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Darboux transformation and soliton solutions of the semi-discrete massive Thirring model

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ABSTRACT

A one-fold Darboux transformation between solutions of the semi-discrete massive Thirring model is derived using the Lax pair and the dressing method. This transformation is used to find the exact expressions for soliton solutions on zero and nonzero backgrounds. It is shown that the discrete solitons have the same properties as their continuous counterparts.

1. Introduction

The massive Thirring model (MTM) in laboratory coordinates [25] is an example of the nonlinear Dirac equation arising in two-dimensional quantum field theory [16,17], optical Bragg gratings [7], and diatomic chains with periodic couplings [1]. This model received much of attention because of its integrability [19] which was used to study the inverse scattering [13–15,18,23,29,30], soliton solutions [2–4,22], spectral and orbital stability of solitons [6,12,24], and construction of rogue waves [8].

Several integrable semi-discretizations of the MTM in characteristic coordinates were proposed in the literature [20,21,26–28] by discretizing one of the two characteristic coordinates. These semi-discretizations are not relevant for the time-evolution problem related to the MTM in laboratory coordinates. It was only recently [11] when the integrable semi-discretization of the MTM in laboratory coordinates was derived. The corresponding semi-discrete MTM is written as the following system of three coupled equations:

$$\begin{cases} 4\mathrm{i}\dfrac{dU_n}{dt} + Q_{n+1} + Q_n + \dfrac{2\mathrm{i}}{h}(R_{n+1} - R_n) + U_n^2(\bar{R}_n + \bar{R}_{n+1}) \\ \qquad -U_n(|Q_{n+1}|^2 + |Q_n|^2 + |R_{n+1}|^2 + |R_n|^2) - \frac{\mathrm{i}h}{2}U_n^2(\bar{Q}_{n+1} - \bar{Q}_n) = 0, \\ -\dfrac{2\mathrm{i}}{h}(Q_{n+1} - Q_n) + 2U_n - |U_n|^2(Q_{n+1} + Q_n) = 0, \\ R_{n+1} + R_n - 2U_n + \dfrac{\mathrm{i}h}{2}|U_n|^2(R_{n+1} - R_n) = 0, \end{cases} \tag{1}$$

where h is the lattice spacing of the spatial discretization and n is the discrete lattice variable. $\bar{R}_n$ and $\bar{Q}_n$ denote the complex conjugate of R_n and Q_n respectively. Only the first equation of the system (1) represents the time evolution problem, whereas the other two equations represent the constraints which define components of $\{R_n\}_{n\in\mathbb{Z}}$ and $\{Q_n\}_{n\in\mathbb{Z}}$ in terms of $\{U_n\}_{n\in\mathbb{Z}}$ instantaneously in time t.

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In the continuum limit $h \to 0$, the slowly varying solutions to the system (1) can be represented by

$$U_n(t) = U(x = hn, t), \quad R_n(t) = R(x = hn, t), \quad Q_n(t) = Q(x = nh, t),$$

where the continuous variables satisfy the following three equations:

$$\begin{cases} 2\mathrm{i}\dfrac{\partial U}{\partial t} + \mathrm{i}\dfrac{\partial R}{\partial x} + Q + U^2\bar{R} - U(|Q|^2 + |R|^2) = 0, \\ -\mathrm{i}\dfrac{\partial Q}{\partial x} + U - |U|^2 Q = 0, \\ R - U = 0. \end{cases} \tag{2}$$

The system (2) in variables $U(x,t) = u(x, t-x)$ and $Q(x,t) = v(x, t-x)$ yields the continuous MTM system in the form:

$$\begin{cases} \mathrm{i}\left(\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x}\right) + v = |v|^2 u, \\ \mathrm{i}\left(\dfrac{\partial v}{\partial t} - \dfrac{\partial v}{\partial x}\right) + u = |u|^2 v. \end{cases} \tag{3}$$

It is shown in [11] that the semi-discrete MTM system (1) is the compatibility condition

$$\frac{d}{dt}N_n(\lambda) = P_{n+1}(\lambda)N_n(\lambda) - N_n(\lambda)P_n(\lambda), \tag{4}$$

of the following Lax pair of two linear equations:

$$\Phi_{n+1}(\lambda) = N_n(\lambda)\Phi_n(\lambda), \quad N_n(\lambda) = \begin{pmatrix} \lambda + \frac{2\mathrm{i}}{h\lambda}\left(\frac{1+\frac{\mathrm{i}}{2}h|U_n|^2}{1-\frac{\mathrm{i}}{2}h|U_n|^2}\right) & \frac{2U_n}{1-\frac{\mathrm{i}}{2}h|U_n|^2} \\ \frac{2\bar{U}_n}{1-\frac{\mathrm{i}}{2}h|U_n|^2} & \frac{2\mathrm{i}}{h\lambda} - \lambda\left(\frac{1+\frac{\mathrm{i}}{2}h|U_n|^2}{1-\frac{\mathrm{i}}{2}h|U_n|^2}\right) \end{pmatrix}, \tag{5a}$$

$$\frac{d}{dt}\Phi_n(\lambda) = P_n(\lambda)\Phi_n(\lambda), \quad P_n(\lambda) = \frac{\mathrm{i}}{2}\begin{pmatrix} \lambda^2 - |R_n|^2 & \lambda R_n - Q_n\lambda^{-1} \\ \lambda\bar{R}_n - \bar{Q}_n\lambda^{-1} & |Q_n|^2 - \lambda^{-2} \end{pmatrix}, \tag{5b}$$

where $\Phi_n(\lambda) \in \mathbb{C}^2$ is defined for $n \in \mathbb{Z}$ and λ is a spectral parameter.

Because the passage from the discrete system (1) to the continuum limit (3) involves the change of the coordinates $U(x,t) = u(x, t-x)$ and $Q(x,t) = v(x, t-x)$, the initial-value problem for the semi-discrete MTM system (1) does not represent the initial-value problem for the continuous MTM system (3) in time variable t. In addition, numerical explorations of the semi-discrete system (1) are challenging because the last two constraints in the system (1) may lead to appearance of bounded but non-decaying sequences $\{R_n\}_{n\in\mathbb{Z}}$ and $\{Q_n\}_{n\in\mathbb{Z}}$ in response to the bounded and decaying sequence $\{U_n\}_{n\in\mathbb{Z}}$. On the other hand, since the semi-discrete MTM system (1) has the Lax pair of linear equations (5), it is integrable by the inverse scattering transform method which implies existence of infinitely many conserved quantities, exact solutions, transformations between different solutions, and reductions to other integrable equations [10]. These properties of integrable systems were not explored for the semi-discrete MTM system (1) in the previous work [11].

The purpose of this work is to derive the one-fold Darboux transformation between solutions of the semi-discrete MTM system (1). We employ the Darboux transformation in order to generate one-soliton and two-soliton solutions on zero background in the exact analytical form. By looking at the continuum limit $h \to 0$, we show that the discrete solitons share many properties with their continuous counterparts. We also construct one-soliton solutions over the nonzero background. Further properties of the model, e.g. conserved quantities and solvability of the initial-value problem, are left for further studies.

The following theorem represents the main result of this work.

Theorem 1. *Let $\Phi_n(\lambda_1) = (f_n, g_n)^T$ be a nonzero solution of the Lax pair (5) with $\lambda = \lambda_1$ and (U_n, R_n, Q_n) be a solution of the semi-discrete MTM system (1). Another solution of the semi-discrete MTM system (1) is given by*

$$U_n^{[1]} = -\frac{2\,\mathrm{i}(\bar{\lambda}_1|f_n|^2 + \lambda_1|g_n|^2)U_n - h|\lambda_1|^2(\lambda_1|f_n|^2 + \bar{\lambda}_1|g_n|^2)U_n + 2\,\mathrm{i}(\lambda_1^2 - \bar{\lambda}_1^2)f_n\bar{g}_n}{2\,\mathrm{i}(\lambda_1|f_n|^2 + \bar{\lambda}_1|g_n|^2) - h|\lambda_1|^2(\bar{\lambda}_1|f_n|^2 + \lambda_1|g_n|^2) + h(\lambda_1^2 - \bar{\lambda}_1^2)\bar{f}_n g_n U_n}, \tag{6a}$$

$$R_n^{[1]} = -\frac{\left(\bar{\lambda}_1|f_n|^2 + \lambda_1|g_n|^2\right)R_n + \left(\lambda_1^2 - \bar{\lambda}_1^2\right)f_n\bar{g}_n}{\lambda_1|f_n|^2 + \bar{\lambda}_1|g_n|^2}, \tag{6b}$$

$$Q_n^{[1]} = -\frac{|\lambda_1|^2\left(\lambda_1|f_n|^2 + \bar{\lambda}_1|g_n|^2\right)Q_n + \left(\lambda_1^2 - \bar{\lambda}_1^2\right)f_n\bar{g}_n}{|\lambda_1|^2(\bar{\lambda}_1|f_n|^2 + \lambda_1|g_n|^2)}. \tag{6c}$$

Theorem 1 is proven in Section 2 using the Lax pair (5) and the dressing method. One-soliton and two-soliton solutions on zero background are obtained in Section 3. One-soliton solutions over the nonzero background are constructed in Section 4. Both zero and nonzero backgrounds are modulationally stable in the evolution of the semi-discrete MTM system (1). A summary and further directions are discussed in Section 5.

2. Proof of the one-fold Darboux transformation

The one-fold Darboux transformation takes an abstract form (see, e.g., [9]):

$$\Phi^{[1]}(\lambda) = T(\lambda)\Phi(\lambda), \tag{7}$$

where $T(\lambda)$ is the Darboux matrix, $\Phi(\lambda)$ is a solution to the system (5), whereas $\Phi^{[1]}(\lambda)$ is a solution of the transformed system

$$\Phi^{[1]}_{n+1}(\lambda) = N^{[1]}_n(\lambda)\Phi^{[1]}_n(\lambda), \quad \frac{d}{dt}\Phi^{[1]}_n(\lambda) = P^{[1]}_n(\lambda)\Phi^{[1]}_n(\lambda), \tag{8}$$

with $N^{[1]}_n(\lambda)$ and $P^{[1]}_n(\lambda)$ having the same form as $N_n(\lambda)$ and $P_n(\lambda)$ except that the potentials $\left(U_n, Q_n, R_n\right)$ are replaced by $\left(U^{[1]}_n, Q^{[1]}_n, R^{[1]}_n\right)$. By substituting (7) into the linear equations (8) and using the linear equations (5), we obtain the following system of equations for the Darboux matrix $T(\lambda)$:

$$T_{n+1}(\lambda)N_n(\lambda) = N^{[1]}_n(\lambda)T_n(\lambda), \tag{9a}$$

$$\frac{d}{dt}T_n(\lambda) + T_n(\lambda)P_n(\lambda) = P^{[1]}_n(\lambda)T_n(\lambda). \tag{9b}$$

Because (5a) represents the Darboux transformation of the continuous MTM hierarchy [11,28] and (9a) represents permutations of two Darboux transformations, the Darboux matrix $T_n(\lambda)$ takes the form similar to the Darboux matrix $N_n(\lambda)$. Therefore, we are looking for the Darboux matrix $T(\lambda)$ in the following form (used in [31] in the context of the semi-discrete nonlocal nonlinear Schrödinger equation):

$$T_n(\lambda) = \begin{pmatrix} \lambda + a_n\frac{|\lambda_1|^2}{\lambda} & b_n \\ c_n & d_n\lambda + \frac{|\lambda_1|^2}{\lambda} \end{pmatrix}, \tag{10}$$

where the coefficients $\{a_n, b_n, c_n, d_n\}$ are to be computed by using the dressing method from Appendix A in [5]. To do so, we use the symmetry properties of the Lax pair (5). This allows us to find simultaneously both the coefficients of $T(\lambda)$ and the transformations between the potentials $\left(U, Q, R\right)$ and $\left(U^{[1]}, Q^{[1]}, R^{[1]}\right)$.

Lemma 2. *Let $\Phi(\lambda_1) = \left(f, g\right)^T$ be a nonzero solution of the Lax pair (5) at $\lambda = \lambda_1$. Then,*

$$[\Phi(\bar\lambda_1)]_n = \Omega_n\begin{pmatrix} -\bar g_n \\ \bar f_n \end{pmatrix}, \quad [\Phi(-\lambda_1)]_n = (-1)^n\begin{pmatrix} -f_n \\ g_n \end{pmatrix}, \quad [\Phi(-\bar\lambda_1)]_n = (-1)^n\Omega_n\begin{pmatrix} \bar g_n \\ \bar f_n \end{pmatrix}, \tag{11}$$

are solutions of the Lax pair (5) at $\lambda = \bar\lambda_1$, $\lambda = -\lambda_1$, and $\lambda = -\bar\lambda_1$ respectively, where $\Omega_n(t)$ satisfies:

$$\Omega_{n+1} = -S_n\Omega_n, \quad S_n := \frac{1 + \frac{\mathrm{i}}{2}h|U_n|^2}{1 - \frac{\mathrm{i}}{2}h|U_n|^2}, \tag{12a}$$

$$\frac{d\Omega_n}{dt} = M_n\Omega_n, \quad M_n := \frac{\mathrm{i}}{2}\left(\bar\lambda_1^2 - \bar\lambda_1^{-2} + |Q_n|^2 - |R_n|^2\right). \tag{12b}$$

Proof. It follows from (5a) that components of $\Phi(\lambda_1)$ satisfy the system of difference equations:

$$\begin{cases} f_{n+1} = \left(\lambda_1 + \frac{2\mathrm{i}}{h\lambda_1}S_n\right)f_n + \frac{2U_n}{1-\frac{\mathrm{i}}{2}h|U_n|^2}g_n, \\ g_{n+1} = \frac{2\bar U_n}{1-\frac{\mathrm{i}}{2}h|U_n|^2}f_n + \left(\frac{2\mathrm{i}}{h\lambda_1} - \lambda_1 S_n\right)g_n, \end{cases} \tag{13}$$

whereas components of $\Phi(\bar\lambda_1)$ satisfy the system of difference equations:

$$\begin{cases} \Omega_{n+1}\bar g_{n+1} = \left(\bar\lambda_1 + \frac{2\mathrm{i}}{h\bar\lambda_1}S_n\right)\Omega_n\bar g_n - \frac{2U_n}{1-\frac{\mathrm{i}}{2}h|U_n|^2}\Omega_n\bar f_n, \\ \Omega_{n+1}\bar f_{n+1} = -\frac{2\bar U_n}{1-\frac{\mathrm{i}}{2}h|U_n|^2}\Omega_n\bar g_n + \left(\frac{2\mathrm{i}}{h\bar\lambda_1} - \bar\lambda_1 S_n\right)\Omega_n\bar f_n. \end{cases} \tag{14}$$

Dividing (14) by Ω_{n+1} and taking the complex conjugation yields (13) if and only if Ω satisfies the difference equation (12a). Similarly, it follows from (5b) that components of $\Phi(\lambda_1)$ satisfy the time evolution equations:

$$\begin{cases} \frac{df_n}{dt} = \frac{\mathrm{i}}{2}\left[(\lambda_1^2 - |R_n|^2)f_n + (\lambda_1 R_n - \lambda_1^{-1}Q_n)g_n\right], \\ \frac{dg_n}{dt} = \frac{\mathrm{i}}{2}\left[(\lambda_1\bar R_n - \lambda_1^{-1}\bar Q_n)f_n + (-\lambda_1^{-2} + |Q_n|^2)g_n\right], \end{cases} \tag{15}$$

whereas components of $\Phi(\bar\lambda_1)$ satisfy the time evolution equations:

$$\begin{cases} \frac{d\Omega_n}{dt}\bar g_n + \Omega_n\frac{d\bar g_n}{dt} = \frac{\mathrm{i}}{2}\left[(\bar\lambda_1^2 - |R_n|^2)\Omega_n\bar g_n - (\bar\lambda_1 R_n - \bar\lambda_1^{-1}Q_n)\Omega_n\bar f_n\right], \\ \frac{d\Omega_n}{dt}\bar f_n + \Omega_n\frac{d\bar f_n}{dt} = \frac{\mathrm{i}}{2}\left[-(\bar\lambda_1\bar R_n - \bar\lambda_1^{-1}\bar Q_n)\Omega_n\bar g_n + (-\bar\lambda_1^{-2} + |Q_n|^2)\Omega_n\bar f_n\right]. \end{cases} \tag{16}$$

Taking the complex conjugation of (16) yields (15) if and only if Ω satisfies the time evolution equation (12b). The other two solutions in (11) are obtained by the symmetry of the system (5) with respect to the reflection $\lambda \to -\lambda$. □

Lemma 3. *Let $\Phi(\lambda_1) = (f, g)^T$ be in the kernel of the Darboux matrix $T(\lambda_1)$ and $\Phi(\bar{\lambda}_1) = \Omega\big(-\bar{g}, \bar{f}\big)^T$ be in the kernel of $T(\bar{\lambda}_1)$. Then, the coefficients of $T(\lambda)$ in (10) are given by*

$$a_n = -\frac{\bar{\Delta}_n}{\Delta_n}, \quad b_n = -\frac{\left(\lambda_1^2 - \bar{\lambda}_1^2\right) f_n \bar{g}_n}{\Delta_n}, \quad c_n = \frac{\left(\lambda_1^2 - \bar{\lambda}_1^2\right) \bar{f}_n g_n}{\Delta_n}, \quad d_n = -\frac{\bar{\Delta}_n}{\Delta_n}, \tag{17}$$

where $\Delta_n := \bar{\lambda}_1 |f_n|^2 + \lambda_1 |g_n|^2$. Furthermore, $\Phi(-\lambda_1)$ and $\Phi(-\bar{\lambda}_1)$ in (11) are in the kernel of $T(-\lambda_1)$ and $T(-\bar{\lambda}_1)$ respectively.

Proof. We rewrite the linear equations for $T(\lambda_1)\Phi(\lambda_1) = 0$ and $T(\bar{\lambda}_1)\Phi(\bar{\lambda}_1) = 0$ in the following explicit form:

$$\begin{cases} (\lambda_1 + a_n \bar{\lambda}_1) f_n + b_n g_n = 0, \\ c_n f_n + (d_n \lambda_1 + \bar{\lambda}_1) g_n = 0, \\ -(\bar{\lambda}_1 + a_n \lambda_1) \bar{g}_n + b_n \bar{f}_n = 0, \\ -c_n \bar{g}_n + (d_n \bar{\lambda}_1 + \lambda_1) \bar{f}_n = 0, \end{cases} \tag{18}$$

where the scalar factor Ω has been canceled out. Solving the linear system (18) with Cramer's rule yields (17). Then, it follows from (10) and (17) that $T_n(\lambda)$ can be written in the form:

$$T_n(\lambda) = \frac{(\lambda^2 - \lambda_1^2)(\lambda^2 - \bar{\lambda}_1^2)}{2\lambda \Delta_n} \hat{T}_n(\lambda), \tag{19}$$

where

$$\hat{T}_n(\lambda) = \frac{1}{\lambda - \lambda_1} \begin{pmatrix} \bar{g}_n \\ \bar{f}_n \end{pmatrix} \begin{pmatrix} g_n & -f_n \end{pmatrix} + \frac{1}{\lambda + \lambda_1} \begin{pmatrix} -\bar{g}_n \\ \bar{f}_n \end{pmatrix} \begin{pmatrix} g_n & f_n \end{pmatrix} + \frac{1}{\lambda - \bar{\lambda}_1} \begin{pmatrix} f_n \\ -g_n \end{pmatrix} \begin{pmatrix} \bar{f}_n & \bar{g}_n \end{pmatrix} + \frac{1}{\lambda + \bar{\lambda}_1} \begin{pmatrix} f_n \\ g_n \end{pmatrix} \begin{pmatrix} -\bar{f}_n & \bar{g}_n \end{pmatrix}.$$

It follows from (19) that

$$T_n(\lambda_1) \begin{pmatrix} f_n \\ g_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad T_n(-\lambda_1) \begin{pmatrix} -f_n \\ g_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad T_n(\bar{\lambda}_1) \begin{pmatrix} -\bar{g}_n \\ \bar{f}_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad T_n(-\bar{\lambda}_1) \begin{pmatrix} \bar{g}_n \\ \bar{f}_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

hence $T(\pm\lambda_1)\Phi(\pm\lambda_1) = 0$ and $T(\pm\bar{\lambda}_1)\Phi(\pm\bar{\lambda}_1) = 0$. □

Lemma 4. *Let the Darboux matrix $T(\lambda)$ be in the form (10) with the coefficients given by Eqs. (17). Then, the determinant of $T(\lambda)$ is given by*

$$\det T_n(\lambda) = -\frac{(\lambda^2 - \lambda_1^2)(\lambda^2 - \bar{\lambda}_1^2)}{\lambda^2} \frac{\bar{\Delta}_n}{\Delta_n}. \tag{20}$$

Proof. Expanding $\det T_n(\lambda)$ given by (10) yields

$$\det T_n(\lambda) = d_n \lambda^2 + a_n d_n |\lambda_1|^2 - b_n c_n + |\lambda_1|^2 + a_n |\lambda_1|^4 \lambda^{-2}. \tag{21}$$

Since $\pm\lambda_1$ and $\pm\bar{\lambda}_1$ are the roots of $\det T(\lambda)$, we obtain (20) by substituting (17) into (21). □

For $\lambda \neq \pm\lambda_1$ and $\lambda \neq \pm\bar{\lambda}_1$, we define

$$\mathrm{ad}T_n(\lambda) = \det T_n(\lambda)[T_n(\lambda)]^{-1} = \begin{pmatrix} d_n \lambda + \frac{|\lambda_1|^2}{\lambda} & -b_n \\ -c_n & \lambda + a_n \frac{|\lambda_1|^2}{\lambda} \end{pmatrix}, \tag{22}$$

and obtain $\mathrm{ad}T_n(\lambda)$ from (10) and (17) in the form:

$$\mathrm{ad}T_n(\lambda) = \frac{(\lambda^2 - \lambda_1^2)(\lambda^2 - \bar{\lambda}_1^2)}{2\lambda \Delta_n} \mathrm{ad}\hat{T}_n(\lambda), \tag{23}$$

where

$$\begin{aligned} \mathrm{ad}\hat{T}_n(\lambda) = & \frac{1}{\lambda - \lambda_1} \begin{pmatrix} f_n \\ g_n \end{pmatrix} \begin{pmatrix} -\bar{f}_n & \bar{g}_n \end{pmatrix} + \frac{1}{\lambda + \lambda_1} \begin{pmatrix} f_n \\ -g_n \end{pmatrix} \begin{pmatrix} \bar{f}_n & \bar{g}_n \end{pmatrix} \\ & + \frac{1}{\lambda - \bar{\lambda}_1} \begin{pmatrix} \bar{g}_n \\ -\bar{f}_n \end{pmatrix} \begin{pmatrix} -g_n & -f_n \end{pmatrix} + \frac{1}{\lambda + \bar{\lambda}_1} \begin{pmatrix} \bar{g}_n \\ \bar{f}_n \end{pmatrix} \begin{pmatrix} g_n & -f_n \end{pmatrix}. \end{aligned}$$

New potentials $N_n^{[1]}(\lambda)$ and $P_n^{[1]}(\lambda)$ are derived from Eqs. (9) by using the Darboux matrix $T(\lambda)$. Assuming $\lambda \neq \pm\lambda_1$ and $\lambda \neq \pm\bar{\lambda}_1$, we obtain from (9) and (23) that

$$N_n^{[1]}(\lambda) = \frac{1}{\det T_n(\lambda)} T_{n+1}(\lambda) N_n(\lambda) \mathrm{ad}T_n(\lambda) = -\frac{\lambda}{2\bar{\Delta}_n} T_{n+1}(\lambda) N_n(\lambda) \mathrm{ad}\hat{T}_n(\lambda) \tag{24}$$

and

$$P_n^{[1]}(\lambda) = \frac{1}{\det T_n(\lambda)} \left[\frac{d}{dt} T_n(\lambda) + T_n(\lambda) P_n(\lambda)\right] \mathrm{ad}T_n(\lambda) = -\frac{\lambda}{2\bar{\Delta}_n} \left[\frac{d}{dt} T_n(\lambda) + T_n(\lambda) P_n(\lambda)\right] \mathrm{ad}\hat{T}_n(\lambda), \tag{25}$$

where the expressions (20) and (23) have been used.

First, we compute the products in the right-hand side of Eq. (24). By Lemma 2 and direct computations, we obtain

$$N_n(\lambda)\begin{pmatrix} f_n \\ g_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ g_{n+1} \end{pmatrix} + (\lambda-\lambda_1)\begin{pmatrix} 1-\frac{2\,\mathrm{i}}{h\lambda\lambda_1}S_n & 0 \\ 0 & -\frac{2\,\mathrm{i}}{h\lambda\lambda_1}-S_n \end{pmatrix}\begin{pmatrix} f_n \\ g_n \end{pmatrix}, \tag{26a}$$

$$N_n(\lambda)\begin{pmatrix} f_n \\ -g_n \end{pmatrix} = \begin{pmatrix} -f_{n+1} \\ g_{n+1} \end{pmatrix} + (\lambda+\lambda_1)\begin{pmatrix} 1+\frac{2\,\mathrm{i}}{h\lambda\lambda_1}S_n & 0 \\ 0 & \frac{2\,\mathrm{i}}{h\lambda\lambda_1}-S_n \end{pmatrix}\begin{pmatrix} f_n \\ -g_n \end{pmatrix}, \tag{26b}$$

$$N_n(\lambda)\begin{pmatrix} \bar g_n \\ -\bar f_n \end{pmatrix} = -S_n\begin{pmatrix} \bar g_{n+1} \\ -\bar f_{n+1} \end{pmatrix} + (\lambda-\bar\lambda_1)\begin{pmatrix} 1-\frac{2\,\mathrm{i}}{h\lambda\bar\lambda_1}S_n & 0 \\ 0 & -\frac{2\,\mathrm{i}}{h\lambda\bar\lambda_1}-S_n \end{pmatrix}\begin{pmatrix} \bar g_n \\ -\bar f_n \end{pmatrix}, \tag{26c}$$

$$N_n(\lambda)\begin{pmatrix} \bar g_n \\ \bar f_n \end{pmatrix} = S_n\begin{pmatrix} \bar g_{n+1} \\ \bar f_{n+1} \end{pmatrix} + (\lambda+\bar\lambda_1)\begin{pmatrix} 1+\frac{2\,\mathrm{i}}{h\lambda\bar\lambda_1}S_n & 0 \\ 0 & \frac{2\,\mathrm{i}}{h\lambda\bar\lambda_1}-S_n \end{pmatrix}\begin{pmatrix} \bar g_n \\ \bar f_n \end{pmatrix}, \tag{26d}$$

where S_n is defined in Eq. (12a). By using this table, we compute the first product in (24):

$$\begin{aligned} N_n(\lambda)\mathrm{ad}\hat T_n(\lambda) &= \frac{1}{\lambda-\lambda_1}\begin{pmatrix} f_{n+1} \\ g_{n+1} \end{pmatrix}\begin{pmatrix} -\bar f_n & \bar g_n \end{pmatrix} + \frac{1}{\lambda+\lambda_1}\begin{pmatrix} -f_{n+1} \\ g_{n+1} \end{pmatrix}\begin{pmatrix} \bar f_n & \bar g_n \end{pmatrix} \\ &+ \frac{1}{\lambda-\bar\lambda_1}S_n\begin{pmatrix} \bar g_{n+1} \\ -\bar f_{n+1} \end{pmatrix}\begin{pmatrix} g_n & f_n \end{pmatrix} + \frac{1}{\lambda+\bar\lambda_1}S_n\begin{pmatrix} \bar g_{n+1} \\ \bar f_{n+1} \end{pmatrix}\begin{pmatrix} g_n & -f_n \end{pmatrix} + \frac{4\mathrm{i}}{h\lambda|\lambda_1|^2}\begin{pmatrix} S_n\Delta_n & 0 \\ 0 & -\bar\Delta_n \end{pmatrix}. \end{aligned}$$

By Lemma 3 and direct computations, we obtain

$$T_n(\lambda)\begin{pmatrix} f_n \\ g_n \end{pmatrix} = (\lambda-\lambda_1)\begin{pmatrix} 1-a_n\frac{\bar\lambda_1}{\lambda} & 0 \\ 0 & d_n-\frac{\bar\lambda_1}{\lambda} \end{pmatrix}\begin{pmatrix} f_n \\ g_n \end{pmatrix}, \tag{27a}$$

$$T_n(\lambda)\begin{pmatrix} f_n \\ -g_n \end{pmatrix} = (\lambda+\lambda_1)\begin{pmatrix} 1+a_n\frac{\bar\lambda_1}{\lambda} & 0 \\ 0 & d_n+\frac{\bar\lambda_1}{\lambda} \end{pmatrix}\begin{pmatrix} f_n \\ -g_n \end{pmatrix}, \tag{27b}$$

$$T_n(\lambda)\begin{pmatrix} \bar g_n \\ -\bar f_n \end{pmatrix} = (\lambda-\bar\lambda_1)\begin{pmatrix} 1-a_n\frac{\lambda_1}{\lambda} & 0 \\ 0 & d_n-\frac{\lambda_1}{\lambda} \end{pmatrix}\begin{pmatrix} \bar g_n \\ -\bar f_n \end{pmatrix}, \tag{27c}$$

$$T_n(\lambda)\begin{pmatrix} \bar g_n \\ \bar f_n \end{pmatrix} = (\lambda+\bar\lambda_1)\begin{pmatrix} 1+a_n\frac{\lambda_1}{\lambda} & 0 \\ 0 & d_n+\frac{\lambda_1}{\lambda} \end{pmatrix}\begin{pmatrix} \bar g_n \\ \bar f_n \end{pmatrix}. \tag{27d}$$

By using this table, we compute the second product in (24):

$$\begin{aligned} T_{n+1}(\lambda)N_n(\lambda)\mathrm{ad}\hat T_n(\lambda) &= 2\begin{pmatrix} -\left(f_{n+1}\bar f_n - S_n\bar g_{n+1}g_n\right) & -\frac{a_{n+1}}{\lambda}\left(\bar\lambda_1 f_{n+1}\bar g_n + S_n\lambda_1\bar g_{n+1}f_n\right) \\ \frac{1}{\lambda}\left(\bar\lambda_1 g_{n+1}\bar f_n + S_n\lambda_1\bar f_{n+1}g_n\right) & d_{n+1}\left(g_{n+1}\bar g_n - S_n\bar f_{n+1}f_n\right) \end{pmatrix} \\ &+ \frac{4\mathrm{i}}{h\lambda|\lambda_1|^2}\begin{pmatrix} \lambda+a_{n+1}\frac{|\lambda_1|^2}{\lambda} & b_{n+1} \\ c_{n+1} & d_{n+1}\lambda+\frac{|\lambda_1|^2}{\lambda} \end{pmatrix}\begin{pmatrix} S_n\Delta_n & 0 \\ 0 & -\bar\Delta_n \end{pmatrix}. \end{aligned}$$

Substituting this expression into (24), we finally obtain

$$N_n^{[1]}(\lambda) = \begin{pmatrix} \delta_0\lambda+\frac{2\,\mathrm{i}}{h\lambda}\delta_1 & \delta_2 \\ \delta_3 & \frac{2\,\mathrm{i}}{h\lambda}-\delta_4\lambda \end{pmatrix}, \tag{28}$$

where

$$\begin{aligned} \delta_0 &= \frac{\bar f_n f_{n+1} - S_n g_n \bar g_{n+1}}{\bar\Delta_n} - \frac{2\,\mathrm{i}}{h}\frac{S_n\Delta_n}{|\lambda_1|^2\bar\Delta_n}, \\ \delta_1 &= -\frac{a_{n+1}S_n\Delta_n}{\bar\Delta_n}, \\ \delta_2 &= a_{n+1}\frac{\bar\lambda_1 f_{n+1}\bar g_n + S_n\lambda_1\bar g_{n+1}f_n}{\bar\Delta_n} + \frac{2\,\mathrm{i}b_{n+1}}{h|\lambda_1|^2}, \\ \delta_3 &= -\frac{\bar\lambda_1 g_{n+1}\bar f_n + S_n\lambda_1\bar f_{n+1}g_n}{\bar\Delta_n} - \frac{2\,\mathrm{i}c_{n+1}S_n\Delta_n}{h|\lambda_1|^2\bar\Delta_n}, \\ \delta_4 &= -\frac{2\,\mathrm{i}d_{n+1}}{h|\lambda_1|^2} + d_{n+1}\frac{g_{n+1}\bar g_n - S_n\bar f_{n+1}f_n}{\bar\Delta_n}. \end{aligned}$$

It follows from substitution of (13) and (14) for f_{n+1}, g_{n+1}, $\bar f_{n+1}$ and $\bar g_{n+1}$ that

$$\bar{f}_n f_{n+1} - S_n g_n \bar{g}_{n+1} = \bar{\Delta}_n + \frac{2\,\mathrm{i} S_n \Delta_n}{h|\lambda_1|^2}$$

and

$$g_{n+1}\bar{g}_n - S_n \bar{f}_{n+1} f_n = -S_n \Delta_n + \frac{2\,\mathrm{i}\bar{\Delta}_n}{h|\lambda_1|^2}.$$

As a result, we verify that $\delta_0 = 1$ and $\delta_1 = \delta_4$. We represent $N_n^{[1]}(\lambda)$ in (28) in the same form as $N_n(\lambda)$ in (5a), therefore, we write

$$\delta_1 = \frac{1+\frac{\mathrm{i}}{2}hW_n}{1-\frac{\mathrm{i}}{2}hW_n}, \quad \delta_2 = \frac{2Y_n}{1-\frac{\mathrm{i}}{2}hW_n}, \quad \delta_3 = \frac{2Z_n}{1-\frac{\mathrm{i}}{2}hW_n} \tag{29}$$

for some Y_n, Z_n, and W_n. Using Eqs. (17) for a_{n+1}, b_{n+1}, and c_{n+1} and solving Eq. (29) for W_n, Y_n, and Z_n yield

$$W_n = \frac{2\,\mathrm{i}(\bar{\Delta}_n\Delta_{n+1} - S_n\bar{\Delta}_{n+1}\Delta_n)}{h(\bar{\Delta}_n\Delta_{n+1} + S_n\bar{\Delta}_{n+1}\Delta_n)}, \tag{30a}$$

$$Y_n = -\frac{h|\lambda_1|^2\bar{\Delta}_{n+1}\left(\lambda_1 S_n f_n \bar{g}_{n+1} + \bar{\lambda}_1 f_{n+1}\bar{g}_n\right) + 2\,\mathrm{i}(\lambda_1^2 - \bar{\lambda}_1^2)\bar{\Delta}_n f_{n+1}\bar{g}_{n+1}}{h|\lambda_1|^2(\bar{\Delta}_n\Delta_{n+1} + S_n\bar{\Delta}_{n+1}\Delta_n)}, \tag{30b}$$

$$Z_n = -\frac{h|\lambda_1|^2\Delta_{n+1}\left(\lambda_1 S_n \bar{f}_{n+1} g_n + \bar{\lambda}_1 \bar{f}_n g_{n+1}\right) + 2\,\mathrm{i}(\lambda_1^2 - \bar{\lambda}_1^2)S_n\Delta_n \bar{f}_{n+1} g_{n+1}}{h|\lambda_1|^2(\bar{\Delta}_n\Delta_{n+1} + S_n\bar{\Delta}_{n+1}\Delta_n)}. \tag{30c}$$

Substituting Eqs. (13) and (14) into Eqs. (30b)–(30c) simplifies Y_n and Z_n to the form:

$$Y_n = \frac{h|\lambda_1|^2\bar{\Delta}_n U_n - 2\mathrm{i}\left(\lambda_1^2 - \bar{\lambda}_1^2\right) f_n \bar{g}_n - 2\mathrm{i}\Delta_n U_n}{h\left(\lambda_1^2 - \bar{\lambda}_1^2\right)\bar{f}_n g_n U_n - h|\lambda_1|^2\Delta_n + 2\mathrm{i}\bar{\Delta}_n}, \tag{31a}$$

$$Z_n = \frac{h|\lambda_1|^2\Delta_n \bar{U}_n - 2\mathrm{i}\left(\lambda_1^2 - \bar{\lambda}_1^2\right) g_n \bar{f}_n + 2\mathrm{i}\bar{\Delta}_n \bar{U}_n}{h\left(\bar{\lambda}_1^2 - \lambda_1^2\right) f_n \bar{g}_n \bar{U}_n - h|\lambda_1|^2\bar{\Delta}_n - 2\mathrm{i}\Delta_n}. \tag{31b}$$

It follows from Eqs. (31) that $Y_n = \bar{Z}_n$. We have checked with the aid of Wolfram's MATHEMATICA from Eq. (30a) that $W_n = Y_n Z_n$ is satisfied. As a result, we conclude that $N_n^{[1]}(\lambda)$ in (28) is the same as that of $N_n(\lambda)$ in (5a) with the correspondence: $U_n^{[1]} = Y_n$, $\overline{U_n^{[1]}} = Z_n = \bar{Y}_n$, and $|U_n^{[1]}|^2 = W_n = |Y_n|^2$. Thus, Eq. (6a) follows from the transformation formula (31a).

Next, we use Eq. (25) and derive the transformations for R_n and Q_n in Eqs. (6b) and (6c). Again, using Lemma 2 and direct computations, we obtain

$$P_n(\lambda)\begin{pmatrix} f_n \\ g_n \end{pmatrix} = \begin{pmatrix} f_{n,t} \\ g_{n,t} \end{pmatrix} + (\lambda - \lambda_1)H_1(\lambda)\begin{pmatrix} f_n \\ g_n \end{pmatrix}, \tag{32a}$$

$$P_n(\lambda)\begin{pmatrix} f_n \\ -g_n \end{pmatrix} = \begin{pmatrix} f_{n,t} \\ -g_{n,t} \end{pmatrix} + (\lambda + \lambda_1)H_2(\lambda)\begin{pmatrix} f_n \\ -g_n \end{pmatrix}, \tag{32b}$$

$$P_n(\lambda)\begin{pmatrix} \bar{g}_n \\ -\bar{f}_n \end{pmatrix} = \begin{pmatrix} \bar{g}_{n,t} \\ -\bar{f}_{n,t} \end{pmatrix} + M_n\begin{pmatrix} \bar{g}_n \\ -\bar{f}_n \end{pmatrix} + (\lambda - \bar{\lambda}_1)H_3(\lambda)\begin{pmatrix} \bar{g}_n \\ -\bar{f}_n \end{pmatrix}, \tag{32c}$$

$$P_n(\lambda)\begin{pmatrix} \bar{g}_n \\ \bar{f}_n \end{pmatrix} = \begin{pmatrix} \bar{g}_{n,t} \\ \bar{f}_{n,t} \end{pmatrix} + M_n\begin{pmatrix} \bar{g}_n \\ \bar{f}_n \end{pmatrix} + (\lambda + \bar{\lambda}_1)H_4(\lambda)\begin{pmatrix} \bar{g}_n \\ \bar{f}_n \end{pmatrix}, \tag{32d}$$

where M_n is defined in Eq. (12b) and matrices $H_{1,2,3,4}(\lambda)$ are given by

$$H_1(\lambda) = \frac{\mathrm{i}}{2}\begin{pmatrix} \lambda + \lambda_1 & R_n + \frac{1}{\lambda\lambda_1}Q_n \\ \bar{R}_n + \frac{1}{\lambda\lambda_1}\bar{Q}_n & \frac{\lambda+\lambda_1}{\lambda^2\lambda_1^2} \end{pmatrix},$$

$$H_2(\lambda) = \frac{\mathrm{i}}{2}\begin{pmatrix} \lambda - \lambda_1 & R_n - \frac{1}{\lambda\lambda_1}Q_n \\ \bar{R}_n - \frac{1}{\lambda\lambda_1}\bar{Q}_n & \frac{\lambda-\lambda_1}{\lambda^2\lambda_1^2} \end{pmatrix},$$

$$H_3(\lambda) = \frac{\mathrm{i}}{2}\begin{pmatrix} \lambda + \bar{\lambda}_1 & R_n + \frac{1}{\lambda\bar{\lambda}_1}Q_n \\ \bar{R}_n + \frac{1}{\lambda\bar{\lambda}_1}\bar{Q}_n & \frac{\lambda+\bar{\lambda}_1}{\lambda^2\bar{\lambda}_1^2} \end{pmatrix},$$

$$H_4(\lambda) = \frac{\mathrm{i}}{2}\begin{pmatrix} \lambda - \bar{\lambda}_1 & R_n - \frac{1}{\lambda\bar{\lambda}_1}Q_n \\ \bar{R}_n - \frac{1}{\lambda\bar{\lambda}_1}\bar{Q}_n & \frac{\lambda-\bar{\lambda}_1}{\lambda^2\bar{\lambda}_1^2} \end{pmatrix}.$$

Based on the results in Eqs. (32), the product in the right-hand side of Eq. (25) can be obtained as

$$
\begin{aligned}
&\left[\frac{d}{dt}T_n(\lambda)+T_n(\lambda)P_n(\lambda)\right]\mathrm{ad}\hat{T}_n(\lambda)\\
&=\frac{1}{\lambda-\lambda_1}\left[T_n(\lambda)\begin{pmatrix} f_n\\ g_n\end{pmatrix}\right]_t\begin{pmatrix}-\bar{f}_n & \bar{g}_n\end{pmatrix}+\frac{1}{\lambda+\lambda_1}\left[T_n(\lambda)\begin{pmatrix} f_n\\ -g_n\end{pmatrix}\right]_t\begin{pmatrix}\bar{f}_n & \bar{g}_n\end{pmatrix}\\
&\quad+\frac{1}{\lambda-\bar{\lambda}_1}\left[T_n(\lambda)\begin{pmatrix} \bar{g}_n\\ -\bar{f}_n\end{pmatrix}\right]_t\begin{pmatrix}-g_n & -f_n\end{pmatrix}+\frac{1}{\lambda+\bar{\lambda}_1}\left[T_n(\lambda)\begin{pmatrix} \bar{g}_n\\ \bar{f}_n\end{pmatrix}\right]_t\begin{pmatrix}g_n & -f_n\end{pmatrix}\\
&\quad+T_n(\lambda)H_1(\lambda)\begin{pmatrix} f_n\\ g_n\end{pmatrix}\begin{pmatrix}-\bar{f}_n & \bar{g}_n\end{pmatrix}+T_n(\lambda)H_2(\lambda)\begin{pmatrix} f_n\\ -g_n\end{pmatrix}\begin{pmatrix}\bar{f}_n & \bar{g}_n\end{pmatrix}\\
&\quad+T_n(\lambda)H_3(\lambda)\begin{pmatrix} \bar{g}_n\\ -\bar{f}_n\end{pmatrix}\begin{pmatrix}-g_n & -f_n\end{pmatrix}+T_n(\lambda)H_4(\lambda)\begin{pmatrix} \bar{g}_n\\ \bar{f}_n\end{pmatrix}\begin{pmatrix}g_n & -f_n\end{pmatrix}\\
&\quad+M_n\left[\frac{1}{\lambda-\bar{\lambda}_1}T_n(\lambda)\begin{pmatrix} \bar{g}_n\\ -\bar{f}_n\end{pmatrix}\begin{pmatrix}-g_n & -f_n\end{pmatrix}+\frac{1}{\lambda+\bar{\lambda}_1}T_n(\lambda)\begin{pmatrix} \bar{g}_n\\ \bar{f}_n\end{pmatrix}\begin{pmatrix}g_n & -f_n\end{pmatrix}\right].
\end{aligned}
$$

Expanding the above equation and substituting it into (25) gives

$$
\begin{aligned}
P_n^{[1]}(\lambda)=&\frac{1}{\bar{\Delta}_n}\begin{pmatrix}-\bar{\lambda}_1\bar{f}_n\,(a_nf_n)_t-\lambda_1 g_n\,(a_n\bar{g}_n)_t & \lambda\left(f_n\bar{g}_{n,t}-f_{n,t}\bar{g}_n\right)\\ \lambda\,d_n\left(\bar{f}_ng_{n,t}-\bar{f}_{n,t}g_n\right) & \lambda_1 f_n\bar{f}_{n,t}+\bar{\lambda}_1\bar{g}_ng_{n,t}\end{pmatrix}\\
&+\frac{\mathrm{i}}{2}\begin{pmatrix}\lambda^2+|\lambda_1|^2a_n+\frac{b_n}{|\lambda_1|^2}\left(\frac{\Delta_n}{\bar{\Delta}_n}\bar{Q}_n-\frac{\lambda_1^2-\bar{\lambda}_1^2}{|\lambda_1|^2\bar{\Delta}_n}\bar{f}_ng_n\right) & -\left(\frac{a_n}{\lambda}+\frac{\lambda}{|\lambda_1|^2}\right)Q_n-\frac{b_n}{\lambda|\lambda_1|^2}\\ \lambda c_n+\left(\frac{1}{\lambda}+\frac{\lambda d_n}{|\lambda_1|^2}\right)\left(\frac{\Delta_n}{\bar{\Delta}_n}\bar{Q}_n-\frac{\lambda_1^2-\bar{\lambda}_1^2}{|\lambda_1|^2\bar{\Delta}_n}g_n\bar{f}_n\right) & -\frac{1}{\lambda^2}-\frac{(d_n+c_nQ_n)}{|\lambda_1|^2}\end{pmatrix}\\
&+M_n\begin{pmatrix}\frac{\lambda_1}{\Delta_n}|g_n|^2 & \frac{\lambda}{\bar{\Delta}_n}f_n\bar{g}_n\\ \frac{\lambda}{\Delta_n}\bar{f}_ng_n & \frac{\lambda_1}{\bar{\Delta}_n}|f_n|^2\end{pmatrix},
\end{aligned}
\tag{33}
$$

where we have used Eq. (17) in obtaining the last term. Thus, $P_n^{[1]}$ can be formally written in the form

$$
P_n^{[1]}(\lambda)=\frac{\mathrm{i}}{2}\begin{pmatrix}\lambda^2-\pi_1\pi_3 & \pi_1\lambda-\pi_2\lambda^{-1}\\ \pi_3\lambda-\pi_4\lambda^{-1} & \pi_2\pi_4-\lambda^{-2}\end{pmatrix}.
\tag{34}
$$

Comparing Eqs. (33) and (34) and using Eqs. (17) together with (15), we can express π_i's ($1\le i\le 4$) as

$$
\pi_1=-\frac{\Delta_nR_n+\left(\lambda_1^2-\bar{\lambda}_1^2\right)f_n\bar{g}_n}{\bar{\Delta}_n},
\tag{35a}
$$

$$
\pi_2=-\frac{|\lambda_1|^2\bar{\Delta}_nQ_n+\left(\lambda_1^2-\bar{\lambda}_1^2\right)f_n\bar{g}_n}{|\lambda_1|^2\Delta_n},
\tag{35b}
$$

$$
\pi_3=-\frac{\bar{\Delta}_n\bar{R}_n+\left(\bar{\lambda}_1^2-\lambda_1^2\right)\bar{f}_ng_n}{\Delta_n},
\tag{35c}
$$

$$
\pi_4=-\frac{|\lambda_1|^2\Delta_n\bar{Q}_n+\left(\bar{\lambda}_1^2-\lambda_1^2\right)\bar{f}_ng_n}{|\lambda_1|^2\bar{\Delta}_n},
\tag{35d}
$$

where Wolfram's MATHEMATICA has been used for simplification. It is obvious from (35) that $\bar{\pi}_1=\pi_3$ and $\bar{\pi}_2=\pi_4$. As a result, we conclude that $P_n^{[1]}(\lambda)$ in (33) is the same as that of $P_n(\lambda)$ in (5b) with the correspondence: $R_n^{[1]}=\pi_1$ and $Q_n^{[1]}=\pi_2$. Thus, Eqs. (6b)–(6c) follow from the transformation formulas (35a)–(35b). Theorem 1 is proven with the algorithmic computations.

3. Soliton solutions on zero background

Here we use the one-fold Darboux transformation of Theorem 1 and construct soliton solutions on zero background. Hence we take zero potentials $(U,R,Q)=(0,0,0)$ in the transformation formulas (6) and obtain

$$
U_n^{[1]}=-\frac{2\,\mathrm{i}(\lambda_1^2-\bar{\lambda}_1^2)f_n\bar{g}_n}{2\,\mathrm{i}(\lambda_1|f_n|^2+\bar{\lambda}_1|g_n|^2)-h|\lambda_1|^2(\bar{\lambda}_1|f_n|^2+\lambda_1|g_n|^2)},
\tag{36a}
$$

$$
R_n^{[1]}=-\frac{(\lambda_1^2-\bar{\lambda}_1^2)f_n\bar{g}_n}{\lambda_1|f_n|^2+\bar{\lambda}_1|g_n|^2},
\tag{36b}
$$

$$
Q_n^{[1]}=-\frac{(\lambda_1^2-\bar{\lambda}_1^2)f_n\bar{g}_n}{|\lambda_1|^2(\bar{\lambda}_1|f_n|^2+\lambda_1|g_n|^2)},
\tag{36c}
$$

where $\Phi_n(\lambda_1)=(f_n,g_n)^T$ is a nonzero solution of the Lax pair (5) with $\lambda=\lambda_1$ at the zero background. First, we prove that the zero background is linearly stable in the semi-discrete MTM system (1). Next, we construct Jost solutions of the Lax pair (5) at the zero background. At last, we obtain and study the exact expressions for one-soliton and two-soliton solutions.

3.1. Stability of zero background

Linearization of the semi-discrete MTM system (1) at the zero background is written as the linear system

$$\begin{cases} 4\mathrm{i}\dfrac{du_n}{dt} + q_{n+1} + q_n + \dfrac{2\mathrm{i}}{h}(r_{n+1} - r_n) = 0, \\ q_{n+1} - q_n + \mathrm{i}hu_n = 0, \\ r_{n+1} + r_n - 2u_n = 0. \end{cases} \tag{37}$$

Thanks to the linear superposition principle, we use the discrete Fourier transform on the lattice,

$$u_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{u}(\theta)e^{\mathrm{i}n\theta}d\theta, \quad n \in \mathbb{Z}, \tag{38}$$

invert the second and third equations of the differential-difference system (37), and obtain the following differential equation with parameter $\theta \in (-\pi, \pi)\backslash\{0\}$:

$$h\frac{d\hat{u}}{dt} = \left(\frac{h^2}{4}\frac{e^{\mathrm{i}\theta}+1}{e^{\mathrm{i}\theta}-1} - \frac{e^{\mathrm{i}\theta}-1}{e^{\mathrm{i}\theta}+1}\right)\hat{u}. \tag{39}$$

Separating variables in $\hat{u} = \hat{u}_0(\theta)e^{-\mathrm{i}t\omega(\theta)}$ yields the dispersion relation for the Fourier mode $\hat{u}_0(\theta)$:

$$\omega(\theta) = \frac{1}{h\sin\theta}\left[\left(\frac{h^2}{4}+1\right) + \left(\frac{h^2}{4}-1\right)\cos\theta\right], \quad \theta \in (-\pi, \pi)\backslash\{0\}. \tag{40}$$

Since $\omega(\theta) \in \mathbb{R}$ for every $\theta \in (-\pi, \pi)\backslash\{0\}$, the zero background is linearly stable. Note however that $|\omega(\theta)| \to \infty$ as $\theta \to 0$ and $\theta \to \pm\pi$. Divergences of the dispersion relation in (40) as $\theta \to 0$ and $\theta \to \pm\pi$ are related to inversion of the second and third difference equations in the linear system (37).

3.2. Solutions of the Lax pair (5) at zero background

Lax pair (5) at the zero background is decoupled into two systems which admit the following two linearly independent solutions:

$$[\Phi_+(\lambda)]_n(t) = \alpha\begin{pmatrix}1\\0\end{pmatrix}\mu_+^n e^{\frac{\mathrm{i}\lambda^2}{2}t}, \quad [\Phi_-(\lambda)]_n(t) = \beta\begin{pmatrix}0\\1\end{pmatrix}\mu_-^n e^{-\frac{\mathrm{i}}{2\lambda^2}t}, \tag{41}$$

where $\alpha, \beta \in \mathbb{C}\backslash\{0\}$ are parameters and

$$\mu_\pm(\lambda) := \frac{2\,\mathrm{i}}{h\lambda} \pm \lambda.$$

We say that $\Phi(\lambda)$ is the Jost function if $\lambda \in \mathbb{C}$ yields either $|\mu_+(\lambda)| = 1$ or $|\mu_-(\lambda)| = 1$, in which case one of the two fundamental solutions in (41) is bounded in the limit $|n| \to \infty$. Constraints $|\mu_\pm(\lambda)| = 1$ for $\lambda = |\lambda|e^{\mathrm{i}\theta/2}$ in the polar form are equivalent to the following equation:

$$|\lambda|^2 \pm \frac{4}{h}\sin(\theta) + \frac{4}{h^2|\lambda|^2} = 1. \tag{42}$$

Roots of Eq. (42) in the complex plane for $\lambda \in \mathbb{C}$ are shown on Fig. 1 for $h < 4$ (left) and $h > 4$ (right). For every λ on each curve of the Lax spectrum, there exists one Jost function in (41) which remains bounded in the limit $|n| \to \infty$. On the other hand, thanks to the time dependence in (41), Jost functions remain bounded also in the limit $|t| \to \infty$ if and only if $\lambda^2 \in \mathbb{R}$. No such Jost functions exist for $h < 4$ as is seen from the left panel of Fig. 1. In other words, all Jost functions diverge exponentially either as $t \to -\infty$ or as $t \to +\infty$ if $h < 4$.

3.3. One-soliton solutions

Fix $\lambda_1 \in \mathbb{C}$ such that $\mu_\pm(\lambda_1) \neq 0$ and $\lambda_1^2 \notin \mathbb{R}$. Taking a general solution for $\Phi(\lambda_1) = (f, g)^T$, we write f and g in the form:

$$f_n(t) = \alpha_1 e^{\xi_{1,n}(t)}, \quad g_n(t) = \beta_1 e^{\eta_{1,n}(t)}, \tag{43}$$

where

$$\xi_{1,n}(t) = n\log\left(\lambda_1 + \frac{2\mathrm{i}}{h\lambda_1}\right) + \frac{\mathrm{i}}{2}\lambda_1^2 t, \quad \eta_{1,n}(t) = n\log\left(-\lambda_1 + \frac{2\mathrm{i}}{h\lambda_1}\right) - \frac{\mathrm{i}t}{2\lambda_1^2}, \tag{44}$$

and $\alpha_1, \beta_1 \in \mathbb{C}\backslash\{0\}$ are parameters. Without loss of generality, we set $\lambda_1 = \delta_1 e^{\mathrm{i}\theta_1/2}$ with some $\delta_1 > 0$ and $\theta_1 \in (0, \pi)$. Substituting Eq. (43) into Eqs. (36) yields the exact one-soliton solution in the form:

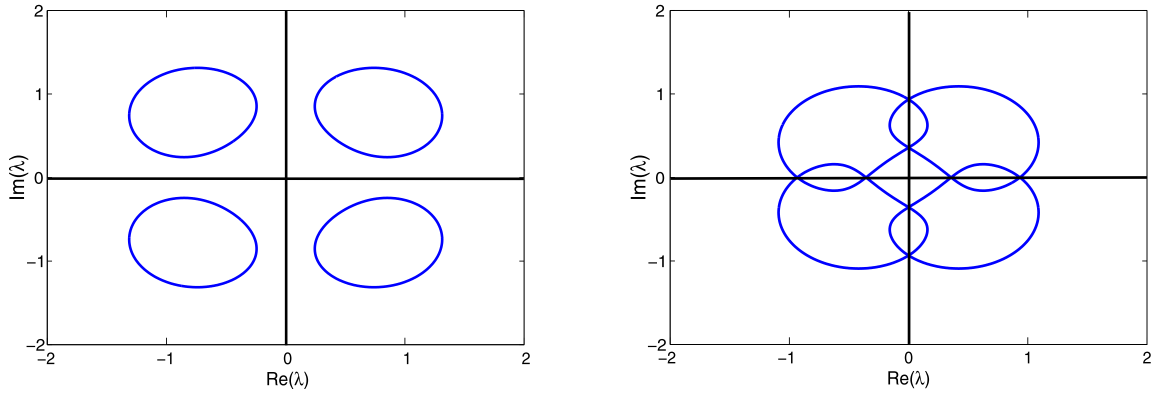

Fig. 1. Solutions to the transcendental equation (42) in the complex plane for $h = 2$ (left) and $h = 6$ (right). Each curve encloses a point λ_0 where either $\mu_+(\lambda_0) = 0$ or $\mu_-(\lambda_0) = 0$.

$$U_n^{[1]} = -\frac{4i\delta_1\alpha_1\bar{\beta}_1 \sin\theta_1 e^{i\theta_1/2}}{|\beta_1|^2(2 + ih\delta_1^2 e^{i\theta_1})e^{\eta_{1,n}-\xi_{1,n}} + |\alpha_1|^2(2e^{i\theta_1} + ih\delta_1^2)e^{-\bar{\eta}_{1,n}+\bar{\xi}_{1,n}}}, \tag{45a}$$

$$R_n^{[1]} = -\frac{2i\delta_1\alpha_1\bar{\beta}_1 \sin\theta_1 e^{i\theta_1/2}}{|\beta_1|^2 e^{\eta_{1,n}-\xi_{1,n}} + |\alpha_1|^2 e^{-\bar{\eta}_{1,n}+\bar{\xi}_{1,n}+i\theta_1}}, \tag{45b}$$

$$Q_n^{[1]} = -\frac{2i\alpha_1\bar{\beta}_1 \sin\theta_1 e^{i\theta_1/2}}{\delta_1(|\beta_1|^2 e^{\eta_{1,n}-\xi_{1,n}+i\theta_1} + |\alpha_1|^2 e^{-\bar{\eta}_{1,n}+\bar{\xi}_{1,n}})}, \tag{45c}$$

where

$$\xi_{1,n}(t) - \eta_{1,n}(t) = n\log\left(\frac{2 - ih\delta_1^2 e^{i\theta_1}}{2 + ih\delta_1^2 e^{i\theta_1}}\right) + \frac{i}{2}\left(\delta_1^2 + \frac{1}{\delta_1^2}\right)\cos\theta_1 t - \frac{1}{2}\left(\delta_1^2 - \frac{1}{\delta_1^2}\right)\sin\theta_1 t.$$

Fig. 2(a)–2(c) presents the one-soliton solutions (45) for $\alpha_1 = 1$, $\beta_1 = 1 + i$, $\delta_1 = 2$, $\theta_1 = 2\pi/5$, and $h = 1$.

Let us check that the discrete solitons (45) recover solitons of the continuous MTM system (2) in the particularly simple case $\delta_1 = 1$. By defining $x_n = hn$, $n \in \mathbb{Z}$ and taking the limit $h \to 0$, we obtain for $\delta_1 = 1$:

$$U_n^{[1]} \to U(x,t) = -\frac{2i\alpha_1\bar{\beta}_1 \sin\theta_1 e^{i\cos\theta_1(t-x)}}{|\alpha_1|^2 e^{\sin\theta_1 x + i\theta_1/2} + |\beta_1|^2 e^{-\sin\theta_1 x - i\theta_1/2}}, \tag{46a}$$

$$R_n^{[1]} \to R(x,t) = -\frac{2i\alpha_1\bar{\beta}_1 \sin\theta_1 e^{i\cos\theta_1(t-x)}}{|\alpha_1|^2 e^{\sin\theta_1 x + i\theta_1/2} + |\beta_1|^2 e^{-\sin\theta_1 x - i\theta_1/2}}, \tag{46b}$$

$$Q_n^{[1]} \to Q(x,t) = -\frac{2i\alpha_1\bar{\beta}_1 \sin\theta_1 e^{i\cos\theta_1(t-x)}}{|\alpha_1|^2 e^{\sin\theta_1 x - i\theta_1/2} + |\beta_1|^2 e^{-\sin\theta_1 x + i\theta_1/2}}, \tag{46c}$$

which agree with the stationary MTM solitons in the continuous system (2). Parameters $\alpha_1, \beta_1 \in \mathbb{C}\backslash\{0\}$ determine translations in space and rotation in time, whereas $\theta_1 \in (0,\pi)$ determines the frequency $\omega_1 := \cos\theta_1 \in (-1,1)$ of the stationary MTM solitons. In the limit $\omega_1 \to 1$ ($\theta_1 \to 0$), the MTM soliton (46) degenerates to the zero solution, whereas in the limit $\omega_1 \to -1$ ($\theta_1 \to \pi$) and $\alpha_1 = \beta_1$, it becomes the algebraic soliton:

$$U(x,t) \to U_a(x,t) = -\frac{e^{-i(t-x)}}{x - i/2}. \tag{47}$$

Discrete solitons (45) enjoy the same properties as the continuous solitons. In particular, let us recover the discrete algebraic soliton for the case $\alpha_1 = \beta_1$ and $\delta_1 = 1$ in the limit $\theta_1 \to \pi$. By setting $\theta_1 = \pi - \epsilon$ and expanding to the first order in ϵ, we obtain from (45a)

$$U_n^{[1]} = \frac{4(\epsilon + \mathcal{O}(\epsilon^2))e^{-it}}{(2 - ih - \epsilon h + \mathcal{O}(\epsilon^2))\left(\frac{2-ih+i\epsilon(2+ih)/2+\mathcal{O}(\epsilon^2)}{2+ih+i\epsilon(2-ih)/2+\mathcal{O}(\epsilon^2)}\right)^n - (2 - ih - 2i\epsilon + \mathcal{O}(\epsilon^2))\left(\frac{2-ih-i\epsilon(2+ih)/2+\mathcal{O}(\epsilon^2)}{2+ih-i\epsilon(2-ih)/2+\mathcal{O}(\epsilon^2)}\right)^n}.$$

This expression yields in the limit $\epsilon \to 0$ the discrete algebraic soliton

$$U_n^{[1]} \to [U_a]_n = -\frac{4e^{-it}}{\frac{8nh(2-ih)}{4+h^2} - 2i + h}\left(\frac{2+ih}{2-ih}\right)^n. \tag{48}$$

If $x_n = hn$, $n \in \mathbb{Z}$, the discrete algebraic soliton (48) reduces in the limit $h \to 0$ to the continuous algebraic soliton (47). Similarly, one can prove that the discrete soliton (45) degenerates to the zero solution in the limit $\theta_1 \to 0$.

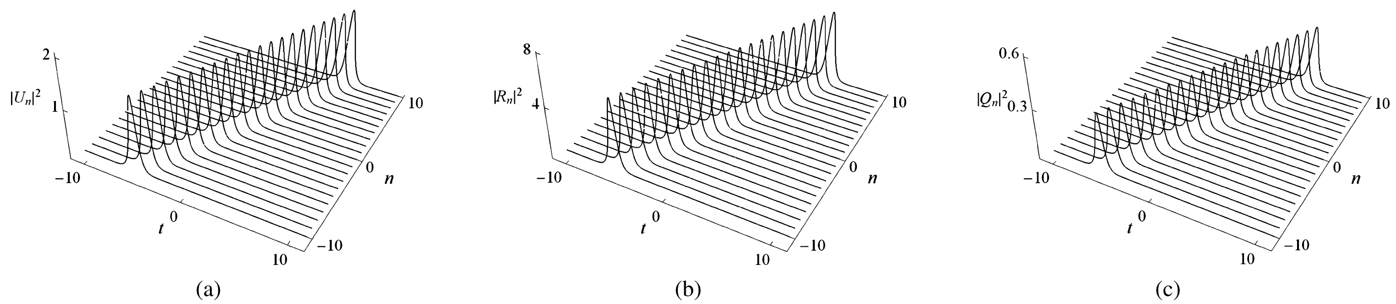

Fig. 2. An example of the one-soliton solutions (45). The following components are shown: $|U_n|^2$ (left), $|R_n|^2$ (middle), and $|Q_n|^2$ (right).

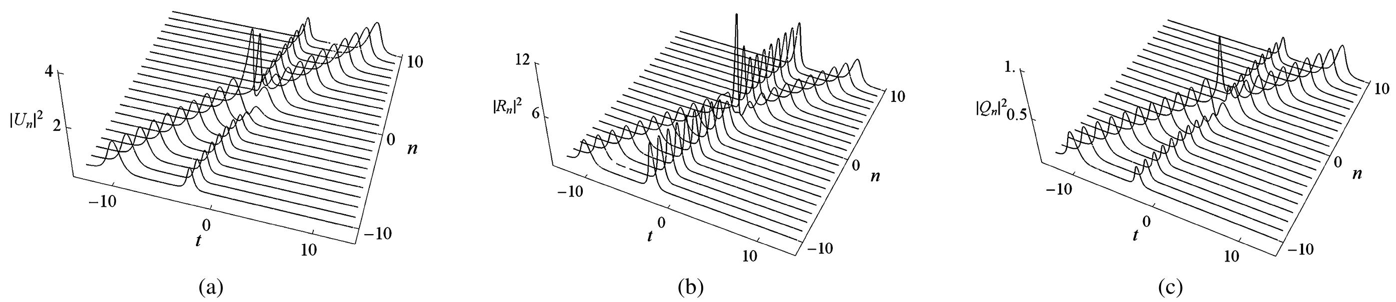

Fig. 3. An example of the two-soliton solutions.

3.4. Two-soliton solutions

In order to construct the two-soliton solutions, one needs to use the one-fold Darboux transformation (6) twice. Fix $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\{0\}$ such that $\mu_\pm(\lambda_{1,2}) \neq 0$, $\lambda_{1,2}^2 \notin \mathbb{R}$, $\lambda_2 \neq \pm\lambda_1$, and $\lambda_2 \neq \pm\bar{\lambda}_1$. A general solution of the Lax pair (5) with $\lambda = \lambda_1$ and $\lambda = \lambda_2$ at zero background is written in the form

$$[\Phi(\lambda_1)]_n(t) = \begin{pmatrix} \alpha_1 e^{\xi_{1,n}(t)} \\ \beta_1 e^{\eta_{1,n}(t)} \end{pmatrix}, \quad [\Phi(\lambda_2)]_n(t) = \begin{pmatrix} \alpha_2 e^{\xi_{2,n}(t)} \\ \beta_2 e^{\eta_{2,n}(t)} \end{pmatrix}, \tag{49}$$

where $\xi_{j,n}$ and $\eta_{j,n}$ with $j = 1, 2$ are given by (44) for $\lambda_{1,2}$, and $\alpha_{1,2}, \beta_{1,2} \in \mathbb{C}\backslash\{0\}$ are parameters.

By using the one-fold Darboux transformation (6) with zero potentials, $\lambda = \lambda_1$, and $\Phi = \Phi(\lambda_1)$, we obtain the one-soliton solutions $(U_n^{[1]}, R_n^{[1]}, Q_n^{[1]})$ in the form (45). The transformed eigenfunction $\Phi^{[1]}(\lambda_2) = T^{[1]}(\lambda_2)\Phi(\lambda_2)$ satisfies the Lax pair (5) with the potentials $(U_n^{[1]}, R_n^{[1]}, Q_n^{[1]})$ and $\lambda = \lambda_2$. By using the one-fold Darboux transformation (6) with (U_n, R_n, Q_n) replaced by $(U_n^{[1]}, R_n^{[1]}, Q_n^{[1]})$, λ_1 replaced by λ_2, and $\Phi(\lambda_1)$ replaced by $\Phi^{[1]}(\lambda_2)$, we obtain the two-soliton solutions $(U_n^{[2]}, R_n^{[2]}, Q_n^{[2]})$ in the explicit form (which is not given here).

Fig. 3(a)–3(c) shows the two-soliton solutions for $\alpha_1 = 1$, $\beta_1 = 1 + \mathrm{i}$, $\alpha_2 = 1$, $\beta_2 = 1$, $\delta_1 = \sqrt{3}$, $\theta = \pi/3$, $\delta_2 = \sqrt{5}$, $\theta_2 = \arctan 2$, and $h = 1$. The two-soliton solutions feature elastic collisions of two individual solitons with preservation of their shapes. Such collisions are very common in integrable equations including the continuous MTM system (3).

4. Soliton solutions over the nonzero background

Here we use the one-fold Darboux transformation of Theorem 1 and construct soliton solutions over the nonzero background $(U, R, Q) = (\rho, \rho, \rho^{-1})$, where $\rho > 0$ is a real parameter. Similarly to Section 3, we prove that the nonzero background is linearly stable in the semi-discrete MTM system (1) for every $\rho > 0$, construct Jost solutions of the Lax pair (5), and then finally obtain the exact expressions for one-soliton solutions.

4.1. Stability of the nonzero background

Linearization of the semi-discrete MTM system (1) at the nonzero background $(U, R, Q) = (\rho, \rho, \rho^{-1})$ with $\rho > 0$ yields the linear system of equations:

$$\begin{cases} 4\mathrm{i}\dfrac{du_n}{dt} + 2\left(\rho^2 - \dfrac{1}{\rho^2}\right)u_n + \left(1 + \dfrac{\mathrm{i}h\rho^2}{2}\right)\left(\dfrac{2\mathrm{i}}{h}r_{n+1} - \bar{q}_{n+1}\right) - \left(1 - \dfrac{\mathrm{i}h\rho^2}{2}\right)\left(\dfrac{2\mathrm{i}}{h}r_n + \bar{q}_n\right) = 0, \\ \left(1 + \dfrac{\mathrm{i}h\rho^2}{2}\right)\bar{q}_{n+1} - \left(1 - \dfrac{\mathrm{i}h\rho^2}{2}\right)\bar{q}_n + \mathrm{i}hu_n = 0, \\ \left(1 + \dfrac{\mathrm{i}h\rho^2}{2}\right)r_{n+1} + \left(1 - \dfrac{\mathrm{i}h\rho^2}{2}\right)r_n - 2u_n = 0. \end{cases} \tag{50}$$

By using the discrete Fourier transform on the lattice (38), we close the linear system (50) at the following differential equation with parameter $\theta \in (-\pi, \pi)$:

$$i h\frac{d\hat{u}}{dt}+\frac{h}{2}\left(\rho^2-\frac{1}{\rho^2}\right)\hat{u}+\left(\frac{h^2}{4}\frac{\cos\frac{\theta}{2}-\frac{h\rho^2}{2}\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}+\frac{h\rho^2}{2}\cos\frac{\theta}{2}}-\frac{\sin\frac{\theta}{2}+\frac{h\rho^2}{2}\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}-\frac{h\rho^2}{2}\sin\frac{\theta}{2}}\right)\hat{u}=0. \tag{51}$$

The dispersion relation following from linear equation (51) is purely real, which implies that the nonzero background is linearly stable for every $\rho > 0$. Note that the linear equation (51) does not reduce to equation (39) in the limit $\rho \to 0$ because the nonzero background $(U, R, Q) = (\rho, \rho, \rho^{-1})$ is singular in this limit, hence the variable q in the linearized system (37) is replaced by $\bar{q}$ in the system (50).

Note that $(u, v) = (\rho, \rho^{-1})$ is also the nonzero solution of the continuous MTM system (3). However, computations similar to those in (50) and (51) show that the nonzero background for any $\rho > 0$ is modulationally unstable. This is different from the conclusion that holds for the semi-discrete MTM system (1).

4.2. Solutions of the Lax pair (5) at nonzero background

Solving Lax pair (5) with the potentials $(U, R, Q) = (\rho, \rho, \rho^{-1})$, we have two linearly independent solutions:

$$[\Phi_+(\lambda)]_n(t)=\alpha\begin{pmatrix}\rho\\-\lambda\end{pmatrix}\mu_+^n e^{\frac{i}{2}\left(\frac{1}{\rho^2}-\rho^2\right)t},\quad [\Phi_-(\lambda)]_n(t)=\beta\begin{pmatrix}\lambda\\\rho\end{pmatrix}\mu_-^n e^{\frac{i}{2}\left(\lambda^2-\frac{1}{\lambda^2}\right)t}, \tag{52}$$

where $\alpha, \beta \in \mathbb{C}\backslash\{0\}$ are parameters and

$$\mu_+(\lambda):=\left(\frac{2\,i}{h\lambda}-\lambda\right)\frac{1+\frac{ih\rho^2}{2}}{1-\frac{ih\rho^2}{2}},\quad \mu_-(\lambda):=\frac{2\,i}{h\lambda}+\lambda.$$

Similarly to the case of zero potentials, we say that $\Phi(\lambda)$ is a Jost function if $\lambda \in \mathbb{C}$ yields either $|\mu_+(\lambda)| = 1$ or $|\mu_-(\lambda)| = 1$. Interestingly, the constraints $|\mu_\pm(\lambda)| = 1$ with $\lambda = |\lambda|e^{i\theta/2}$ yield the same equation (42). Hence, any point on each curve of the Lax spectrum shown on Fig. 1 gives one Jost function in (52) which remains bounded in the limit $|n| \to \infty$. The function of $\Phi_+(\lambda)$ is always bounded in the limit $|t| \to \infty$ since $\rho > 0$. On the other hand, $\Phi_-(\lambda)$ is bounded as $|t| \to \infty$ if and only if $\lambda^2 \in \mathbb{R}$, and no such Jost functions exist for $\Phi_-(\lambda)$ if $h < 4$.

4.3. One-soliton solutions

Fix $\lambda_1 \in \mathbb{C}$ such that $\mu_\pm(\lambda_1) \neq 0$ and $\lambda_1^2 \notin \mathbb{R}$. Let $\Phi(\lambda_1) = (f, g)^T$ be the general solution of Lax pair (5) with $(U, R, Q) = (\rho, \rho, \rho^{-1})$ and $\lambda = \lambda_1$. We write f and g in the form

$$f_{1,n}=\alpha_1\rho\, e^{\mu_{1,n}(t)}+\beta_1\lambda_1 e^{\nu_{1,n}(t)},\quad g_{1,n}=-\alpha_1\lambda_1 e^{\mu_{1,n}(t)}+\beta_1\rho\, e^{\nu_{1,n}(t)}, \tag{53}$$

with

$$\mu_{1,n}(t)=n\log\left[\left(\frac{2\,i}{h\lambda_1}-\lambda_1\right)\frac{1+\frac{ih\rho^2}{2}}{1-\frac{ih\rho^2}{2}}\right]+\frac{i}{2}\left(\frac{1}{\rho^2}-\rho^2\right)t,\quad \nu_{1,n}(t)=n\log\left(\lambda_1+\frac{2i}{h\lambda_1}\right)+\frac{i}{2}\left(\lambda_1^2-\frac{1}{\lambda_1^2}\right)t,$$

where $\alpha_1, \beta_1 \in \mathbb{C}\backslash\{0\}$ are parameters. Substituting Eq. (53) into Eqs. (6), we obtain the one-soliton solutions over the nonzero background in the form:

$$U_n^{[1]}=-\frac{|\alpha_1|^2\rho\bar{\lambda}_1 h_{\lambda_1}\bar{\chi}_1 e^{\Theta_{1,n}}+|\beta_1|^2\rho\lambda_1 h_{\bar{\lambda}_1}\chi_1 e^{-\Theta_{1,n}}+\bar{\alpha}_1\beta_1|\lambda_1|^2 h_\rho(\lambda_1^2-\bar{\lambda}_1^2)e^{-i\,\Xi_{1,n}}}{|\alpha_1|^2\lambda_1 h_{\lambda_1}\bar{\chi}_1 e^{\Theta_{1,n}}+|\beta_1|^2\bar{\lambda}_1 h_{\bar{\lambda}_1}\chi_1 e^{-\Theta_{1,n}}-\bar{\alpha}_1\beta_1\rho h_\rho(\lambda_1^2-\bar{\lambda}_1^2)e^{-i\,\Xi_{1,n}}}, \tag{54a}$$

$$R_n^{[1]}=-\frac{|\alpha_1|^2\rho\bar{\lambda}_1\bar{\chi}_1 e^{\Theta_{1,n}}+|\beta_1|^2\rho\lambda_1\chi_1 e^{-\Theta_{1,n}}-\bar{\alpha}_1\beta_1|\lambda_1|^2(\lambda_1^2-\bar{\lambda}_1^2)e^{-i\,\Xi_{1,n}}}{|\alpha_1|^2\lambda_1\bar{\chi}_1 e^{\Theta_{1,n}}+|\beta_1|^2\bar{\lambda}_1\chi_1 e^{-\Theta_{1,n}}+\bar{\alpha}_1\beta_1\rho(\lambda_1^2-\bar{\lambda}_1^2)e^{-i\,\Xi_{1,n}}}, \tag{54b}$$

$$Q_n^{[1]}=-\frac{|\alpha_1|^2\bar{\lambda}_1^3\chi_1 e^{\Theta_{1,n}}+|\beta_1|^2\lambda_1^3\bar{\chi}_1 e^{-\Theta_{1,n}}+\alpha_1\bar{\beta}_1\rho^3(\lambda_1^2-\bar{\lambda}_1^2)e^{i\,\Xi_{1,n}}}{\rho|\lambda_1|^2\left[|\alpha_1|^2\bar{\lambda}_1\chi_1 e^{\Theta_{1,n}}+|\beta_1|^2\lambda_1\bar{\chi}_1 e^{-\Theta_{1,n}}-\alpha_1\bar{\beta}_1\rho(\lambda_1^2-\bar{\lambda}_1^2)e^{i\,\Xi_{1,n}}\right]}, \tag{54c}$$

where

$$\Theta_{1,n}=\mathrm{Re}(\mu_{1,n}-\nu_{1,n}),\quad \Xi_{1,n}=\mathrm{Im}(\mu_{1,n}-\nu_{1,n}),$$
$$\chi_1=\rho^2+\lambda_1^2,\quad \bar{\chi}_1=\rho^2+\bar{\lambda}_1^2,\quad h_{\lambda_1}=-2i+h\lambda_1^2,\quad h_{\bar{\lambda}_1}=-2i+h\bar{\lambda}_1^2,\quad h_\rho=2i+h\rho^2.$$

Due to the presence of the oscillatory terms $e^{i\,\Xi_{1,n}}$ and $e^{-i\,\Xi_{1,n}}$, solutions (54), in general, exhibit localized solitons which oscillate periodically both in n and t. Fig. 4(a)–4(c) illustrates the one-soliton solutions (54) over the nonzero background for $\alpha_1 = 1$, $\beta_1 = 1 + i$, $\rho = 1$, $\delta_1 = 2$, $\theta = \pi/4$, and $h = 3/4$.

No periodic oscillations occur in the one-soliton solutions (54) if and only if $\Xi_{1,n} = 0$. In this case, solutions (54) describe traveling solitons illustrated on Fig. 5(a)–5(c) for $\alpha_1 = 1$, $\beta_1 = 1 + i$, $\rho = 2^{3/4}/7^{1/4}$, $\delta_1 = 7^{1/4}/6^{1/4}$, $\theta_1 = 5\pi/6$, and $h = \sqrt{3}$.

We show that the one-soliton solutions (54) feature no periodic oscillations if the modulus and argument of λ_1 are given by

$$|\lambda_1|=\frac{1}{\rho}\sqrt{\frac{2}{h}},\quad \arg(\lambda_1)=\frac{1}{2}\arccos\left(2h\frac{1-\rho^4}{4-h^2\rho^4}\right) \tag{55}$$

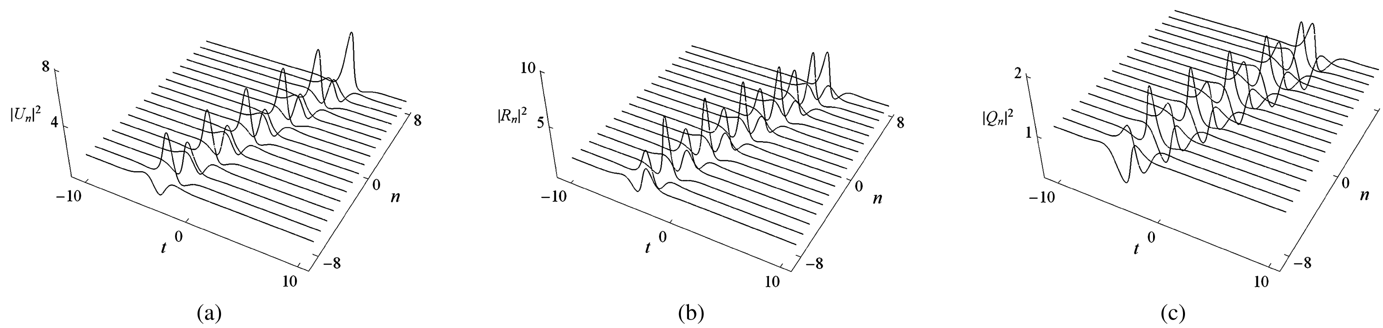

Fig. 4. An example of the one-soliton solutions (54) with periodic oscillations.

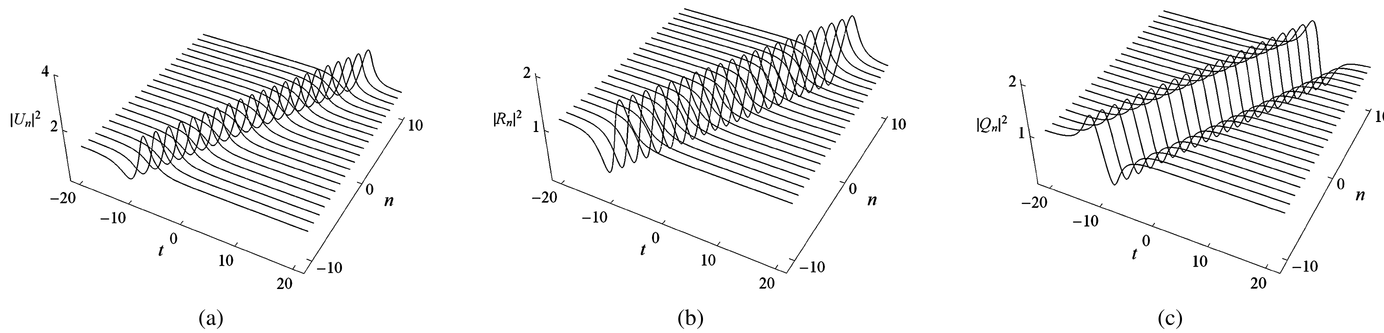

Fig. 5. An example of the one-soliton solutions (54) without periodic oscillations.

in the two regions described by

$$\text{either } h > \frac{2}{\rho^4}, \;\; \rho < 1, \;\text{ or } 0 < h < \frac{2}{\rho^4}, \;\; \rho > 1. \tag{56}$$

Note that the two regions intersect at $\rho = 1$, $h = 2$, for which $|\lambda_1| = 1$ whereas $\arg(\lambda_1)$ is not determined. In fact, we show that $\arg(\lambda_1) \in (\frac{\pi}{4}, \frac{\pi}{2})$. The existence region for non-oscillating one-soliton solutions (54) on the (h, ρ) plane is displayed in Fig. 6.

In order to verify (55), we note that the condition $\Xi_{1,n} = 0$ is equivalent to the system of two equations

$$\begin{cases} \frac{2}{\rho^2} - 2\rho^2 - \bar{\lambda}_1^2 - \lambda_1^2 + \frac{1}{\bar{\lambda}_1^2} + \frac{1}{\lambda_1^2} = 0, \\ \frac{4\rho^2}{h|\lambda_1|^2} - h|\lambda_1|^2\rho^2 + \frac{2}{h}\left(\frac{\bar{\lambda}_1}{\lambda_1} + \frac{\lambda_1}{\bar{\lambda}_1}\right)\left(1 - \frac{h^2\rho^4}{4}\right) = 0, \end{cases} \tag{57}$$

subject to the constraint

$$\left(\frac{4}{h^2|\lambda_1|^2} - |\lambda_1|^2\right)\left(1 - \frac{h^2\rho^4}{4}\right) - 2\rho^2\left(\frac{\bar{\lambda}_1}{\lambda_1} + \frac{\lambda_1}{\bar{\lambda}_1}\right) > 0. \tag{58}$$

By using the polar form $\lambda_1 = \delta_1 e^{\mathrm{i}\theta_1/2}$ with $\delta_1 > 0$ and $\theta_1 \in (0, \pi)$, we rewrite the constraints (57)–(58) in the form:

$$\begin{cases} \frac{1}{\rho^2} - \rho^2 + \left(\frac{1}{\delta_1^2} - \delta_1^2\right)\cos\theta_1 = 0, \\ \delta_1^4 h^2\rho^2 + \delta_1^2\left(h^2\rho^4 - 4\right)\cos\theta_1 - 4\rho^2 = 0, \end{cases} \tag{59}$$

subject to the constraint

$$\frac{\left(\delta_1^4 h^2 - 4\right)\left(h^2\rho^4 - 4\right)}{4\delta_1^2 h^2} - 4\rho^2\cos\theta_1 > 0. \tag{60}$$

Let us first assume that $\delta_1 \neq 1$, in which case the first equation in (59) gives a unique solution for θ_1:

$$\cos\theta_1 = \frac{\rho^2 - \rho^{-2}}{\delta_1^{-2} - \delta_1^2}. \tag{61}$$

Substituting (61) into the second equation in (59) yields the following equation

$$\delta_1^8 h^2\rho^4 - \delta_1^4(h^2\rho^8 + 4) + 4\rho^4 = 0$$

with two roots $\delta_1^4 = \rho^4$ and $\delta_1^4 h^2\rho^4 = 4$. Since $\delta_1 = \rho$ implies $\cos\theta_1 = -1$ in (61), which is not admissible, we only have one positive root for δ_1 given by

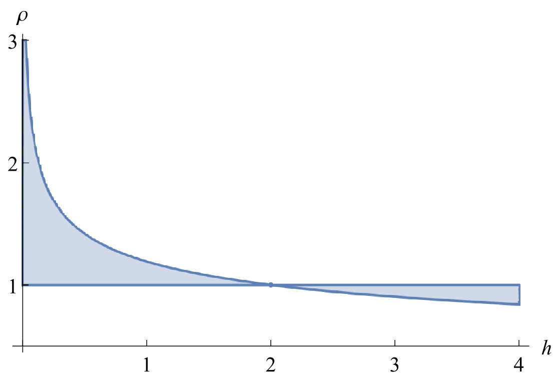

Fig. 6. Region on the (h, ρ) plane given by (56).

$$\delta_1 = \frac{\sqrt{2}}{\rho\sqrt{h}}, \tag{62}$$

which implies

$$\cos\theta_1 = 2h\frac{1-\rho^4}{4-h^2\rho^4} \tag{63}$$

thanks to (61). Solutions (62) and (63) are equivalent to (55). The constraint (60) with the solutions (62)–(63) is rewritten in the form

$$\frac{(1-\rho^4)(h^2\rho^4+4)^2}{2h\rho^2(h^2\rho^4-4)} > 0,$$

from which the two regions in (56) follow. In the exceptional case, $\delta_1 = 1$, we have from the first equation in (59) that $\rho = 1$ whereas $\cos\theta_1$ is not determined. Then, the second equation in (59) implies that $h = 2$ since $\cos\theta_1 = -1$ is not admissible. The constraint (60) yields $\cos\theta_1 < 0$ so that $\theta_1 \in \left(\frac{\pi}{2}, \pi\right)$.

5. Conclusion

We derived the one-fold Darboux transformation between solutions of the semi-discrete MTM system using the Lax pair and the dressing method. When one solution of the semi-discrete MTM system is either zero or nonzero, the one-fold Darboux transformation generates one-soliton solution on zero or nonzero backgrounds, respectively. When the one-fold Darboux transformation is used recursively, it also allows us to construct two-soliton solutions and generally multi-soliton solutions. We showed that properties of the discrete solitons in the semi-discrete MTM system are very similar to properties of the continuous MTM solitons.

Among further problems related to the semi-discrete MTM system, we mention construction of conserved quantities which may clarify orbital stability of the discrete MTM solitons, similar to the work [24]. Another direction is to develop the inverse scattering transform for solutions of the Cauchy problem associated with the semi-discrete MTM system, similar to the work [23]. Since numerical simulations of the semi-discrete MTM system (1) present serious challenges, it may be interesting to look for another version of the integrable semi-discretization of the continuous MTM system (3).

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