.15}$$

satisfies the ordering $|\lambda_1| < 1 < |\lambda_1^{-1}|$.

To summarize, the two families of traveling periodic waves of the dmKdV equation are given by the dnoidal wave (3.5) and the cnoidal wave (3.13). We have shown that the dnoidal waves are related to four pairs of real roots of $P(\lambda)$ symmetric about the unit circle and that the cnoidal waves are related to two quadruplets of complex roots of $P(\lambda)$ symmetric about the unit circle.

4. Modulational instability of traveling periodic waves

Let $\{u_n(t)\}_{n\in\mathbb{Z}}$ be a solution to the dmKdV equation (1.1). By adding a perturbation $\{v_n(t)\}_{n\in\mathbb{Z}}$ and expanding (1.1) up to the linear terms in $\{v_n(t)\}_{n\in\mathbb{Z}}$, we obtain the linearized dmKdV equation in the form:

$$\dot{v}_n = (1+u_n^2)(v_{n+1} - v_{n-1}) + 2u_n(u_{n+1} - u_{n-1})v_n, \quad n \in \mathbb{Z}. \tag{4.1}$$

A useful property of integrable equations is the explicit relation between solutions of the linearized equations and solutions of the linear Lax equations. The following lemma specifies this relation for the linearized dmKdV equation (4.1).

Lemma 4. *Let $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ with $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ be an arbitrary solution of the Lax system (2.1) and (2.2) with some arbitrary λ. Then,*

$$v_n = \lambda p_n^2 - \lambda^{-1}q_n^2 + 2u_n p_n q_n \tag{4.2}$$

is a solution to the linearized dmKdV equation (4.1).

Proof. The proof is based on direct computations. It follows from (2.1) that

$$\varphi_{n-1} = \frac{1}{\sqrt{1+u_{n-1}^2}}\begin{pmatrix} \lambda^{-1} & -u_{n-1} \\ u_{n-1} & \lambda \end{pmatrix}\varphi_n. \tag{4.3}$$

Differentiating of (4.2) in t and using the dmKdV equation (1.1) and thelinear equation (2.2) yield

$$\begin{aligned}\dot{v}_n =&(\lambda^2-\lambda^{-2})(\lambda p_n^2+\lambda^{-1}q_n^2)\\&+2u_n^2(\lambda q_n^2-\lambda^{-1}p_n^2)+2u_nu_{n-1}(\lambda^{-1}q_n^2-\lambda p_n^2)\\&+2p_nq_n[(\lambda^2+\lambda^{-2})u_n+u_{n+1}(1+u_n^2)+u_{n-1}(1-u_n^2)].\end{aligned}$$

By using (2.1) and (4.3), we obtain

$$\begin{aligned}v_{n+1}=&\frac{(\lambda^3-\lambda^{-1}u_n^2-2\lambda u_nu_{n+1})p_n^2+(\lambda u_n^2-\lambda^{-3}+2\lambda^{-1}u_nu_{n+1})q_n^2}{1+u_n^2}\\&+\frac{2(\lambda^2u_n+\lambda^{-2}u_n+u_{n+1}-u_n^2u_{n+1})p_nq_n}{1+u_n^2}\end{aligned}$$

and

$$v_{n-1}=\lambda^{-1}p_n^2-\lambda q_n^2-2u_{n-1}p_nq_n,$$

which yield

$$\begin{aligned}&(1+u_n^2)(v_{n+1}-v_{n-1})+2u_n(u_{n+1}-u_{n-1})v_n\\&=(\lambda^2-\lambda^{-2})(\lambda p_n^2+\lambda^{-1}q_n^2)\\&\quad+2u_n^2(\lambda q_n^2-\lambda^{-1}p_n^2)+2u_nu_{n-1}(\lambda^{-1}q_n^2-\lambda p_n^2)\\&\quad+2p_nq_n[(\lambda^2+\lambda^{-2})u_n+u_{n+1}(1+u_n^2)+u_{n-1}(1-u_n^2)].\end{aligned}$$

Hence $v_n(t)$ obtained from (4.2) satisfies (4.1). □

Remark 4. The constraint (2.3) and the squared eigenfunction relation (4.2) look similar but we remind readers that p_n and q_n in (2.3) are computed for a root λ_1 of $P(\lambda)$, whereas p_n and q_n in (4.2) are computed for arbitrary λ. We do not know how the relation for solutions of the linearized equations can be deduced from the nonlinearization method. In the particular case of the dmKdV equation, we have found the relation (4.2) by brutal computations.

Remark 5. If we use the Lax pair in Remark 1, then we are not able to connect the squared eigenfunctions of the linear equations with solutions to the linearized dmKdV equation.

For the traveling periodic wave of the form $u_n(t)=\phi(\alpha n+ct)$ with some $\alpha>0$ and $c>0$, we can separate variables of the linearized mKdV equation in the form

$$v_n(t)=e^{\Lambda t}w(\xi),\qquad \xi:=\alpha n+ct.$$

where Λ is the spectral parameter and $w(\xi)$ is an eigenfunction. The pair (Λ,w) is a solution of the following spectral stability problem

$$\begin{aligned}\Lambda w(\xi)+cw'(\xi)=&(1+\phi^2(\xi))[w(\xi+\alpha)-w(\xi-\alpha)]\\&+2\phi(\xi)[\phi(\xi+\alpha)-\phi(\xi-\alpha)]w(\xi),\end{aligned}\tag{4.4}$$

where w is assumed to be a bounded function of $\xi\in\mathbb{R}$.

We give the analytical solution of the linearized dmKdV equation for the constant wave solution $\phi(\xi)=\tan(\alpha)$ in Appendix B, from which it follows that the constant wave is modulationally stable. Fig. 3 shows the Lax spectrum for the constant wave with $\alpha=\frac{\pi}{6}$. The Lax spectrum is obtained from the explicit expression (B.5) in Appendix B. The red dots show eigenvalues $\{\pm\lambda_1,\pm\lambda_1^{-1},\pm\lambda_2,\pm\lambda_2^{-1}\}$ with λ_1 given by (3.11) and $\lambda_2=1$.

In what follows, we construct the numerical solutions for the Lax spectrum and the modulation stability spectrum for the dnoidal and cnoidal periodic waves (3.5) and (3.13).

4.1. Dnoidal waves

In order to obtain the Lax spectrum of the spectral problem (2.1) associated with the dnoidal wave (3.5), we set $\alpha=K(k)/M$

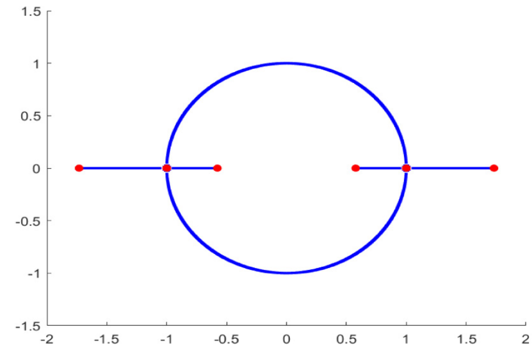

Fig. 3. Lax spectrum for the constant wave with $\alpha=\frac{\pi}{6}$.

with integer $M\in\mathbb{N}$. Then, the dnoidal wave $u_n(t)=\phi(\alpha n+ct)$ becomes periodic in $n\in\mathbb{Z}$ with the period $2M$ for any fixed $t\in\mathbb{R}$.

Combining (2.1) and (4.3) for $\varphi_n=(p_n,q_n)^{\mathrm{T}}$ yields the eigenvalue problem in the form

$$\left\{\begin{array}{l}\sqrt{1+u_n^2}p_{n+1}+\sqrt{1+u_{n-1}^2}p_{n-1}-(u_n-u_{n-1})q_n=zp_n,\\(u_n-u_{n-1})p_n+\sqrt{1+u_n^2}q_{n+1}+\sqrt{1+u_{n-1}^2}q_{n-1}=zq_n,\end{array}\right.\tag{4.5}$$

where $z:=\lambda+\lambda^{-1}$. We use the Floquet theorem for difference equations and look for the Bloch solution of the eigenvalue problem (4.5) in the form

$$\begin{array}{ll}p_n=\hat{p}_n(\theta)e^{i\theta n},& q_n=\hat{q}_n(\theta)e^{i\theta n},\\p_{n+2M}=p_n,& q_{n+2M}=q_n,\end{array}\tag{4.6}$$

where $\theta\in[0,\pi/M]$ is a continuous parameter. Once the spectral bands $z(\theta)$ are obtained from (4.5) and (4.6), the Lax spectrum $\lambda(\theta)$ is obtained from solutions of the quadratic equation $\lambda(\theta)+\lambda(\theta)^{-1}=z(\theta)$ for $\theta\in[0,\pi/M]$.

Fig. 4 shows the Lax spectrum for the dnoidal wave (3.5) for $\alpha=K(k)/M$ with $M=10$. Two cases are shown: $k=0.7$ (left) and $k=0.95$ (right). For $k=0.7$ the Lax spectrum is similar to the one shown in Fig. 3. However, the spectral bands on the real axis are now disjoined from the unit circle. The real spectral bands are located for $|\lambda|\in[\lambda_1,\lambda_2]$ and $|\lambda|\in[\lambda_2^{-1},\lambda_1^{-1}]$. The red dots show eigenvalues $\{\pm\lambda_1,\pm\lambda_1^{-1},\pm\lambda_2,\pm\lambda_2^{-1}\}$, where λ_1 and λ_2 are given by (3.9) and (3.10). For $k=0.95$, the real spectral bands becomes narrow, and as $k\to 1$ they shrink to four points $\pm\lambda_1$ and $\pm\lambda_1^{-1}$, where λ_1 is given by (3.12).

In order to solve the spectral stability problem (4.4) with the squared eigenfunction relation (4.2), we separate the variables for the time evolution of eigenfunctions:

$$\begin{array}{l}p_n(t)=\hat{p}(\xi,\theta)e^{i\theta\alpha^{-1}(\xi-ct)+\gamma(\theta)t},\\q_n(t)=\hat{q}(\xi,\theta)e^{i\theta\alpha^{-1}(\xi-ct)+\gamma(\theta)t},\end{array}\tag{4.7}$$

where $\hat{p}(\xi+2K(k),\theta)=\hat{p}(\xi,\theta)$ and $\hat{q}(\xi+2K(k),\theta)=\hat{q}(\xi,\theta)$. The definition of θ in (4.7) coincides with the one in (4.6). After the separation of variables, we can reduce the time evolution problem (2.2) to the eigenvalue problem

$$\left\{\begin{array}{l}-c\partial_\xi\hat{p}+\frac{1}{2}(\lambda^2-\lambda^{-2})\hat{p}+\left[\lambda\phi(\xi)+\lambda^{-1}\phi(\xi-\alpha)\right]\hat{q}=\gamma\hat{p},\\-\left[\lambda\phi(\xi-\alpha)+\lambda^{-1}\phi(\xi)\right]\hat{p}-c\partial_\xi\hat{q}-\frac{1}{2}(\lambda^2-\lambda^{-2})\hat{q}=\gamma\hat{q},\end{array}\right.\tag{4.8}$$

where the value of $\lambda=\lambda(\theta)$ is defined by the Lax spectrum for the same value of $\theta\in[0,\pi/M]$ and the value of $\gamma=\gamma(\theta)$ is computed from the spectral problem (4.8) with periodic coefficients. The spectral problem (4.8) has been solved numerically by

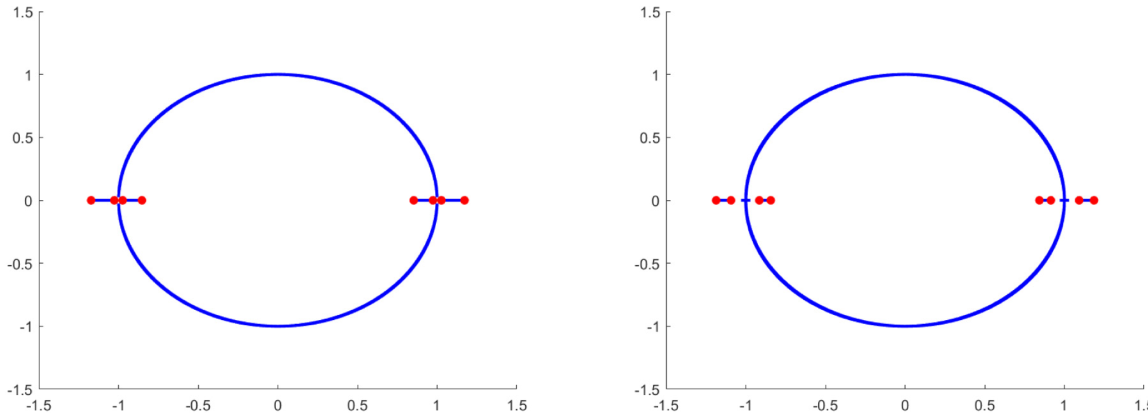

Fig. 4. Lax spectrum for the dnoidal wave with $\alpha = K(k)/M$ with $M = 10$ for $k = 0.7$ (left) and $k = 0.95$ (right).

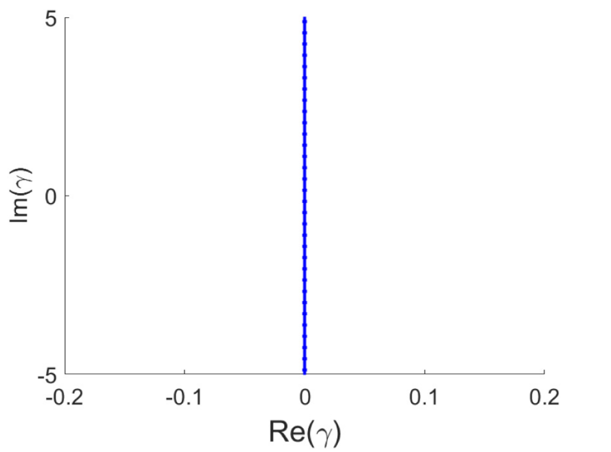

Fig. 5. Stability spectrum for the dnoidal wave with $\alpha = K(k)/M$ with $M = 20$ for $k = 0.8$.

using the Fourier interpolation method [42]. The stability spectrum in the spectral problem (4.4) is obtained from the squared eigenfunction relation (4.2) by $\Lambda = 2\gamma(\theta)$.

Fig. 5 shows the results of numerical approximations for the dnoidal wave (3.5) with $k = 0.8$. The spectrum is located on the imaginary axis, which suggests that the dnoidal wave is spectrally (and modulationally) stable, similarly to the constant wave in Appendix B.

Remark 6. The accuracy of numerical approximation deteriorates for larger values of k and we have observed spurious unstable eigenvalues for $k \geq 0.9$.

4.2. Cnoidal waves

For the cnoidal waves (3.13), the period of Jacobi elliptic function changes to $4K(k)$. When we fix $\alpha = K(k)/M$ with $M \in \mathbb{N}$, the potential $u_n(t)$ becomes periodic in $n \in \mathbb{Z}$ with the period $4M$ for any fixed $t \in \mathbb{R}$.

Fig. 6 shows the Lax spectrum for the cnoidal wave (3.13) for $\alpha = K(k)/M$ with $M = 20$ for the same values of k as in Fig. 4. For $k = 0.7$ the spectral bands are complex and they cross the unit circle. The eight end points of the complex spectral bands are determined by λ_1 given by (3.15) and they are shown by red dots. For $k = 0.95$, the complex spectral bands are disjoint from the unit circle, and as $k \to 1$ they shrink to four points $\pm\lambda_1$ and $\pm\lambda_1^{-1}$, where λ_1 is given by (3.12).

Fig. 7 shows the stability spectrum for the cnoidal wave (3.13) with $k = 0.8$. There are two unstable bands of the spectrum outside the imaginary axis which are connected near the origin. As a result, the cnoidal wave is spectrally (and modulationally) unstable.

To summarize, the Lax spectrum of the dnoidal and cnoidal waves in the dmKdV equation (Figs. 4 and 6) resembles the Lax spectrum of the dnoidal and cnoidal waves in the continuous mKdV equation (left panels on Figs. 1 and 2). The only difference is that the spectral bands of the dmKdV equation occupy the unit circle with four bands outside the unit circle, whereas the spectral bands of the mKdV equation occupy the imaginary axis with two bands outside the imaginary axis. Also similarly to the case of the continuous mKdV equation, the dnoidal waves of the dmKdV equation are spectrally stable, whereas the cnoidal waves of the dmKdV equation are spectrally unstable.

5. One-fold Darboux transformation

The one-fold Darboux transformation (1-fold DT) with a real eigenvalue can be used to construct a new solution to the dmKdV equation from a given solution. The dnoidal wave (3.5) is associated with real eigenvalues $\{\pm\lambda_1, \pm\lambda_1^{-1}, \pm\lambda_2, \pm\lambda_2^{-1}\}$ given by roots of $P(\lambda)$ in (3.4), where $\lambda_1, \lambda_2 \in \mathbb{R}$ are given by (3.9) and (3.10). 1-fold DT will be applied to construct algebraic solitons (spatially decaying traveling waves with an algebraic rate of decay) propagating on the dnoidal wave background.

The following lemma presents the explicit form of 1-fold DT. We prove the explicit transformation by direct methods similarly to the recent work [43].

Lemma 5. *Let $\{u_n(t)\}_{n\in\mathbb{Z}}$ be a solution of the dmKdV equation (1.1) and $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ with $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ be a nontrivial real solution to the linear system (2.1)–(2.2) with $\lambda = \lambda_1 \in \mathbb{R}$. Then,*

$$\hat{u}_n = -\frac{p_n^2 + \lambda_1^2 q_n^2}{\lambda_1^2 p_n^2 + q_n^2} u_n + \frac{(1-\lambda_1^4) p_n q_n}{\lambda_1(\lambda_1^2 p_n^2 + q_n^2)} \tag{5.1}$$

presents a new solution of the dmKdV equation (1.1).

Proof. Let $\{\varphi_n\}_{n\in\mathbb{Z}}$ be a solution to the linear Eqs. (2.1) and (2.2) with arbitrary λ for $\{u_n(t)\}_{n\in\mathbb{Z}}$. We will show that $\{\hat{\varphi}_n\}_{n\in\mathbb{Z}}$ is a solution to the same equations with $\{\hat{u}_n(t)\}_{n\in\mathbb{Z}}$ given by (5.1) and the same λ if $\hat{\varphi}_n = M_n(\lambda)\varphi_n$, where the Darboux matrix $M_n(\lambda)$ is given by

$$M_n(\lambda) = \frac{\sqrt{p_n^2 + \lambda_1^2 q_n^2}}{\sqrt{\lambda_1^2 p_n^2 + q_n^2}} \begin{pmatrix} \lambda + \lambda^{-1} a_n & b_n \\ -b_n & \lambda a_n + \lambda^{-1} \end{pmatrix} \tag{5.2}$$

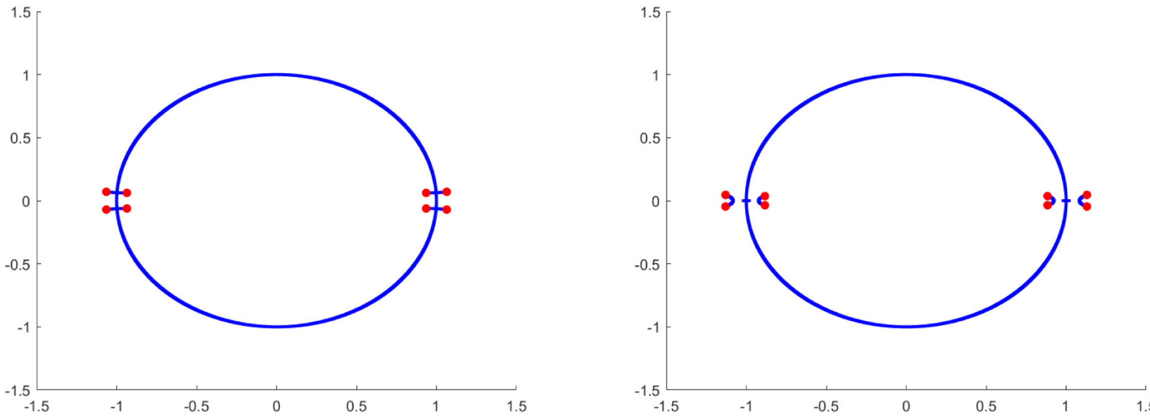

Fig. 6. Lax spectrum for the cnoidal wave with $\alpha = K(k)/M$ with $M = 20$ for $k = 0.7$ (left) and $k = 0.95$ (right).

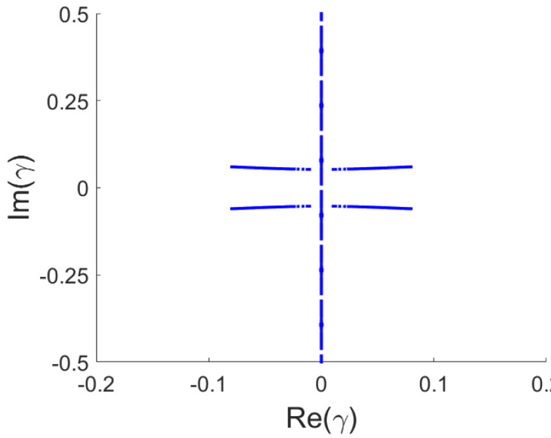

Fig. 7. Stability spectrum for the cnoidal wave with $\alpha = K(k)/M$ with $M = 20$ for $k = 0.8$.

with

$$a_n = -\frac{\lambda_1^2 p_n^2 + q_n^2}{p_n^2 + \lambda_1^2 q_n^2}, \qquad b_n = \frac{(1-\lambda_1^4) p_n q_n}{\lambda_1 (p_n^2 + \lambda_1^2 q_n^2)}. \tag{5.3}$$

For this, we need to show validity of the Darboux equations

$$U(\hat{u}_n, \lambda) M_n(\lambda) = M_{n+1}(\lambda) U(u_n, \lambda) \tag{5.4}$$

and

$$V(\hat{u}_n, \lambda) M_n(\lambda) = \dot{M}_n(\lambda) + M_n(\lambda) V(u_n, \lambda). \tag{5.5}$$

Substituting (2.1) and (5.2) into (5.4) and collecting different powers with respect to λ yields the following system of equations

$$\begin{cases} a_{n+1} - a_n - b_{n+1} u_n + b_n \hat{u}_n = 0, \\ a_n \hat{u}_n + b_n - u_n = 0, \\ \hat{u}_n - a_{n+1} u_n - b_{n+1} = 0, \\ a_{n+1}(1+u_n^2) - a_n (1+\hat{u}_n^2) = 0. \end{cases} \tag{5.6}$$

It follows from (2.1) that

$$\begin{cases} p_{n+1}^2 + \lambda_1^2 q_{n+1}^2 = \lambda_1^2 p_n^2 + q_n^2, \\ (1+u_n^2)(\lambda_1^2 p_{n+1}^2 + q_{n+1}^2) = (\lambda_1^4 + u_n^2) p_n^2 + 2(\lambda_1^3 - \lambda_1^{-1}) u_n p_n q_n \\ \quad + (\lambda_1^2 u_n^2 + \lambda_1^{-2}) q_n^2, \\ (1+u_n^2) p_{n+1} q_{n+1} = -\lambda_1 u_n p_n^2 + (1-u_n^2) p_n q_n + \lambda_1^{-1} u_n q_n^2. \end{cases}$$

Using these equations together with (5.1) and (5.3), we have confirmed validity of each equation in system (5.6). Substituting (2.2) and (5.2) into (5.5) and collecting different powers with respect to λ yields the system of equations

$$\begin{cases} b_n \hat{u}_n - b_n u_{n-1} - \frac{1}{2} a_n^{-1} \dot{a}_n = 0, \\ b_n \hat{u}_{n-1} - b_n u_n + \frac{1}{2} \dot{a}_n = 0, \\ u_n - b_n - a_n \hat{u}_n = 0, \\ \dot{b}_n + u_{n-1} + a_n u_n - \frac{1}{2} a_n^{-1} \dot{a}_n b_n - \hat{u}_n - a_n \hat{u}_{n-1} = 0, \\ \hat{u}_{n-1} - a_n u_{n-1} - b_n = 0, \end{cases} \tag{5.7}$$

where the third and fifth equations repeat the last two equations in (5.6) and the first two equations are equivalent to each other. Thus, system (5.7) is equivalent to just two equations:

$$\begin{cases} \dot{a}_n = -2 b_n (a_n u_{n-1} + b_n - u_n), \\ a_n \dot{b}_n = (a_n^2 - b_n^2 - 1)(a_n u_{n-1} + b_n - u_n). \end{cases} \tag{5.8}$$

It follows from (2.2) that

$$\begin{cases} \frac{d}{dt}(\lambda_1^2 p_n^2 + q_n^2) = (\lambda_1^2 - \lambda_1^{-2})(\lambda_1^2 p_n^2 - q_n^2) + 2 u_n p_n q_n (\lambda_1^3 - \lambda_1^{-1}), \\ \frac{d}{dt}(p_n^2 + \lambda_1^2 q_n^2) = (\lambda_1^2 - \lambda_1^{-2})(p_n^2 - \lambda_1^2 q_n^2) + 2 u_{n-1} p_n q_n (\lambda_1^{-1} - \lambda_1^3), \\ \frac{d}{dt}(p_n q_n) = u_n (\lambda_1 q_n^2 - \lambda_1^{-1} p_n^2) + u_{n-1}(\lambda_1^{-1} q_n^2 - \lambda_1 p_n^2). \end{cases}$$

Together with (5.3) these equations yield

$$\dot{a}_n = \frac{2(1-\lambda_1^4) p_n q_n}{\lambda_1 (p_n^2 + \lambda_1^2 q_n^2)^2} \left[\lambda_1 (\lambda_1^2 - \lambda_1^{-2}) p_n q_n \right. \\ \left. + \; u_n (p_n^2 + \lambda_1^2 q_n^2) + u_{n-1}(\lambda_1^2 p_n^2 + q_n^2)\right]$$

and

$$\dot{b}_n = \frac{1-\lambda_1^4}{\lambda_1 (p_n^2 + \lambda_1^2 q_n^2)^2} [(\lambda_1 q_n^2 - \lambda_1^{-1} p_n^2) \\ \times \; [u_n (p_n^2 + \lambda_1^2 q_n^2) + u_{n-1}(\lambda_1^2 p_n^2 + q_n^2)] \\ - \; (\lambda_1^2 - \lambda_1^{-2}) p_n q_n (p_n^2 - \lambda_1^2 q_n^2)].$$

Using these equations and (5.3), we have confirmed validity of each equation in system (5.8). □

Remark 7. Let $\{u_n(t)\}_{n\in\mathbb{Z}}$ be the dnoidal wave given by (3.5) and $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ with $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ be the associated eigenfunction of the linear system (2.1)–(2.2) with $\lambda = \lambda_1$, where λ_1 is a root of $P(\lambda)$ in (3.4). The 1-fold DT yields

$$\hat{u}_n = -F_1 u_n^{-1} \\ = -\frac{\sigma_1 \mathrm{sn}(\alpha; k)}{\mathrm{cn}(\alpha; k)} \frac{\sqrt{1-k^2}}{\mathrm{dn}(\xi; k)} \\ = -\frac{\sigma_1 \mathrm{sn}(\alpha; k)}{\mathrm{cn}(\alpha; k)} \mathrm{dn}(\xi + K(k); k) \\ = -\sigma_1 u_n (t + c^{-1} K(k)),$$

where we have used (3.8) for F_1. The new solution is a translation of the dnoidal wave (3.5) to a half-period and the flip of the sign if $\sigma_1 = +1$. These transformations are allowed by the symmetries of the dmKdV equation (1.1).

In order to construct a nontrivial new solution $\{\hat{u}_n(t)\}_{n\in\mathbb{Z}}$ on the background of the dnoidal wave (3.5) in Remark 7, we need to construct the second, linearly independent solution of the linear system (2.1) and (2.2) for the same eigenvalue $\lambda = \lambda_1$. This solution is denoted by $\{\hat{\varphi}_n(t)\}_{n\in\mathbb{Z}}$ with $\hat{\varphi}_n = (\hat{p}_n, \hat{q}_n)^{\mathrm{T}}$. The following lemma gives the explicit construction of the second solution.

Lemma 6. *Let $\{u_n(t)\}_{n\in\mathbb{Z}}$ be a solution of the dmKdV equation (1.1) satisfying the reduction (2.3) and $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ with $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ be the associated eigenfunction of the linear system (2.1)–(2.2) with $\lambda = \lambda_1$, where λ_1 is given by a root of the polynomial $P(\lambda)$ in (3.4). The second solution $\{\hat{\varphi}_n(t)\}_{n\in\mathbb{Z}}$ with $\hat{\varphi}_n = (\hat{p}_n, \hat{q}_n)^{\mathrm{T}}$ of the same linear system (2.1)–(2.2) with $\lambda = \lambda_1$ is given by*

$$\hat{p}_n = p_n\theta_n - \frac{q_n}{p_n^2 + q_n^2}, \quad \hat{q}_n = q_n\theta_n + \frac{p_n}{p_n^2 + q_n^2}, \tag{5.9}$$

where the scalar sequence $\{\theta_n(t)\}_{n\in\mathbb{Z}}$ satisfies

$$\theta_{n+1} - \theta_n = \frac{(\lambda_1 + \lambda_1^{-1})^2(u_n^2 - F_1)}{(1 + u_n^2)(u_n + u_{n-1})(u_n + u_{n+1})} \tag{5.10}$$

and

$$\dot{\theta}_n = \frac{(\lambda_1 + \lambda_1^{-1})^2(u_n^2 + u_{n-1}^2 - 2F_1)}{(u_n + u_{n-1})^2}. \tag{5.11}$$

Proof. Substitution of (5.9) into the linear system (2.1) and elimination of p_{n+1} and q_{n+1} from the same linear system (2.1) yield the first-order difference equation

$$\theta_{n+1} - \theta_n = \frac{(\lambda_1 - \lambda_1^{-1})u_n(p_n^2 - q_n^2) + (\lambda_1^{-2} - \lambda_1^2)p_nq_n}{(p_n^2 + q_n^2)\left[\lambda_1^2p_n^2 + \lambda_1^{-2}q_n^2 + u_n^2(p_n^2 + q_n^2) + 2(\lambda_1 - \lambda_1^{-1})u_np_nq_n\right]}.$$

This equation is simplified after elimination of the squared eigenfunctions with the inverse relations (2.6) and (2.10) to the form:

$$\theta_{n+1} - \theta_n = \frac{(\lambda_1 + \lambda_1^{-1})^2(u_n^2 - F_1)}{(u_n + u_{n-1})\left[(\lambda_1^2 + \lambda_1^{-2} + 2F_1 + 1)u_n + u_n^3 - u_{n-1}(1 + u_n^2)\right]}.$$

By using (2.11), we rewrite this equation in the symmetric form (5.10).

Similarly, substitution of (5.9) into the linear system (2.2) and elimination of $\dot{p}_n$ and $\dot{q}_n$ from the same linear system (2.2) yields the first-order differential equation

$$\dot{\theta}_n = \frac{(\lambda_1 - \lambda_1^{-1})(u_n - u_{n-1})(p_n^2 - q_n^2) - 2(\lambda_1^2 - \lambda_1^{-2})p_nq_n}{(p_n^2 + q_n^2)^2}.$$

This equation is simplified after elimination of the squared eigenfunctions with the inverse relations (2.6) and (2.10) to the form (5.11). □

Remark 8. The Wronskian between the two solutions $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ and $\{\hat{\varphi}_n(t)\}_{n\in\mathbb{Z}}$ with $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ and $\hat{\varphi}_n = (\hat{p}_n, \hat{q}_n)^{\mathrm{T}}$ is normalized to unity as follows:

$$p_n(t)\hat{q}_n(t) - \hat{p}_n(t)q_n(t) = 1, \quad \forall n \in \mathbb{Z}, \quad \forall t \in \mathbb{R}.$$

Remark 9. System (5.10) and (5.11) is compatible if $\{u_n(t)\}_{n\in\mathbb{Z}}$ satisfies (2.11) and (2.12). Indeed, taking derivative of (5.10) in t and substituting (5.11) gives a constraint on $\{u_n(t)\}_{n\in\mathbb{Z}}$, which is equivalent to system (2.11) and (2.12).

We are now ready to construct a new solution on the background of the dnoidal wave. Hence we take

$$u_n(t) = \frac{\mathrm{sn}(\alpha; k)}{\mathrm{cn}(\alpha; k)}\mathrm{dn}(\xi; k), \quad \xi = \alpha n + ct.$$

By separation of variables, the solution of system (5.10) and (5.11) can be written in the form

$$\theta_n(t) = an + bt + \chi(\xi), \tag{5.12}$$

where $\chi(\xi + 2K(k)) = \chi(\xi)$ is periodic with the same period as $\mathrm{dn}(\xi; k)$. It follows from (5.11) with the help of (3.8) that b is uniquely found in the form

$$b = \frac{(\lambda_1 + \lambda_1^{-1})^2}{2K(k)} \times \int_0^{2K(k)} \frac{\mathrm{dn}^2(\xi; k) + \mathrm{dn}^2(\xi - \alpha; k) - 2\sigma_1\sqrt{1 - k^2}}{[\mathrm{dn}(\xi; k) + \mathrm{dn}(\xi - \alpha; k)]^2}d\xi,$$

and χ with normalization $\chi(0) = 0$ is uniquely found in the form

$$c\chi(\xi) = (\lambda_1 + \lambda_1^{-1})^2 \times \int_0^{\xi} \frac{\mathrm{dn}^2(\xi'; k) + \mathrm{dn}^2(\xi' - \alpha; k) - 2\sigma_1\sqrt{1 - k^2}}{[\mathrm{dn}(\xi'; k) + \mathrm{dn}(\xi' - \alpha; k)]^2}d\xi' - b\xi,$$

where σ_1 is defined by (3.8). It follows from (5.10) evaluated at $\xi = 0$ that a is uniquely found from

$$a + \chi(\alpha) = \frac{(\lambda_1 + \lambda_1^{-1})^2\ \mathrm{cn}^2(\alpha; k)\ [1 - \sigma_1\sqrt{1 - k^2}]}{[1 + \mathrm{dn}(\alpha; k)]^2}.$$

The 1-fold DT with the second solution (5.9), where $\theta_n(t)$ is given by (5.12), yields a new solution to the dmKdV equation (1.1) in the form:

$$\hat{u}_n = -\frac{\hat{p}_n^2 + \lambda_1^2\hat{q}_n^2}{\lambda_1^2\hat{p}_n^2 + \hat{q}_n^2}u_n + \frac{(1 - \lambda_1^4)\hat{p}_n\hat{q}_n}{\lambda_1(\lambda_1^2\hat{p}_n^2 + \hat{q}_n^2)}. \tag{5.13}$$

Fig. 8 shows the solution surface for the new solution (5.13) associated with the dnoidal wave (3.5) for the eigenvalues given by either (3.9) (top) or (3.10) (bottom). In both cases, we have selected $\alpha = K(k)/4$ and $k = 0.95$ for illustration. The solutions are shown in the continuous coordinates ξ and t, where $\xi = \alpha n + ct$. The background wave is obtained from the limit $|\theta_n| \to \infty$, for which

$$\hat{u}_n(t) \to -\sigma_1 u_n(t + c^{-1}K(k)).$$

For λ_1, the background wave is positive since $\sigma_1 = -1$. If λ_1 is replaced by λ_2 given by (3.10), then the background wave is negative since $\sigma_1 = +1$. In both cases, an algebraic soliton propagates on the background wave. The algebraic soliton is found from (5.12), which can be rewritten as

$$\theta_n(t) = \alpha^{-1}a\xi + \chi(\xi) + (b - \alpha^{-1}ac)t.$$

We have detected numerically that $a, b > 0$ with $a^{-1}b > \alpha^{-1}c$ for λ_1 (top) and $a, b < 0$ with $a^{-1}b < \alpha^{-1}c$ for λ_2 (bottom). As a result, the algebraic soliton moves to the left relative to the dnoidal wave (which also moves to the left since $\alpha > 0$ and $c > 0$) in the former case and it moves to the right relative to the dnoidal wave in the latter case.

To summarize, the 1-fold DT has been used for the dnoidal wave (3.5) with the real eigenvalues λ_1 and λ_2 found from roots of the polynomial $P(\lambda)$ in (3.4). We have shown that two distinct algebraic solitons propagate on the dnoidal wave background for two quadruplets of real eigenvalues associated with λ_1 and λ_2 given by (3.9) and (3.10). Stable propagation of algebraic solitons is related to the modulational stability of the dnoidal waves.

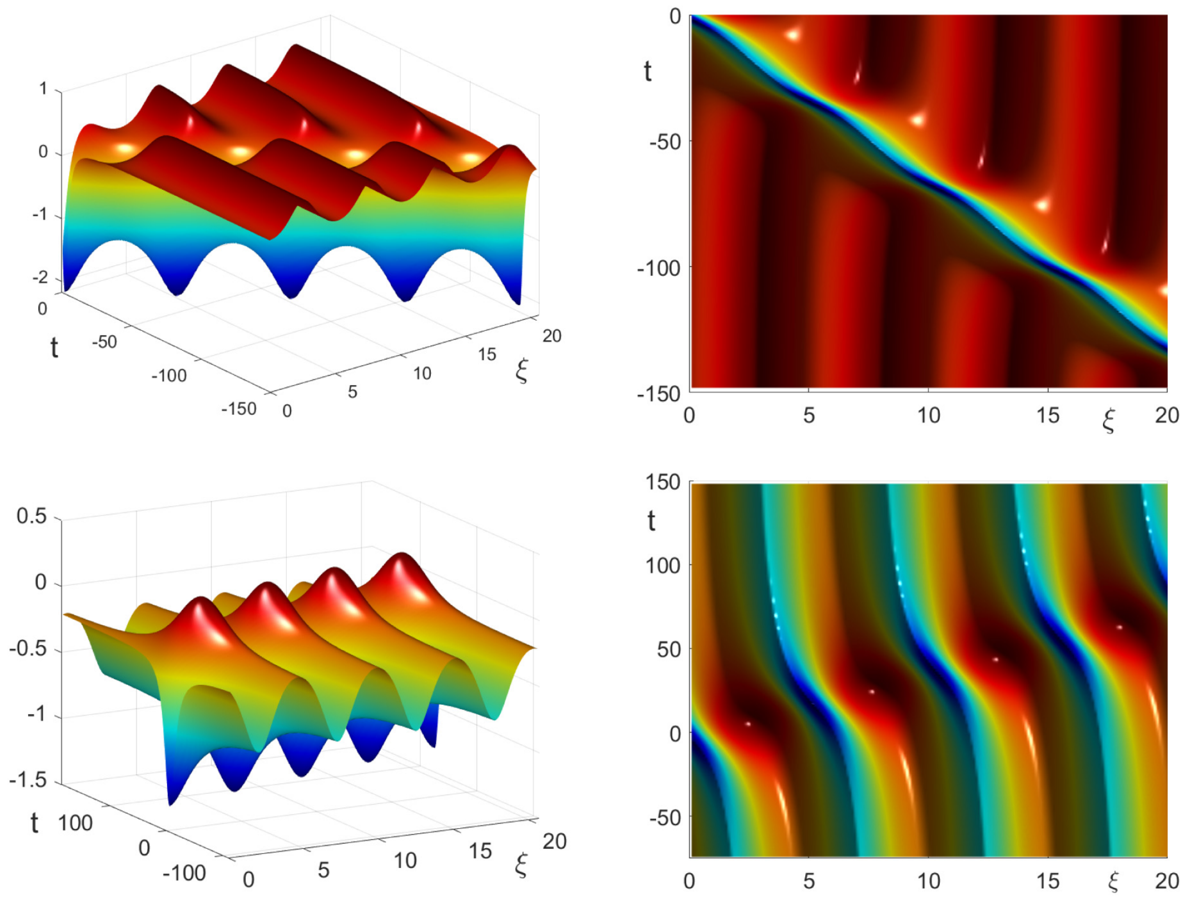

Fig. 8. The solution surface (left: sideview, right: topview) for the algebraic soliton propagating on the background of the dnoidal wave with $\alpha = K(k)/4$ and $k = 0.95$ for eigenvalues λ_1 (top) and λ_2 (bottom).

6. Two-fold Darboux transformation

The two-fold Darboux transformation (2-fold DT) with two eigenvalues can be used to generate a new real solution to the dmKdV equation both if the two eigenvalues are real and if they are complex-conjugate to each other. Therefore, it will be used to obtain two algebraic solitons propagating on the dnoidal wave background and a rogue wave arising on the cnoidal wave background.

The following lemma presents the explicit form of 2-fold DT. Although similar expressions for 2-fold DT have appeared in the literature [22] based on a different Lax pair, we have checked computationally how to generalize the 2-fold DT from the Lax pair (2.1) and (2.2). The technical details of computations are presented in Appendix C.

Lemma 7. *Let $\{u_n(t)\}_{n\in\mathbb{Z}}$ be a solution of the dmKdV equation (1.1). Let $\{\varphi_{1n}(t)\}_{n\in\mathbb{Z}}$ with $\varphi_{1n} = (p_{1n}, q_{1n})^{\mathrm{T}}$ and $\{\varphi_{2n}(t)\}_{n\in\mathbb{Z}}$ with $\varphi_{2n} = (p_{2n}, q_{2n})^{\mathrm{T}}$ be nontrivial solutions to the linear system (2.1)–(2.2) with $\lambda = \lambda_1$ and $\lambda = \lambda_2$ such that $\lambda_1^2 \neq \lambda_2^2$ and $\lambda_1^2\lambda_2^2 \neq 1$. Then, a new solution of the dmKdV equation (1.1) is given by*

$$\hat{u}_n = \frac{\Upsilon_n}{\Delta_n}u_n - \frac{\Sigma_n}{\lambda_1\lambda_2\Delta_n}, \tag{6.1}$$

where

$$\begin{aligned}
\Upsilon_n =&\lambda_2^2(q_{2n}^2+\lambda_2^2p_{2n}^2)(p_{1n}^2+\lambda_1^6q_{1n}^2)+\lambda_1^2(q_{1n}^2+\lambda_1^2p_{1n}^2)(p_{2n}^2+\lambda_2^6q_{2n}^2)\\
&-2\lambda_1^2\lambda_2^2(p_{1n}^2+\lambda_1^2q_{1n}^2)(p_{2n}^2+\lambda_2^2q_{2n}^2)\\
&-2p_{1n}q_{1n}p_{2n}q_{2n}\lambda_1\lambda_2(\lambda_1^4-1)(\lambda_2^4-1),\\
\Sigma_n =&(\lambda_1^2-\lambda_2^2)(\lambda_1^2\lambda_2^2-1)\left[\lambda_1(\lambda_2^4-1)p_{2n}q_{2n}(q_{1n}^2+\lambda_1^2p_{1n}^2)\right.\\
&\left.-\lambda_2(\lambda_1^4-1)p_{1n}q_{1n}(q_{2n}^2+\lambda_2^2p_{2n}^2)\right],\\
\Delta_n =&(\lambda_1^2\lambda_2^2-1)^2(\lambda_1^2p_{1n}^2q_{2n}^2+\lambda_2^2p_{2n}^2q_{1n}^2)\\
&+(\lambda_1^2-\lambda_2^2)^2(\lambda_1^2\lambda_2^2p_{1n}^2p_{2n}^2+q_{1n}^2q_{2n}^2)\\
&-2p_{1n}q_{1n}p_{2n}q_{2n}\lambda_1\lambda_2(\lambda_1^4-1)(\lambda_2^4-1).
\end{aligned}$$

Let us first apply the 2-fold DT to the bounded eigenfunctions satisfying the reduction (2.3) and (2.5) with the eigenvalues λ_1 and λ_2 found from the roots of $P(\lambda)$ in (3.4).

Remark 10. By using (2.6) and (2.10), we eliminate squared eigenfunctions from Υ_n, Σ_n, and Δ_n in terms of u_n and u_{n-1}. The resulting expressions can be simplified due to the conserved quantity (2.12) with F_1 for λ_1 and F_2 for λ_2:

$$\begin{aligned}
\Upsilon_n &= (F_1-F_2)^2\lambda_1^3\lambda_2^3,\\
\Sigma_n &= (F_2-F_1)\lambda_1^2\lambda_2^2(\lambda_1^2-\lambda_2^2)(\lambda_1^2\lambda_2^2-1)u_n,\\
\Delta_n &= (F_1-F_2)^2\lambda_1^3\lambda_2^3.
\end{aligned}$$

The 2-fold DT yields

$$\hat{u}_n = u_n + \frac{(\lambda_1^2-\lambda_2^2)(\lambda_1^2\lambda_2^2-1)}{\lambda_1^2\lambda_2^2(F_1-F_2)}u_n.$$

For the dnoidal wave (3.5), we use (3.8), (3.9), and (3.10) to obtain $\hat{u}_n = u_n - 2u_n = -u_n$ so that the new solution is a flip of the sign of the dnoidal wave. For the cnoidal wave (3.13), we use (3.14) and (3.15) with $\lambda_2 = \bar{\lambda}_1$ to obtain the same result: $\hat{u}_n = u_n - 2u_n = -u_n$.

When bounded periodic solutions of the linear system (2.1) and (2.2) for $\lambda = \lambda_1$ and $\lambda = \lambda_2$ in Remark 10 are replaced

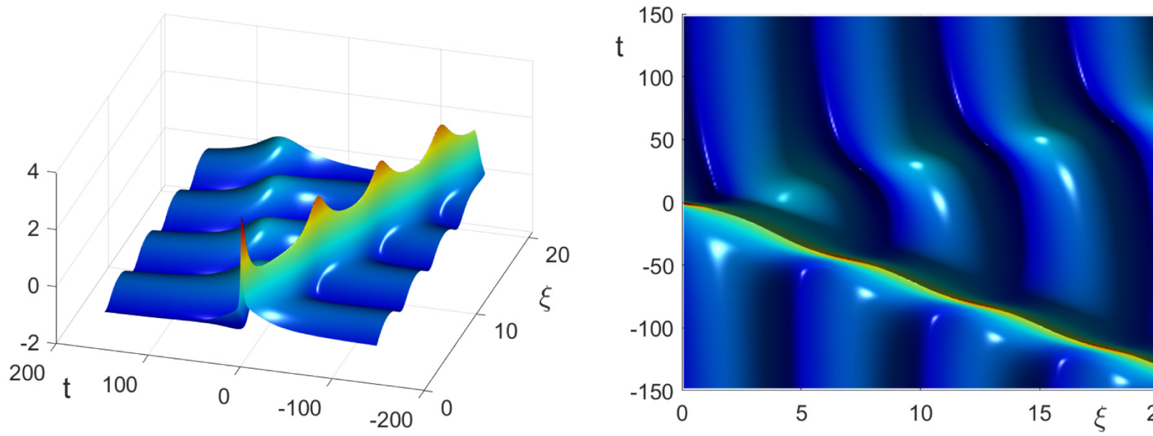

Fig. 9. The solution surface (left: sideview, right: topview) for two algebraic solitons propagating on the dnoidal wave background with $\alpha = K(k)/4$ and $k = 0.95$ for eigenvalues λ_1 and λ_2.

by the unbounded solutions constructed in Lemma 6, the 2-fold DT produces a new solution to the dmKdV equation in the form of two algebraic solitons propagating on the dnoidal wave background. Fig. 9 shows the solution surface for the new solution with $\alpha = K(k)/4$ and $k = 0.95$. On comparison with Fig. 8, we can recognize the same two algebraic solitons associated with the eigenvalues λ_1 and λ_2 but superposed together and flipped in their signs. They are propagating on the background given by the flipped dnoidal wave $-u_n$ according to Remark 10.

Remark 11. Eigenvalues λ_1 and λ_2 determine two particular algebraic solitons propagating on the dnoidal wave background. We do not know if there exist other algebraic solitons which propagate on the dnoidal wave background with different characteristic speeds.

For the cnoidal wave (3.13), the expressions (5.9) for unbounded second solutions of the linear system (2.1) and (2.2) are no longer useful because

$$p_n^2 + q_n^2 = \frac{u_n + u_{n-1}}{\lambda_1 + \lambda_1^{-1}}$$

with $u_n + u_{n-1}$ being sign-indefinite. Therefore, $(\hat{p}_n, \hat{q}_n)$ given by (5.9) are singular if u_n is given by (3.13). In order to obtain a non-singular expression for the second solution, we follow our work in [8,15] and modify the representation of the second solution. The following lemma gives the explicit construction.

Lemma 8. *Let $\{u_n(t)\}_{n\in\mathbb{Z}}$ be a solution of the dmKdV equation (1.1) satisfying the constraint (2.3) and $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ with $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ be the associated eigenfunction of the linear system (2.1)–(2.2) with $\lambda = \lambda_1$ given by a root of the polynomial $P(\lambda)$ in (3.4). The second solution $\{\hat{\varphi}_n(t)\}_{n\in\mathbb{Z}}$ with $\hat{\varphi}_n = (\hat{p}_n, \hat{q}_n)^{\mathrm{T}}$ of the same linear system (2.1)–(2.2) with $\lambda = \lambda_1$ is given by*

$$\hat{p}_n = p_n\theta_n - \frac{1}{2q_n}, \quad \hat{q}_n = q_n\theta_n + \frac{1}{2p_n}, \tag{6.2}$$

where the scalar sequence $\{\theta_n(t)\}_{n\in\mathbb{Z}}$ satisfies

$$\theta_{n+1} - \theta_n = \frac{(\lambda_1^2 - \lambda_1^{-2})^2 u_n^2}{2(1+u_n^2)(F_1 - u_n u_{n-1})(F_1 - u_{n+1}u_n)} \tag{6.3}$$

and

$$\dot{\theta}_n = \frac{(\lambda_1^2 - \lambda_1^{-2})^2 u_n u_{n-1}}{(F_1 - u_n u_{n-1})^2}. \tag{6.4}$$

Proof. Substitution of (6.2) into the linear system (2.1) and elimination of p_{n+1} and q_{n+1} from the same linear system (2.1) yield the first-order difference equation

$$\theta_{n+1} - \theta_n = \frac{u_n(\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2)}{2p_n q_n\left[(1-u_n^2)p_n q_n + u_n(-\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2)\right]},$$

which together with (2.6) and (2.10) yields

$$\theta_{n+1} - \theta_n$$
$$= \frac{(\lambda_1^2 - \lambda_1^{-2})^2 u_n^2}{2(F_1 - u_n u_{n-1})[F_1(1-u_n^2) + u_n u_{n-1}(1+u_n^2) - u_n^2(\lambda_1^2 + \lambda_1^{-2})]}.$$

By using (2.11), we rewrite this equation in the symmetric form (6.3).

Substitution of (6.2) into the linear system (2.2) and elimination of $\dot{p}_n$ and $\dot{q}_n$ from the same linear system (2.2), we arrive at

$$\dot{\theta}_n = \frac{\lambda_1 u_{n-1} + \lambda_1^{-1} u_n}{2q_n^2} + \frac{\lambda_1 u_n + \lambda_1^{-1} u_{n-1}}{2p_n^2},$$

which together with (2.6) and (2.10) yields

$$\dot{\theta}_n = \frac{(\lambda_1^2 - \lambda_1^{-2})^2 u_n u_{n-1}}{(\lambda_1^2 + \lambda_1^{-2})u_n u_{n-1} - u_n^2 - u_{n-1}^2}.$$

By using (2.12), we rewrite this equation in the symmetric form (6.4). □

Remark 12. The Wronskian between the two solutions $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ and $\{\hat{\varphi}_n(t)\}_{n\in\mathbb{Z}}$ with $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ and $\hat{\varphi}_n = (\hat{p}_n, \hat{q}_n)^{\mathrm{T}}$ is again normalized to unity as below:

$$p_n(t)\hat{q}_n(t) - \hat{p}_n(t)q_n(t) = 1, \quad \forall n \in \mathbb{Z}, \quad \forall t \in \mathbb{R}.$$

Remark 13. System (6.3) and (6.4) is compatible if $\{u_n(t)\}_{n\in\mathbb{Z}}$ satisfies (2.11) and (2.12).

We take now the cnoidal wave in the form

$$u_n(t) = k\frac{\mathrm{sn}(\alpha;k)}{\mathrm{dn}(\alpha;k)}\mathrm{cn}(\xi;k), \quad \xi = \alpha n + ct.$$

By separation of variables, the solution of system (6.3) and (6.4) can be written in the form (5.12) with periodic $\chi(\xi + 4K(k)) = \chi(\xi)$ of the same period as $\mathrm{cn}(\xi;k)$. Since $\lambda_1 \in \mathbb{C}$, a, b, and χ are also complex-valued. It follows from (6.4) with the help of (3.14) that b is uniquely found in the form

$$b = \frac{(\lambda_1^2 - \lambda_1^{-2})^2}{4K(k)} \times \int_0^{4K(k)} \frac{\mathrm{dn}^2(\alpha;k)\mathrm{cn}(\xi;k)\mathrm{cn}(\xi-\alpha;k)}{\mathrm{sn}^2(\alpha;k)[i\sqrt{1-k^2} - k\mathrm{cn}(\xi;k)\mathrm{cn}(\xi-\alpha;k)]^2}d\xi,$$

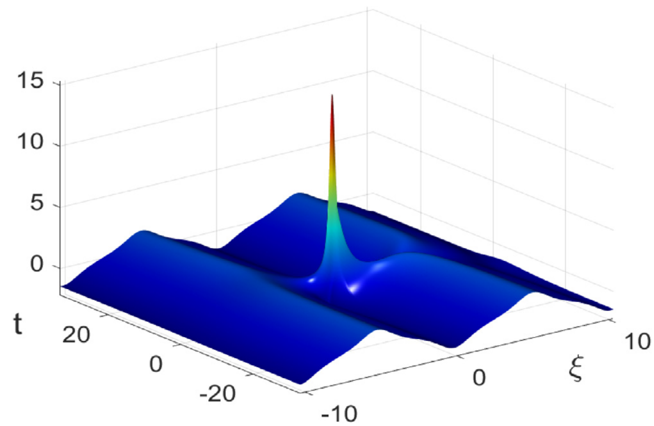

Fig. 10. The solution surface for the rogue wave arising on the background of the cnoidal wave with $\alpha = K(k)/2$ and $k = 0.95$ for eigenvalues $\lambda_2 = \bar{\lambda}_1$.

and χ with normalization $\chi(0) = 0$ is uniquely found in the form

$$c\chi(\xi) = (\lambda_1^2 - \lambda_1^{-2})^2 \times \int_0^{\xi} \frac{\mathrm{dn}^2(\alpha;k)\mathrm{cn}(\xi';k)\mathrm{cn}(\xi'-\alpha;k)}{\mathrm{sn}^2(\alpha;k)[i\sqrt{1-k^2} - k\mathrm{cn}(\xi';k)\mathrm{cn}(\xi'-\alpha;k)]^2} d\xi' - b\xi.$$

It follows from (6.3) evaluated at $\xi = 0$ that a is uniquely found in the form

$$a + \chi(\alpha) = \frac{(\lambda_1^2 - \lambda_1^{-2})^2 \mathrm{dn}^4(\alpha;k)}{2\mathrm{sn}^2(\alpha;k)[i\sqrt{1-k^2} - k\mathrm{cn}(\alpha;k)]^2}.$$

Now we take the unbounded solutions constructed in Lemma 8 in the 2-fold DT with $\lambda_2 = \bar{\lambda}_1$ for the cnoidal wave (3.13). The new solution shown in Fig. 10 is a proper rogue wave arising and disappearing on background of the flipped cnoidal wave $-u_n$. The solution surface was constructed for $\alpha = K(k)/2$ and $k = 0.95$. The rogue wave provides four-times magnification of the amplitude of the cnoidal wave.

Remark 14. Higher-order rogue waves may also exist on the cnoidal wave background similar to the higher-order rogue waves existing on the constant background [20].

To summarize, the 2-fold DT has been used for the dnoidal wave (3.5) to construct a solution with two algebraic solitons propagating in two different directions. By using the 2-fold DT with complex-conjugate eigenvalues for the cnoidal wave (3.13), we have shown that the algebraically decaying solutions correspond to a rogue wave with high magnification of the wave amplitude. Generation of rogue waves is related to the modulational instability of the cnoidal waves.

7. Conclusion

We have developed the nonlinearization method in order to characterize the traveling periodic waves of the dmKdV equation. We obtained the characteristic polynomial $P(\lambda)$ with eight roots for the end points of the spectral bands of the Lax spectrum outside the unit circle. We showed with numerical approximations that the dnoidal waves are spectrally and modulationally stable and the cnoidal waves are spectrally and modulationally unstable. With the help of Darboux transformations, we constructed localized waves on the background of the traveling periodic waves. In the case of dnoidal waves, we found two algebraic solitons with steady propagation speeds. In the case of cnoidal waves, we found a rogue wave which appears from nowhere and disappears without any trace on the wave background.

Several questions are opened for further studies. First, it is unclear if there exists a completeness result which would guarantee that the set of traveling periodic wave solutions of the dmKdV equation coincides with the set of real-valued standing periodic wave solutions of the AL equation expressed by the second-order difference equation (2.11). Second, compared to the continuous mKdV equation (Appendix A), we do not know if there exists a relation between the spectral parameter Λ of the spectral stability problem (4.4) and the characteristic polynomial $P(\lambda)$. Third, the question is open if other algebraic solitons or higher-order rogue waves can be constructed on the background of traveling periodic waves in the same way as they exist on the constant background [20]. Finally, numerical accuracy of computations of the stability spectrum (Figs. 5 and 7) is low and more sophisticated numerical methods need to be elaborated to obtain better numerical approximations of the modulation instability of the cnoidal waves.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix A. Lax and stability spectra in the continuous case

Let $\mathfrak{u} = \mathfrak{u}(\xi, \tau)$ be a solution of the continuous mKdV equation neqrefmKdV-cont. Then, it is a compatibility condition of the Lax pair of linear equations:

$$\varphi_\xi = \begin{pmatrix} \lambda & \mathfrak{u} \\ -\mathfrak{u} & -\lambda \end{pmatrix} \varphi \quad \text{(A.1)}$$

and

$$\varphi_\tau = \begin{pmatrix} 4\lambda^3 + 2\lambda\mathfrak{u}^2 & 4\lambda^2\mathfrak{u} + 2\lambda\mathfrak{u}_\xi + 2\mathfrak{u}^3 + \mathfrak{u}_{\xi\xi} \\ -4\lambda^2\mathfrak{u} + 2\lambda\mathfrak{u}_\xi - 2\mathfrak{u}^3 - \mathfrak{u}_{\xi\xi} & -4\lambda^3 - 2\lambda\mathfrak{u}^2 \end{pmatrix} \varphi. \quad \text{(A.2)}$$

By adding a perturbation $\mathfrak{v} = \mathfrak{v}(\xi, \tau)$ to $\mathfrak{u} = \mathfrak{u}(\xi, \tau)$ and expanding (1.3) up to the linear terms, we obtain the linearized mKdV equation

$$\mathfrak{v}_\tau = 6(\mathfrak{u}^2\mathfrak{v})_\xi + \mathfrak{v}_{\xi\xi\xi}. \quad \text{(A.3)}$$

Direct computations show that if $\varphi = \varphi(\xi, \tau)$ with $\varphi = (p, q)^{\mathrm{T}}$ is a solution of the Lax system (A.1) and (A.2), then $\mathfrak{v} = p^2 - q^2$ is a solution of the linearized mKdV equation (A.3). This yields the squared eigenfunction relation which can be used for relating the Lax spectrum, which consists of the admissible values of λ in the spectral problem (A.1) for which the eigenfunctions φ are bounded, to the stability spectrum of the traveling periodic waves.

Traveling periodic waves $\mathfrak{u} = \phi(\xi + c\tau)$ with the wave speed c satisfy

$$\phi''' - c\phi' + 6\phi^2\phi' = 0,$$

which can be integrated twice to obtain

$$\phi'' - c\phi + 2\phi^3 = b$$

and

$$(\phi')^2 - c\phi^2 + \phi^4 = 2b\phi + d,$$

where b and d are integration constants. As is well-known, see [15], the dnoidal wave in (1.4) corresponds to $b = 0$, $c = 2 - k^2$, $d = k^2 - 1$, whereas the cnoidal wave in (1.4) corresponds to $b = 0$, $c = 2k^2 - 1$, $d = k^2(1 - k^2)$.

For stability analysis of traveling periodic waves (see, e.g., [44]), we separate the variables as $\varphi = \hat{\varphi}(\xi + c\tau)e^{\gamma\tau}$ and obtain from (A.2) the linear homogeneous system, nontrivial solutions of which exists if and only if the following characteristic equation is satisfied:

$$\begin{vmatrix} 4\lambda^3 + 2\lambda\phi^2 - c\lambda - \gamma & 4\lambda^2\phi + 2\lambda\phi' + 2\phi^3 + \phi'' - c\phi \\ -4\lambda^2\phi + 2\lambda\phi' - 2\phi^3 - \phi'' + c\phi & -4\lambda^3 - 2\lambda\phi^2 + c\lambda - \gamma \end{vmatrix} = 0.$$

Expanding the characteristic equation and using the integration constants for the traveling periodic waves, we obtain $\gamma^2 = P(\lambda)$, where the characteristic polynomial for the traveling periodic waves is

$$P(\lambda) := 16\lambda^6 - 8c\lambda^4 + (4d + c^2)\lambda^2 - b^2.$$

Thus, if λ belongs to the Lax spectrum, which is the Floquet spectrum of the spectral problem (A.1) associated with the periodic wave $u = \phi(\xi + c\tau)$, then the perturbation $v = \psi(\xi + c\tau)e^{\Lambda\tau}$ satisfies the linearized mKdV equation (A.3) with $\Lambda = 2\gamma = \pm 2\sqrt{P(\lambda)}$.

Lax spectrum is shown in the left panels of Figs. 1 and 2, where the end points are nonzero roots of the polynomial $P(\lambda)$. Lax spectrum is computed numerically from the spectral problem (A.1) by using the 12-point central discretization of the first derivative as in [26]. The stability spectrum is obtained from the image of the curve $\gamma = \pm\sqrt{P(\lambda)}$, where λ belongs to the Lax spectrum. It is shown in the right panels of Figs. 1 and 2.

Appendix B. Modulation stability of the constant wave solution

Let $A = \tan(\alpha)$ with $\alpha \in (0, \frac{\pi}{2})$ be the amplitude of the constant wave $\phi(\xi) = A$. This coincides with the limiting solution (3.6) of the dnoidal periodic wave (3.5) as $k \to 0$. The linearized dmKdV equation (4.1) becomes

$$\dot{v}_n = (1 + A^2)(v_{n+1} - v_{n-1}), \quad n \in \mathbb{Z}.$$

With the band-limited Fourier transform,

$$v_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{v}(t,\theta)e^{i\theta n}d\theta,$$

we can solve the linearized dmKdV equation in the exact form:

$$\hat{v}(t,\theta) = \hat{v}_0(\theta)e^{2i(1+A^2)\sin(\theta)t}. \tag{B.1}$$

The spectrum of the difference operator $(1 + A^2)(E - E^{-1})$ is purely imaginary and can be parameterized by $\theta \in [-\pi, \pi]$ of the band-limited Fourier transform as follows:

$$\Lambda = \Lambda(\theta) := 2i(1 + A^2)\sin(\theta), \quad \theta \in [-\pi, \pi], \tag{B.2}$$

in agreement with the solution (B.1). Hence we conclude that

The constant wave is linearly and spectrally stable.

Let us recover the exact solution (B.1) and the stability spectrum (B.2) from Lemma 4. With explicit parameterization $A = \tan(\alpha)$, the spectral problem (2.1) is given by

$$\varphi_{n+1} = \begin{pmatrix} \lambda\cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \lambda^{-1}\cos(\alpha) \end{pmatrix}\varphi_n. \tag{B.3}$$

By using the band-limited Fourier transform, the first-order map (B.3) reduces to an algebraic system

$$\begin{pmatrix} e^{i\theta} - \lambda\cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & e^{i\theta} - \lambda^{-1}\cos(\alpha) \end{pmatrix}\hat{\varphi}(t,\theta) = 0. \tag{B.4}$$

A nonzero solution for $\hat{\varphi}(t,\theta)$ exists if and only if λ belongs to the Lax spectrum defined from the quadratic equation

$$\lambda + \lambda^{-1} = 2\frac{\cos(\theta)}{\cos(\alpha)}.$$

The Lax spectrum is computed explicitly as

$$\lambda = \lambda_{\pm}(\theta) := \frac{\cos(\theta) \pm \sqrt{\cos^2(\theta) - \cos^2(\alpha)}}{\cos(\alpha)}, \tag{B.5}$$

where $\theta \in [-\pi, \pi]$. The Lax spectrum fills the unit circle for $|\theta| \in (\alpha, \pi - \alpha)$ and the real segments between $[\lambda_1, \lambda_1^{-1}]$ for $\theta \in [-\alpha, \alpha]$ and $[-\lambda_1^{-1}, -\lambda_1]$ for $|\theta| \in [\pi - \alpha, \pi]$, where λ_1 is given by (3.11). Lax spectrum is shown in Fig. 3 for $\alpha = \frac{\pi}{6}$.

Since

$$\lambda(\theta) - \lambda(\theta)^{-1} = \pm 2\frac{\sqrt{\cos^2(\theta) - \cos^2(\alpha)}}{\cos(\alpha)},$$

the time evolution of the eigenfunction $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ is given by the system (2.2) defined at the Lax spectrum (B.5). This yields

$$\dot{\varphi}_n = 2\frac{\cos(\theta)}{\cos^2(\alpha)} \times \begin{pmatrix} \pm\sqrt{\cos^2(\theta) - \cos^2(\alpha)} & \sin(\alpha) \\ -\sin(\alpha) & \mp\sqrt{\cos^2(\theta) - \cos^2(\alpha)} \end{pmatrix}\varphi_n. \tag{B.6}$$

Solving (B.3) and (B.6) yields the solution in the Fourier form

$$\hat{\varphi}(t,\theta) = \hat{\varphi}_0(\theta)e^{\gamma(\theta)t},$$

where $\hat{\varphi}_0(\theta)$ is the eigenvector of the homogeneous system (B.4) with $\lambda = \lambda_{\pm}(\theta)$. Since the eigenvector is defined up to scalar multiplication, we write

$$\hat{\varphi}_0(\theta) = \begin{pmatrix} \sin(\alpha) \\ \mp\sqrt{\cos^2(\theta) - \cos^2(\alpha)} + i\sin(\theta) \end{pmatrix}$$

and obtain the unique solution for $\gamma(\theta)$ in the form

$$\gamma(\theta) = i\frac{\sin(2\theta)}{\cos^2(\alpha)}, \quad \theta \in [-\pi, \pi].$$

Comparing this expressions with the one given by (B.2) shows that

$$\Lambda(2\theta) = 2\gamma(\theta),$$

which represents the squared eigenfunction relation (4.2) for a single Fourier mode $\varphi_n(t) = \hat{\varphi}_0(\theta)e^{i\theta n + \gamma(\theta)t}$, which gives a single Fourier mode $v_n(t) = \hat{v}_0(\theta)e^{2i\theta n + 2\gamma(\theta)t}$ with some $\hat{v}_0(\theta)$ defined up to the scalar multiplication.

Appendix C. Verification of the two-fold Darboux transformation

Let $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ be a solution to the linear system (2.1) and (2.2) with $\{u_n(t)\}_{n\in\mathbb{Z}}$ for an arbitrary λ. We will show that $\{\hat{\varphi}_n(t)\}_{n\in\mathbb{Z}}$ is a solution of the same equations with $\{\hat{u}_n(t)\}_{n\in\mathbb{Z}}$ given by (6.1) and the same λ if $\hat{\varphi}_n = N_n(\lambda)\varphi_n$, where the Darboux matrix $N_n(\lambda)$ is given by

$$N_n(\lambda) = |B_n|^{-\frac{1}{2}} \begin{pmatrix} \lambda^2 + A_n + \lambda^{-2}B_n & \lambda C_n + \lambda^{-1}D_n \\ -\lambda D_n - \lambda^{-1}C_n & \lambda^2 B_n + A_n + \lambda^{-2} \end{pmatrix}, \tag{C.1}$$

where A_n, B_n, C_n, and D_n are properly chosen.

We need to show validity of the Darboux equations

$$U_n(\hat{u}_n, \lambda)N_n(\lambda) = N_{n+1}(\lambda)U_n(u_n, \lambda), \tag{C.2}$$

and

$$V_n(\hat{u}_n, \lambda)N_n(\lambda) = \dot{N}_n(\lambda) + N_n(\lambda)V_n(u_n, \lambda). \tag{C.3}$$

Substituting (2.1) and (C.1) into (C.2) and collecting different powers with respect to λ yields the system of equations

$$A_{n+1} - u_n C_{n+1} - A_n + \hat{u}_n D_n = 0, \tag{C.4}$$

$$B_{n+1} - u_n D_{n+1} - B_n + \hat{u}_n C_n = 0, \tag{C.5}$$

$$u_n - \hat{u}_n B_n - C_n = 0, \tag{C.6}$$

$$u_n A_{n+1} + C_{n+1} - \hat{u}_n A_n - D_n = 0, \tag{C.7}$$

$$u_n B_{n+1} + D_{n+1} - \hat{u}_n = 0, \tag{C.8}$$

$$|B_{n+1}|(1 + u_n^2) - |B_n|(1 + \hat{u}_n^2) = 0. \tag{C.9}$$

On the other hand, inserting (2.2) and (C.1) into the Darboux Eq. (C.3) and collecting different powers with respect to λ yields the system of equations

$$\hat{u}_n D_n - u_{n-1} C_n - \frac{1}{2} B_n^{-1} \dot{B}_n = 0, \tag{C.10}$$

$$\hat{u}_n C_n + \hat{u}_{n-1} D_n - u_n C_n - u_{n-1} D_n + \dot{A}_n - \frac{1}{2} A_n B_n^{-1} \dot{B}_n = 0, \tag{C.11}$$

$$\hat{u}_{n-1} C_n - u_n D_n + \frac{1}{2} \dot{B}_n = 0, \tag{C.12}$$

$$\hat{u}_n B_n - u_n + C_n = 0, \tag{C.13}$$

$$\hat{u}_{n-1} - u_{n-1} B_n - D_n = 0, \tag{C.14}$$

$$\hat{u}_n A_n + \hat{u}_{n-1} B_n - u_{n-1} - u_n A_n + D_n - \dot{C}_n + \frac{1}{2} C_n B_n^{-1} \dot{B}_n = 0, \tag{C.15}$$

$$\hat{u}_n + \hat{u}_{n-1} A_n - u_n B_n - u_{n-1} A_n - C_n - \dot{D}_n + \frac{1}{2} D_n B_n^{-1} \dot{B}_n = 0. \tag{C.16}$$

We claim that systems (C.4)–(C.9) and (C.10)–(C.16) are satisfied by the explicit expressions

$$\begin{aligned} A_n &= \frac{\mathcal{A}_n}{\lambda_1\lambda_2\Upsilon_n}, & B_n &= \frac{\Delta_n}{\Upsilon_n}, \\ C_n &= \frac{\Sigma_n}{\lambda_1\lambda_2\Upsilon_n}, & D_n &= \frac{\mathcal{D}_n}{\lambda_1\lambda_2\Upsilon_n}, \end{aligned} \tag{C.17}$$

where Υ_n, Σ_n, and Δ_n are given in Lemma 7, whereas $\mathcal{A}_n$ and $\mathcal{D}_n$ are given by

$$\begin{aligned} \mathcal{A}_n &= p_{1n}q_{1n}p_{2n}q_{2n}(\lambda_1^4 - 1)(\lambda_2^4 - 1)(\lambda_1^2 + \lambda_2^2)(\lambda_1^2\lambda_2^2 + 1) \\ &\quad - \lambda_1\lambda_2(\lambda_1^2 - \lambda_2^2)^2(\lambda_1^2 + \lambda_2^2)(p_{1n}^2 p_{2n}^2 + q_{1n}^2 q_{2n}^2) \\ &\quad - \lambda_1\lambda_2(\lambda_1^2\lambda_2^2 - 1)^2(\lambda_1^2\lambda_2^2 + 1)(p_{2n}^2 q_{1n}^2 + p_{1n}^2 q_{2n}^2), \\ \mathcal{D}_n &= (\lambda_1^2 - \lambda_2^2)(\lambda_1^2\lambda_2^2 - 1)\left[p_{1n}q_{1n}\lambda_2(p_{2n}^2 + \lambda_2^2 q_{2n}^2)(\lambda_1^4 - 1)\right. \\ &\quad \left. - p_{2n}q_{2n}\lambda_1(p_{1n}^2 + \lambda_1^2 q_{1n}^2)(\lambda_2^4 - 1)\right]. \end{aligned}$$

Thanks to (C.17), the identity (C.6) gives the two-fold Darboux transformation (6.1). By using (2.1), the identity (C.6), and the transformation (C.17), we have confirmed validity of Eqs. (C.4), (C.5), (C.7), (C.8), and (C.9) by using Wolfram's Mathematica symbolic computations.

It is further obvious that (C.13) and (C.14) are the same as (C.6) and (C.8). With the aid of (C.13) and (C.14), the identities (C.10) and (C.12) are equivalent to

$$\dot{B}_n = 2[u_n D_n - C_n(D_n + u_{n-1}B_n)], \tag{C.18}$$

and the formulas (C.11), (C.15), and (C.16) are reduced to

$$\begin{aligned} \dot{A}_n &= u_n[C_n + B_n^{-1}(A_n D_n - C_n)] + u_{n-1}(D_n - A_n C_n - B_n D_n) \\ &\quad + B_n^{-1} C_n (C_n - A_n D_n) - D_n^2, \end{aligned} \tag{C.19}$$

$$\begin{aligned} \dot{C}_n &= u_n[B_n^{-1}(A_n + C_n D_n) - A_n] + u_{n-1}(B_n^2 - C_n^2 - 1) \\ &\quad + D_n(B_n + 1) - B_n^{-1} C_n (A_n + C_n D_n), \end{aligned} \tag{C.20}$$

$$\begin{aligned} \dot{D}_n &= u_n[B_n^{-1}(D_n^2 + 1) - B_n] + u_{n-1}[A_n(B_n - 1) - C_n D_n] \\ &\quad - B_n^{-1} C_n (D_n^2 + 1) + A_n D_n - C_n. \end{aligned} \tag{C.21}$$

By using (2.2) and the transformation (C.17), we have confirmed validity of Eqs. (C.18), (C.19), (C.20), and (C.21) by using Wolfram's Mathematica symbolic computations.

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