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Sharp bounds on enstrophy growth in the viscous Burgers equation

By Dmitry Pelinovsky*

Department of Mathematics, McMaster University, Hamilton, Ontario, Canada L8S 4K1

We use the Cole–Hopf transformation and the Laplace method for the heat equation to justify the numerical results on enstrophy growth in the viscous Burgers equation on the unit circle. We show that the maximum enstrophy achieved in the time evolution is scaled as $\mathcal{E}^{3/2}$, where $\mathcal{E}$ is the large initial enstrophy, whereas the time needed for reaching the maximal enstrophy is scaled as $\mathcal{E}^{-1/2}$. These bounds are sharp for initial conditions given by odd C^3 functions that are convex on half-period.

Keywords: viscous Burgers equation; enstrophy; Laplace method; Cole–Hopf transformation

1. Introduction

Existence and regularity of solutions of the three-dimensional Navier–Stokes equations are challenging problems that recently attracted many researchers (Kreiss & Lorenz 2004; Doering 2009). One possibility to continue local solutions globally in time is to control enstrophy of the Navier–Stokes equations during the time evolution. Enstrophy is an integral quantity in space, which may diverge in a finite evolution time, indicating singularities in the Navier–Stokes equations. To control the growth of enstrophy, Lu & Doering (2008) studied bounds on the instantaneous growth rate of enstrophy and showed numerically that these bounds are sharp in the limit of large enstrophy. However, these bounds on the instantaneous growth rate do not imply that enstrophy blows up in a finite time, because solutions of the Cauchy problem associated with the Navier–Stokes equation may deviate away from the maximizers of the bounds even if the initial data are close to the maximizers.

To deal with this problem, Ayala & Protas (2011) looked at a toy model, the one-dimensional viscous Burgers equation. They integrated the bounds on the instantaneous growth rate of enstrophy in time and showed numerically that the integrated bounds are not sharp. Limited accuracy of numerical results did not allow them to conclude whether better estimates on the enstrophy growth over a finite time interval can be justified within this context.

*dmpeli@math.mcmaster.ca

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To address this question, we consider the Cauchy problem for the one-dimensional viscous Burgers equation (Burgers 1948),

$$\left.\begin{aligned} u_t + 2uu_x &= u_{xx}, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}_+ \\ u|_{t=0} &= u_0, \quad x \in \mathbb{T}, \end{aligned}\right\} \tag{1.1}$$

where $\mathbb{T} = [-\frac{1}{2}, \frac{1}{2}]$ is the unit circle equipped with the periodic boundary conditions for the real-valued function u. Local well-posedness of the initial-value problem (1.1) holds for $u_0 \in H^s_{\text{per}}(\mathbb{T})$ with $s > -\frac{1}{2}$ (Dix 1996). Global existence holds in $H^s_{\text{per}}(\mathbb{T})$ for any integer $s \geq 0$, and we consider here global solutions of the viscous Burgers equation (1.1) in $H^1_{\text{per}}(\mathbb{T})$. For simplicity, we will assume that u_0 is odd in x, which implies that $u(-x,t) = -u(x,t)$ holds for all $t \geq 0$ and $x \in \mathbb{T}$.

All L^p_{per} norms with even p are monotonically decreasing in the time evolution of the viscous Burgers equation (1.1). In particular, the energy dissipation equation follows from (1.1) after integration by parts:

$$K(u) = \frac{1}{2}\int_{\mathbb{T}} u^2 \,\mathrm{d}x \Rightarrow \frac{\mathrm{d}K(u)}{\mathrm{d}t} = \int_{\mathbb{T}} u(u_{xx} - 2uu_x)\,\mathrm{d}x = -2E(u), \tag{1.2}$$

where $E(u) = \frac{1}{2}\int_{\mathbb{T}} u_x^2 \,\mathrm{d}x$ is a positive definite enstrophy. For a smooth solution $u \in C(\mathbb{R}_+, H^3_{\text{per}}(\mathbb{T}))$, the enstrophy changes according to the equation

$$\frac{\mathrm{d}E(u)}{\mathrm{d}t} = \int_{\mathbb{T}} u_x(u_{xxx} - 2uu_{xx} - 2u_x^2)\,\mathrm{d}x = -\int_{\mathbb{T}} (u_{xx}^2 + u_x^3)\,\mathrm{d}x \equiv R(u), \tag{1.3}$$

where $R(u)$ is the rate of change of $E(u)$. We can see from (1.3) that $R(u)$ is a sum of negative definite quadratic part and a sign-indefinite cubic part. The quadratic part corresponds to the diffusion term of the viscous Burgers equation, and the cubic part corresponds to the nonlinear advection term. It is the latter term that may lead to the enstrophy growth during the initial time evolution.

Lu & Doering (2008) showed that the rate of change $R(u)$ in (1.3) can be estimated by

$$R(u) \leq \tfrac{3}{2} E^{5/3}(u), \tag{1.4}$$

and illustrated numerically that the growth $R(u) = \mathcal{O}(\mathcal{E}^{5/3})$ is achieved in the limit of large $\mathcal{E} := E(u)$. If bound (1.4) is sharp on the time interval $[0, T]$ for some $T > 0$, then integration of the enstrophy equation (1.3) with the use of the energy dissipation equation (1.2) implies

$$E^{1/3}(u(T)) - E^{1/3}(u_0) \leq \frac{1}{2}\int_0^T E(u(t))\,\mathrm{d}t = \frac{1}{4}[K(u_0) - K(u(T))]. \tag{1.5}$$

Using the Poincaré inequality for periodic functions with zero mean,

$$K(u_0) \leq \frac{1}{4\pi^2} E(u_0), \tag{1.6}$$

and neglecting $K(u(T))$ in (1.5), we can obtain

$$E(u(T)) \leq \left(\mathcal{E}^{1/3} + \frac{1}{16\pi^2}\mathcal{E}\right)^3, \quad \mathcal{E} := E(u_0). \tag{1.7}$$

Ayala & Protas (2011) showed numerically that the integral bound (1.7) is not sharp even in the limit of large $\mathcal{E}$. Instead, they obtained the following numerical result:

$$T_* = \mathcal{O}(\mathcal{E}^{-1/2}), \quad E(u(T_*)) = \mathcal{O}(\mathcal{E}^{3/2}), \quad K(u(T_*)) = \mathcal{O}(\mathcal{E}), \tag{1.8}$$

where T_* is the value of time t, at which $E(u(t))$ is maximal. They also wanted to show that $K(u_0) - K(u(T_*)) = \mathcal{O}(\mathcal{E}^{1/2})$, so that the full integral bound (1.5) could be useful, but numerical approximations of this quantity suffered from large errors:

$$K(u_0) - K(u(T_*)) = \mathcal{O}(\mathcal{E}^{0.68\pm0.25}). \tag{1.9}$$

In the previous work (Pelinovsky 2012), we used dynamical system methods to study analytically the enstrophy growth in the viscous Burgers equation. Our technique, based on the self-similar transformation and analysis of asymptotic stability of viscous shocks in an unbounded domain, did not rely on the remarkable properties of the viscous Burgers equation such as the Cole–Hopf transformation (Hopf 1950; Cole 1951) of equation (1.1) to the linear heat equation. On the other hand, a weaker version of bounds (1.8) modified by logarithmic factors was justified as a result of this approach. No estimate on $K(u_0) - K(u(T_*))$ has been obtained.

In this work, we shall rely on the Cole–Hopf transformation and use the Laplace method for the heat equation. The Laplace method is typically used to recover solutions of the inviscid Burgers equations from solutions of the viscous Burgers equation in the limit of vanishing viscosity (Whitham 1974, ch. 2; Ablowitz & Fokas 1997, example 6.5.2; Miller 2006, §3.6). Applications of this method to statistical properties of the Burgers turbulence can be found in Frachebourg & Martin (2000).

Note that the limit of vanishing viscosity corresponds to the limit of large enstrophy in the context of our work. We shall implement the Laplace method to justify numerical results (1.8) and to estimate $K(u_0) - K(u(T_*))$ as $\mathcal{E} \to \infty$. Our main result is the following theorem.

Theorem 1.1. *Consider the initial-value problem (1.1) with initial data* $u_0(x) = kf(x)$*, where* $f \in C^3_{\text{per}}(\mathbb{T})$ *is an arbitrary odd non-zero function such that* $f''(x) \geq 0$ *for all* $x \in [0, \frac{1}{2}]$*. Consider the limit* $k \to \infty$ *and denote the initial enstrophy by* $\mathcal{E} = E(u_0) = \mathcal{O}(k^2)$*. There exists* $T_* > 0$ *such that the enstrophy* $E(u)$ *achieves its maximum at* $u_* = u(\cdot, T_*)$ *with*

$$T_* = \mathcal{O}(\mathcal{E}^{-1/2}), \quad E(u_*) = \mathcal{O}(\mathcal{E}^{3/2}), \quad K(u_*) = \mathcal{O}(\mathcal{E}) \tag{1.10}$$

and

$$K(u_0) - K(u_*) = \mathcal{O}(\mathcal{E}), \tag{1.11}$$

where all bounds are sharp as $\mathcal{E} \to \infty$*.*

Because $f \in C^3_{\text{per}}(\mathbb{T})$ is an odd function with $f''(x) \geq 0$ for all $x \in [0, \frac{1}{2}]$, it implies necessarily that $f(0) = f(\frac{1}{2}) = 0$, $f(x) < 0$ for all $x \in (0, \frac{1}{2})$, and $f'(y)$ is a

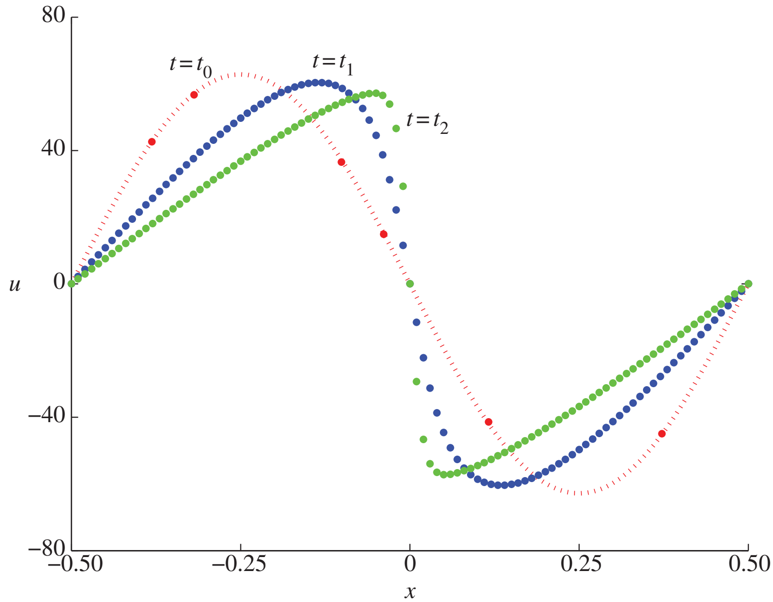

Figure 1. Initial condition at t_0 and numerically obtained solutions at t_1 and t_2 of the viscous Burgers equation (1.1) on the unit circle. (Online version in colour.)

monotonically increasing function from $f'(0) < 0$ to $f'(\frac{1}{2}) > 0$ with a unique zero at x_* such that $f'(x_*) = 0$. We will actually show that

$$T_* = \frac{x_*}{2k|f(x_*)|}, \tag{1.12}$$

$$E(u_*) = \tfrac{1}{2}k^3|f(x_*)|^3 + \mathcal{O}(k^2), \quad \text{as } k \to \infty \tag{1.13}$$

and

$$K(u_0) - K(u_*) = k^2\left(\int_0^{x_*} f^2(y)\,\mathrm{d}y - \frac{1}{3}x_* f^2(x_*)\right) + \mathcal{O}(k), \quad \text{as } k \to \infty, \tag{1.14}$$

where the leading-order terms are all non-zero. Sharp bounds (1.12)–(1.14) rule out the hope of using the integral bound (1.5) that follows from the instantaneous estimate (1.4) and the balance equations (1.2) and (1.3).

As an example, we can consider $f(x) = -2\pi \sin(2\pi x)$ that saturates the Poincaré inequality (1.6) and satisfies the conditions of theorem 1.1. Then, $x_* = \frac{1}{4}$, where $f'(x_*) = 0$, and the sharp bounds (1.12)–(1.14) yield the explicit expressions $T_* = 1/16\pi k$ and

$$\begin{cases} E(u_*) = 4\pi^3 k^3 + \mathcal{O}(k^2), \\ K(u_0) - K(u_*) = \dfrac{\pi^2}{6}k^2 + \mathcal{O}(k), \end{cases} \quad \text{as } k \to \infty. \tag{1.15}$$

Figure 1 shows the initial condition $u(x, t_0) = kf(x)$ for $t_0 = 0$ and $k = 10$ and the numerically obtained solution $u(x, t_{1,2})$ for $t_1 = T_*/2$ and $t_2 = T_*$. A viscous shock is formed at $t = T_*$, where enstrophy $E(u)$ is maximal. Dynamics for $t > T_*$

is represented by the relaxation of the solution to the zero equilibrium state, during which enstrophy $E(u)$ is monotonically decreasing. Note that the energy difference $K(u_0) - K(u)$ becomes visible on the figure scale.

A generalization of theorem 1.1 can be developed for any $f \in C^3_{\text{per}}(\mathbb{T})$ with finitely many changes in the sign of f'' on $[0, \frac{1}{2}]$ by the price of lengthier technical computations. On the other hand, the obtained results do not imply that the bounds (1.10) and (1.11) remain sharp for initial conditions with limited regularity, say for $f \in H^1_{\text{per}}(\mathbb{T})$, for which $E(u)$ and $K(u)$ are still well defined for all $t \in \mathbb{R}_+$.

2. Proof of theorem 1.1

In the remainder of this paper, we shall prove theorem 1.1. Using the Cole–Hopf transformation (Hopf 1950; Cole 1951),

$$u(x,t) = -\frac{\partial}{\partial x} \log \psi(x,t), \quad \psi(x,t) > 0 \quad \text{for all } (x,t), \tag{2.1}$$

we rewrite the Cauchy problem (1.1) in the equivalent form

$$\left.\begin{aligned} \psi_t &= \psi_{xx}, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}_+ \\ \psi|_{t=0} &= \psi_0, \quad x \in \mathbb{T}, \end{aligned}\right\} \tag{2.2}$$

where

$$\psi_0(x) = \mathrm{e}^{-\int_0^x u_0(s)\,\mathrm{d}s} = \mathrm{e}^{-kF(x)}, \quad F(x) := \int_0^x f(s)\,\mathrm{d}s. \tag{2.3}$$

If $f \in C^3_{\text{per}}(\mathbb{T})$ is odd in x, then $\psi_0 \in C^4_{\text{per}}(\mathbb{T})$ is even in x.

Although the Cauchy problem (2.2) is posed on the periodic domain, we can still construct the solution as a convolution of the initial data ψ_0 with the heat kernel $G_t : \mathbb{R} \to \mathbb{R}_+$ defined on the entire axis,

$$G_t(x) = \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-x^2/4t}, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+. \tag{2.4}$$

The solution to the Cauchy problem (2.2) is expressed in the explicit form:

$$\psi(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \mathrm{e}^{-kF(y)-(x-y)^2/4t}\,\mathrm{d}y. \tag{2.5}$$

This representation is justified by the generalized Young inequality,

$$\|G_t \star \psi_0\|_{L^r(\mathbb{R})} \le \|G_t\|_{L^p(\mathbb{R})} \|\psi_0\|_{L^q(\mathbb{R})}, \quad 1 \le p, q, r \le \infty : \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \tag{2.6}$$

where $\star$ denotes the convolution integral. If $\psi_0 \in L^\infty_{\text{per}}(\mathbb{R})$, then $G_t \star \psi_0 \in L^\infty_{\text{per}}(\mathbb{R})$ for all $t \in \mathbb{R}_+$.

Let us define the parametrization of the time variable by $t = 1/2ka$, where $a \in \mathbb{R}_+$ is a new parameter. The solution of the viscous Burgers equation can now be written in the explicit form:

$$u(x,t) = -\frac{\partial}{\partial x} \log I_{x,a}(k), \quad I_{x,a}(k) := \int_{-\infty}^{\infty} \mathrm{e}^{-k\phi_{x,a}(y)}\,\mathrm{d}y, \tag{2.7}$$

where $\phi_{x,a}(y) := F(y) + \frac{1}{2}a(x-y)^2$. Note that we have cancelled the t-dependent factor in the expression (2.5) because it plays no role in the logarithmic derivative with respect to x in (2.7).

Integral $I_{x,a}(k)$ in the explicit expression (2.7) can be studied in the limit $k \to \infty$ by means of the Laplace method (§3.4 in Miller 2006). We recall the main result of the Laplace method.

Proposition 2.1. *For any $-\infty \le a < b \le \infty$, assume that $\phi \in C^4(a,b)$ has a global minimum at $c \in (a,b)$ such that $\phi'(c) = 0$ and $\phi''(c) > 0$. Then, for any $\theta \in C^2(a,b) \cap L^1(a,b)$, we have the following asymptotic expansion:*

$$I(k) := \int_a^b \theta(y)\,\mathrm{e}^{-k\phi(y)}\,\mathrm{d}y = \left(\frac{2\pi}{k\phi''(c)}\right)^{1/2} \theta(c)\,\mathrm{e}^{-k\phi(c)}\left[1 + \mathcal{O}\left(\frac{1}{k}\right)\right] \quad \text{as } k \to \infty. \tag{2.8}$$

Here and in what follows, we use the following notations. Let X be a Banach space. We write $A = \mathcal{O}_X(k^p)$ as $k \to \infty$, if there exist constants $C_\pm$ such that $0 \le C_- < C_+ < \infty$ and $C_-\mathcal{E}^p \le \|A\|_X \le C_+\mathcal{E}^p$. If $X = \mathbb{R}$, we write $A = \mathcal{O}(\mathcal{E}^p)$.

Applying proposition 2.1, we obtain the following.

Lemma 2.2. *Under assumptions of theorem 1.1, the expansion*

$$u(x,t) = kf(s_{x,a}) + \mathcal{O}_{L^\infty_{\mathrm{per}}(\mathbb{T})}(1), \quad \text{as } k \to \infty, \tag{2.9}$$

holds for all $t \in [0, t_0)$, where $t_0 := 1/2k|f'(0)|$, and $s_{x,a}$ is the unique root of $a(s-x) + f(s) = 0$.

Proof. Under assumptions of theorem 1.1, in particular that $f \in C^3_{\mathrm{per}}(\mathbb{T})$ is an odd non-zero function, we compute

$$\phi'_{x,a}(y) = f(y) + a(y-x), \quad \phi''_{x,a}(y) = f'(y) + a. \tag{2.10}$$

For every $a > \max_{y\in\mathbb{T}}(-f'(y)) = |f'(0)|$ and every $x \in \mathbb{T}$, there exists exactly one root of $f(y) + a(y-x) = 0$. Let us denote this root by $s_{x,a}$. Conditions of proposition 2.1 are satisfied, so that as $k \to \infty$, we have

$$I_{x,a}(k) = \left(\frac{2\pi}{k(f'(s_{x,a}) + a)}\right)^{1/2} \mathrm{e}^{-k\phi_{x,a}(s_{x,a})}\left[1 + \mathcal{O}\left(\frac{1}{k}\right)\right] \tag{2.11}$$

and

$$\begin{aligned}\partial_x I_{x,a}(k) &= ak\int_{-\infty}^{\infty}(y-x)\mathrm{e}^{-k\phi_{x,a}(y)}\,\mathrm{d}y \\ &= ak(s_{x,a}-x)\left(\frac{2\pi}{k(f'(s_{x,a}) + a)}\right)^{1/2} \mathrm{e}^{-k\phi_{x,a}(s_{x,a})}\left[1 + \mathcal{O}\left(\frac{1}{k}\right)\right].\end{aligned} \tag{2.12}$$

Using these expansions, the representation $u = -\partial_x I_{x,a}(k)/I_{x,a}(k)$ and the equation $a(x - s_{x,a}) = f(s_{x,a})$, we obtain the assertion (2.9) of the lemma. ■

Remark 2.3. The result of lemma 2.2 is known in the limit of vanishing viscosity, when a smooth solution of the viscous Burgers equation is shown to converge to the classical solution of the inviscid Burger equation before a shock is formed Whitham (1974, ch. 2).

Using the representation

$$u_x(x,t) = u^2(x,t) - \frac{\partial_x^2 I_{x,a}(k)}{I_{x,a}(k)} \tag{2.13}$$

and the Laplace method for $\partial_x^2 I_{x,a}(k)$, the result of lemma 2.2 can be extended to show for all $t \in [0, t_0)$ that

$$u_x(x,t) = k\frac{af'(s_{x,a})}{a + f'(s_{x,a})} + \mathcal{O}_{L^\infty_{\text{per}}(\mathbb{T})}(1) \quad \text{as } k \to \infty. \tag{2.14}$$

As a result, the energy and enstrophy are expanded as $k \to \infty$ as follows

$$K(u(t)) = \frac{1}{2}k^2 \int_{\mathbb{T}} f^2(s_{x,a})\,\mathrm{d}x + \mathcal{O}(k) \tag{2.15}$$

and

$$E(u(t)) = \frac{1}{2}k^2 \int_{\mathbb{T}} \frac{a^2(f'(s_{x,a}))^2}{(a + f'(s_{x,a}))^2}\,\mathrm{d}x + \mathcal{O}(k). \tag{2.16}$$

Because $f(0) = f(\frac{1}{2}) = 0$, we have $s_{x,a} = x$ for $x = 0$ and $x = \frac{1}{2}$. Moreover, the map $\mathbb{T} \ni x \mapsto s_{x,a} \in \mathbb{T}$ is one-to-one and onto, and hence $y = s_{x,a}$ gives

$$\mathrm{d}y = \partial_x s_{x,a}\,\mathrm{d}x = \frac{a\,\mathrm{d}x}{a + f'(y)}. \tag{2.17}$$

This argument gives energy conservation at the leading order $\mathcal{O}(k^2)$ for all $t \in [0, t_0)$:

$$\begin{aligned} K(u(t)) &= \frac{1}{2}k^2 \int_{\mathbb{T}} f^2(y)\left(1 + \frac{1}{a}f'(y)\right)\mathrm{d}y + \mathcal{O}(k) \\ &= \frac{1}{2}k^2 \int_{\mathbb{T}} f^2(y)\mathrm{d}y + \mathcal{O}(k) \\ &= K(u_0) + \mathcal{O}(k). \end{aligned} \tag{2.18}$$

On the other hand, the enstrophy grows initially but remains within the $\mathcal{O}(k^2)$ order for all $t \in [0, t_0)$:

$$E(u(t)) = \frac{1}{2}k^2 \int_{\mathbb{T}} \frac{a(f'(y))^2}{a + f'(y)}\,\mathrm{d}y + \mathcal{O}(k). \tag{2.19}$$

At $t = t_0$, that is, for $a = \max_{y\in\mathbb{T}}(-f'(y)) = |f'(0)|$, a local (pitchfork) bifurcation occurs among the roots of $f(s) + as = 0$ near $s = 0$. For $t > t_0$, that is, for $a < |f'(0)|$, we have $f'(0) + a < 0$, so that for all x near 0, a local maximum $s_{x,a}$ of $\phi_{x,a}$ exists near 0, whereas two local minima $s^+_{x,a}$ and $s^-_{x,a}$ of $\phi_{x,a}$ exist for $s^+_{x,a} \in (0, \frac{1}{2})$ and $s^-_{x,a} \in (-\frac{1}{2}, 0)$. Figure 2 shows roots s of the equation $f(s) + a(s - x) = 0$ for $f(s) = -2\pi\sin(2\pi s)$, $a = 2\pi^2$ and three different values of x in $[0, \frac{1}{2}]$. At $x = 0$, the values of roots are symmetric: $s_{0,a} = 0$ and $s^-_{0,a} = -s^+_{0,a}$. At $x = x_0 \in (0, \frac{1}{2})$, the roots $s_{x,a}$ and $s^-_{x,a}$ coalesce at $x = x_0$ and disappear as a result of the saddle-node

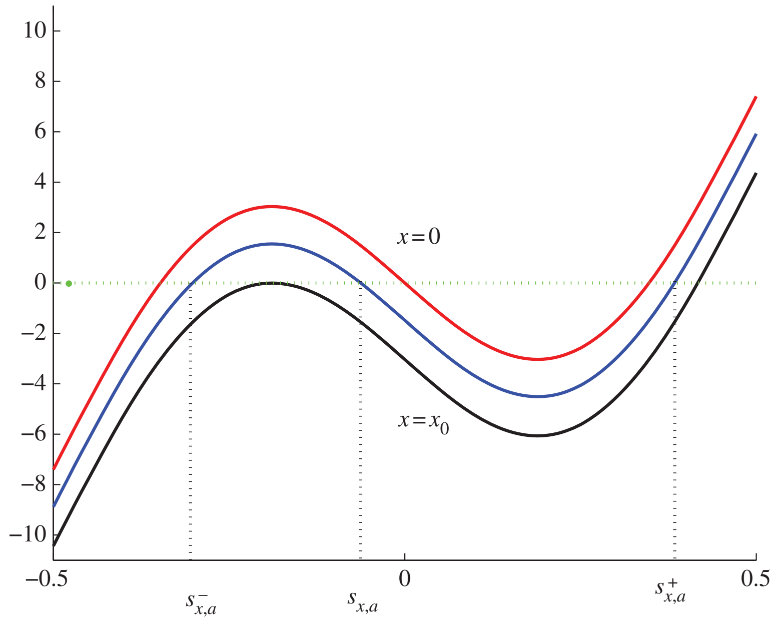

Figure 2. Roots s of the equation $f(s)+a(s-x)=0$ for three different values of x. (Online version in colour.)

bifurcation. The value of x_0 exists in $(0,\frac{1}{2})$ for a slightly below $|f'(0)|$ but moves beyond $(0,\frac{1}{2})$ for sufficiently small values of a.

In what follows, we consider the values of $x\in[0,\frac{1}{2}]$ and use the odd symmetry of $u(x,t)$ for $x\in[-\frac{1}{2},0]$. Note that $\phi_{0,a}(s^+_{0,a})=\phi_{0,a}(s^-_{0,a})$ is due to the pitchfork symmetry. Assume for simplicity that the saddle–node bifurcation happens at $x_0\in(0,\frac{1}{2})$. When x is increased in $(0,x_0)$, then $\phi_{x,a}(s^+_{x,a})<\phi_{x,a}(s^-_{x,a})$ and the difference is growing with the values of x. For $x\in(x_0,\frac{1}{2}]$, only one minimum of $\phi_{x,a}(s)$ exists in $[-\frac{1}{2},\frac{1}{2}]$ at $s=s^+_{x,a}$. To prove these claims, we denote

$$\varphi_{x,a}=\phi_{x,a}(s^-_{x,a})-\phi_{x,a}(s^+_{x,a})=\frac{1}{2}a(s^-_{x,a}-s^+_{x,a})(s^+_{x,a}+s^-_{x,a}-2x)-\int_{s^-_{x,a}}^{s^+_{x,a}}f(s)\,\mathrm{d}s \tag{2.20}$$

and

$$\chi_{x,a}=\left(\frac{f'(s^+_{x,a})+a}{f'(s^-_{x,a})+a}\right)^{1/2}. \tag{2.21}$$

We have the following.

Lemma 2.4. *For every $t>t_0$, where $t_0:=1/2k|f'(0)|$, there is $x_0>0$ such that for all $x\in[0,x_0)$, there exist three roots of the algebraic equation $a(s-x)+f(s)=0$ in the following order*:

$$-\tfrac{1}{2}<s^-_{x,a}<s_{x,a}\le 0<s^+_{x,a}<\tfrac{1}{2}. \tag{2.22}$$

Moreover, $\varphi_{x,a}$ and $\chi_{x,a}$ are C^1 monotonically increasing functions of x with $\varphi_{0,a}=0$, $\chi_{0,a}=1$ and $\chi_{x,a}\to+\infty$ as $x\to x_0$. The point x_0 marks the saddle–node bifurcation among the roots of $a(s-x)+f(s)=0$ such that $s^-_{x_0,a}=s_{x_0,a}$ and $a+f'(s_{x_0,a})=0$.

Proof. The presence of three roots $s^-_{0,a}=-s^+_{0,a}$ and $s_{0,a}=0$ follows for $x=0$ because $a+f'(0)<0$ for $t>t_0$ and f is an odd function. By continuity, three roots persist and, by the implicit function theorem, the three roots are C^1 functions of x as long as $a+f'(s^{\pm}_{x,a})>0$. We now compute

$$\partial_x\varphi_{x,a}=a(s^+_{x,a}-s^-_{x,a})>0, \tag{2.23}$$

and

$$\partial_x\chi_{x,a}=\chi_{x,a}\left[\frac{f''(s^+_{x,a})}{(a+f'(s^+_{x,a}))^2}-\frac{f''(s^-_{x,a})}{(a+f'(s^-_{x,a}))^2}\right]\geq 0, \tag{2.24}$$

where the last inequality is due to $f''(x)\geq 0$ for $x\in[0,\frac{1}{2}]$ and the fact that f'' is an odd function. In addition, we note that

$$\varphi_{0,a}=\frac{1}{2}a(s^-_{0,a}-s^+_{0,a})(s^+_{0,a}+s^-_{0,a})-\int_{s^-_{0,a}}^{s^+_{0,a}}f(s)\,\mathrm{d}s=0 \tag{2.25}$$

and

$$\chi_{0,a}=\left(\frac{f'(s^+_{0,a})+a}{f'(s^-_{0,a})+a}\right)^{1/2}=1, \tag{2.26}$$

because $s^-_{0,a}=-s^+_{0,a}$ and f is odd. The statement of the lemma is proved. ∎

Remark 2.5. When a is reduced further, additional saddle–node bifurcations occur among the roots of $f(s)+as=0$ owing to periodicity of f. For all x near 0, these bifurcations give rise to new maxima and minima of $\phi_{x,a}$ outside $\mathbb{T}$, and the values of $\phi_{x,a}$ at new local minima are larger than the values of $\phi_{x,a}(s^{\pm}_{x,a})$. Therefore, we can simply neglect the presence of these additional bifurcations in the applications of the Laplace method for all $a\in(0,|f'(0)|)$.

We shall apply proposition 1.2 to obtain the following.

Lemma 2.6. *Let $s^{\pm}_{x,a}$ and x_0 be described in lemma 1.5. Under assumptions of theorem 1.1, for every $t>t_0$, where $t_0:=1/2k|f'(0)|$, there is $x_1\in(0,\min\{x_0,\frac{1}{2}\})$ such that for all $x\in[0,x_1]$,*

$$u(x,t)=k\frac{f(s^+_{x,a})+\chi_{x,a}f(s^-_{x,a})\,\mathrm{e}^{-k\varphi_{x,a}}}{1+\chi_{x,a}\,\mathrm{e}^{-k\varphi_{x,a}}}+\mathcal{O}_{L^\infty(0,x_1)}(1)\quad \text{as } k\to\infty \tag{2.27}$$

whereas for all $x\in[x_1,\frac{1}{2}]$,

$$u(x,t)=kf(s^+_{x,a})+\mathcal{O}_{L^\infty\left(x_1,\frac{1}{2}\right)}(1)\quad \text{as } k\to\infty. \tag{2.28}$$

Proof. We split the integral in the explicit solution (2.7) into two parts

$$I_{x,a}(k) := \int_{-\infty}^{0} \mathrm{e}^{-k\phi_{x,a}(y)}\,\mathrm{d}y + \int_{0}^{\infty} \mathrm{e}^{-k\phi_{x,a}(y)}\,\mathrm{d}y, \tag{2.29}$$

and apply the Laplace method of proposition 1.2 separately for each integral as $k \to \infty$:

$$\begin{aligned} I_{x,a}(k) = {} & \left(\frac{2\pi}{k(f'(s^-_{x,a}) + a)}\right)^{1/2} \mathrm{e}^{-k\phi_{x,a}(s^-_{x,a})}\left[1 + \mathcal{O}\left(\frac{1}{k}\right)\right] \\ & + \left(\frac{2\pi}{k(f'(s^+_{x,a}) + a)}\right)^{1/2} \mathrm{e}^{-k\phi_{x,a}(s^+_{x,a})}\left[1 + \mathcal{O}\left(\frac{1}{k}\right)\right] \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} \partial_x I_{x,a}(k) = {} & ak(s^-_{x,a} - x)\left(\frac{2\pi}{k(f'(s^-_{x,a}) + a)}\right)^{1/2} \mathrm{e}^{-k\phi_{x,a}(s^-_{x,a})}\left[1 + \mathcal{O}\left(\frac{1}{k}\right)\right] \\ & + ak(s^+_{x,a} - x)\left(\frac{2\pi}{k(f'(s^+_{x,a}) + a)}\right)^{1/2} \mathrm{e}^{-k\phi_{x,a}(s^+_{x,a})}\left[1 + \mathcal{O}\left(\frac{1}{k}\right)\right]. \end{aligned} \tag{2.31}$$

For $x = 0$, both leading-order terms in these expansions have equal magnitude resulting in (2.27), whereas for any fixed $x > 0$, the first integral is exponentially small compared with the second integral resulting in (2.28). ■

Remark 2.7. The two different expansions (2.27) and (2.28) are asymptotically equivalent for $x = x_1$ (and near $x = x_1$), because $\varphi_{x,a} > 0$ for any $x > 0$ and, therefore, the term involving $\mathrm{e}^{-k\varphi_{x,a}}$ is exponentially small as $k \to \infty$ for any fixed $x > 0$.

Using the representation (2.13) and the Laplace method for $\partial_x^2 I_{x,a}(k)$, the result of lemma 2.6 can be extended to prove for all $t > t_0$ and all $x \in [0, x_1]$

$$u_x(x,t) = -k^2 \frac{\chi_{x,a}(f(s^+_{x,a}) - f(s^-_{x,a}))^2\,\mathrm{e}^{-k\varphi_{x,a}}}{(1 + \chi_{x,a}\,\mathrm{e}^{-k\varphi_{x,a}})^2} + \mathcal{O}_{L^\infty(0,x_1)}(k) \quad \text{as } k \to \infty, \tag{2.32}$$

whereas for all $x \in [x_1, \frac{1}{2}]$,

$$u_x(x,t) = \mathcal{O}_{L^\infty\left(x_1,\frac{1}{2}\right)}(k) \quad \text{as } k \to \infty. \tag{2.33}$$

We shall now compute the leading order of the enstrophy for $t > t_0$:

$$E(u(t)) = \int_0^{x_1} u_x^2(x,t)\,\mathrm{d}x + \mathcal{O}(k^2), \quad \text{as } k \to \infty. \tag{2.34}$$

Because $\varphi_{x,a}$ is monotonically growing for all $x \in [0, x_1]$, we can use another version of the Laplace method (§3.3 in Miller 2006).

Proposition 2.8. *For any $c>0$, assume that $\phi \in C^2(0,c)$ is monotonically growing such that $\phi'(0)>0$. Then, for any $\theta \in C^1(0,c)\cap L^1(0,c)$, we have the following asymptotic expansion*:

$$I(k) := \int_0^c \theta(y)\,\mathrm{e}^{-k\phi(y)}\,\mathrm{d}y = \frac{\theta(0)}{k\phi'(0)}\,\mathrm{e}^{-k\phi(0)}\left[1+\mathcal{O}\left(\frac{1}{k}\right)\right] \quad \textit{as } k\to\infty. \tag{2.35}$$

We have computed previously: $\varphi_{0,a}=0$, $\chi_{0,a}=1$ and

$$\partial_x\varphi_{0,a} = a(s^+_{0,a}-s^-_{0,a}) = 2as^+_{0,a} = -2f(s^+_{0,a}) > 0. \tag{2.36}$$

Therefore, a straightforward application of proposition 2.8 yields

$$E(u(t)) = \tfrac{1}{2}k^3|f(s^+_{0,a})|^3 + \mathcal{O}(k^2), \quad \text{as } k\to\infty. \tag{2.37}$$

For $a=|f'(0)|$, $s^+_{0,a}=0$ and the $\mathcal{O}(k^3)$ term of (2.37) vanishes because $f(0)=0$. When a is decreased from $|f'(0)|$ to 0, the value of $s^+_{0,a}$ grows from 0 to $\frac{1}{2}$ and it passes the value x_*, where $f'(x_*)=0$ and $|f(s^+_{0,a})|$ is maximal. The corresponding value of a_* is found from the equation $s^+_{0,a_*}=x_*$, or, explicitly, $a_*=|f(x_*)|/x_*$. This argument completes the proof of the first two bounds (1.10) of theorem 1.1 with $T_*=1/2ka_*=x_*/2k|f(x_*)|$ (recall that $k=\mathcal{O}(\mathcal{E}^{1/2})$).

Remark 2.9. An application of the Laplace method to the values of t near t_0 that depends on k is much more delicate. An extension of the Laplace method for integrals with vanishing $\phi''(c)$ at the point c of the global minimum of ϕ was studied under the assumption that $\phi'''(c)\neq 0$ in the pioneer work (Chester *et al.* 1957; see Wong 2001 for recent development). In our case, we have the situation when both $\phi''(c)$ and $\phi'''(c)$ vanish at $t=t_0$ and $x=0$ that is even more delicate. We are very fortunate that the main result of theorem 1.1 can be proved without knowing the behaviour of the solution near $t=t_0$.

To complete the proof of theorem 1.1, we need to compute the energy $K(u(t))$ for all $t>t_0$. By lemma 2.6, we represent for all $x\in\mathbb{T}$:

$$u(x,t) = kf(s^+_{x,a}) + \tilde{u}(x,t), \tag{2.38}$$

where $\tilde{u}$ is found from the asymptotic expansions (2.27) and (2.28). By proposition 2.8, we have

$$K(u(t)) = k^2\int_0^{1/2} f^2(s^+_{x,a})\,\mathrm{d}x + \mathcal{O}(k) \quad \text{as} \quad k\to\infty. \tag{2.39}$$

For all $a<|f'(0)|$, we have $s^+_{0,a}>0$. Nevertheless, we still have $s^+_{x,a}=x$ for $x=\frac{1}{2}$. The map $[0,\frac{1}{2}]\ni x\mapsto s^+_{x,a}\in[s^+_{0,a},\frac{1}{2}]$ is one-to-one and onto. As a result, for all $t>t_0$:

$$\begin{aligned} K(u(t)) &= k^2\int_{s^+_{0,a}}^{1/2} f^2(y)\left(1+\frac{1}{a}f'(y)\right)\mathrm{d}y + \mathcal{O}(k) \\ &= k^2\left(\int_{s^+_{0,a}}^{1/2} f^2(y)\,\mathrm{d}y + \frac{1}{3a}|f(s^+_{0,a})|^3\right) + \mathcal{O}(k). \end{aligned} \tag{2.40}$$

This argument completes the proof of the third bound (1.10) of theorem 1.1. To prove the bound (1.11), we need to show that the $\mathcal{O}(k^2)$ term in the expansion for $K(u(t))$ is different from the one for $K(u_0)$ (it can only be smaller), or equivalently, that

$$\int_0^{s_{0,a}^+} f^2(y)\,dy > \frac{1}{3a}|f(s_{0,a}^+)|^3 = \frac{1}{3} s_{0,a}^+ f^2(s_{0,a}^+). \tag{2.41}$$

To prove (2.41), we define two functions $F, H : [0, x_*] \to \mathbb{R}$ by

$$G(x) := \int_0^x f^2(y)\,dy - \frac{1}{3} x f^2(x), \quad H(x) := f(x) - x f'(x). \tag{2.42}$$

By the conditions on f, it is clear that $G \in C^3([0, x_*])$ and $H \in C^2([0, x_*])$. Furthermore, $G(0) = 0$, $H(0) = 0$ and

$$G'(x) = \frac{2}{3} f(x) H(x), \quad H'(x) = -x f''(x). \tag{2.43}$$

Because $f''(x) \geq 0$ for all $x \in [0, \frac{1}{2}]$, H is a monotonically decreasing function from $H(0) = 0$ to $H(x) < 0$ for all $x \in (0, x_*]$. Therefore, G is a monotonically increasing function from $G(0) = 0$ to $G(x) > 0$ for all $x \in (0, x_*]$. Therefore, inequality (2.41) is proved and, hence, bound (2.37) is verified. Note that (2.37) and (2.41) yield explicit bounds (1.12) and (1.14). The proof of theorem 1.1 is complete.

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