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TWO-PULSE SOLUTIONS IN THE FIFTH-ORDER KDV EQUATION: RIGOROUS THEORY AND NUMERICAL APPROXIMATIONS

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Abstract. We revisit existence and stability of two-pulse solutions in the fifth-order Korteweg–de Vries (KdV) equation with two new results. First, we modify the Petviashvili method of successive iterations for numerical (spectral) approximations of pulses and prove convergence of iterations in a neighborhood of two-pulse solutions. Second, we prove structural stability of embedded eigenvalues of negative Krein signature in a linearized KdV equation. Combined with stability analysis in Pontryagin spaces, this result completes the proof of spectral stability of the corresponding two-pulse solutions. Eigenvalues of the linearized problem are approximated numerically in exponentially weighted spaces where embedded eigenvalues are isolated from the continuous spectrum. Approximations of eigenvalues and full numerical simulations of the fifth-order KdV equation confirm stability of two-pulse solutions associated with the minima of the effective interaction potential and instability of two-pulse solutions associated with the maxima points.

1. **Introduction.** One-pulse solutions (solitons) are commonly met in many nonlinear evolution equations where dispersive terms (represented by unbounded differential operators) and nonlinear terms (represented by power functions) are taken in a certain balance. Typical examples of such nonlinear evolution equations with one-pulse solutions are given by the NLS (nonlinear Schrödinger) equation, the Klein-Gordon (nonlinear wave) equation and the KdV (Korteweg-de Vries) equation, as well as their countless generalizations.

One-pulse solutions are the only stationary (traveling) localized solutions of the simplest nonlinear evolution equations. However, uniqueness is not a generic property and bound states of spatially separated pulses can represent other stationary (traveling) localized solutions of the same evolution equation. For instance, two-pulse, three-pulse, and generally N-pulse solutions exist in nonlinear evolution equations with a higher-order dispersion (represented by a higher-order differential operator). The prototypical example of such situation is the fifth-order KdV

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equation in the form,

$$u_t + u_{xxx} - u_{xxxxx} + 2uu_x = 0, \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}_+, \tag{1}$$

where $u : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$ and all coefficients of the nonlinear PDE are normalized by a scaling transformation. See Bridges & Derks [4] for a review of history and applications of the fifth-order KdV equation (1) to magneto-acoustic waves in plasma and capillary-gravity water waves.

Traveling localized solutions $u(x,t) = \phi(x - ct)$ of the fifth-order KdV equation (1) satisfy the fourth-order ODE

$$\phi^{(\mathrm{iv})} - \phi'' + c\phi = \phi^2, \qquad z \in \mathbb{R}, \tag{2}$$

where $z = x - ct$ is the traveling coordinate and one integration of the fifth-order ODE in z is performed subject to zero boundary conditions on $\phi(z)$ and its derivatives as $|z| \to \infty$. Existence of localized solutions (homoclinic orbits) to the fourth-order ODE (2) was considered by methods of the dynamical system theory. See Champneys [10] for a review of various results on existence of homoclinic orbits in the ODE (2).

In particular, it is proved with the variational method by Buffoni & Sere [7] and Groves [22] (see references to earlier works in [10]) that the fourth-order ODE (2) has the one-pulse solution $\phi(z)$ for $c > 0$, which is the only localized solution of the ODE (2) for $0 < c < \frac{1}{4}$ up to the translation $\phi(z - s)$ for any $s \in \mathbb{R}$. The analytical expression for the one-pulse solution is only available for $c = \frac{36}{169} < \frac{1}{4}$ with

$$\phi(z) = \frac{105}{338} \operatorname{sech}^4\left(\frac{z}{2\sqrt{13}}\right). \tag{3}$$

For $c > \frac{1}{4}$, the fourth-order ODE (2) has infinitely many multi-pulse solutions in addition to the one-pulse solution [7, 22]. The multi-pulse solutions look like multiple copies of the one-pulse solutions separated by finitely many oscillations close to the zero equilibrium $\phi = 0$. Stability and evolution of multi-pulse solutions are beyond the framework of the fourth-order ODE (2) and these questions were considered by two theories in the recent past.

The pioneer work of Gorshkov & Ostrovsky explains multi-pulse solutions of the fifth-order KdV equation (1) from the effective interaction potential computed from the one-pulse solution [19, 20]. When the interaction potential has an alternating sequence of maxima and minima (which corresponds to the case when the one-pulse solution $\phi(z)$ has oscillatory decaying tails at infinity), an infinite countable sequence of two-pulse solutions emerge with the property that the distance between the pulses occurs near the extremal points of the interaction potential. Three-pulse solutions can be constructed as a bi-infinite countable sequence of three one-pulse solutions where each pair of two adjacent pulses is located approximately at a distance defined by the two-pulse solution. Similarly, N-pulse solutions can be formed by a $(N-1)$-infinite countable sequence of N copies of one-pulse solutions. The perturbative procedure in [19] has the advantages that both the linear and nonlinear stability of multi-pulse solutions can be predicted from analysis of the approximate ODE system derived for distances between the individual pulses. Numerical evidences of validity of this procedure in the context of the fifth-order KdV equation are reported in [8].

A different theory was developed by Sandstede [36] who extended the Lin's work on the Lyapunov–Schmidt reductions for nonlinear evolution equations [28]. In this method, a linear superposition of N one-pulse solutions $\phi(z) = \sum_{j=1}^{N} \Phi(z - s_j)$ is a

solution of the ODE (2) in the case when the distances between pulses are infinite (i.e. $|s_{j+1} - s_j| = \infty$, $\forall j$). The Jacobian of the nonlinear ODE (2) defines a linear self-adjoint operator from $H^4(\mathbb{R})$ to $L^2(\mathbb{R})$:

$$\mathcal{H} = c - \partial_z^2 + \partial_z^4 - 2\phi(z), \qquad c > 0, \tag{4}$$

where the unbounded differential part $c-\partial_z^2+\partial_z^4$ is positive and bounded away from zero while the exponentially decaying potential term $-2\phi(z)$ is a relatively compact perturbation. When $\phi(z)$ is a linear superposition of N infinitely-separated one-pulse solutions $\Phi(z - s_j)$, the Jacobian $\mathcal{H}$ has N zero eigenvalues related to the eigenfunctions $\Phi'(z - s_j)$ due to the translational invariance of the ODE (2). The Lyapunov–Schmidt method leads to a system of bifurcation equations for the distances between individual pulses. When $\phi(z)$ is the N-pulse solution with finitely separated pulses (i.e. $|s_{j+1}-s_j| < \infty$, $\forall j$), one zero eigenvalue of the Jacobian operator $\mathcal{H}$ survives beyond the reductive procedure due to the translational invariance of the N-pulse solution $\phi(z)$, while $N-1$ real eigenvalues bifurcate from zero. The reduction method may give not only information about existence of multi-pulse solutions but also prediction of their spectral stability in the linearized time-evolution problem [36]. The linearized problem for the fifth-order KdV equation takes the form

$$\partial_z \mathcal{H} v = \lambda v, \qquad z \in \mathbb{R}, \tag{5}$$

where $v : \mathbb{R} \mapsto \mathbb{C}$ is an eigenfunction for a small perturbation of $\phi(z)$ in the reference frame $z = x - ct$ and $\lambda \in \mathbb{C}$ is an eigenvalue. We say that the eigenvalue λ is *unstable* if $\mathrm{Re}(\lambda) > 0$. We say that the eigenvalue λ is of *negative Krein signature* if $\mathrm{Re}(\lambda) = 0$, $\mathrm{Im}(\lambda) > 0$, $v \in H^2(\mathbb{R})$ and $(\mathcal{H}v, v) < 0$.

Our interest to this well-studied problem is revived by the recent progress in the spectral theory of non-self-adjoint operators arising from linearizations of nonlinear evolution equations [12]. These operators can be defined as self-adjoint operators into Pontryagin space where they have a finite-dimensional negative invariant subspace. Two physically relevant problems for the fifth-order KdV equation (1) have been solved recently by using the formalism of operators in Pontryagin spaces. First, convergence of the numerical iteration method (known as the Petviashvili method) for one-pulse solutions of the ODE (2) was proved using the contraction mapping principle in a weighted Hilbert space (which is equivalent to Pontryagin space with zero index) [34]. Second, eigenvalues of the spectral stability problem in a linearization of the fifth-order KdV equation (1) were characterized in Pontryagin space with a non-zero index defined by the finite number of negative eigenvalues of $\mathcal{H}$ using the invariant subspace theorem [26, 12].

Both recent works rise some open problems when the methods are applied to the N-pulse solutions in the fifth-order KdV equation (1), even in the case of two-pulse solutions ($N = 2$). The successive iterations of the Petviashvili method do not converge for two-pulse solutions. The iterative sequence with two pulses leads either to a single pulse or to a spurious solution with two pulses located at an arbitrary distance (see Remark 6.5 in [34]). This numerical problem arises due to the presence of small eigenvalues of $\mathcal{H}$. A modification of the Petviashvili method is needed to suppress these eigenvalues similarly to the work of Demanet & Schlag [16] where the zero eigenvalue associated to the translational invariance of the three-dimensional NLS equation is suppressed. We shall present the modification of the iterative Petviashvili method in this article. See also [11, 30] and [2, 3] for alternative

numerical techniques for approximations of multi-pulse solutions of the fifth-order KdV equation.

Another open question arises when spectral stability of multi-pulse solutions is considered within the linear eigenvalue problem (5). By either the Gorshkov–Ostrovsky perturbative procedure or the Sandstede–Lin reduction method, the small eigenvalues of the Jacobian operator $\mathcal{H}$ result in small eigenvalues of the linearized operator $\partial_z \mathcal{H}$, which are either pairs of real eigenvalues (one of which is unstable) or pairs of purely imaginary eigenvalues of negative Krein signature (which are neutrally stable but potentially unstable). Both cases are possible in the fifth-order KdV equation in agreement with the count of unstable eigenvalues in Pontryagin spaces (see Theorem 6 in [12]). (Similar count of unstable eigenvalues and eigenvalues of negative Krein signatures was developed for the NLS equations in recent papers [24, 33].) Since the real eigenvalues are isolated from the continuous spectrum of the eigenvalue problem (5), they are structurally stable and persist with respect to parameter continuations. However, the purely imaginary eigenvalues are embedded into the continuous spectrum of the eigenvalue problem (5) and their destiny remains unclear within the reduction methods. It is well known for the NLS-type and Klein–Gordon-type equations that embedded eigenvalues are structurally unstable to the parameter continuations [21]. If a certain Fermi golden rule related to the perturbation term is nonzero, the embedded eigenvalues of negative Krein signature bifurcate off the imaginary axis to complex eigenvalues inducing instabilities of pulse solutions [15]. (The embedded eigenvalues of positive Krein signature simply disappear upon a generic perturbation [15].) This bifurcation does not contradict the count of unstable eigenvalues [24, 33] and it is indeed observed in numerical approximations of pulse solutions of the coupled NLS equations [35].

From a heuristic point of view, we would expect that the time evolution of an energetically stable superposition of stable one-pulse solutions remains stable. (Stability of one-pulse solutions in the fifth-order KdV equation (1) was established with the variational theory [27] and the multi-symplectic Evans function method [4, 5].) According to the Gorshkov-Ostrovsky perturbative procedure, dynamics of well-separated pulses is represented by the Newton law for particle dynamics which describes nonlinear stability of oscillations near the minima of the effective interaction potential [20]. Therefore, we would rather expect (on the contrary to embedded eigenvalues in the linearized NLS and Klein–Gordon equations) that the embedded eigenvalues of negative Krein signature are structurally stable in the linear eigenvalue problem (5) and persist beyond the leading order of the perturbative procedure. (Multi-pulse solutions of the NLS and Klein–Gordon equations with well-separated individual pulses are always linearly stable since the small purely imaginary eigenvalues of the Lyapunov–Schmidt reductions are isolated from the continuous spectrum of the corresponding linearized problems [40].)

Since the count of unstable eigenvalues in [12] does not allow us to prove structural stability of embedded eigenvalues of negative Krein signature, we address this problem separately by using different analytical and numerical techniques. In particular, we present an analytical proof of persistence (structural stability) of embedded eigenvalues of negative Krein signature in the linearized problem (5). We also apply the Fourier spectral method and illustrate the linearized stability of the corresponding two-pulse solutions numerically. Our analytical and numerical methods are based on the construction of exponentially weighted spaces for the linear eigenvalue problem (5). (See [32] for analysis of exponentially weighted spaces

in the context of the generalized KdV equation.) See [9] for computations of the Maslov index for two-pulse solutions of the fifth-order KdV equation (1) and [39] for stability analysis of two-pulse solutions of the coupled KdV equations.

This article is structured as follows. Section 2 contains summary of available results on existence and stability of one-pulse and two-pulse solutions of the fifth-order KdV equation (1). Section 3 presents a modification of the iterative Petviashvili method for convergent numerical approximations of the two-pulse solutions in the fourth-order ODE (2). Section 4 develops the proof of structural stability of embedded eigenvalues in the eigenvalue problem (5) and numerical approximations of unstable and stable eigenvalues in an exponentially weighted space. Section 5 describes full numerical simulations of the fifth-order KdV equation (1) to study nonlinear dynamics of two-pulse solutions.

2. Review of available results on two-pulse solutions.

Linearization of the ODE (2) at the critical point $(0,0,0,0)$ leads to the eigenvalues κ given by roots of the quartic equation,

$$\kappa^4 - \kappa^2 + c = 0. \tag{6}$$

When $c < 0$, one pair of roots κ is purely imaginary and the other pair is purely real. When $0 < c < \frac{1}{4}$, two pairs of roots κ are real-valued. When $c > \frac{1}{4}$, the four complex-valued roots κ are located symmetric about the axes. We will use notations $k_0 = \text{Im}(\kappa) > 0$ and $\kappa_0 = \text{Re}(\kappa) > 0$ for a complex root of (6) in the first quadrant for $c > \frac{1}{4}$. The following two theorems summarize known results on existence of one-pulse and two-pulse solutions of the ODE (2).

Theorem 2.1 (One-pulse solutions)**.**

(i) There exists a one-pulse solution $\phi(z)$ of the ODE (2) for $c > 0$ such that $\phi \in H^2(\mathbb{R}) \cap C^5(\mathbb{R})$, $\phi(-z) = \phi(z)$, and $\phi(z) \to 0$ exponentially as $|z| \to \infty$. Moreover, $\phi(z)$ is $C^m(\mathbb{R})$ for any $m \geq 0$.

(ii) The Jacobian operator $\mathcal{H}$ in (4) associated with the one-pulse solution $\phi(z)$ has exactly one negative eigenvalue with an even eigenfunction and a simple kernel with the odd eigenfunction $\phi'(z)$.

(iii) Assume that the map $\phi(z)$ from $c > 0$ to $H^2(\mathbb{R})$ is $C^1(\mathbb{R}_+)$ and that $P'(c) > 0$, where $P(c) = \|\phi\|_{L^2}^2$. The linearized operator $\partial_z \mathcal{H}$ has a two-dimensional algebraic kernel in $L^2(\mathbb{R})$ and no unstable eigenvalues with $\text{Re}(\lambda) > 0$.

Proof. (i) Existence of a symmetric solution $\phi(z)$ in $H^2(\mathbb{R})$ follows by the mountain-pass lemma and the concentration-compactness principle (see Theorem 8 in [22] and Theorem 2.3 in [27]). The equivalence between weak solutions of the variational theory and strong solutions of the ODE (2) is established in Lemma 1 of [22] and Lemma 2.4 of [27]. The exponential decay of $\phi(z)$ follows from the Stable Manifold Theorem in Appendix A of [7]. Finally, the smoothness of the function $\phi(z)$ is proved from the ODE (2) by the bootstrapping principle [14].

(ii) The Jacobian operator $\mathcal{H}$ coincides with the Hessian of the energy functional $J(u)$ used in the constrained variational problem in [22]. By Proposition 16 in [22], the one-pulse solution $\phi(z)$ is a global minimizer of $J(u)$ subject to the constraint $K(u) = K_0$, where $K(u) = \int_{\mathbb{R}} u^3 dx$. By Lemma 2.3 in [34], ϕ is a minimizer of the constrained variational problem if $\mathcal{H}$ has exactly one negative eigenvalue. Since the negative eigenvalue corresponds to the ground state of $\mathcal{H}$, the corresponding eigenfunction is even. The kernel of $\mathcal{H}$ includes an eigenvalue with the odd eigenfunction

$\phi'(z)$ due to the space translation. The one-pulse solution is isolated, and the kernel of $\mathcal{H}$ is hence simple, due to the duality principle in Theorem 4.1 of [7]. If it is not simple, then global two-dimensional stable and unstable manifolds coincide and the time for a homoclinic orbit to go from the local unstable manifold to the local stable manifold is uniformly bounded. However, a sequence of homoclinic solutions $\{u_n\}_{n\in\mathbb{N}}$ was constructed in [6] such that the time between local manifolds grows linearly in n. By the duality principle, no second even eigenfunction exists in the kernel of $\mathcal{H}$.

(iii) Smoothness of the map $\phi(z)$ from $c > 0$ to $H^2(\mathbb{R})$ is a standard assumption (see Assumption 5.1 in [27]). If $P'(c) > 0$, the one-pulse solution is stable, according to Theorem 4.1 of [27] and Theorem 8.1 of [4]. Therefore, no eigenvalues of $\partial_z\mathcal{H}$ with $\mathrm{Re}(\lambda) > 0$ exist. The two-dimensional algebraic kernel of $\partial_z\mathcal{H}$ follows from the derivatives of the ODE (2) in z and c:

$$\mathcal{H}\phi'(z) = 0, \qquad \mathcal{H}\partial_c\phi(z) = -\phi(z). \tag{7}$$

The algebraic kernel of $\partial_z\mathcal{H}$ is exactly two-dimensional under the condition $P'(c) \neq 0$ [31]. □

Theorem 2.2 (Two-pulse solutions)**.** *There exists an infinite countable set of two-pulse solutions $\phi(z)$ of the ODE (2) for $c > \frac{1}{4}$ such that $\phi \in H^2(\mathbb{R}) \cap C^5(\mathbb{R})$, $\phi(-z) = \phi(z)$, $\phi(z) \to 0$ exponentially as $|x| \to \infty$, and $\phi(z)$ resembles two copies of the one-pulse solutions described in Theorem 2.1 which are separated by small-amplitude oscillatory tails. The members of the set are distinguished by the distance L between individual pulses which takes the discrete values $\{L_n\}_{n\in\mathbb{N}}$. Moreover, for any small $\delta > 0$ there exists $\gamma > 0$ such that*

$$\left| L_n - \frac{2\pi n}{k_0} - \gamma \right| < \delta, \qquad n \in \mathbb{N}. \tag{8}$$

Proof. Existence of an infinite sequence of geometrically distinct two-pulse solutions with the distances distributed by (8) follows by the variational theory in Theorem 1.1 of [7] under the assumption that the single-pulse solution $\phi(z)$ is isolated (up to the space translations). This assumption is satisfied by Theorem 2.1(ii). □

The following theorem describes an asymptotic construction of the two-pulse solutions, which is used in the rest of our paper.

Theorem 2.3. *Let $c > \frac{1}{4}$ and $\Phi(z)$ denote the one-pulse solution described by Theorem 2.1. Let $L = 2s$ be the distance between two copies of the one-pulse solutions of the ODE (2) in the decomposition*

$$\phi(z) = \Phi(z - s) + \Phi(z + s) + \varphi(z), \tag{9}$$

where $\varphi(z)$ is a remainder term. Let $W(L)$ be a $C^2(\mathbb{R}_+)$ function defined by

$$W(L) = \int_{\mathbb{R}} \Phi^2(z)\Phi(z + L)dz. \tag{10}$$

There exists an infinite countable set of extrema of $W(L)$, which is denoted by $\{L_n\}_{n\in\mathbb{N}}$.

(i) Assume that $W''(L_n) \neq 0$ for a given $n \in \mathbb{N}$. There exists a unique symmetric two-pulse solution $\phi(z)$ described by Theorem 2.2, such that

$$|L - L_n| \leq C_n e^{-\kappa_0 L}, \qquad \|\varphi\|_{H^2(\mathbb{R})} \leq \tilde{C}_n e^{-\kappa_0 L}, \tag{11}$$

for some $C_n, \tilde{C}_n > 0$.

(ii) The Jacobian $\mathcal{H}$ *associated with the two-pulse solution* $\phi(z)$ *has exactly two finite negative eigenvalues with even and odd eigenfunctions, a simple kernel with the odd eigenfunction* $\phi'(z)$ *and a small eigenvalue* μ *with an even eigenfunction, such that*

$$\left| \mu + \frac{2W''(L_n)}{Q(c)} \right| \leq D_n e^{-2\kappa_0 L} \tag{12}$$

for some $D_n > 0$, *where* $Q(c) = \|\Phi'\|_{L^2}^2 > 0$. *In particular, the small eigenvalue* μ *is negative when* $W''(L_n) > 0$ *and positive when* $W''(L_n) < 0$.

(iii) There exists a pair of small eigenvalues λ *of the linearized operator* $\partial_z \mathcal{H}$ *associated with the two-pulse solution* $\phi(z)$, *such that*

$$\left| \lambda^2 + \frac{4W''(L_n)}{P'(c)} \right| \leq \tilde{D}_n e^{-2\kappa_0 L}, \tag{13}$$

for some $\tilde{D}_n > 0$, *where* $P(c) = \|\Phi\|_{L^2}^2$ *and* $P'(c) > 0$. *In particular, the pair is real when* $W''(L_n) < 0$ *and purely imaginary (up to the leading order) with negative Krein signature when* $W''(L_n) > 0$.

Proof. When the tails of the one-pulse solution $\Phi(z)$ are decaying and oscillatory (i.e. when $c > \frac{1}{4}$), the function $W(L)$ in (10) is decaying and oscillatory in L and an infinite set of extrema $\{L_n\}_{n\in\mathbb{N}}$ exists. Let us pick L_n for a fixed value of $n \in \mathbb{N}$ such that $W'(L_n) = 0$ and $W''(L_n) \neq 0$.

(i) When the decomposition (9) is substituted into the ODE (2), we find the ODE for $\varphi(z)$:

$$\left(c - \partial_z^2 + \partial_z^4 - 2\Phi(z-s) - 2\Phi(z+s)\right)\varphi - \varphi^2 = 2\Phi(z-s)\Phi(z+s). \tag{14}$$

Let $\epsilon = e^{-\kappa_0 L}$ be a small parameter that measures the L^∞-norm of the overlapping term $\Phi(z-s)\Phi(z+s)$ in the sense that for each $\epsilon > 0$ there exist constants $C_0, s_0 > 0$ such that

$$\|\Phi(z-s)\Phi(z+s)\|_{L^\infty} \leq C_0\epsilon \qquad \forall s \geq s_0. \tag{15}$$

Denote $L = 2s$ and $\epsilon\Psi(z;L) = 2\Phi(z)\Phi(z+L)$ and rewrite the ODE (14) for $\tilde{\varphi}(z) = \varphi(z+s)$:

$$\left(c - \partial_z^2 + \partial_z^4 - 2\Phi(z)\right)\tilde{\varphi} - 2\Phi(z+L)\tilde{\varphi} - \tilde{\varphi}^2 = \epsilon\Psi(z;L). \tag{16}$$

The vector field of the ODE (16) is closed in function space $H^2(\mathbb{R})$, while the Jacobian for the one-pulse solution

$$\mathcal{H} = c - \partial_z^2 + \partial_z^4 - 2\Phi(z)$$

has a simple kernel with the odd eigenfunction $\Phi'(z)$ by Theorem 2.1(ii). By the Lyapunov–Schmidt reduction method (see [18]), there exists a unique solution $\tilde{\varphi} = \tilde{\varphi}_\epsilon(z;L) \in H^2(\mathbb{R})$, such that $(\Phi', \tilde{\varphi}_\epsilon) = 0$, $\tilde{\varphi}_0(z;L) = 0$ and $\tilde{\varphi}_\epsilon(z;L)$ is smooth in ϵ, provided L solves the bifurcation equation $F_\epsilon(L) = 0$, where

$$\begin{aligned} F_\epsilon(L) &= \epsilon\left(\Phi'(z), \Psi(z;L)\right) + 2\left(\Phi'(z), \Phi(z+L)\tilde{\varphi}_\epsilon(z;L)\right) + \left(\Phi'(z), \tilde{\varphi}_\epsilon^2(z;L)\right) \\ &= \epsilon\left(\Phi'(z), \Psi(z;L)\right) - \epsilon\left(\partial_L\Psi(z;L), \tilde{\varphi}_\epsilon(z;L)\right) - \epsilon\left(\Psi(z;L), \partial_z\tilde{\varphi}_\epsilon(z;L)\right) \\ &\quad + \left(\Phi'(z), \tilde{\varphi}_\epsilon^2(z;L)\right). \end{aligned}$$

Since $\tilde{\varphi}_\epsilon(z;L)$ is smooth in ϵ and $\tilde{\varphi}_0(z;L) = 0$, then $\|\tilde{\varphi}_\epsilon(z;L)\|_{H^2(\mathbb{R})} \leq C\epsilon$ for some $C > 0$ such that

$$F_\epsilon(L) = -W'(L) + \tilde{F}_\epsilon(L),$$

where $|W'(L)| \leq C_1\epsilon$ and $|\tilde{F}_\epsilon(L)| \leq C_2\epsilon^2$ for some $C_1, C_2 > 0$. The statement follows by the Implicit Function Theorem applied to the scalar equation $\frac{1}{\epsilon}F_\epsilon(L) = 0$ under the assumption that the root L_n of $W'(L)$ is simple.

(ii) The Jacobian $\mathcal{H}$ associated with the two-pulse solution $\phi(z)$ in (9) has the form:

$$\mathcal{H} = c - \partial_z^2 + \partial_z^4 - 2\Phi(z-s) - 2\Phi(z+s) - 2\varphi(z).$$

In the limit $s \to \infty$, the Jacobian $\mathcal{H}$ has a double negative eigenvalue and a double zero eigenvalue. By a linear combination of eigenfunctions, one can construct one even and one odd eigenfunctions for each of the double eigenvalues. By continuity of eigenvalues of self-adjoint operators, the double negative eigenvalue splits and the two simple eigenvalues remain negative for sufficiently large s. By reversibility of the system, eigenfunctions for simple eigenvalues are either even or odd and by continuity of eigenfunctions, there is exactly one even and one odd eigenfunctions for the two negative eigenvalues. By the translation invariance, the double zero eigenvalue splits into a simple zero eigenvalue which corresponds to the odd eigenfunction $\phi'(z)$ and a small non-zero eigenvalue that corresponds to an even eigenfunction. The splitting of the double zero eigenvalue in the problem $\mathcal{H}v = \mu v$ is considered by the perturbation theory,

$$v(z) = \alpha_1\Phi'(z-s) + \alpha_2\Phi'(z+s) + V(z), \tag{17}$$

where (α_1, α_2) are coordinates of the projections to the kernel of $\mathcal{H}$ in the limit $s \to \infty$ and $V(z)$ is the remainder term. By projecting the eigenvalue problem $\mathcal{H}v = \mu v$ to the kernel of $\mathcal{H}$ and neglecting the higher-order terms, we obtain a reduced eigenvalue problem:

$$\mu Q(c)\alpha_1 = -\tilde{W}\alpha_1 + W''(L_n)\alpha_2, \qquad \mu Q(c)\alpha_2 = W''(L_n)\alpha_1 - \tilde{W}\alpha_2,$$

where $Q(c) = \|\Phi'\|_{L^2}^2 > 0$, $W''(L_n)$ is computed from (10) and

$$\tilde{W} = 2\left([\Phi'(z-s)]^2, \varphi(z) + \Phi(z+s)\right) = 2\left([\Phi'(z)]^2, \tilde{\varphi}(z) + \Phi(z+L)\right).$$

Since one eigenvalue must be zero with the odd eigenfunction $\phi'(z)$, the zero eigenvalue corresponds to the eigenfunction (17) with $\alpha_1 = \alpha_2$ up to the leading order. By looking at the linear system, we find that the zero eigenvalue corresponding to $\alpha_1 = \alpha_2$ exists only if $\tilde{W} = W''(L_n)$. The other eigenvalue at the leading order is $\mu = -2W''(L_n)/Q(c)$ and it corresponds to the even eigenfunction (17) with $\alpha_1 = -\alpha_2$. By continuity of isolated eigenvalues of $\mathcal{H}$ with respect to perturbation terms and estimates of Theorem 2.3(i), we obtain the result (12).

(iii) In the limit $s \to \infty$, the linearized operator $\partial_z\mathcal{H}$ for the two-pulse solution $\phi(z)$ has a four-dimensional algebraic kernel according to the two-dimensional kernel of the one-pulse solution (7). By the translation invariance, the two-dimensional algebraic kernel survives for any s with the eigenfunctions $\{\phi'(z), \partial_c\phi(z)\}$. Two eigenvalues λ of the operator $\partial_z\mathcal{H}$ may bifurcate from the zero eigenvalue. The splitting of the zero eigenvalue in the problem $\partial_z\mathcal{H}v = \lambda v$ is considered by the perturbation theory,

$$v(z) = -\alpha_1\Phi'(z-s) - \alpha_2\Phi'(z+s) + \beta_1\partial_c\Phi(z-s) + \beta_2\partial_c\Phi(z+s) + V(z), \tag{18}$$

where $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ are coordinates of the projections to the algebraic kernel of $\partial_z\mathcal{H}$ in the limit $s \to \infty$ and $V(z)$ is the remainder term. By projecting the

eigenvalue problem $\partial_z \mathcal{H} v = \lambda v$ to the algebraic kernel of the adjoint operator $-\mathcal{H}\partial_z$ and neglecting the higher-order terms, we find at the leading order that $\beta_j = \lambda \alpha_j$, $j = 1, 2$ and (α_1, α_2) satisfy a reduced eigenvalue problem:

$$\frac{1}{2}\lambda^2 P'(c)\alpha_1 = -\tilde{W}\alpha_1 + W''(L_n)\alpha_2, \qquad \frac{1}{2}\lambda^2 P'(c)\alpha_2 = W''(L_n)\alpha_1 - \tilde{W}\alpha_2,$$

where $P(c) = \|\Phi\|_{L^2}^2$ and $\tilde{W} = W''(L_n)$. The non-zero squared eigenvalue λ^2 at the leading order is

$$\lambda^2 = \frac{2Q(c)\mu}{P'(c)} = -\frac{4W''(L_n)}{P'(c)}.$$

Isolated eigenvalues $\partial_z \mathcal{H}$ are continuous with respect to perturbation terms, so that we immediately obtain the result (13) for $\lambda \in \mathbb{R}$ when $W''(L_n) < 0$. In order to prove (13) for $\lambda \in i\mathbb{R}$ when $W''(L_n) > 0$, we compute the energy quadratic form at the leading order

$$(\mathcal{H}v, v) = -4W''(L_n) - P'(c)|\lambda|^2,$$

where $v(z)$ is given by the eigenfunction (18) with $\alpha_1 = -\alpha_2 = 1$ and $\beta_j = \lambda\alpha_j$, $j = 1, 2$. When $\lambda \in i\mathbb{R}$ and $W''(L_n) > 0$, we have $(\mathcal{H}v, v) < 0$ up to the leading order, such that $\lambda \in i\mathbb{R}$ is an eigenvalue of negative Krein signature. Persistence of the eigenvalues of negative Krein signature (even although the eigenvalues $\lambda \in i\mathbb{R}$ are embedded into the continuous spectrum of $\partial_z \mathcal{H}$) follows from Theorem 1 in [12]. In the exponentially weighted spaces [32], the eigenvalues of negative Krein signature are isolated and hence continuous, such that they satisfy the bound (13). □

Remark 1. Theorem 2.3 is a modification of more general Theorems 1 and 2 in [36] (see also [28]). We note that the persistence of eigenvalues (13) on the imaginary axis for $W''(L_n) > 0$ cannot be proved with the Lyapunov–Schmidt reduction method since the continuous spectrum of $\partial_z \mathcal{H}$ occurs on the imaginary axis (contrary to the standard assumption of Theorem 2 in [36] that the continuous spectrum is located in the left half-plane.)

The following conjecture from the Gorshkov–Ostrovsky perturbative procedure [19, 20] illustrates the role of $W(L)$ as the effective interaction potential for the slow dynamics of a two-pulse solution:

Conjecture: *Let C_1, C_2 be some positive constants. For the initial time interval $0 \le t \le C_1 e^{\kappa_0 L/2}$ and up to the leading order $\mathrm{O}(e^{-\kappa_0 L})$, the two-pulse solutions of the fifth-order KdV equation (1) can be written as the decomposition*

$$u(x,t) = \Phi(x - ct - s(t)) + \Phi(x - ct + s(t)) + U(x,t),$$

where $\|U\|_{L^\infty} \le C_2 e^{-\kappa_0 L}$ and the slow dynamics of $L(t) = 2s(t)$ is represented by the Newton law:

$$P'(c)\ddot{L} = -4W'(L). \tag{19}$$

Although rigorous bounds on the time interval and the truncation error of the Newton law were recently found in the context of NLS solitons in external potentials (see [17]), the above conjecture was not proved yet in the context of two-pulse solutions of the fifth-order KdV equation (1). We note that perturbation analysis that leads to the Newton law (19) cannot be used to claim persistence and topological equivalence of dynamics of the second-order ODE (19) to the full dynamics of two-pulse solutions in the fifth-order KdV equation (1).

According to Theorem 2.3, an infinite set of extrema of $W(L)$ generates a sequence of equilibrium configurations for the two-pulse solutions in Theorem 2.2. If

$P'(c) > 0$, the maxima points of $W(L)$ correspond to a pair of real eigenvalues λ of the spectral problem (5), while the minima points of $W(L)$ correspond to a pair of purely imaginary eigenvalues λ. The two-pulse solutions at the maxima points are thus expected to be linearly and nonlinearly unstable. The two-pulse solutions at the minima points are stable within the leading-order approximation (13) and within the Newton law (19) (a particle with the coordinate $L(t)$ performs a periodic oscillation in the potential well). Correspondence of these predictions to the original PDE (1) is a subject of the present article. We will compute the interaction potential $W(L)$ and the sequence of its extrema points $\{L_n\}_{n\in\mathbb{N}}$, as well as the numerical approximations of the two-pulse solutions of the ODE (2) and of the eigenvalues of the operator $\partial_z\mathcal{H}$ in (5).

3. **Iterations of the Petviashvili method for two-pulse solutions.** We address the Petviashvili method for numerical approximations of solutions of the fourth-order ODE (2) with $c > 0$. See review of literature on the Petviashvili method in [34]. By using the standard Fourier transform

$$\hat{\phi}(k) = \int_{\mathbb{R}} \phi(z) e^{-ikz} dz, \qquad k \in \mathbb{R},$$

we reformulate the ODE (2) as a fixed-point problem in the Sobolev space $H^2(\mathbb{R})$:

$$\hat{\phi}(k) = \frac{\widehat{\phi^2}(k)}{(c + k^2 + k^4)}, \qquad k \in \mathbb{R}, \tag{20}$$

where $\widehat{\phi^2}(k)$ can be represented by the convolution integral of $\hat{\phi}(k)$ to itself. An even real-valued solution $\phi(-z) = \phi(z)$ of the ODE (2) in $H^2(\mathbb{R})$ is equivalent to the even real-valued solution $\hat{\phi}(-k) = \hat{\phi}(k)$ of the fixed-point problem (20). Let us denote the space of all even functions in $H^2(\mathbb{R})$ by $H^2_{\rm ev}(\mathbb{R})$ and consider solutions of the fixed-point problem (20) in $H^2_{\rm ev}(\mathbb{R})$.

Let $\{\hat{u}_n(k)\}_{n=0}^{\infty}$ be a sequence of Fourier transforms in $H^2_{\rm ev}(\mathbb{R})$ defined recursively by

$$\hat{u}_{n+1}(k) = M_n^2 \frac{\widehat{u_n^2}(k)}{(c + k^2 + k^4)}, \tag{21}$$

where $\hat{u}_0(k) \in H^2_{\rm ev}(\mathbb{R})$ is a starting approximation and $M_n \equiv M[\hat{u}_n]$ is the Petviashvili factor defined by

$$M[\hat{u}] = \frac{\int_{\mathbb{R}} (c + k^2 + k^4) \left[\hat{u}(k)\right]^2 dk}{\int_{\mathbb{R}} \hat{u}(k) \widehat{u^2}(k) dk}. \tag{22}$$

If $u_n \in H^2(\mathbb{R})$, then $u \in L^3(\mathbb{R})$ due to the Sovolev embedding theorem, and both the nominator and denominator of $M[\hat{u}]$ are bounded. It follows from the fixed-point problem (20) that $M[\hat{\phi}] = 1$ for any solution $\hat{\phi} \in H^2_{\rm ev}(\mathbb{R})$. The following theorem was proved in [34] and reviewed in [16].

Theorem 3.1. *Let $\hat{\phi}(k)$ be a solution of the fixed-point problem (20) in $H^2_{\rm ev}(\mathbb{R})$. Let $\mathcal{H}$ be the Jacobian operator (4) evaluated at the corresponding solution $\phi(z)$ of the ODE (2). If $\mathcal{H}$ has exactly one negative eigenvalue and a simple zero eigenvalue and if*

$$\text{either} \quad \phi(z) \geq 0 \qquad \text{or} \quad \left| \inf_{z\in\mathbb{R}} \phi(z) \right| < \frac{c}{2}, \tag{23}$$

then there exists an open neighborhood of $\hat{\phi}$ in $H^2_{\rm ev}(\mathbb{R})$, in which $\hat{\phi}$ is the unique fixed point and the sequence of iterations $\{\hat{u}_n(k)\}_{n=0}^{\infty}$ in (21)–(22) converges to $\hat{\phi}$.

Proof. We review the basic steps of the proof, which is based on the contraction mapping principle in a local neighborhood of $\hat{\phi}$ in $H^2_{\rm ev}(\mathbb{R})$. The linearization of the iteration map (21) at the solution ϕ is rewritten in the physical space $z \in \mathbb{R}$ as follows:

$$v_{n+1}(z) = -2\alpha_n \phi(z) + v_n(z) - (c - \partial_z^2 + \partial_z^4)^{-1} \mathcal{H} v_n(z), \tag{24}$$

where α_n is a projection of v_n onto ϕ^2 in $L^2(\mathbb{R})$:

$$\alpha_n = \frac{(\phi^2, v_n)}{(\phi^2, \phi)},$$

such that $u_n = \phi + v_n$ and $M_n = 1 - \alpha_n$ to the linear order. The operator $\mathcal{T} = (c - \partial_z^2 + \partial_z^4)^{-1} \mathcal{H}$ is a self-adjoint operator in Pontryagin space Π_0 defined by the inner product

$$\forall f, g \in \Pi_0 : \quad [f, g] = ((c - \partial_z^2 + \partial_z^4) f, g).$$

See [12] for review of Pontryagin spaces and the invariant subspace theorem. Since $c > 0$, the Pontryagin space Π_0 has zero index and, by the invariant subspace theorem, the operator $\mathcal{T}$ in Π_0 has exactly one negative eigenvalue, a simple kernel and infinitely many positive eigenvalues. (Since $\mathcal{T}$ is an identity operator with a compact perturbation, the spectrum of $\mathcal{T}$ is purely discrete.) The eigenfunctions for the negative and zero eigenvalues are known exactly as

$$\mathcal{T}\phi = -\phi, \qquad \mathcal{T}\phi'(z) = 0.$$

Due to orthogonality of the eigenfunctions in the Pontryagin space Π_0 and the relation

$$\phi^2 = (c - \partial_z^2 + \partial_z^4)\phi,$$

we observe that α_n is a projection of v_n to ϕ in Π_0, which satisfies the trivial iteration map:

$$\alpha_{n+1} = 0, \qquad n \geq 1,$$

no matter what the value of α_0 is. In addition, projection of v_n to ϕ' in Π_0 is zero since $v_n \in H^2_{\rm ev}(\mathbb{R})$. As a result, the linearized iteration map (24) defines a contraction map if the maximal positive eigenvalue of $\mathcal{T}$ in $L^2(\mathbb{R})$ is smaller than 2. However,

$$\sigma\left(\mathcal{T}\Big|_{L^2}\right) - 1 \leq -2 \inf_{\|u\|_{L^2}=1} \left(u, (c - \partial_z^2 + \partial_z^4)^{-1} \phi(z) u\right). \tag{25}$$

If $\phi(z) \geq 0$ on $z \in \mathbb{R}$, the right-hand-side of (25) is zero. Otherwise, the right-hand-side of (25) is bounded from above by $\frac{2}{c} |\inf_{z\in\mathbb{R}} \phi(z)|$, which leads to the condition (23). □

Corollary 1. *Let $\phi(z)$ be a one-pulse solution of the ODE (2) with $c > 0$ defined by Theorem 2.1. Then, the iteration method (21)–(22) converges to $\phi(z)$ in a local neighborhood of ϕ in $H^2_{\rm ev}(\mathbb{R})$ provided that the condition (23) is met.*

The condition (23) is satisfied for the positive exact solution (3), which exists for $c = \frac{36}{169}$. Since the one-pulse solution is positive definite for $0 < c < \frac{1}{4}$ [1], it is also satisfied for all values of $c \in \left(0, \frac{1}{4}\right)$. However, the solution is sign-indefinite for $c \geq \frac{1}{4}$, such that the condition (23) must be checked *a posteriori*, after a numerical approximation of the solution is obtained.

Besides the convergence criterion described in Theorem 3.1, there are additional factors in the numerical approximation of the one-pulse solution of the ODE (2) which comes from the discretization of the Fourier transform, truncation of the resulting Fourier series, and termination of iterations within the given tolerance bound. These three numerical factors are accounted by three parameters:

(i) d - the half-period of the computational interval $z \in [-d, d]$ where the solution $\phi(z)$ is represented by the Fourier series for periodic functions;

(ii) N - the number of terms in the partial sum for the truncated Fourier series such that the grid size h of the discretization is $h = 2d/N$;

(iii) ε - the small tolerance distance that measures deviation of M_n from 1 and the distance between two successive approximations, such that the method can be terminated at the iteration n if

$$E_M \equiv |M_n - 1| < \varepsilon \qquad \text{and} \qquad E_\infty \equiv \|u_{n+1} - u_n\|_{L^\infty} < \varepsilon.$$

Then, $\tilde{\phi} \equiv u_n(z)$ can be taken as the numerical approximation of the solution $\phi(z)$.

The numerical approximation depends weakly of the three parameters, provided (i) d is much larger than the half-width of the one-pulse solution, (ii) N is sufficiently large for convergence of the Fourier series, and (iii) ε is sufficiently small above the level of the round-off error. Indeed, the constraint (i) ensures that the truncation error is exponentially small when the one-pulse solution is replaced by the periodic sequence of one-pulse solutions in the trigonometric approximation [37]. The constraint (ii) ensures that the remainder of the Fourier partial sum is smaller than any inverse power of N (by Theorem 2.1(i), all derivatives of the function $\phi(z)$ are continuous) [38]. The constraint (iii) specifies the level of accuracy achieved when the iterations of the method (21)–(22) are terminated. While we do not proceed with formal analysis of the three numerical factors (see [16] for an example of this analysis), we illustrate the weak dependence of three numerical factors on the example of the numerical approximation $\tilde{\phi}(z)$ of the exact one-pulse solution (3), which exists for $c = \frac{36}{169}$. Numerical implementation of the iteration method (21)–(22) was performed in MATLAB according to a standard spectral method [38].

Figure 1 displays the distance $E = \|\tilde{\phi} - \phi\|_{L^\infty}$ versus the three numerical factors d, h, and ε described above. The left panel shows that the error E converges to the numerical zero, which is $\mathrm{O}(10^{-15})$ in MATLAB under the Windows platform, when the step size h is reduced, while $d = 50$ and $\varepsilon = 10^{-15}$ are fixed. The middle panel computed for $h = 1$ and $\varepsilon = 10^{-15}$ shows that the error E converges to the level $\mathrm{O}(10^{-13})$ when the half-width d is enlarged. The numerical zero is not reached in this case because the step size h is not sufficiently small. The right panel computed for $h = 1$ and $d = 50$ shows that the error E converges to the same level $\mathrm{O}(10^{-13})$ as the tolerance bound ε is reduced. In all approximations that follow, we will specify $h = 0.01$, $d = 50$ and $\varepsilon = 10^{-15}$ to ensure that the error of the iteration method (21)–(22) for one-pulse solutions is on the level of the numerical zero $\mathrm{O}(10^{-15})$.

Figure 2 (left) shows the numerical approximation of the one-pulse solutions for $c = 4$, where the small-amplitude oscillations of the exponentially decaying tail are visible. We check a posteriori the condition (23) for non-positive one-pulse solutions $|\inf_{z\in\mathbb{R}} \phi(z)| < 2$ for $c = 4$. Figure 2 (right) displays convergence of the errors $E_M = |M_n - 1|$ and $E_\infty = \|u_{n+1} - u_n\|_{L^\infty}$ computed dynamically at each n as n increases. We can see that the error E_M converges to zero much faster than the error E_∞, in agreement with the decomposition of the linearized iterative map

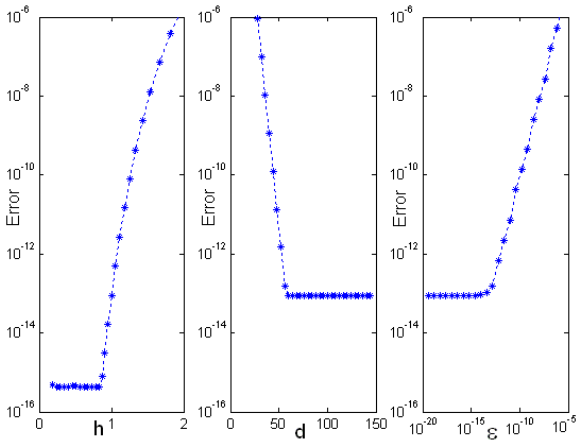

FIGURE 1. The distance $E = \|\tilde{\phi} - \phi\|_{L^\infty}$ for the ODE (2) with $c = \frac{36}{169}$ versus the half-period d of the computational interval, the step size h of the discretization, and the tolerance bound ε.

(24) into the one-dimensional projection α_n and the infinite-dimensional orthogonal complement (see the proof of Theorem 3.1). In all further approximations, we will use the error E_∞ for termination of iterations and detecting its minimal values since E_∞ is more sensitive compared to E_M.

Figure 3 shows the dependence of $\tilde{P}(c) = \|\tilde{\phi}\|^2_{L^2(\mathbb{R})}$ on $c > 0$. Since the dependence of $\tilde{P}(c)$ is strictly increasing and the approximation error is controlled in the numerical method, the assumption of Theorem 2.1(iii) that $P'(c) > 0$ is verified.

Since the numerical approximations $\tilde{\phi}(z)$ of one-pulse solutions can be computed for any value of $c > 0$, one can use $\tilde{\phi}(z)$ for a given c and compute the effective interaction potential (10), which defines the extremal values $\{L_n\}_{n\in\mathbb{N}}$. Theorem 2.3 guarantees that the two-pulse solution $\phi(z)$ consists of two copies of the one-pulse solutions separated by the distance L near the point L_n where $W'(L_n) = 0$ and $W''(L_n) \neq 0$. Table 1 shows the first four values of the sequence $\{L_n\}_{n=1}^\infty$ for $c = 1$ (where $s_n = L_n/2$ is the half-distance between the pulses). It also shows the corresponding values from the first four numerical approximations of two-pulse solutions $\phi(z)$ (obtained below) and the computational error computed from the difference of the two numerical approximations. We can see that the error decreases for larger indices n in the sequence $\{L_n\}_{n\in\mathbb{N}}$ since the Lyapunov–Schmidt reductions of Theorem 2.3 become more and more accurate in this limit.

solution	effective potential	root finding	error
$s = s_1$	5.058733328146916	5.079717398028492	0.02098406988158
$s = s_2$	8.196800619090793	8.196620796452045	$1.798226387474955 \cdot 10^{-4}$
$s = s_3$	11.338414567609066	11.338406246900558	$8.320708507980612 \cdot 10^{-6}$
$s = s_4$	14.479997655627219	14.479996635578457	$1.020048761901649 \cdot 10^{-6}$

Table 1: The first four members of the sequence of two-pulse solutions for $c = 1$.

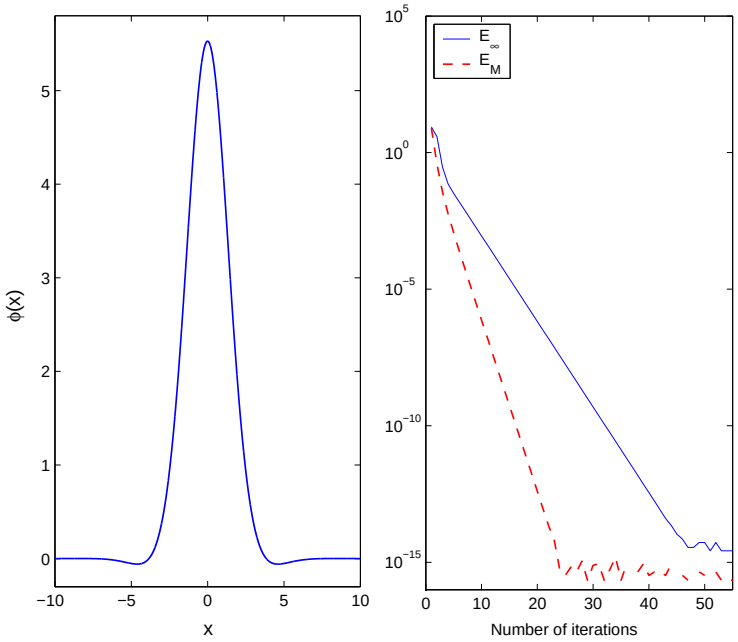

FIGURE 2. One-pulse solutions of the ODE (2) with $c = 4$ (left) and convergence of the errors E_M and E_∞ to zero versus the number of iterations n.

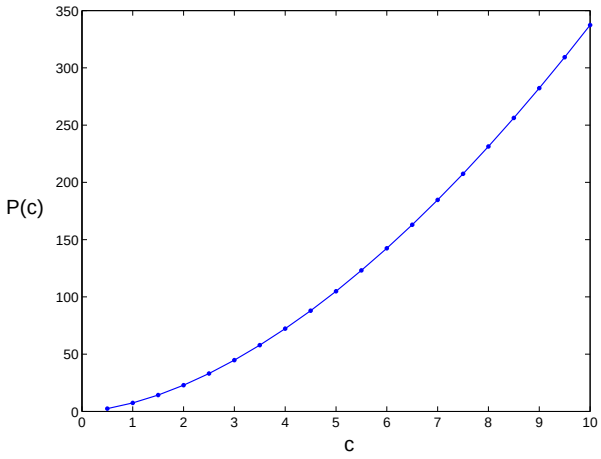

FIGURE 3. The squared L^2-norm of the one-pulse solutions of the ODE (2) versus c.

By Theorem 2.3(ii), the Jacobian operator $\mathcal{H}$ associated with a two-pulse solution $\phi(z)$ has one finite negative eigenvalue in the space of even functions and one small eigenvalue which is either negative or positive depending on the sign of $W''(L_n)$. This small eigenvalue leads to either weak divergence or weak convergence of the Petviashvili method in a local neighborhood of ϕ in $H^2_{\rm ev}(\mathbb{R})$. Even if the small

eigenvalue is positive and the algorithm is weakly convergent, the truncation error from the numerical discretization may push the small eigenvalue to a negative value and lead thus to weak divergence of the iterations.

Figure 4 illustrates typical behaviors of the errors E_M and E_∞ versus n for the starting approximation

$$u_0(z) = U_0(z-s) + U_0(z+s), \tag{26}$$

where $U_0(z)$ is a starting approximation of a sequence $\{u_n(z)\}_{n\in\mathbb{N}}$ which converges to the one-pulse solution $\Phi(z)$ and s is a parameter defined near $L_n/2$ for the two-pulse solution $\phi(z)$. The left panel shows iterations for s near s_1 and the right panel shows iterations for s near s_2. Since $W''(L_1) > 0$ and $W''(L_2) < 0$, the iteration method (21)–(22) diverges weakly near the former solution, while it converges weakly near the latter solution.

At the initial stage of iterations, both errors E_M and E_∞ quickly drops to small values, since the starting iterations $U_0(z \mp s)$ converge to the one-pulse solutions $\Phi(z \mp s)$ while the contribution from the overlapping tails of $\Phi(z \mp s)$ is negligible. However, at the later stage of iterations, both errors either start to grow (the left panel of Figure 4) or stop to decrease (the right panel). As it is explained above, this phenomenon is related to the presence of zero eigenvalue of $\mathcal{H}$ in the space of even functions which bifurcates to either positive or negative values due to overlapping tails of $\Phi(z \mp s)$ and due to the truncation error. At the final stage of iterations on the left panel of Figure 4, the numerical approximation $u_n(z)$ converges to the one-pulse solution $\Phi(z)$ centered at $z = 0$ and both errors quickly drop to the numerical zero, which occurs similarly to the right panel of Figure 2. No transformation of the solution shape occurs for large n on the right panel of Figure 4.

The following theorem defines an effective numerical algorithm, which enables us to compute the two-pulse solutions from the weakly divergent iterations of the Petviashvili method (21)–(22).

Theorem 3.2. *Let $\phi(z)$ be the two-pulse solution of the ODE (2) defined by Theorems 2.2 and 2.3. There exists $s = s_*$ near $s = L_n/2$ such that the iteration method (21)–(22) with the starting approximation $u_0(z) = \Phi(z-s) + \Phi(z+s)$ converges to $\phi(z)$ in $H^2_{\rm ev}(\mathbb{R})$.*

Proof. The iteration operator (21)–(22) in a neighborhood of the two-pulse solution $\phi(z)$ in $H^2_{\rm ev}(\mathbb{R})$ can be represented into an abstract form

$$v_{n+1} = M(\epsilon)v_n + N(v_n,\epsilon), \qquad n \in \mathbb{N},$$

where the linear operator $M(\epsilon)$ has a unit eigenvalue at $\epsilon = 0$ and the nonlinear vector field $N(v_n,\epsilon)$ is C^∞ in $v_n \in H^2_{\rm ev}$ and $\epsilon \in \mathbb{R}$, such that $N(0,0) = D_v N(0,0) = 0$. Here v_n is a distance between u_n and the fixed point ϕ and ϵ is a small parameter for two-pulse solutions defined in Theorem 2.3. By the Center Manifold Reduction for quasi-linear discrete systems (Theorem 1 in [23]), there exists a one-dimensional smooth center manifold of the discrete system above in a local neighborhood of ϕ in $H^2_{\rm ev}(\mathbb{R})$. Let ξ be a coordinate of the center manifold such that $\xi \in \mathbb{R}$, $\xi = 0$ corresponds to $v = 0$, and the dynamics on the center manifold is

$$\xi_{n+1} = \mu(\epsilon)\xi_n + f(\xi_n,\epsilon), \qquad n \in \mathbb{N},$$

where $\mu(\epsilon)$ satisfies $\mu(0) = 1$ and $f(\xi_n,\epsilon)$ is C^∞ in $\xi \in \mathbb{R}$ and $\epsilon \in \mathbb{R}$, such that $f(0,0) = \partial_\xi f(0,0) = 0$. Consider the one-parameter starting approximation $u_0(z) = \Phi(z-s) + \Phi(z+s)$ in a neighborhood of ϕ in $H^2_{\rm ev}(\mathbb{R})$, where s is

close to the value $s = s_n$ defined in Theorem 2.3. By the time evolution of the hyperbolic component of v_n (see Lemma 2 in [23]), the sequence v_n approaches to the center manifold with the coordinate ξ_n. Iterations of ξ_n are sign-definite in a neighborhood of $\xi = 0$. Moreover, there exists $s_1 < s_n$ and $s_2 > s_n$, such that the sequences $\{\xi_n(s_1)\}_{n\in\mathbb{N}}$ and $\{\xi_n(s_2)\}_{n\in\mathbb{N}}$ are of opposite signs. By smoothness of v_n and ξ_n from parameter s, there exists a root s_* in between $s_1 < s_* < s_2$ such that $\xi_n(s_*) = 0$ for all $n \in \mathbb{N}$. □

Remark 2. The proof of Theorem 3.2 does not require that the root s_* be unique for the one-parameter starting approximation $u_0(z) = \Phi(z - s) + \Phi(z + s)$. Our numerical computations starting with a more general approximation (26) show, however, that the root s_* is unique near $s = s_n$.

To capture the two-pulse solutions according to Theorem 3.2, we compute the minimum of the error E_∞ for different values of s and find numerically a root $s = s_*$ of the function

$$f(s) = \min_{0\le n\le n_0}(E_\infty),$$

where n_0 is the number of the iteration after which the value of E_∞ increases (in case of the left panel of Figure 4) or remains unchanged (in case of the right panel of Figure 4). The numerical root $s = s_*$ is found by using the secant method:

$$s_k = \frac{s_{k-2}f(s_{k-1}) - s_{k-1}f(s_{k-2})}{f(s_{k-1}) - f(s_{k-2})}. \tag{27}$$

The Petviashvili method (21)–(22) with the starting approximation (26), where s is close to the root $s = s_*$ near the point $s = s_n$, converges to the two-pulse solution $\phi(z)$ within the accuracy of the round-off error.

Figure 5 shows the graph of $f(s)$ near the value $s = s_1$ for $c = 1$. (The graph of $f(s)$ near $s = s_2$ as well as other values of s_n look similar to Figure 5.) The left panel shows uniqueness of the root, while the right panel shows the linear behavior of $f(s)$ near $s = s_*$, which indicates that the root is simple. Numerical approximations for the first four values of the sequence $\{s_n\}_{n\in\mathbb{N}}$ obtained in this root finding algorithm are shown in Table 1. We note that the number of iterations $N_{\rm h}$ of the secant method (27) decreases with larger values of n, such that $N_{\rm h} = 14$ for $n = 1$, $N_{\rm h} = 12$ for $n = 2$, $N_{\rm h} = 10$ for $n = 3$ and $N_{\rm h} = 9$ for $n = 4$, while the number of iterations of the Petviashili method for each computation does not exceed 100 iterations.

Figure 6 shows numerical approximations of the two-pulse solutions for $c = 1$ and $c = 4$. We can see from the right panel that two-pulse solutions with $c = 4$ resemble the two copies of the one-pulse solutions from the left panel of Figure 2, separated by the small-amplitude oscillatory tails.

The three-pulse and multi-pulse solutions of the fixed-point problem (20) cannot be approximated numerically with the use of the Petviashili method (21)–(22). The Jacobian operator $\mathcal{H}$ associated with the three-pulse solution has two finite negative eigenvalues and one small eigenvalue in the space of even functions, while the stabilizing factor of Theorem 3.1 and the root finding algorithm of Theorem 3.2 can only be useful for one finite negative eigenvalue and one zero eigenvalue. The additional finite negative eigenvalue introduces a *strong* divergence of the iterative method, which leads to failure of numerical approximations for three-pulse solutions. This numerical problem remains open for further analysis.

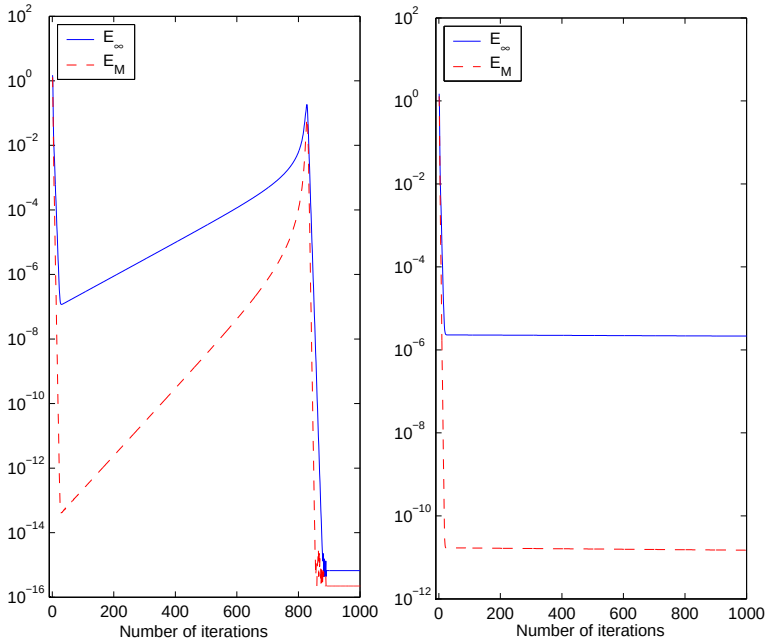

FIGURE 4. Errors E_M and E_∞ versus the number of iterations n for the starting approximation (26) with $s = 5.079$ (left panel) and $s = 8.190$ (right panel). The other parameters are: $c = 1$, $d = 50$, $h = 0.01$ and $\varepsilon = 10^{-15}$.

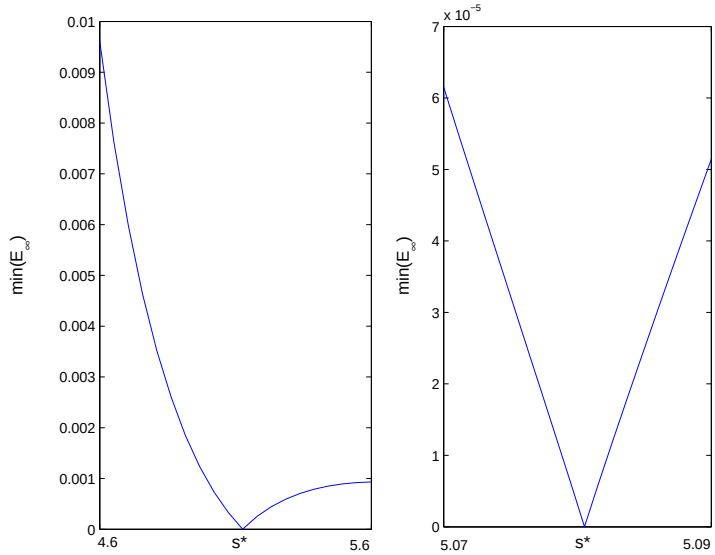

FIGURE 5. Minimal value of E_∞ versus s near $s_1 = 5.080$ (left panel) and the zoom of the graph, which shows the linear behavior of $f(s)$ near the root (right panel).

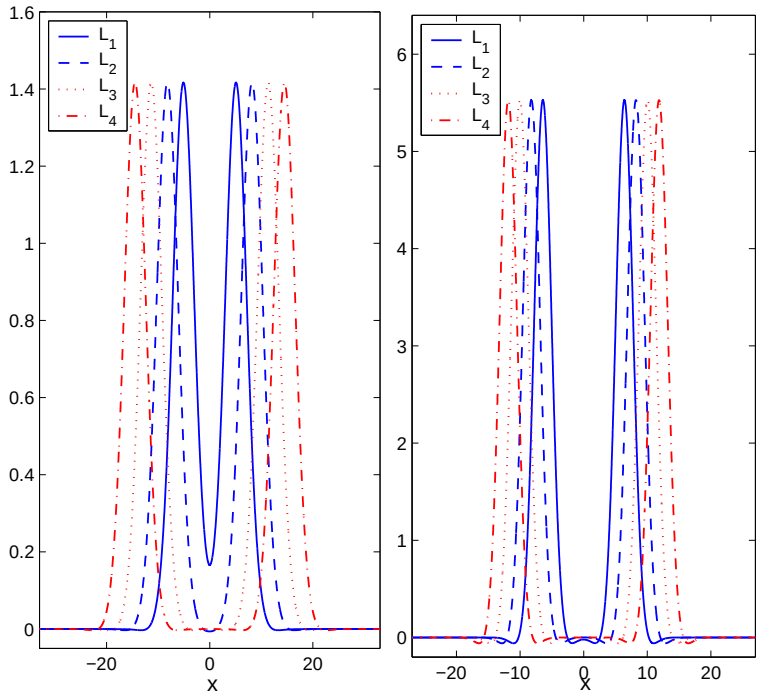

FIGURE 6. Numerical approximation of the first four two-pulse solutions of the ODE (2) for $c = 1$ (left) and $c = 4$ (right).

4. **Eigenvalues of the stability problem for two-pulse solutions.** We address spectral stability of the two-pulse solution by analyzing the linearized problem (5), where the operator $\mathcal{H} : H^4(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is the Jacobian operator (4) evaluated at the two-pulse solution $\phi(z)$.

By Theorem 2.3(ii), operator $\mathcal{H}$ has two finite negative eigenvalue, a simple kernel and one small eigenvalue, which is negative when $W''(L_n) > 0$ and positive when $W''(L_n) < 0$. Persistence (structural stability) of these isolated eigenvalues beyond the leading order (12) is a standard property of perturbation theory of self-adjoint operators in Hilbert spaces (see Section IV.3.5 in [25]).

By Theorem 2.3(iii), operator $\partial_z \mathcal{H}$ has a pair of small eigenvalues, which are purely imaginary when $W''(L_n) > 0$ and real when $W''(L_n) < 0$. We first prove that no other eigenvalues may induce instability of two-pulse solutions (i.e. no other bifurcations of eigenvalues of $\partial_z \mathcal{H}$ with $\mathrm{Re}(\lambda) > 0$ may occur). We then prove persistence (structural stability) of the purely imaginary eigenvalues beyond the leading order (13). Combined together, these two results lead to the theorem on spectral stability of the two-pulse solution $\phi(z)$ that corresponds to L_n with $W''(L_n) > 0$.

Theorem 4.1. *Let $N_{\rm real}$ be the number of real positive eigenvalues of the linearized problem (5), $N_{\rm comp}$ be the number of complex eigenvalues in the first open quadrant, and $N^-_{\rm imag}$ be the number of simple positive imaginary eigenvalues with $(\mathcal{H}v, v) \leq 0$, where $v(x)$ is the corresponding eigenfunction for $\lambda \in i\mathbb{R}_+$. Assume that the kernel*

of $\mathcal{H}$ is simple and $P'(c) > 0$, where $P = \|\phi\|_{L^2}^2$. Then,

$$N_{\rm real} + 2N_{\rm comp} + 2N_{\rm imag}^- = n(\mathcal{H}) - 1, \tag{28}$$

where $n(\mathcal{H})$ is the number of negative eigenvalues of $\mathcal{H}$.

Proof. The statement is equivalent to Theorem 6 in [12] in the case $(\mathcal{H}^{-1}\phi, \phi) = -(\partial_c \phi, \phi) = -\frac{1}{2}P'(c) < 0$. The result follows from the invariant subspace theorem in the Pontryagin space Π_κ, where $\kappa = n(\mathcal{H})$. □

Corollary 2. *Let $\phi(z) \equiv \Phi(z)$ be a one-pulse solution defined by Theorem 2.1. Then, it is a spectrally stable ground state in the sense that $N_{\rm real} = N_{\rm comp} = N_{\rm imag}^- = 0$.*

Corollary 3. *Let $\phi(z)$ be a two-pulse solution defined by Theorem 2.3. Then,*

(i) the solution corresponding to L_n with $W''(L_n) < 0$ is spectrally unstable in the sense that $N_{\rm real} = 1$ and $N_{\rm comp} = N_{\rm imag}^- = 0$ for sufficiently large L_n

(ii) the solution corresponding to L_n with $W''(L_n) > 0$ satisfies $N_{\rm real} = 0$ and $N_{\rm comp} + N_{\rm imag}^- = 1$ for sufficiently large L_n.

Proof. It follows from Theorem 2.3 for sufficiently large L_n that the kernel of $\mathcal{H}$ is simple for $W''(L_n) \neq 0$ and the only pair of imaginary eigenvalues with $(\mathcal{H}v, v) < 0$ in the case $W_n''(L_n) > 0$ is simple. Therefore, assumptions of Theorem 4.1 are satisfied for the two-pulse solutions $\phi(z)$ with $W''(L_n) \neq 0$. By the count of Theorem 2.3(ii), $n(\mathcal{H}) = 3$ for $W''(L_n) > 0$ and $n(\mathcal{H}) = 2$ for $W''(L_n) < 0$. Furthermore, persistence (structural stability) of simple real eigenvalues of the operator $\partial_z \mathcal{H}$ follows from the perturbation theory of isolated eigenvalues of non-self-adjoint operators (see Section VIII.2.3 in [25]). □

There exists one uncertainty in Corollary 3(ii) since it is not clear if the eigenvalue of negative Krein signature in Theorem 2.3(iii) remains imaginary in $N_{\rm imag}^-$ or bifurcates to a complex eigenvalue in $N_{\rm comp}$. This question is important for spectral stability of the corresponding two-pulse solutions since the former case implies stability while the latter case implies instability of solutions. We will remove the uncertainty and prove that $N_{\rm imag}^- = 1$ and $N_{\rm comp} = 0$ for sufficiently large L_n. To do so, we rewrite the linearized problem (5) in the exponentially weighted space [32]:

$$L^2_\alpha = \left\{ v \in L^2_{\rm loc}(\mathbb{R}) : \quad e^{\alpha z} v(z) \in L^2(\mathbb{R}) \right\}. \tag{29}$$

The linearized operator $\partial_z \mathcal{H}$ transforms to the form

$$\mathcal{L}_\alpha = (\partial_z - \alpha)\left(c - (\partial_z - \alpha)^2 + (\partial_z - \alpha)^4 - 2\phi(z) \right), \tag{30}$$

which acts on the eigenfunction $v_\alpha(z) = e^{\alpha z} v(z) \in L^2(\mathbb{R})$. The absolute continuous part of the spectrum of $\mathcal{L}_\alpha$ is located at $\lambda = \lambda_\alpha(k)$, where

$$\lambda_\alpha(k) = (ik - \alpha)(c - (ik - \alpha)^2 + (ik - \alpha)^4), \qquad k \in \mathbb{R}. \tag{31}$$

A simple analysis shows that

$$\begin{aligned} \frac{d}{dk}{\rm Re}(\lambda_\alpha(k)) &= -2\alpha k(10k^2 - 10\alpha^2 + 3), \\ \frac{d}{dk}{\rm Im}(\lambda_\alpha(k)) &= c - 3\alpha^2 + 5\alpha^4 + 3k^2(1 - 10\alpha^2) + 5k^4. \end{aligned}$$

The following lemma gives a precise location of the dispersion relation $\lambda = \lambda_\alpha(k)$ on $\lambda \in \mathbb{C}$.

Lemma 4.2. *The dispersion relation $\lambda = \lambda_\alpha(k)$ is a simply-connected curve located in the left half-plane of $\lambda \in \mathbb{C}$ if*

$$0 < \alpha < \frac{1}{\sqrt{10}}, \qquad c > \frac{1}{4}. \tag{32}$$

Proof. The mapping $k \mapsto \text{Im}(\lambda_\alpha)$ is one-to-one provided that $c - 3\alpha^2 + 5\alpha^4 > 0$ and $1 - 10\alpha^2 > 0$. Since $c - 3\alpha^2 + 5\alpha^4$ reaches the minimum value on $\alpha \in \left[0, \frac{1}{\sqrt{10}}\right]$ at the right end $\alpha = \frac{1}{\sqrt{10}}$ and the minimum value is positive if $c > \frac{1}{4}$, the first inequality is satisfied under (32). The second inequality is obviously satisfied if $|\alpha| < \frac{1}{\sqrt{10}}$. The mapping $k \mapsto \text{Re}(\lambda_\alpha)$ has a single extremal point at $k = 0$ provided $3 - 10\alpha^2 > 0$, which is satisfied if $|\alpha| < \frac{1}{\sqrt{10}}$. The extremal point is the point of maximum and the entire curve is located in the left half-plane of $\lambda \in \mathbb{C}$ if $0 < \alpha < \frac{1}{\sqrt{10}}$. □

The following two lemmas postulate properties of eigenfunctions corresponding to embedded eigenvalues of negative Krein signature.

Lemma 4.3. *Let $v_0(z)$ be an eigenfunction of $\partial_z \mathcal{H}$ for a simple eigenvalue $\lambda_0 \in i\mathbb{R}_+$ in $L^2(\mathbb{R})$. Then, $\lambda_0 \in i\mathbb{R}_+$ is also an eigenvalue in $L^2_\alpha(\mathbb{R})$ for sufficiently small α.*

Proof. Let $k = k_0 \in \mathbb{R}$ be the unique real root of the dispersion relation $\lambda_0(k) = \lambda_0$ (with $\alpha = 0$) for a given eigenvalue $\lambda_0 \in i\mathbb{R}_+$. The other four roots $k = k_{1,2,3,4}$ for a given $\lambda_0 \in i\mathbb{R}_+$ are complex with $|\text{Re}(k_j)| \geq \kappa_0 > 0$. By the Stable and Unstable Manifolds Theorem in linearized ODEs [13], the decaying eigenfunction $v_0(z) \in L^2(\mathbb{R})$ is exponentially decaying with the decay rate greater than $\kappa_0 > 0$ and it does not include the bounded term $e^{ik_0 z}$ as $z \to \pm\infty$. By construction, $v_\alpha(z) = e^{\alpha z} v_0(z)$ is also exponentially decaying as $z \to \pm\infty$ for sufficiently small $|\alpha| < \kappa_0$. Since $v_0 \in L^2(\mathbb{R})$ and due to the exponential decay of $v_\alpha(z)$ as $|z| \to \infty$, we have $v_\alpha \in L^2(\mathbb{R})$ for any small α. □

Lemma 4.4. *Let $v_0 \in H^2(\mathbb{R})$ be an eigenfunction of $\partial_z \mathcal{H}$ for a simple eigenvalue $\lambda_0 \in i\mathbb{R}_+$ with $(\mathcal{H}v_0, v_0) < 0$. Then, there exists $w_0 \in H^2(\mathbb{R})$, such that $v_0 = w_0'(x)$ and $w_0(z)$ is an eigenfunction of $\mathcal{H}\partial_z$ for the same eigenvalue λ_0. Moreover, $(w_0, v_0) \in i\mathbb{R}_+$.*

Proof. Since $\mathcal{H} : H^4(\mathbb{R}) \mapsto L^2(\mathbb{R})$, the eigenfunction $v_0(z)$ of the eigenvalue problem $\partial_z \mathcal{H} v_0 = \lambda_0 v_0$ for any $\lambda_0 \neq 0$ must satisfy the constraint $\int_{\mathbb{R}} v_0(z) dz = 0$. Let $v_0 = w_0'(z)$. Since $v_0(z)$ decays exponentially as $|z| \to \infty$ and $(1, v_0) = 0$, then $w_0(z)$ decays exponentially as $|z| \to \infty$, so that $w_0 \in H^2(\mathbb{R})$. By construction, $\mathcal{H}\partial_z w_0 = \mathcal{H} v_0 = \lambda_0 \int v_0(z) dz = \lambda w_0$. The values of (w_0, v_0) are purely imaginary as

$$\overline{(w_0, v_0)} = \int_{\mathbb{R}} \bar{w}_0 v_0 dz = \int_{\mathbb{R}} \bar{w}_0 \partial_z w_0 dz = -\int_{\mathbb{R}} w_0 \partial_z \bar{w}_0 dz = -\int_{\mathbb{R}} w_0 \bar{v}_0 dz = -(w_0, v_0).$$

Since $\mathcal{H} v_0 = \lambda_0 w_0$ with $\lambda_0 \in i\mathbb{R}_+$ and $(\mathcal{H}v_0, v_0) < 0$, we have (w_0, v_0) $= \lambda_0^{-1} (\mathcal{H} v_0, v_0) \in i\mathbb{R}_+$. □

The following theorem states that the embedded eigenvalues of negative Krein signature are structurally stable in the linearized problem (5).

Theorem 4.5. *Let $\lambda_0 \in i\mathbb{R}_+$ be a simple eigenvalue of $\partial_z \mathcal{H}$ with the eigenfunction $v_0 \in H^2(\mathbb{R})$ such that $(\mathcal{H}v_0, v_0) < 0$. Then, it is structurally stable to parameter*

continuations, e.g. for any $V \in L^\infty(\mathbb{R})$ and sufficiently small δ, there exists an eigenvalue $\lambda_\delta \in i\mathbb{R}_+$ of $\partial_z(\mathcal{H} + \delta V(z))$ in $L^2(\mathbb{R})$, such that $|\lambda_\delta - \lambda_0| \leq C\delta$ for some $C > 0$.

Proof. By Lemma 4.3, λ_0 is also an eigenvalue of $\mathcal{L}_\alpha$ in $L^2(\mathbb{R})$ for sufficiently small α. Let α be fixed in the bound (32). There exists a small neighborhood of λ_0, which is isolated from the absolute continuous part of the spectrum of $\mathcal{L}_\alpha$. By the perturbation theory of isolated eigenvalues of non-self-adjoint operators (see Section VIII.2.3 in [25]), there exists a simple eigenvalue λ_δ of $\partial_z(\mathcal{H} + \delta V(z))$ in $L^2_\alpha(\mathbb{R})$ for the same value of α and sufficiently small δ in a local neighborhood of λ_0, such that $|\lambda_\delta - \lambda_0| \leq C\delta$ for some $C > 0$.

It remains to show that the simple eigenvalue λ_δ is purely imaginary for the same value of $\alpha > 0$. Denote the eigenfunction of $\partial_z(\mathcal{H} + \delta V(z))$ in $H^2_\alpha(\mathbb{R})$ for the eigenvalue λ_δ by $v_\delta(z)$, such that $e^{\alpha z} v_\delta \in H^2(\mathbb{R})$. If $v_\delta \notin H^2(\mathbb{R})$, then the count of eigenvalues (28) is discontinuous at $\delta = 0$: the eigenvalue λ_0 in the number N^-_{imag} at $\delta = 0$ disappears from the count for $\delta \neq 0$. If $v_\delta \in H^2(\mathbb{R})$, then $(1, v_\delta) = 0$ and since $v_\delta(z)$ is exponentially decaying as $|z| \to \infty$, there exists $w_\delta(z) \in H^2(\mathbb{R})$ such that $v_\delta = w'_\delta(z)$. The 2-form (w_δ, v_δ) is invariant with respect to the weight α since if $e^{\alpha z} v_\delta(z)$ is an eigenfunction of $\partial_z(\mathcal{H} + \delta V(z))$ for the eigenvalue λ_δ (i.e. $v_\delta \in H^2_\alpha(\mathbb{R})$), then $e^{-\alpha z} w_\delta(z)$ is an eigenfunction of $(\mathcal{H} + \delta V(z))\partial_z$ for the same eigenvalue λ_δ (i.e. $w_\delta \in H^2_{-\alpha}(\mathbb{R})$). Computing (w_δ, v_δ) at $\alpha = 0$, we have

$$\lambda_\delta (w_\delta, v_\delta) = (\mathcal{H} v_\delta, v_\delta) \in \mathbb{R}.$$

Since (w_δ, v_δ) is continuous in δ and $(w_\delta, v_\delta) \in i\mathbb{R}$ by Lemma 4.4, then $\lambda_\delta \in i\mathbb{R}$ for any $\delta \neq 0$. □

Corollary 4. *Let $\phi(z)$ be a two-pulse solution defined by Theorem 2.3 that corresponds to L_n with $W''(L_n) > 0$. Then, it is spectrally stable in the sense that $N_{\text{real}} = N_{\text{comp}} = 0$ and $N^-_{\text{imag}} = 1$ for sufficiently large L_n.*

Remark 3. Using perturbation theory in exponentially weighted spaces for a fixed value $\alpha > 0$, one cannot a priori exclude the shift of eigenvalue λ_0 to λ_δ with $\text{Re}(\lambda_\delta) > 0$. Even if $v_0(z)$ for λ_0 contains no term $e^{ik_0 z}$ as $z \to -\infty$ (see Lemma 4.3), the eigenfunction $v_\delta(z)$ for λ_δ may contain the term $e^{ik_\delta z}$ as $z \to -\infty$ with $\text{Im}(k_\delta) < 0$ and $\lim_{\delta \to 0} k_\delta = k_0 \in \mathbb{R}$. However, when Theorem 4.5 holds (that is under the assumptions that $v_0 \in H^2(\mathbb{R})$ and $(\mathcal{H} v_0, v_0) < 0$), the eigenvalue λ_δ remains on $i\mathbb{R}$ and the eigenfunction $v_\delta(z)$ must have no term $e^{ik_\delta z}$ as $z \to -\infty$ for any sufficiently small δ. The hypothetical bifurcation above can however occur if $v_0 \notin H^2(\mathbb{R})$ but $v_0 \in H^2_\alpha(\mathbb{R})$ with $\alpha > 0$. We do not know any example of such a bifurcation.

Remark 4. When the potential is symmetric (i.e. $\phi(-z) = \phi(z)$), the stability problem $\partial_z \mathcal{H} v = \lambda v$ admits a symmetry reduction: if $v(z)$ is an eigenfunction for λ, then $\overline{v(-z)}$ is the eigenfunction for $-\bar{\lambda}$. If $\lambda_0 \in i\mathbb{R}$ is a simple eigenvalue and $v_0 \in H^2_\alpha(\mathbb{R})$ with $\alpha \geq 0$, the above symmetry shows that $v_0 \in H^2_{-\alpha}(\mathbb{R})$ with $-\alpha \leq 0$. If $\text{Re}(\lambda_\delta) > 0$ and $v_\delta \in H^2_\alpha(\mathbb{R})$, then $-\overline{v_\delta(-z)} \in H^2_{-\alpha}(\mathbb{R})$ is an eigenfunction of the same operator for eigenvalue $\text{Re}(-\overline{\lambda_\delta}) = -\text{Re}(\lambda_\delta)$ and $\text{Im}(-\overline{\lambda_\delta}) = \text{Im}(\lambda_\delta)$. Thus, the hypothetical bifurcation in Remark 3 implies that the embedded eigenvalue $\lambda_0 \in i\mathbb{R}$ may split into two isolated eigenvalues λ_δ and $-\overline{\lambda_\delta}$ as $\delta \neq 0$. Theorem 4.5 shows that such splitting is impossible if $v_0 \in H^2(\mathbb{R})$ and $(\mathcal{H} v_0, v_0) < 0$.

We confirm results of Corollaries 3 and 4 with numerical computations of eigenvalues in the linearized problem (5). Throughout computations, we use the values $\alpha = 0.04$ and $c = 1$, which satisfy the constraint (32). The spectra of the operators $\mathcal{H}$ in $L^2(\mathbb{R})$ and $\partial_z \mathcal{H}$ in $L^2_\alpha(\mathbb{R})$ are computed by using the Fourier spectral method. This method is an obvious choice since the solution $\phi(z)$ is obtained by using the spectral approximations in the iterative scheme (21)–(22). As in the previous section, we use numerical parameters $d = 100$, $h = 0.01$ and $\varepsilon = 10^{-15}$ for the Petviashvili method (21)–(22).

Eigenvalues of the discretized versions of the operators $\mathcal{H}$ and $\mathcal{L}_\alpha$ are obtained with the MATLAB eigenvalue solver `eig`. The spectra are shown on Figure 7 for the two-pulse solution $\phi_1(z)$ and on Figure 8 for the two-pulse solution $\phi_2(z)$. The inserts show zoomed eigenvalues around the origin and the dotted line connects eigenvalues of the discretized operators that belong to the absolutely continuous part of the spectra. Figures 7 and 8 clearly illustrate that the small eigenvalue of $\mathcal{H}$ is negative for $\phi_1(z)$ and positive for $\phi_2(z)$, while the pair of small eigenvalues of $\mathcal{L}_\alpha$ is purely imaginary for $\phi_1(z)$ and purely real for $\phi_2(z)$. This result is in agreement with Corollaries 3 and 4. We have observed the same alternation of small eigenvalues for two-pulse solutions $\phi_3(z)$ and $\phi_4(z)$, as well as for other values of parameters c and α.

The numerical discretization based on the Fourier spectral method shifts eigenvalues of the operators $\mathcal{H}$ and $\mathcal{L}_\alpha$. In order to measure the numerical error introduced by the discretization, we compute the numerical value for the "zero" eigenvalue corresponding to the simple kernel of $\mathcal{H}$ and the double zero eigenvalue of $\mathcal{L}_\alpha$. Table II shows numerical values for the "zero" and small eigenvalues for two-pulse solutions $\phi_n(z)$ with $n = 1, 2, 3, 4$. It is obvious from the numerical data that the small eigenvalues are still distinguished (several orders higher) than the numerical approximations for zero eigenvalues for $n = 1, 2, 3$ but they become comparable for the two-pulse solution with $n = 4$. This behavior is understood from Theorem 2.3 since the small eigenvalues becomes exponentially small for larger values of s (larger n) in the two-pulse solution (9) and the exponentially small contribution is negligible compared to the numerical error of discretization.

	$\phi_1(z)$	$\phi_2(z)$	$\phi_3(z)$	$\phi_4(z)$
"Zero" EV of $\mathcal{H}$	$1.216 \cdot 10^{-9}$	$2.668 \cdot 10^{-9}$	$1.474 \cdot 10^{-9}$	$1.894 \cdot 10^{-9}$
Small EV of $\mathcal{H}$	$1.785 \cdot 10^{-2}$	$7.664 \cdot 10^{-5}$	$3.334 \cdot 10^{-7}$	$2.921 \cdot 10^{-9}$
"Zero" EVs of $\mathcal{L}_\alpha$	$0.365 \cdot 10^{-5}$	$0.532 \cdot 10^{-5}$	$0.783 \cdot 10^{-5}$	$1.237 \cdot 10^{-5}$
Re of small EVs of $\mathcal{L}_\alpha$	$4.529 \cdot 10^{-6}$	$3.285 \cdot 10^{-3}$	$6.326 \cdot 10^{-5}$	$1.652 \cdot 10^{-5}$
Im of small EVs of $\mathcal{L}_\alpha$	$0.502 \cdot 10^{-1}$	$1.152 \cdot 10^{-8}$	$2.167 \cdot 10^{-4}$	$5.444 \cdot 10^{-6}$

Table II: Numerical approximations of the zero and small eigenvalues (EVs) of operators $\mathcal{H}$ and $\mathcal{L}_\alpha$ for the first four two-pulse solutions with $c = 1$, $\alpha = 0.04$, $d = 100$, $h = 0.01$ and $\varepsilon = 10^{-15}$. The absolute values are shown.

We have confirmed numerically the analytical predictions that all two-pulse solutions corresponding to the points L_n with $W''(L_n) < 0$ (which are maxima of the effective interaction potential) are unstable with a simple real positive eigenvalue, while all two-pulse solutions corresponding to the points L_n with $W''(L_n) > 0$ (which are minima of the effective interaction potential) are spectrally stable. The stable two-pulse solutions are not, however, ground states since the corresponding linearized problem has a pair of eigenvalues of negative Krein signature.

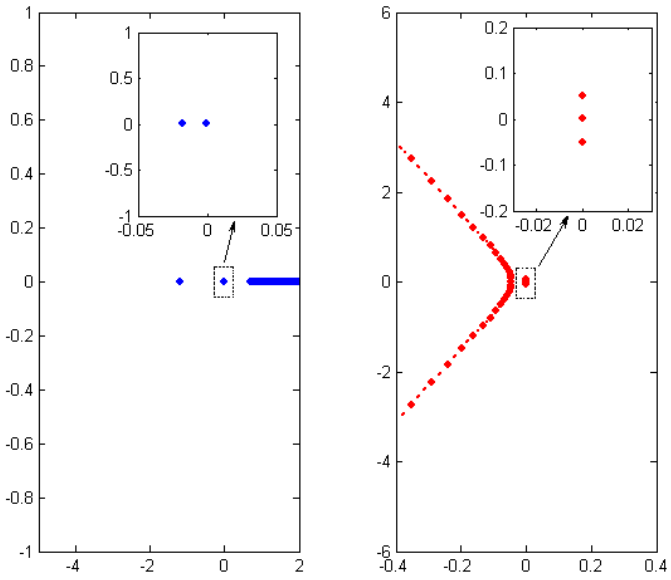

FIGURE 7. Numerical approximations of the spectra of operators $\mathcal{H}$ and $\mathcal{L}_\alpha$ for the two-pulse solution $\phi_1(z)$ with $c = 1$ and $\alpha = 0.04$. The insert shows zoom of small eigenvalues and the dotted curve connects eigenvalues of the continuous spectrum of $\mathcal{L}_\alpha$.

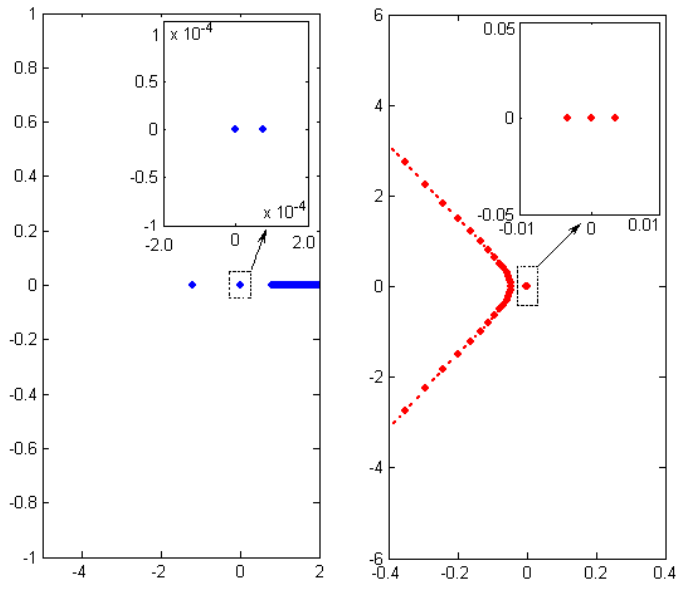

FIGURE 8. The same as Figure 7 but for the two-pulse solution $\phi_2(z)$.

5. **Nonlinear dynamics of two-pulse solution.** The Newton law (19) is a useful qualitative tool to understand the main results of our article. Existence of an infinite countable sequence of two-pulse solution $\{\phi_n(z)\}_{n\in\mathbb{N}}$ is related to existence of extremal points $\{L_n\}_{n\in\mathbb{N}}$ of the effective potential function $W(L)$, while alternation of stability and instability of the two-pulse solutions is related to the alternation of minima and maxima points of $W(L)$. It is natural to ask if the Newton law (19) extends beyond the existence and spectral stability analysis. In particular, one can ask if the purely imaginary (embedded) eigenvalues of the linearized problem (5) lead to nonlinear asymptotic stability of two-pulse solutions or at least to their nonlinear stability in the sense of Lyapunov. From a more technical point of view, one can ask whether the Newton law (19) serves as the center manifold reduction for slow nonlinear dynamics of two-pulse solutions in the PDE (1) and whether solutions of the full problem are topologically equivalent to solutions of the Newton law. While we do not attempt to develop mathematical analysis of these questions, we illustrate nonlinear dynamics of two-pulse solutions with explicit numerical simulations.

The numerical pseudo-spectral method for solutions of the fifth-order KdV equation (1) is described in details in [29]. The main idea of this method is to compute analytically the linear part of the PDE (1) by using the Fourier transform and to compute numerically its nonlinear part by using an ODE solver. Let $\hat{u}(k,t)$ denote the Fourier transform of $u(x,t)$ and rewrite the PDE (1) in the Fourier domain:

$$\hat{u}_t = i(k^3 + k^5)\hat{u} - ik\widehat{u^2}. \tag{33}$$

In order to compute $\widehat{u^2}(k,t)$ we evaluate $u^2(x,t)$ on $x \in \mathbb{R}$ and apply the discrete Fourier transform. Substitution $\hat{u} = s(k,t)e^{i(k^3+k^5)t}$ transforms the evolution equation (33) to the form:

$$s_t = -ike^{-i(k^3+k^5)t}\widehat{u^2}(k,t). \tag{34}$$

The fourth-order Runge-Kutta method is used to integrate the evolution equation (34) in time with time step $\triangle t$. To avoid large variations of the exponent for large values of k and t, the substitution above is updated after m time steps as follows:

$$\hat{u} = s_m(k,t)e^{i(k^3+k^5)(t-m\triangle t)}, \qquad m\triangle t \le t \le (m+1)\triangle t. \tag{35}$$

The greatest advantage of this numerical method is that no stability restriction arising from the linear part of (33) is posed on the timestep of numerical integration. On contrast, the standard explicit method for the fifth-order KdV equation (1) has a serious limitation on the timestep of the numerical integration since the fifth-order derivative term brings stiffness to the evolution problem. The small timestep is, however, an obstacle for the long time integration of the evolution problem due to accumulation of computational errors.

Numerical simulations of the PDE (33) are started with the initial condition:

$$u(x,0) = \Phi(x-s) + \Phi(x+s), \tag{36}$$

where $\Phi(x)$ is the one-pulse solution and $2s$ is the initial separation between the two pulses. The one-pulse solution $\Phi(x)$ is constructed with the iteration method (21)–(22) for $c = 4$. The numerical factors of the spectral approximation are $L = 100$, $N = 2^{12}$, $\varepsilon = 10^{-15}$, while the timestep is set to $\triangle t = 10^{-4}$.

Figure 9 shows six individual simulations of the initial-value problem (33) and (36) with $s = 2.3$, $s = 2.8$, $s = 3.6$, $s = 4.2$, $s = 4.5$ and $s = 4.7$. Figure 10 brings these six individual simulations on the effective phase plane $(L, \dot{L})$ computed from the distance $L(t)$ between two local maxima (humps) of the two-pulse solutions.

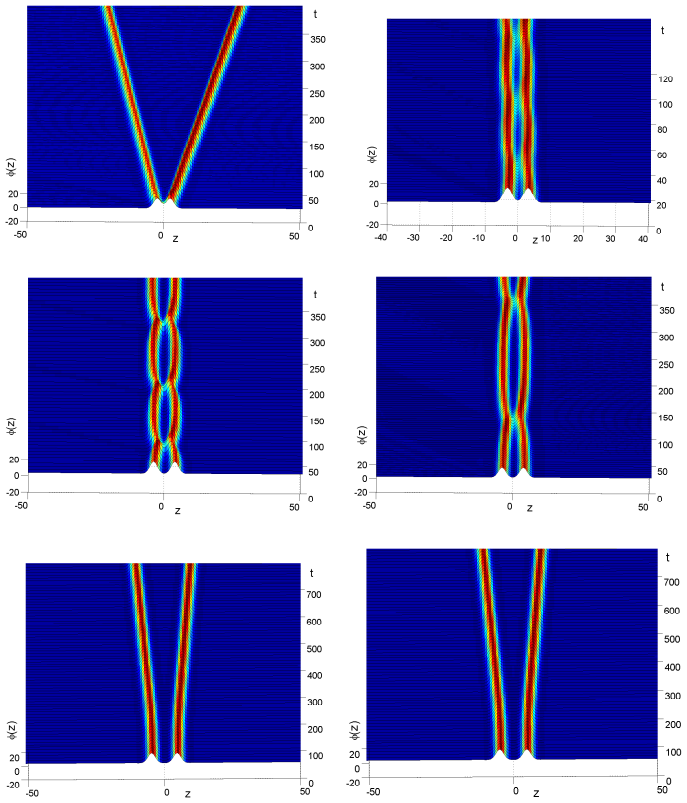

FIGURE 9. Individual simulations of the initial data (36) with $s = 2.3$ (top left), $s = 2.8$ (top right), $s = 3.6$ (middle left), $s = 4.2$ (middle right), $s = 4.5$ (bottom left) and 4.7 (bottom right).

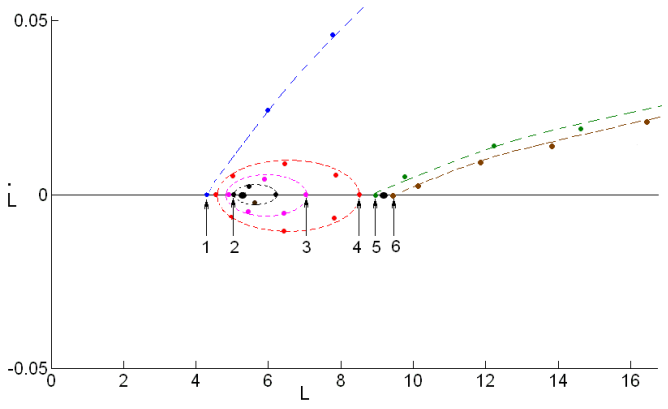

FIGURE 10. The effective phase plane $(L, \dot{L})$ for six simulations on Figure 9, where L is the distance between two pulses. The black dots denote stable and unstable equilibrium points which correspond to the two-pulse solutions $\phi_1(x)$ and $\phi_2(x)$.

When the initial distance ($s = 2.3$) is taken far to the left from the stable equilibrium point (which corresponds to the two-pulse solution $\phi_1(x)$), the two pulses repel and diverge from each other (trajectory 1). When the initial distance ($s = 2.8$) is taken close to the left from the stable equilibrium point, we observe small-amplitude oscillations of two pulses relative to each other (trajectory 2). When the initial distances ($s = 3.6$) and ($s = 4.2$) are taken to the right from the stable equilibrium point, we continue observing stable oscillations of larger amplitudes and larger period (trajectories 3 and 4). The oscillations are destroyed when the initial distances are taken close to the unstable equilibrium point (which corresponds to the two-pulse solution $\phi_2(x)$) from either left ($s = 4.5$) or right ($s = 4.7$). In either case, the two pulses repel and diverge from each other (trajectories 5 and 6). Ripples in the pictures are due to radiation effect and the numerical integration does not make sense after $t \approx 500$ because the ripples reach the left end of the computational interval and appear from the right end due to periodic boundary conditions.

The numerical simulations of the full PDE problem (1) indicate the validity of the Newton law (19). Due to the energy conservation, all equilibrium points in the Newton law are either centers or saddle points and the center points are surrounded by closed periodic orbits in the interior of homoclinic loops from the stable and unstable manifolds of the saddle points. Trajectories 2,3, and 4 are taken inside the homoclinic orbit from the saddle point corresponding to $\phi_2(x)$ and these trajectories represent periodic oscillations of two-pulse solutions near the center point corresponding to $\phi_1(x)$. Trajectories 1 and 6 are taken outside the homoclinic orbit and correspond to unbounded dynamics of two-pulse solutions. The only exception from the Newton law (19) is trajectory 5, which is supposed to occur inside the homoclinic loop but turns out to occur outside the loop. This discrepancy can be explained by the fact that the Newton law (19) does not *exactly* represent the dynamics of the PDE (33) generated by the initial condition (36) but it corresponds to an *asymptotic* solution after the full solution is projected into the discrete and continuous parts and the projection equations are truncated (see details in [17] in the context of the NLS equations).

Summarizing, we have studied existence, spectral stability and nonlinear dynamics of two-pulse solutions of the fifth-order KdV equation. We have proved that the two-pulse solutions can be numerically approximated by the Petviashili method supplemented with a root finding algorithm. We have also proved structural stability of embedded eigenvalues with negative Krein signature and this result completes the proof of spectral stability of two-pulse solutions related to the minima points of the effective interaction potential. The validity of the Newton law is illustrated by the full numerical simulations of the fifth-order KdV equation (1).

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