J. Nonlinear Sci. Vol. 16: pp. 1–44 (2006)
DOI: 10.1007/s00332-005-0652-7

Journal of Nonlinear Science

Evans Function for Lax Operators with Algebraically Decaying Potentials

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Received June 29, 2004; revised version accepted for publication July 20, 2005
Online publication October 7, 2005
Communicated by T. Fokas

Summary. We study the instability of algebraic solitons for integrable nonlinear equations in one spatial dimension that include modified KdV, focusing NLS, derivative NLS, and massive Thirring equations. We develop the analysis of the Evans function that defines eigenvalues in the corresponding Lax operators with algebraically decaying potentials. The standard Evans function generically has singularities in the essential spectrum, which may include embedded eigenvalues with algebraically decaying eigenfunctions. We construct a renormalized Evans function and study bifurcations of embedded eigenvalues, when an algebraically decaying potential is perturbed by a generic potential with a faster decay at infinity. We show that the bifurcation problem for embedded eigenvalues can be reduced to cubic or quadratic equations, depending on whether the algebraic potential decays to zero or approaches a nonzero constant. Roots of the bifurcation equations define eigenvalues which correspond to nonlinear waves that are formed from unstable algebraic solitons.

Our results provide precise information on the transformation of unstable algebraic solitons in the time-evolution problem associated with the integrable nonlinear equation. Algebraic solitons of the modified KdV equation are shown to transform to either travelling solitons or time-periodic breathers, depending on the sign of the perturbation. Algebraic solitons of the derivative NLS and massive Thirring equations are shown to transform to travelling and rotating solitons for either sign of the perturbation. Finally, algebraic homoclinic orbits of the focusing NLS equation are destroyed by the perturbation and evolve into time-periodic space-decaying solutions.

1. Introduction

Nonlinear evolution equations in one and two spatial dimensions, which are integrable by means of the inverse scattering transform method [AC91], typically have algebraic solitons as special solutions. Algebraic solitons decay to zero at infinity or approach nonzero boundary values at an algebraic rate. Depending on properties of the nonlinear equation, the solution evolves in time as a travelling wave, a travelling and rotating wave, or a time-homoclinic orbit. Algebraic solitons are stable under the time evolution of the nonlinear system if they are separated from the linear wave spectrum, as happens for the KPI equation [PS00a]. Algebraic solitons are unstable if they are embedded into the wave spectrum, as happens for the DSII equation [PS00b]. Stability of algebraic solitons in the initial-value problem can be studied using the spectral analysis of Lax operators associated with the integrable evolution equation.

In this paper, we consider integrable nonlinear evolution equations in one spatial dimension [AC91] that have algebraic solitons. The list includes the modified Korteweg–de Vries (KdV) equation:

$$u_t + 6u^2 u_x + u_{xxx} = 0, \quad u \in \mathbb{R}; \tag{1.1}$$

the focusing nonlinear Schrödinger (NLS) equation:

$$i u_t = u_{xx} + 2|u|^2 u, \quad u \in \mathbb{C}; \tag{1.2}$$

the derivative NLS equation:

$$i u_t = u_{xx} + i\left(|u|^2 u\right)_x, \quad u \in \mathbb{C}; \tag{1.3}$$

and the massive Thirring model (MTM) system in characteristic coordinates:

$$i v_t + w - 2|w|^2 v = 0, \qquad -i w_x + v - 2|v|^2 w = 0, \quad v, w \in \mathbb{C}. \tag{1.4}$$

Algebraic solitons occur in all the above nonlinear equations. Factoring out arbitrary parameters related to translations in x and t and gauge invariance, the list of algebraic solitons includes the travelling wave of the modified KdV equation (1.1) [O76], [PG97]:

$$u(x,t) = \frac{4(x-6t)^2 - 3}{4(x-6t)^2 + 1}; \tag{1.5}$$

the homoclinic orbit of the focusing NLS equation (1.2) [KI78a], [EKK86]:

$$u(x,t) = e^{-2it} \frac{4x^2 + 16t^2 + 16it - 3}{4x^2 + 16t^2 + 1}; \tag{1.6}$$

the travelling and rotating wave of the derivative NLS equation (1.3) [KN78], [M78]:

$$u(x,t) = 4\delta e^{2i\delta^2(x+2\delta^2 t)} \frac{4\delta^2(x+4\delta^2 t) + i}{(4\delta^2(x+4\delta^2 t) - i)^2}; \tag{1.7}$$

and the travelling and rotating wave of the MTM system (1.4) [KN77], [BPZ98]:

$$\begin{aligned} v(x,t) &= \frac{2\delta}{4\delta^2(x+\tau t) - i} e^{2i\delta^2(x-\tau t)}, \\ w(x,t) &= -\frac{\delta^{-1}}{4\delta^2(x+\tau t) + i} e^{2i\delta^2(x-\tau t)}, \quad \tau = \frac{1}{4\delta^4}, \end{aligned} \tag{1.8}$$

where δ is a parameter. All the above algebraic solutions are nonsingular for $x \in \mathbb{R}$ and $t \in \mathbb{R}$ and are relevant for applications of solitary waves in various physical sciences. The two solutions (1.5)–(1.6) satisfy the nonzero boundary conditions $\lim_{|x|\to\infty} |u|(x, t) = 1$, while the other two solutions (1.7)–(1.8) satisfy $\lim_{|x|\to\infty} |u|(x, t) = 0$.

Nonlinear stability of algebraic solitons with respect to the time evolution of the nonlinear equation is a difficult analytical problem (see [PG97]). It is typical for integrable equations that algebraic solitons are weakly spectrally stable and squared eigenfunctions of the Lax operator define a complete set of neutrally stable eigenfunctions for the linearized evolution problem [AKNS74]. Nevertheless, algebraic solitons in nonlinear equations cannot be stable from the energetic point of view (see [PAK96], [PG97] and references therein). For instance, numerical simulations display a decay of algebraic solitons in the NLS-type equations [PAK96] or transformation of algebraic solitons to time-dependent breathers in the mKdV equation [PG97].

While a nonlinear analysis of instability of algebraic solitons has not yet been developed, we offer a simple analytical solution of this problem for the class of integrable nonlinear evolution equations. Using the Lax operators for the integrable equations, we develop the spectral analysis of algebraically decaying potentials that correspond to algebraic solitons. We show that these potentials typically have embedded eigenvalues so that the spectral data are singular across the continuous spectrum. When algebraically decaying potentials are perturbed by variations in the initial data, embedded eigenvalues bifurcate from the continuous spectrum of the Lax operator as isolated eigenvalues. Isolated eigenvalues correspond to proper (exponentially decaying) nonlinear waves of the integrable equations. Therefore, by studying bifurcation equations for shifts of embedded eigenvalues, we predict how algebraic solitons transform in the time evolution of the integrable equations, such as the modified KdV, focusing NLS, derivative NLS, and MTM system (1.1)–(1.4).

In particular, we show that algebraic solitons of the modified KdV equation (1.1) transform to either travelling solitons or time-periodic breathers, depending on the sign of perturbations (see also [PG97]). On the other hand, algebraic homoclinic orbits of the focusing NLS equation (1.2) are destroyed by the perturbation. They transform into time-periodic space-decaying solutions, similar to the transformation of homoclinic space-periodic solutions into both time- and space-periodic solutions [AHS96]. Finally, algebraic solitons of the derivative NLS equation (1.3) transform smoothly to travelling and rotating solitons for either sign of the perturbation. The latter scenario holds also for the MTM equation (1.4) and is explained by the integrability of the MTM equation. In comparison, algebraic solitons are spectrally unstable in nonintegrable generalizations of the MTM equation (see [BPZ98], [KS02]).

We also note that the perturbation theory for exponentially decaying solitons of the nonlinear equations (1.1)–(1.4) is well studied (see [AH90], [CY03], [S02], [WM84]). However, projection formulas of the perturbation theory diverge in the limit when an exponentially decaying potential of the Lax operator transforms to an algebraically decaying potential. Results of our analysis can be used for an improved perturbation theory for algebraically decaying solitons.

The paper is organized as follows. We review the Lax operators for nonlinear evolution equations (1.1)–(1.4) in Section 2. The Evans function for the AKNS spectral problem with nonzero boundary values [AKNS74] is introduced in Section 3. The behavior of the

Evans function for algebraically decaying potentials is studied in Section 4. Section 5 considers bifurcations of embedded eigenvalues for the algebraic solitons (1.5) and (1.6). The Evans function for the KN spectral problem with zero boundary values [KN78] is introduced in Section 6. The behavior of the Evans function and bifurcations of embedded eigenvalues for the algebraic solitons (1.7)–(1.8) are studied in Section 7. Section 8 gives a summary of our main results. Appendix A contains the proofs of three lemmas stated in Section 4. Appendix B presents results from perturbation theory used in Section 5.

2. Lax Operators for Integrable Nonlinear Equations

We consider the standard Lax formulation of the inverse scattering transform [AC91],

$$\psi_x = \mathcal{L}(\lambda; u)\psi, \qquad \psi_t = \mathcal{A}(\lambda; u)\psi, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{2.1}$$

where $\psi \in \mathbb{C}^2$ is an eigenfunction (eigenvector), $\lambda \in \mathbb{C}$ is a spectral parameter (eigenvalue), and $\mathcal{L}(\lambda; u)$ and $\mathcal{A}(\lambda; u)$ are 2-by-2 matrix operators that depend analytically on λ and $u = u(x, t)$. The spectral parameter λ is independent of t if and only if $u(x, t)$ solves the Lax equation that follows from the compatibility of (2.1):

$$\frac{\partial \mathcal{L}}{\partial t} - \frac{\partial \mathcal{A}}{\partial x} + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L} = 0. \tag{2.2}$$

The modified KdV and NLS equations are related to the Ablowitz-Kaup-Newell-Segur (AKNS) spectral problem [AKNS74], given by the operator $\mathcal{L}(\lambda; u)$ in the form

$$\mathcal{L}(\lambda; u) = Q(u) + \lambda J, \tag{2.3}$$

where

$$Q(u) = \begin{bmatrix} 0 & -u \\ \bar{u} & 0 \end{bmatrix}, \qquad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.4}$$

The other operator $\mathcal{A}(\lambda; u)$ in the Lax formalism (2.1) is given for the modified KdV equation (1.1) by

$$\mathcal{A}(\lambda; u) = -4\lambda^3 J - 2\lambda u^2 J - 4\lambda^2 Q(u) - 2\lambda J Q(u_x) - 2Q(u^3) - Q(u_{xx}), \tag{2.5}$$

and for the focusing NLS equation (1.2) by

$$\mathcal{A}(\lambda; u) = -2i\lambda^2 J - i|u|^2 J - 2i\lambda Q(u) - iJQ(u_x). \tag{2.6}$$

The derivative NLS and MTM system are related to the Kaup-Newell (KN) spectral problem [KN78]:

$$\mathcal{L}(\lambda; u) = \lambda Q(u) - i\lambda^2 J, \tag{2.7}$$

where $Q(u)$ and J are given by (2.4). The operator $\mathcal{A}(\lambda; u)$ in the Lax formalism (2.1) is given for the derivative NLS equation (1.3) by

$$\mathcal{A}(\lambda; u) = 2i\lambda^4 J - i\lambda^2 |u|^2 J - 2\lambda^3 Q(u) - i\lambda J Q(u_x) + \lambda |u|^2 Q(u), \tag{2.8}$$

and for the MTM system (1.4) by

$$\mathcal{A}(\lambda; u) = \frac{i}{4\lambda^2} J - 2i|w|^2 J + \frac{1}{2\lambda} Q \left(i \int_x^\infty u \, dx \right), \tag{2.9}$$

where

$$\begin{aligned} v &= \frac{1}{2} u(x,t) \exp\left(-\frac{i}{2} \int_x^\infty |u|^2(x',t)\, dx' \right), \\ w &= \frac{i}{2} \left(\int_x^\infty u(x',t)\, dx' \right) \exp\left(-\frac{i}{2} \int_x^\infty |u|^2(x',t)\, dx' \right). \end{aligned} \tag{2.10}$$

The spectral analysis of Lax operators (2.3) and (2.7) depends on the localization of the potential $u(x)$. When the potential $u(x)$ decays exponentially at infinity, the spectral data include the continuous and discrete spectra [AKNS74]. The continuous spectrum Σ_{con} defines a continuous curve of the spectral parameter λ. The discrete spectrum Σ_{dis} consists of a finite number of isolated eigenvalues of finite multiplicity.

Isolated eigenvalues of the discrete spectrum correspond to zeros of the inverse transmission coefficient $a(\lambda)$ associated with the Jost eigenfunctions of the Lax operator. The inverse transmission coefficient $a(\lambda)$ is defined for $\lambda \in \Sigma_{\text{con}}$. When it is extended analytically into an appropriate domain of $\lambda \in \mathbb{C}$ [AKNS74], one can trace multiple eigenvalues λ and their bifurcations as zeros of $a(\lambda)$, taking account of their multiplicity. It is typical for exponentially decaying potentials that no isolated eigenvalues may come from $\lambda = \infty$ upon a small perturbation of $u(x)$ [AKNS74].

A *branch point* or *edge* bifurcation occurs when new eigenvalues detach from the branch points of $\lambda \in \Sigma_{\text{con}}$. Since the inverse transmission coefficient $a(\lambda)$ typically diverges at the branch points [PKA98], a better characterization of the branch point bifurcation is given by the Evans function $E(\lambda)$ [KS98]. The Evans function $E(\lambda)$ is defined for $\lambda \in \mathbb{C}\backslash\Sigma_{\text{con}}$ as an intersection of unstable and stable manifolds. Zeros of $E(\lambda)$ define the number and location of isolated eigenvalues $\lambda \in \Sigma_{\text{dis}}$, taking account of their multiplicity. By the Gap Lemma [GZ98], [KS98], the Evans function $E(\lambda)$ can be continued across $\lambda \in \Sigma_{\text{con}}$, such that zeros of $E(\lambda)$ at the branch points characterize branch point bifurcations [KS02].

If an exponentially decaying perturbation $\epsilon U(x)$ is added to the exponentially decaying potential $u(x)$, the Evans function $E(\lambda; \epsilon)$ remains analytic in both λ and ϵ. The convergent Taylor series of $E(\lambda; \epsilon)$ in λ and ϵ describes perturbations of isolated eigenvalues and branch point bifurcations as functions of ϵ [KS02]. If the potential $u(x)$ decays algebraically at infinity, the Gap Lemma fails and the Evans function $E(\lambda; \epsilon)$ may have some singularities on $\lambda \in \Sigma_{\text{con}}$ [SS04]. In such cases, the spectral analysis critically depends on the class of algebraically decaying potentials $u(x)$ and the perturbation functions $\epsilon U(x)$ [N86], [K88a], [K88b].

Algebraic solitons (1.5)–(1.8) are explicit examples of the algebraically decaying potential $u(x)$ in the spectral problems (2.3) and (2.7). Although the standard Evans function $E(\lambda) \equiv E(\lambda; u)$ is bounded for the exact algebraic soliton potential $u(x)$ on $\lambda \in \Sigma_{\text{con}}$, a perturbation $\epsilon U(x)$ typically gives rise to singularities of $E(\lambda; \epsilon) \equiv E(\lambda; u + \epsilon U)$ at some points of $\lambda \in \Sigma_{\text{con}}$ as $\epsilon \neq 0$.

In this paper we study algebraic potentials $u(x)$ that exhibit embedded eigenvalues for $\epsilon = 0$ and address the following main question: *How can we modify the Evans* *function* $E(\lambda;\epsilon)$ *so that it becomes a useful tool for the analysis of bifurcations of such embedded eigenvalues when* $\epsilon \neq 0$? Technically, the paper develops rigorous methods of the perturbation theory for Lax operators with algebraically decaying potentials, treated in [PG97] only formally. There, an asymptotic multiscale expansion method was used to study instability of algebraic solitons in the modified KdV equation (1.1), without studies of convergence of asymptotic series and bounds on the error terms. Here, we exploit the Implicit Function Theorem applied to the renormalized Evans function $\hat{E}(\lambda;\epsilon)$ and justify the formal asymptotic results of [PG97]. Our analysis relies on similar results for the Schrödinger spectral problem with algebraically decaying potentials [N86], [K88a], [K88b].

3. Evans Function for the AKNS Spectral Problem

We consider the AKNS spectral problem (2.3), when the potential $u(x)$ satisfies nonzero (normalized) boundary conditions:

$$\lim_{x\to\pm\infty} u(x) = 1. \tag{3.1}$$

Using the transformation $u(x) = 1 + w(x)$, we explicitly rewrite the AKNS spectral problem in the form:

$$\begin{aligned} \psi_1' &= -(1 + w(x))\psi_2 + \lambda\psi_1, \\ \psi_2' &= (1 + \overline{w}(x))\psi_1 - \lambda\psi_2. \end{aligned} \tag{3.2}$$

In the case $w(x) \equiv 0$, two fundamental solutions of the problem (3.2) exist for $\lambda \in \mathbb{C}\backslash\{\pm 1\}$ as follows:

$$\psi(x) = \mathbf{e}_{\pm}(\lambda) e^{\pm\kappa(\lambda)x}, \qquad \mathbf{e}_{\pm}(\lambda) = \begin{bmatrix} \lambda \pm \kappa(\lambda) \\ 1 \end{bmatrix}, \quad \kappa(\lambda) = \sqrt{\lambda^2 - 1}. \tag{3.3}$$

The complex plane is decomposed as $\mathbb{C} = \mathcal{D}_+ \cup \Sigma_{\mathrm{con}} \cup \mathcal{D}_-$, where

$$\Sigma_{\mathrm{con}} = \{\lambda \in \mathbb{C}\colon \ \mathrm{Re}(\kappa(\lambda)) = 0\} = \Gamma_+ \cup \{\lambda \in i\mathbb{R}\} \cup \Gamma_-, \tag{3.4}$$

$$\Gamma_+ = \{\lambda \in \mathbb{R}\colon \ 0 < \lambda \leq 1\}, \qquad \Gamma_- = \{\lambda \in \mathbb{R}\colon \ -1 \leq \lambda < 0\}, \tag{3.5}$$

and

$$\mathcal{D}_{\pm} = \{\lambda \in \mathbb{C}\colon \ \mathrm{Re}(\kappa(\lambda)) \gtrless 0\} = \{\lambda\colon \ \mathrm{Re}(\lambda) \gtrless 0\} \setminus \Gamma_{\pm}. \tag{3.6}$$

The decomposition of $\mathbb{C}$ into domains $\mathcal{D}_+$ and $\mathcal{D}_-$ is shown schematically in Figure 1. There are two branch points at $\lambda = \pm\lambda_b$, $\lambda_b = 1$, where $\kappa'(\pm\lambda_b) = \infty$, and one crossing point at $\lambda = \lambda_c = 0$, where $\kappa'(\lambda_c) = 0$. The stars show isolated eigenvalues of the discrete spectrum $\lambda = \pm\lambda_p \in \Sigma_{\mathrm{dis}}$. Due to the symmetry,

$$\psi_1(x,\lambda) \mapsto \bar{\psi}_2(x,-\bar{\lambda}), \qquad \psi_2(x,\lambda) \mapsto -\bar{\psi}_1(x,-\bar{\lambda}), \tag{3.7}$$

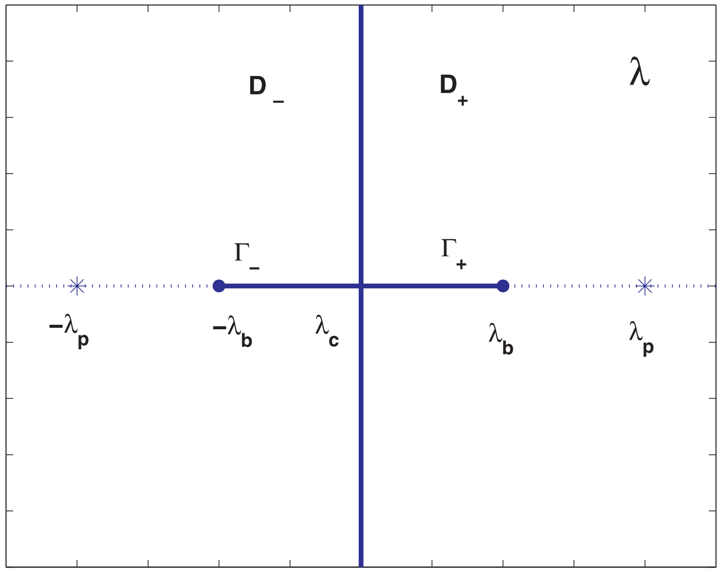

Fig. 1. Decomposition of $\mathbb{C}$ into domains $\mathcal{D}_\pm$ for the AKNS spectral problem (3.2).

isolated eigenvalues $\lambda \in \Sigma_{\text{dis}}$ are symmetric with respect to the line $\text{Re}(\lambda) = 0$. We shall hence consider eigenvalues in the domain $\lambda \in \mathcal{D}_+$ only, where $\text{Re}\,(\kappa(\lambda)) > 0$.

We define the weighted L^1-space as

$$u \in L^1_s(\mathbb{R}): \int_{-\infty}^{\infty} (1 + |x|)^s |u(x)|\, dx < \infty, \tag{3.8}$$

such that $L^1_0 \equiv L^1$. By Proposition 8.1 from Coddington and Levinson [CL55, p. 92] (see also Problem 29 on p. 104), the following result will be used throughout our analysis:

Lemma 3.1. *Let* $\mathbf{a}(x)$ *be a solution of the system*

$$\mathbf{a}'(x) = (A + R(x))\,\mathbf{a}(x), \quad \mathbf{a} \in \mathbb{C}^2, \tag{3.9}$$

where A *is a constant matrix with distinct eigenvalues and* $R(x) \in L^1(\mathbb{R})$. *Then, there exist two sets* $(\mathbf{a}_1^+, \mathbf{a}_2^+)$ *and* $(\mathbf{a}_1^-, \mathbf{a}_2^-)$ *of fundamental solutions of the system (3.9), such that*

$$\lim_{x \to \pm\infty} \mathbf{a}_j^\pm(x)\, e^{-\mu_j x} = \boldsymbol{\alpha}_j, \quad j = 1, 2, \tag{3.10}$$

where (μ_1, μ_2) *and* $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$ *are eigenvalues and eigenvectors of* A, *respectively.*

Since the matrix A in the system (3.2) has two eigenvalues $\mu_{1,2} = \pm\kappa(\lambda) = \pm\sqrt{\lambda^2 - 1}$, which are distinct for $\lambda \in \mathbb{C} \setminus \{\pm 1\}$, the existence of fundamental solutions of the system (3.2) follows by Lemma 3.1.

Proposition 3.2. *Let $w \in L^1(\mathbb{R})$ and $\lambda \in \mathbb{C} \setminus \{\pm 1\}$. There exist two sets of fundamental solutions $\phi^{\pm}(x;\lambda)$ and $\psi^{\pm}(x;\lambda)$ of the AKNS problem (3.2), such that*

$$\lim_{x \to -\infty} \phi^{\pm}(x;\lambda) e^{\mp\kappa(\lambda)x} = \mathbf{e}_{\pm}(\lambda), \qquad \lim_{x \to +\infty} \psi^{\pm}(x;\lambda) e^{\mp\kappa(\lambda)x} = \mathbf{e}_{\pm}(\lambda). \tag{3.11}$$

By linear independence of fundamental solutions, the two sets of solutions in Proposition 3.2 are related for all $\lambda \in \mathbb{C} \setminus \{\pm 1\}$ as follows:

$$\begin{aligned} \phi^{+}(x;\lambda) &= a(\lambda)\psi^{+}(x;\lambda) + b(\lambda)\psi^{-}(x;\lambda), \\ \phi^{-}(x;\lambda) &= c(\lambda)\psi^{+}(x;\lambda) + d(\lambda)\psi^{-}(x;\lambda), \end{aligned} \tag{3.12}$$

where (a, b, c, d) are suitable coefficients. If $\lambda \in \Sigma_{\text{con}}$, the coefficients $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, and $d(\lambda)$ are referred to as scattering coefficients, while $a(\lambda)$ is called *the inverse transmission coefficient*. For $\lambda \in \mathbb{C} \setminus \{\pm 1\}$, these coefficients satisfy the constraint

$$a(\lambda)d(\lambda) - b(\lambda)c(\lambda) = 1, \tag{3.13}$$

such that

$$\begin{aligned} \psi^{+}(x;\lambda) &= d(\lambda)\phi^{+}(x;\lambda) - b(\lambda)\phi^{-}(x;\lambda), \\ \psi^{-}(x;\lambda) &= -c(\lambda)\phi^{+}(x;\lambda) + a(\lambda)\phi^{-}(x;\lambda). \end{aligned} \tag{3.14}$$

Indeed, the Wronskian of two solutions of the problem (3.2) is independent of x, so that the Wronskian of $\phi^{+}(x;\lambda)$ and $\phi^{-}(x;\lambda)$ is evaluated by using the boundary conditions (3.11) and the relations (3.12):

$$\det(\phi^{+}(x;\lambda), \phi^{-}(x;\lambda)) = E_0(\lambda) = E_0(\lambda)\left(a(\lambda)d(\lambda) - b(\lambda)c(\lambda)\right), \tag{3.15}$$

where $E_0(\lambda) = \det(\mathbf{e}_{+}(\lambda), \mathbf{e}_{-}(\lambda)) = 2\kappa(\lambda)$. The discrete spectrum $\lambda \in \Sigma_{\text{dis}}$ of the AKNS problem (3.2) in the domain $\lambda \in \mathcal{D}_{+}$ is defined by the zeros of $E(\lambda)$, where

$$E(\lambda) = \det(\phi^{+}(x;\lambda), \psi^{-}(x;\lambda)) = 2\kappa(\lambda)a(\lambda), \quad \operatorname{Re}(\kappa(\lambda)) > 0, \tag{3.16}$$

which follows from the Wronskians. The function $E(\lambda)$ is referred to as *the Evans function*. When $E(\lambda) = 0$ at $\lambda \in \Sigma_{\text{dis}} \in \mathcal{D}_{+}$, the eigenvector $\psi(x) = c_0\phi^{+}(x;\lambda) = d_0\psi^{-}(x;\lambda)$, where $c_0, d_0 \neq 0$, decays exponentially as $|x| \to \infty$.

A standard analysis shows that the functions $a(\lambda)$ and $E(\lambda)$ are analytic in $\lambda \in \mathcal{D}_{+}$ off the continuous spectrum (see Proposition 2.7 in [KS98]). On the continuous spectrum Σ_{con}, there is a branch cut on $\lambda \in \Gamma_{+} \cup \Gamma_{-}$ between the two branch points $\lambda = \pm 1$. Due to the symmetry, we shall consider only the branch cut Γ_{+} and the branch point $\lambda = 1$. We shall define a two-sheet Riemann surface near $\lambda = 1$ as follows:

$$\operatorname{Re}(\kappa(\lambda)) > 0: \quad -\pi < \arg(\lambda - 1) < \pi, \tag{3.17}$$

$$\operatorname{Re}(\kappa(\lambda)) < 0: \quad \pi < \arg(\lambda - 1) < 3\pi. \tag{3.18}$$

The two sheets (3.17) and (3.18) are connected at $\lambda \in \Gamma_{+}$, where $\arg(\lambda - 1) = \pi$. Let k, $0 \leq k < 1$ be the parameter, such that $\lambda = \sqrt{1 - k^2}$ and $\kappa(\lambda) = ik$ on $\lambda \in \Gamma_{+}$. By Proposition 3.2, the eigenvectors $\phi^{\pm}(x;k)$ and $\psi^{\pm}(x;k)$ of the AKNS problem (3.2) exist for $w \in L^1(\mathbb{R})$ and $\lambda \in \Gamma_{+} \setminus \{1\}$, when $0 < k < 1$. By standard Green's function methods [AC91, p. 106], Volterra integral equations hold for these eigenvectors.

Proposition 3.3. *Let $w \in L^1(\mathbb{R})$ and $0 < k < 1$. There exist two sets of solutions $\phi^\pm(x;k)$ and $\psi^\pm(x;k)$ of the AKNS problem (3.2) with $\lambda = \sqrt{1-k^2}$, such that*

$$\phi^\pm(x;k) = \mathbf{e}_\pm(k)e^{\pm ikx} - \int_{-\infty}^{x} K(x,s;k)\phi^\pm(s;k)\,ds,$$
$$\psi^\pm(x;k) = \mathbf{e}_\pm(k)e^{\pm ikx} + \int_{x}^{\infty} K(x,s;k)\psi^\pm(s;k)\,ds, \tag{3.19}$$

where $\mathbf{e}_\pm(k) = (\sqrt{1-k^2} \pm ik, 1)^T$ and

$$K(x,s;k) = \begin{pmatrix} \bar{w}(s) & w(s)\sqrt{1-k^2} \\ \bar{w}(s)\sqrt{1-k^2} & w(s) \end{pmatrix} \frac{\sin k(x-s)}{k} + \begin{pmatrix} 0 & w(s) \\ -\bar{w}(s) & 0 \end{pmatrix} \cos k(x-s). \tag{3.20}$$

Furthermore, the integral equations also hold for $\phi^+(x;k)$ and $\psi^-(x;k)$ provided that $\mathrm{Im}(k) < 0$, *i.e.* $\lambda \in \mathcal{D}_+$.

The Evans function $E(\lambda)$ is defined on the first sheet (3.17) of the Riemann surface, where $\mathrm{Re}\,(\kappa(\lambda)) > 0$. Across $\lambda \in \Gamma_+$, we fix $\arg(\lambda - 1) = \pi$ and define the function $G(k)$ from the matching condition:

$$G(k) = E(\lambda) = \det\left(\phi^+(x;k), \psi^-(x;k)\right), \quad \lambda = \sqrt{1-k^2}, \quad 0 < k < 1. \tag{3.21}$$

By Proposition 3.3, we have $0 \le |G(k)| < \infty$ for $0 < k < 1$. When $G(k)$ vanishes for some $k = k_0$, $0 < k_0 < 1$, the point $k = k_0$ is referred to as *an embedded resonance*. Embedded resonances will not be studied here, since they are not generic for algebraic potentials. When $G(0)$ exists and $G(0) = 0$, the point $k = 0$ is referred to as *a branch point resonance*. We study the point $k = 0$ in the next section.

4. Singularities of $G(k)$ as $k \to 0$

We focus here on the behavior of the function $G(k)$ as $k \to 0$, depending on the power of the algebraic decay in the potential function $w(x)$:

$$\lim_{x\to\pm\infty} |x|^p w(x) = b_\infty^\pm, \qquad \mathrm{Re}(b_\infty^-) = \mathrm{Re}(b_\infty^+) \equiv b_\infty, \tag{4.1}$$

where $0 < |b_\infty| < \infty$ and $p > 1$. By the statement $k \to 0$, we mean that k approaches the origin along any curve in the closed complex lower k half-plane. Note that this corresponds to λ approaching the point $\lambda = 1$ along any curve in $\mathcal{D}_+$. Algebraically decaying solitons were found in [O76], [KI78a] for $p = 2$ and $b_\infty = -1$. We will be working only on $x > 0$. A similar analysis holds for $x < 0$.

Definition 4.1. If there exists an L^2-eigenvector $\psi(x)$ of the AKNS problem (3.2) for $\lambda \in \Gamma_+\backslash\{1\}$, the point λ is referred to as an embedded eigenvalue. If there exists an

L^2-eigenvector $\psi(x)$ of the AKNS problem (3.2) for $\lambda = 1$, the point $\lambda = 1$ is referred to as a branch point eigenvalue.

By Proposition 3.3, the nondecaying fundamental solutions $\phi^{\pm}(x; k)$ exist for $0 < k < 1$, such that no *embedded eigenvalues* may exist on $\Gamma_+ \backslash \{1\}$. On the other hand, Proposition 3.3 does not rule out a *branch point eigenvalue* at $\lambda = 1$. We show that the branch point eigenvalue may exist only if the algebraically decaying potential $w(x)$ has the decay rate (4.1) with $1 < p \leq 2$ and $b_\infty < 0$.

Lemma 4.2. *Let $w(x)$ satisfy (4.1). The point $\lambda = 1$ is not a branch point eigenvalue of the AKNS problem (3.2) if $p > 2$ or if $p = 2$ and $b_\infty > -\frac{3}{8}$. The point $\lambda = 1$ can be a branch point eigenvalue if $p = 2$ and $b_\infty < -\frac{3}{8}$.*

Proof. Using variables $\varphi_1 = \psi_1 + \psi_2$ and $\varphi_2 = \psi_1 - \psi_2$, we transform the AKNS problem (3.2) with $\lambda = 1$ to the form

$$\begin{aligned} \varphi_1' &= 2\varphi_2 - w_-(x)\varphi_1 + w_+(x)\varphi_2, \\ \varphi_2' &= -w_+(x)\varphi_1 + w_-(x)\varphi_2, \end{aligned} \tag{4.2}$$

where

$$w_\pm(x) = \frac{w(x) \pm \overline{w}(x)}{2}.$$

Let $x \geq x_0 > 0$, and define

$$w(x) = \frac{b(x)}{x^p}, \qquad \varphi_1 = \frac{a_1(x)}{x^q}, \qquad \varphi_2 = \frac{a_2(x)}{x^{q+1}}, \tag{4.3}$$

where $p > 1$ and $q > 0$. The system (4.2) can be rewritten in the form

$$x\frac{d\mathbf{a}}{dx} = (A + R(x))\,\mathbf{a}, \tag{4.4}$$

where $\mathbf{a} = (a_1, a_2)^T$ and

$$A = \begin{pmatrix} q & 2 \\ 0 & q+1 \end{pmatrix}, \qquad R(x) = \frac{1}{2x^{p-2}} \begin{pmatrix} (\bar{b}(x) - b(x))/x & (\bar{b}(x) + b(x))/x^2 \\ -(b(x) + \bar{b}(x)) & (b(x) - \bar{b}(x))/x \end{pmatrix}. \tag{4.5}$$

The system (4.4) can be mapped to the form (3.9) with the transformation $x = e^z$. The eigenvalues of A are distinct, while $\tilde{R}(z) = R(e^z)$ is in $L^1(z \geq z_0)$ for $p > 2$. By Lemma 3.1, there exist two solutions of the system (4.4), such that

$$\lim_{x\to+\infty} \mathbf{a}_1(x)x^{-q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \lim_{x\to+\infty} \mathbf{a}_2(x)x^{-q-1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

These two solutions give nondecaying eigenvectors of the system (4.2) as $x \to +\infty$:

$$\boldsymbol{\varphi}_1(x) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{\varphi}_2(x) \to \begin{pmatrix} 2x \\ 1 \end{pmatrix}. \tag{4.6}$$

Therefore, no decaying bound state $\psi(x)$ of the AKNS problem (3.2) exists for $p > 2$. When $p = 2$, the matrices A and $R(x)$ can be reshuffled as follows:

$$\hat{A} = \begin{pmatrix} q & 2 \\ -b_\infty & q+1 \end{pmatrix}, \tag{4.7}$$

$$\hat{R}(x) = \frac{1}{2x} \begin{pmatrix} (\bar{b}(x) - b(x)) & (\bar{b}(x) + b(x))/x \\ x(2b_\infty - b(x) - \bar{b}(x)) & (b(x) - \bar{b}(x)) \end{pmatrix}. \tag{4.8}$$

It is clear that $x(2b_\infty - b(x) - \bar{b}(x)) = \mathrm{o}(x)$ as $x \to +\infty$. Then, conditions of Lemma 3.1 are satisfied again and there exist two solutions of the problem (4.4), such that

$$\lim_{x\to+\infty} \mathbf{a}_1(x)x^{-(q+\alpha_+)} = \begin{pmatrix} 2 \\ \alpha_+ \end{pmatrix}, \qquad \lim_{x\to+\infty} \mathbf{a}_2(x)x^{-(q-\alpha_-)} = \begin{pmatrix} 2 \\ -\alpha_- \end{pmatrix},$$

where

$$\alpha_\pm = \frac{\sqrt{1-8b_\infty} \pm 1}{2}.$$

These two solutions correspond to eigenvectors of the system (4.2) as $x \to +\infty$:

$$\varphi_1(x) \to \begin{pmatrix} 2x^{\alpha_+} \\ \alpha_+ x^{\alpha_-} \end{pmatrix}, \qquad \varphi_2(x) \to \begin{pmatrix} 2x^{-\alpha_-} \\ -\alpha_- x^{-\alpha_+} \end{pmatrix}. \tag{4.9}$$

The decaying bound state $\psi(x)$ of the AKNS problem (3.2) may exist only for $b_\infty < 0$, when $\mathrm{sign}(\alpha_+) = \mathrm{sign}(\alpha_-)$. When it exists, the decaying bound state $\psi^{(0)}(x)$ has the asymptotic form as $x \to +\infty$:

$$\psi^{(0)}(x) \to \frac{1}{x^q} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1 + \mathrm{o}(1)) - \frac{q}{2x^{q+1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (1 + \mathrm{o}(1)), \tag{4.10}$$

where $q = \alpha_-$ and $q + 1 = \alpha_+$. The bound state $\psi^{(0)}(x)$ belongs to $L^2(\mathbb{R})$, when $q > \frac{1}{2}$, which is equivalent to $b_\infty < -\frac{3}{8}$. □

Lemma 4.3. *Let $w(x)$ satisfy (4.1) and $w^{(k)}(x) = \mathrm{O}(|x|^{-p-k})$ as $x \to \pm\infty$ for $k = 1, 2, 3, 4$. The point $\lambda = 1$ can be a branch point eigenvalue of the AKNS problem (3.2) if $1 < p < 2$ and $b_\infty < 0$.*

Proof. When $p < 2$, the method of Lemma 4.2 is not applicable, since the matrix $R(x)$ in (4.5) diverges as $x \to +\infty$. We use an equivalent transformation of the system (4.2):

$$\sqrt{w_+}\varphi_1 = -\phi'(x) - \frac{w_+'\phi(x)}{2w_+} + w_-\phi(x), \quad \varphi_2 = \sqrt{w_+}\phi(x). \tag{4.11}$$

Then the system (4.2) reduces to a scalar problem for $\phi(x)$:

$$\phi'' + Q(x)\phi = 0, \tag{4.12}$$

where

$$Q(x) = w_+(2 + w_+) + \frac{w_+''}{2w_+} - \frac{3}{4}\left(\frac{w_+'}{w_+}\right)^2 - w_-' - w_-^2 + \frac{w_+' w_-}{w_+}. \qquad (4.13)$$

When $1 < p < 2$ and under assumptions on $w(x)$, the dominant term in the potential $Q(x)$ as $x \to +\infty$ is $Q(x) \to 2b_\infty/x^p$. Therefore, the potential $Q(x)$ satisfies the conditions

$$Q^{1/2}(x) \notin L^1(\mathbb{R}), \qquad Q^{-1/4}(x)\left[Q^{-1/4}(x)\right]'' \in L^1(\mathbb{R}). \qquad (4.14)$$

By Theorem 2.5.1 in [E89], there exist two solutions of the problem (4.12) with the limits as $x \to +\infty$:

$$\phi(x) \to Q^{-1/4}(x) \exp\left(\pm \int_{x_0}^{x} \sqrt{-Q(x')}\, dx'\right). \qquad (4.15)$$

When $b_\infty < 0$, $-Q(x)$ is positive for large $x \geq x_0 > 0$ and the exponential factor in (4.15) diverges. In this case, the decaying bound state $\boldsymbol{\psi}(x)$ of the AKNS problem (3.2) may exist. When it exists, the decaying bound state $\boldsymbol{\psi}^{(0)}(x)$ has the exponential factor in the asymptotic form as $x \to +\infty$,

$$\boldsymbol{\psi}^{(0)}(x) \to x^{p/4} e^{-ax^r}\left[\begin{pmatrix}1\\1\end{pmatrix}(1 + \mathrm{o}(1)) - \frac{ar}{2x^{p/2}}\begin{pmatrix}1\\-1\end{pmatrix}(1 + \mathrm{o}(1))\right], \qquad (4.16)$$

where

$$r = 1 - \frac{p}{2}, \qquad a = \frac{\sqrt{2|b_\infty|}}{r}.$$

When $b_\infty > 0$, the exponential factor in (4.15) oscillates and no decaying bound states exist. □

Lemma 4.4. *Let $w(x)$ satisfy (4.1) with $p = 2$ and $b_\infty < -\frac{3}{8}$. Let $\lambda = 1$ be a branch point eigenvalue of the AKNS problem (3.2). The geometric multiplicity of $\lambda = 1$ is one and the algebraic multiplicity of $\lambda = 1$ is finite.*

Proof. Only one decaying bound state $\boldsymbol{\psi}^{(0)}(x)$ may exist for $\lambda = 1$, since the other, linearly independent solution of the problem (3.2) grows in x, such that the Wronskian of the two fundamental solutions is a nonzero constant. If $\lambda = 1$ is a multiple eigenvalue, there exist bound states $\boldsymbol{\psi}^{(n)}(x)$ with $n > 0$ of the generalized problem,

$$\begin{aligned}
\psi_1^{(n)\prime} &= -(1 + w(x))\psi_2^{(n)} + \psi_1^{(n)} + \psi_1^{(n-1)}(x),\\
\psi_2^{(n)\prime} &= (1 + \bar{w}(x))\psi_1^{(n)} - \psi_2^{(n)} - \psi_2^{(n-1)}(x),
\end{aligned} \qquad (4.17)$$

where $n = 1, \ldots, N$. We show that $N < \infty$. The decaying bound state $\boldsymbol{\psi}^{(0)}(x)$ has the limiting behavior (4.10). It follows from the balance of algebraically decaying terms in the problem (4.17) that the nonhomogeneous solutions for $\boldsymbol{\psi}^{(n)}(x)$ with $n > 0$ has the asymptotic form, as $x \to +\infty$,

$$\boldsymbol{\psi}^{(n)}(x) \to \frac{c_n}{x^{q-2n}}\begin{pmatrix}1\\1\end{pmatrix}(1 + \mathrm{o}(1)) + \frac{(2n - q)c_n}{2x^{q-2n+1}}\begin{pmatrix}1\\-1\end{pmatrix}(1 + \mathrm{o}(1)), \qquad (4.18)$$

where c_n satisfies the recurrence relation:

$$c_n = -\frac{c_{n-1}}{n(2q-2n+1)}, \qquad c_0 = 1, \quad n \geq 1, \tag{4.19}$$

and we have used that $q(q+1) + 2b_\infty = 0$. The generalized eigenfunctions $\psi^{(n)}(x)$ decay algebraically when $q > 2n$. Under this condition, the recurrence relations (4.19) are nonsingular for any n. Therefore, given any $q < \infty$, there exists $N < \infty$, such that solutions of (4.17) with $n = N + 1$ become nondecaying as $x \to +\infty$. □

We now consider the behavior of $G(k)$ as $k \to 0$ ($\mathrm{Im}(k) \leq 0$), depending on the power $p > 1$ of the algebraically decaying potential $w(x)$ in (4.1). When $p > 3$, we prove that $G(k)$ is continuously differentiable as $k \to 0$ in $\overline{\mathbb{C}^-}$. When $2 < p < 3$, we prove that $G(k)$ is continuous as $k \to 0$, but $G'(k)$ has a power singularity as $k \to 0$ in $\overline{\mathbb{C}^-}$. When $p = 2$, the point $\lambda = 1$ can be an embedded eigenvalue and $G(0)$ does not generally exist. We prove however that $G(k)$ can be replaced by a function $\hat{G}(k)$, which is continuous at $k = 0$. We remark that it suffices to prove these results for real positive k as $k \to 0^+$. This follows from the fact that for $k \to 0^-$ the proofs are essentially the same so that we can appeal to a Phragmén-Lindelöf theorem (see p. 237 in [E66]) and conclude that the results hold when $k \to 0$ from the lower half-plane.

Lemma 4.5. *The function $G(k)$ is continuous at $k = 0$ if $w(x) \in L^1_1(\mathbb{R})$ and is continuously differentiable at $k = 0$ if $w(x) \in L^1_2(\mathbb{R})$.*

Proof. Neumann series expansions for the nonhomogeneous Volterra equations (3.19) with the kernel (3.20) converge to a unique solution if $w(x) \in L^1_1(\mathbb{R})$ [N86]. The solutions $\phi^\pm(x; k)$ and $\psi^\pm(x; k)$ are continuous functions of the parameter k. Therefore, the function $G(k)$, defined in (3.21), is a continuous function of k, including the limit $k \to 0^+$. Similarly, the Neumann series expansions for the k-derivative of the solutions in (3.19) converge if $w(x) \in L^1_2(\mathbb{R})$, such that $G'(k)$ is continuous at $k = 0$. □

Lemma 4.6. *Let $w(x)$ satisfy (4.1) with $2 < p < 3$. The function $G(k)$ has the leading-order behavior*

$$G(k) = G(0)\left(1 + \alpha b_\infty k^{p-2}\right) + \mathrm{o}(k^{p-2}), \tag{4.20}$$

where $\alpha = 2^p\, \Gamma(1-p)\, e^{\frac{\pi i(p-2)}{2}}$.

Lemma 4.7. *Let $w(x)$ satisfy (4.1) with $p > 2$. Let $\lambda = 1$ be a branch point resonance such that $G(0) = 0$ and $\psi^-(x; 0) = \gamma\phi^+(x; 0)$, $\gamma \neq 0$. The function $G(k)$ has the leading-order behavior*

$$G(k) = ik\frac{1+\gamma^2}{\gamma} + \mathrm{o}(k). \tag{4.21}$$

The following Schrödinger equation plays a role in the next lemma:

$$-\chi'' + U(x)\chi = k^2\chi, \tag{4.22}$$

where $U(x) = -2w(x) - w^2(x) + iw'(x)$ and $w(x)$ is assumed to be real-valued. We shall prove that for $k = 0$, this equation has two solutions, $F^+(x)$ and $F^-(x)$, defined by

$$F^{\pm}(x) = \frac{\hat{\psi}_1^{\pm}(x) + i\hat{\psi}_2^{\pm}(x)}{(1+i)}, \tag{4.23}$$

where

$$\hat{\psi}_{1,2}^{+} = \lim_{k\to 0} k^q \psi_{1,2}^{-}(x;k), \qquad \hat{\psi}_{1,2}^{-} = \lim_{k\to 0} k^q \phi_{1,2}^{+}(x;k),$$

q is a positive root of $q(q+1) = 2|b_\infty|$ and the solutions $\boldsymbol{\phi}^+(x;k)$ and $\boldsymbol{\psi}^-(x;k)$ are defined in Proposition 3.3. It follows that the vector $(\hat{\psi}_1^{\pm}, \hat{\psi}_2^{\pm})^T$ solves the AKNS system (3.2) with $\lambda = 1$.

Lemma 4.8. *Let $w(x)$ be real, satisfy (4.1) with $p = 2$ and $b_\infty < -\frac{3}{8}$, and $w'(x) \in L_1^1(\mathbb{R})$. Let q be the positive root of $q(q+1) = 2|b_\infty|$, so that $q > \frac{1}{2}$. Then, the renormalized function $\hat{G}(k) = k^{2q}G(k)$ is continuous at $k = 0$ and has the leading-order behavior*

$$\hat{G}(k) = \alpha_0 + \mathrm{o}(1), \qquad \alpha_0 = W[F^+, F^-], \tag{4.24}$$

where $W[F^+, F^-]$ is the Wronskian of $F^+(x)$ and $F^-(x)$, defined in (4.22)–(4.23). Moreover, $\lambda = 1$ is a branch point eigenvalue if and only if $\alpha_0 = 0$, in which case there exists $\gamma \neq 0$ such that $F^-(x) = \gamma F^+(x)$, while $\hat{G}(k)$ has the leading-order behavior

$$\hat{G}(k) = \alpha_2 k^2 + \mathrm{o}(k^2), \qquad \alpha_2 = -\int_{-\infty}^{\infty} F^+(x)F^-(x)\,dx. \tag{4.25}$$

Proofs of Lemmas 4.6–4.8 are given in Appendix A. When $p = 2$ and $b_\infty < -\frac{3}{8}$, the branch point bifurcations are studied in Section 5 with the renormalized function $\hat{G}(k)$. We note that the coefficient α_2 in (4.25) can be zero, because $F^+(x)$ need not be real-valued. We also note that the statement of Lemma 4.8 can be extended to the case of complex-valued potentials $w(x)$, but the proof becomes fairly long.

5. Examples of Branch Point Bifurcations

We consider two families of exponentially decaying potentials:

$$w = w_p(x) = \frac{2\sigma p^2}{\sqrt{1+p^2}\cosh(2px) + \sigma}, \tag{5.1}$$

where $\sigma = \pm 1$ and $p > 0$. The two families correspond to *the solitons* of the modified KdV equation (1.1) [PG97] and to *the time-periodic orbits* of the focusing NLS equation (1.2) [AH90]. (Homoclinic orbits in [AH90] become the time-periodic orbits (5.1) after parameters of solutions are extended to complex values.) The fundamental solutions of the AKNS problem (3.2) with the potentials (5.1) take the explicit form (see [PG97])

$$\phi^+(x;\lambda) = \frac{e^{\kappa(\lambda)x}}{\kappa(\lambda)+p}\left[\left(\kappa(\lambda) - \frac{\sigma\sqrt{1+p^2}}{2p}\sinh(2px)w_p(x)\right)\mathbf{e}_+(\lambda) - \frac{1}{2}w_p(x)\boldsymbol{\xi}_+(\lambda)\right], \tag{5.2}$$

$$\psi^-(x;\lambda) = \frac{e^{-\kappa(\lambda)x}}{\kappa(\lambda)+p}\left[\left(\kappa(\lambda) + \frac{\sigma\sqrt{1+p^2}}{2p}\sinh(2px)w_p(x)\right)\mathbf{e}_-(\lambda) + \frac{1}{2}w_p(x)\boldsymbol{\xi}_-(\lambda)\right], \tag{5.3}$$

where

$$\mathbf{e}_\pm(\lambda) = \begin{bmatrix}\lambda \pm \kappa(\lambda) \\ 1\end{bmatrix}, \qquad \boldsymbol{\xi}_\pm(\lambda) = \begin{bmatrix}1 \\ -\lambda \mp \kappa(\lambda)\end{bmatrix}, \tag{5.4}$$

and $\kappa(\lambda) = \sqrt{\lambda^2 - 1}$. The inverse transmission coefficient $a(\lambda)$ and the Evans function $E(\lambda)$ can be found from (3.12), (3.16) and (5.2)–(5.3) as follows:

$$a(\lambda) = \frac{\kappa(\lambda)-p}{\kappa(\lambda)+p}, \qquad E(\lambda) = 2\kappa(\lambda)\frac{\kappa(\lambda)-p}{\kappa(\lambda)+p}. \tag{5.5}$$

The Evans function $E(\lambda)$ has a simple zero in $\mathcal{D}_+$ at $\lambda = \lambda_p = \sqrt{1+p^2}$, where *a simple eigenvalue* resides. It has also a zero at the branch point $\lambda = \lambda_b = 1$, where *the branch point resonance* resides, according to Lemma 4.7 ($\gamma = -1$ in (4.21)).

We consider deformations of the exponentially decaying potential $w_p(x)$ of the form $w_\epsilon(x) = w_p(x) + \epsilon W(x)$, where ϵ is a small real parameter and $W(x) = W_r(x) + iW_i(x)$ is a smooth, exponentially decaying function on $x \in \mathbb{R}$. The structure of the perturbed Evans function $E(\lambda;\epsilon) = E(\lambda; w_\epsilon)$ and the changes in the discrete spectrum of the AKNS problem (3.2) are described in the following two propositions.

Proposition 5.1. *Let* $E(\lambda) \equiv E(\lambda; 0)$ *be given by (5.5) such that* $E(\lambda_p) = 0$. *There exist* $\epsilon_0 > 0$ *and* $C > 0$ *such that the function* $E(\lambda;\epsilon)$ *has a simple zero at* $\lambda = \lambda_p(\epsilon)$ *in* $\mathcal{D}_+$, *where* $0 \le |\lambda_p(\epsilon) - \lambda_p| \le C\epsilon_0$ *and* $0 \le |\epsilon| \le \epsilon_0$. *The leading-order behavior for* $\lambda_p(\epsilon)$ *is given explicitly as*

$$\lambda_p(\epsilon) = \lambda_p\left(1 + \frac{\epsilon}{2p}\int_{-\infty}^{\infty} w_p(x)W_{\rm r}(x)\,dx\right) + \frac{i\epsilon}{4p}\int_{-\infty}^{\infty} w_p(x)W_{\rm i}'(x)\,dx + {\rm o}(\epsilon). \tag{5.6}$$

Proof. By Lemmas B.1 and B.3 of Appendix B, the function $E(\lambda;\epsilon)$ is continuously differentiable near $\lambda = \lambda_p$ and $\epsilon = 0$ whenever $w_\epsilon \in L^1(\mathbb{R})$, such that

$$E(\lambda;\epsilon) = \frac{\partial E}{\partial\lambda}(\lambda_p; 0)(\lambda - \lambda_p) + \frac{\partial E}{\partial\epsilon}(\lambda_p; 0)\epsilon + E_R(\lambda;\epsilon), \tag{5.7}$$

where $E_R(\lambda;\epsilon)$ is the remainder term. The root $\lambda = \lambda_p(\epsilon)$ of the equation $E(\lambda;\epsilon) = 0$ has the leading-order behavior

$$\lambda_p(\epsilon) - \lambda_p = -\frac{\frac{\partial E}{\partial \epsilon}(\lambda_p; 0)}{\frac{\partial E}{\partial \lambda}(\lambda_p; 0)}\epsilon + \mathrm{o}(\epsilon). \tag{5.8}$$

Using (B.9) and (B.10) of Appendix B, we compute explicitly that

$$\frac{\partial E}{\partial \lambda}(\lambda_p; 0) = \frac{\sqrt{1+p^2}}{p}, \tag{5.9}$$

and

$$\begin{aligned} \frac{\partial E}{\partial \epsilon}(\lambda_p; 0) \;=\; & -\frac{1+p^2}{2p^2}\int_{-\infty}^{\infty} w_p(x) W_{\mathrm{r}}(x)\,dx \\ & -\frac{i\sqrt{1+p^2}}{4p^2}\int_{-\infty}^{\infty} w_p(x) W_{\mathrm{i}}'(x)\,dx. \end{aligned} \tag{5.10}$$

The leading-order behavior (5.6) for $\lambda_p(\epsilon)$ follows from (5.8)–(5.10). □

Proposition 5.2. *Let $E(\lambda) \equiv E(\lambda; 0)$ be given by (5.5) such that $E(\lambda_b) = 0$. There exist $\epsilon_0 > 0$ and $C > 0$ such that the function $E(\lambda;\epsilon)$ has a simple zero at $\lambda = \lambda_b(\epsilon)$ in $\mathcal{D}_+$, where $0 < |\lambda_b(\epsilon) - \lambda_b| \le C\epsilon_0^2$ for $0 < |\epsilon| \le \epsilon_0$, if*

$$-\frac{\pi}{2} < \arg\left(\epsilon\int_{-\infty}^{\infty}\left[\left(p^2 - w_p(x)\right)W_{\mathrm{r}}(x) - \frac{i}{2}w_p(x)W_{\mathrm{i}}'(x)\right]dx\right) < \frac{\pi}{2}. \tag{5.11}$$

The leading-order behavior for $\lambda_b(\epsilon) = \sqrt{1+\kappa_b^2(\epsilon)}$ is given explicitly by

$$\kappa_b(\epsilon) = \frac{\epsilon}{p^2}\int_{-\infty}^{\infty}\left(p^2 - w_p(x)\right)W_{\mathrm{r}}(x)\,dx - \frac{i\epsilon}{2p^2}\int_{-\infty}^{\infty} w_p(x)W_{\mathrm{i}}'(x)\,dx + \mathrm{o}(\epsilon). \tag{5.12}$$

Proof. The Evans function $E(\lambda;\epsilon)$ is continued across the line segment $\lambda \in \Gamma_+$ by the function $G(k,\epsilon)$ in (3.21). Eigenvalues in $\mathcal{D}_+$ correspond to zeros of $G(\kappa;\epsilon)$ on the first sheet of the Riemann surface (3.17):

$$G(\kappa;\epsilon) = E(\sqrt{1+\kappa^2};\epsilon), \quad -\frac{\pi}{2} < \arg(\kappa) < \frac{\pi}{2}. \tag{5.13}$$

Using (5.5), we have an explicit expression for $G(\kappa; 0)$,

$$G(\kappa; 0) = 2\kappa\frac{\kappa - p}{\kappa + p}, \tag{5.14}$$

which shows a simple zero at $\kappa = 0$. By Lemma 4.5, the function $G(\kappa;\epsilon)$ is continuously differentiable in κ near $\kappa = 0$ whenever $w_\epsilon \in L_2^1(\mathbb{R})$. By Corollary B.4 of Appendix B, the function $G(\kappa;\epsilon)$ is entire in ϵ near $\epsilon = 0$ whenever $w_\epsilon \in L^1_1(\mathbb{R})$. By the Implicit Function Theorem, the zero $\lambda_b(\epsilon) = \sqrt{1+\kappa_b^2(\epsilon)}$ has the leading-order behavior:

$$\kappa_b(\epsilon) = -\frac{\frac{\partial G}{\partial \epsilon}(0;0)}{\frac{\partial G}{\partial \kappa}(0;0)}\epsilon + \mathrm{o}(\epsilon), \qquad -\frac{\pi}{2} < \arg(\kappa_b(\epsilon)) < \frac{\pi}{2}. \tag{5.15}$$

We compute explicitly from (5.14) and (B.10) of Appendix B that

$$\frac{\partial G}{\partial \kappa}(0,0) = -2, \tag{5.16}$$

and

$$\frac{\partial G}{\partial \epsilon}(0,0) = \frac{2}{p^2}\int_{-\infty}^{\infty}\left(p^2 - w_p(x)\right) W_{\mathrm{r}}(x)\,dx - \frac{i}{p^2}\int_{-\infty}^{\infty} w_p(x) W_{\mathrm{i}}'(x)\,dx. \tag{5.17}$$

Under the constraint (5.11), the zero of $G(\kappa,0)$ shifts to the first sheet of the Riemann surface, such that the branch point resonance becomes a simple isolated eigenvalue $\lambda_b(\epsilon)$, with the leading-order behavior given by (5.12). ☐

The family of exponentially decaying potentials (5.1) with $\sigma = -1$ converges as $p \to 0$ to the algebraically decaying potential,

$$w_0(x) = -\frac{4}{1+4x^2}. \tag{5.18}$$

The special solution (5.18) corresponds to *the algebraic soliton* in the modified KdV equation (1.1) [PG97] and to *the algebraic time-homoclinic orbit* in the focusing NLS equation (1.2) [EKK86]. The two fundamental solutions of the AKNS problem follow from (5.2)–(5.3) in the limit $p \to 0$ for $\sigma = -1$:

$$\boldsymbol{\phi}^+(x;\lambda) = \frac{1}{\kappa(\lambda)}e^{\kappa(\lambda)x}\left[(\kappa(\lambda) + x w_0(x))\,\mathbf{e}_+(\lambda) - \frac{1}{2}w_0(x)\boldsymbol{\xi}_+(\lambda)\right], \tag{5.19}$$

$$\boldsymbol{\psi}^-(x;\lambda) = \frac{1}{\kappa(\lambda)}e^{-\kappa(\lambda)x}\left[(\kappa(\lambda) - x w_0(x))\,\mathbf{e}_-(\lambda) + \frac{1}{2}w_0(x)\boldsymbol{\xi}_-(\lambda)\right]. \tag{5.20}$$

The fundamental solutions $\boldsymbol{\phi}^+(x;\lambda)$ and $\boldsymbol{\psi}^-(x;\lambda)$ diverge at the branch point $\lambda = 1$. These singularities indicate that $\lambda = 1$ is a branch point eigenvalue with bound state $\boldsymbol{\psi}^{(0)}(x)$ given by

$$\boldsymbol{\psi}^{(0)}(x) = \begin{pmatrix} 2x-1 \\ 2x+1 \end{pmatrix} w_0(x). \tag{5.21}$$

The bound state $\boldsymbol{\psi}^{(0)}(x)$ decays algebraically and has $q = 1$ in (4.10). These results are in agreement with Lemmas 4.2 and 4.4, since the potential $w_0(x)$ has the algebraic decay (4.1) with $p = 2$ and $b_\infty = -1$. The inverse transmission coefficient $a(\lambda)$ and the Evans function $E(\lambda)$ for the algebraically decaying potential $w_0(x)$ take the explicit form

$$a(\lambda) = 1, \qquad E(\lambda) = 2\kappa(\lambda). \tag{5.22}$$

We regularize the Evans function $E(\lambda; \epsilon)$ according to Lemma 4.8:

$$\hat{E}(\lambda; \epsilon) = \kappa^2(\lambda) E(\lambda; \epsilon). \tag{5.23}$$

By using the correspondence (4.23), we find that

$$F^{\pm}(x) = \frac{\pm 2}{1 + 2ix},$$

such that $\gamma = -1$ and $\alpha_2 = 0$ in the integral (4.25). The renormalized Evans function $\hat{E}(\lambda; \epsilon)$ has a triple zero in the variable $\sqrt{\lambda - 1}$ at $\lambda = 1$ and $\epsilon = 0$. The triple zero defines the algebraic structure of the branch point resonance and eigenvalue. The structure of the perturbed function $\hat{E}(\lambda; \epsilon) = E(\lambda; w_\epsilon)$ and the changes in the discrete spectrum of the AKNS problem (3.2) are described in the following proposition.

Proposition 5.3. *Let $\hat{E}(\lambda) = \hat{E}(\lambda; 0)$ be given by (5.22) and (5.23), such that $\hat{E}(\lambda_b) = 0$. Let $W(x)$ be real and $W(x), W'(x) \in L^1_2(\mathbb{R})$. If*

$$\epsilon \int_{-\infty}^{\infty} w_0(x) W(x)\, dx > 0, \tag{5.24}$$

there exist $\epsilon_0 > 0$ and $C > 0$ such that the function $\hat{E}(\lambda; \epsilon)$ has a simple zero at $\lambda = \lambda_b(\epsilon)$ in $\mathcal{D}_+$, where $0 < |\lambda_b(\epsilon) - \lambda_b| \le C\epsilon_0^{2/3}$ and $0 < |\epsilon| \le \epsilon_0$. If

$$\epsilon \int_{-\infty}^{\infty} w_0(x) W(x)\, dx < 0, \tag{5.25}$$

there exist $\epsilon_0 > 0$ and $C > 0$ such that the function $\hat{E}(\lambda; \epsilon)$ has a pair of simple zeros at $\lambda = \lambda_{c1}(\epsilon)$ and $\lambda = \lambda_{c2}(\epsilon)$ in $\mathcal{D}_+$, where $0 < |\lambda_{c1,2}(\epsilon) - \lambda_b| \le C\epsilon_0^{2/3}$ and $0 < |\epsilon| \le \epsilon_0$.

Proof. We need to study the equation

$$\hat{G}(\kappa; \epsilon) = 0,$$

which we write as

$$\hat{G}(\kappa; \epsilon) = \hat{G}(\kappa; 0) + \frac{\partial \hat{G}}{\partial \epsilon}(0; 0)\epsilon + \left(\frac{\partial \hat{G}}{\partial \epsilon}(\kappa; 0) - \frac{\partial \hat{G}}{\partial \epsilon}(0; 0) \right) \epsilon + \epsilon^2 \hat{R}(\kappa; \epsilon), \tag{5.26}$$

thereby defining the remainder term $\hat{R}(\kappa; \epsilon)$. Note that $\hat{G}(\kappa; \epsilon)$ is an entire function of ϵ (see the proof of Lemma B.3 for a justification) so that $\epsilon^2 \hat{R}(\kappa; \epsilon)$ comprises all the terms whose orders are quadratic or higher in ϵ. Here

$$\hat{G}(\kappa; 0) = 2\kappa^3, \tag{5.27}$$

and

$$\frac{\partial \hat{G}}{\partial \epsilon}(0; 0) = -2 \int_{-\infty}^{\infty} w_0(x) W(x)\, dx. \tag{5.28}$$

We seek solutions of the form

$$\kappa(\epsilon) = \kappa_b(\epsilon) = \alpha^{1/3}\epsilon^{1/3}(1+\eta)^{1/3}, \qquad \alpha = \int_{-\infty}^{\infty} w_0(x) W(x)\, dx, \tag{5.29}$$

and we show below that $\eta = \eta(\epsilon) \to 0$ as $\epsilon \to 0$. There are three complex values of $\alpha^{1/3}$. The following reasoning applies to each choice of value separately. Inserting the ansatz (5.29) in (5.26) gives

$$2\alpha\eta = -\left(\frac{\partial \hat{G}}{\partial \epsilon}(\alpha^{1/3}\epsilon^{1/3}(1+\eta)^{1/3};0) - \frac{\partial \hat{G}}{\partial \epsilon}(0;0)\right) - \epsilon \hat{R}(\alpha^{1/3}\epsilon^{1/3}(1+\eta)^{1/3};\epsilon). \tag{5.30}$$

By Lemma B.6, the derivative with respect to η of the right-hand side goes to zero as $\epsilon \to 0$; note that $\partial \hat{R}(\kappa;\epsilon)/\partial\kappa = o(1/\kappa)$ in view of Lemma B.6 and (5.26). Therefore by the Implicit Function Theorem (or a direct contraction mapping argument) the equation (5.30) has a unique solution $\eta(\epsilon)$ such that $\eta(\epsilon) \to 0$ as $\epsilon \to 0$. When the solutions $\kappa(\epsilon)$ exist in $-\frac{\pi}{2} < \arg(\kappa(\epsilon)) < \frac{\pi}{2}$, they correspond to eigenvalues of the problem (3.2). Under the constraint (5.24), only one zero corresponds to a simple isolated eigenvalue $\lambda_b(\epsilon)$ in $\mathcal{D}_+$. Under the constraint (5.25), two zeros correspond to a pair of simple isolated eigenvalues $\lambda_{c1}(\epsilon)$ and $\lambda_{c2}(\epsilon)$ in $\mathcal{D}_+$. □

In applications to the modified KdV equation (1.1), the leading-order behavior (5.6) with $W_r(x) \equiv W(x)$ and $W_i(x) = 0$ was obtained in [PG97] by means of a formal perturbation series expansion. The real-valued eigenvalue $\lambda_p(\epsilon)$ increases under the perturbation $\epsilon W(x)$ if $\delta P = \epsilon \int_{-\infty}^{\infty} w_p(x) W(x)\, dx > 0$ and decreases if $\delta P < 0$. The correction term δP is related to the first variation of the momentum $P = \int_{-\infty}^{\infty} (u-1)^2\, dx$ of the modified KdV equation (1.1), evaluated at $u = 1 + w_p(x) + \epsilon W(x)$.

The branch point bifurcation, given in Proposition 5.2, was not studied previously. Numerical results clearly display (see Fig. 1 in [PG97]) the second real-valued eigenvalue $\lambda_b(\epsilon)$ bifurcating under the perturbation $\epsilon W(x)$. Since $p^2 - w_p(x) > 0$ for either sign $\sigma = \pm 1$, it follows from (5.11) that branch point bifurcation always takes place for $\epsilon W(x) > 0$, when $\arg(\kappa_b(\epsilon)) = 0$. The new eigenvalue $\lambda_b(\epsilon)$ moves to larger positive values of λ in the domain $\mathcal{D}_+$. When $\sigma = -1$ and $\delta P < 0$, the first eigenvalue $\lambda_p(\epsilon)$ shifts to smaller positive values of λ in the domain $\mathcal{D}_+$. As ϵ increases, the two eigenvalues $\lambda_p(\epsilon)$ and $\lambda_b(\epsilon)$ coalesce and then split into a complex pair of eigenvalues in $\mathcal{D}_+$ (see Fig. 1 in [PG97]). The branch point bifurcation does not take place for $\epsilon W(x) < 0$, when $\arg(\kappa_b(\epsilon)) = \pi$.

The branch point bifurcation, given in Proposition 5.3, was analyzed in [PG97], where the cubic equation (5.29) was found with a heuristic asymptotic method. We note that the number of eigenvalues of the discrete spectrum Σ_{dis} changes as a result of the branch point bifurcation. In the case of (5.24), when $\arg(\kappa_b(\epsilon)) = 0$, the discrete spectrum has only one real-valued eigenvalue $\lambda_b(\epsilon)$ in $\mathcal{D}_+$, while in the case of (5.25), when $\arg(\kappa_b(\epsilon)) = \pm\pi/3$, it has two complex-valued eigenvalues, $\lambda_{c1} = \lambda_c(\epsilon)$ and $\lambda_{c2} = \bar{\lambda}_c(\epsilon)$ in $\mathcal{D}_+$. In the marginal case $\int_{-\infty}^{\infty} w_0(x) W(x)\, dx = 0$, the branch point bifurcation may occur at higher orders of ϵ.

We illustrate the transformation of algebraic solitons of the modified KdV equation in Figure 2, which shows two numerical computations of the initial-value problem:

$$w_t + 12ww_x + 6w^2 w_x + w_{xxx} = 0, \qquad w(x,0) = -\frac{4(1+\epsilon)}{1+4x^2}. \tag{5.31}$$

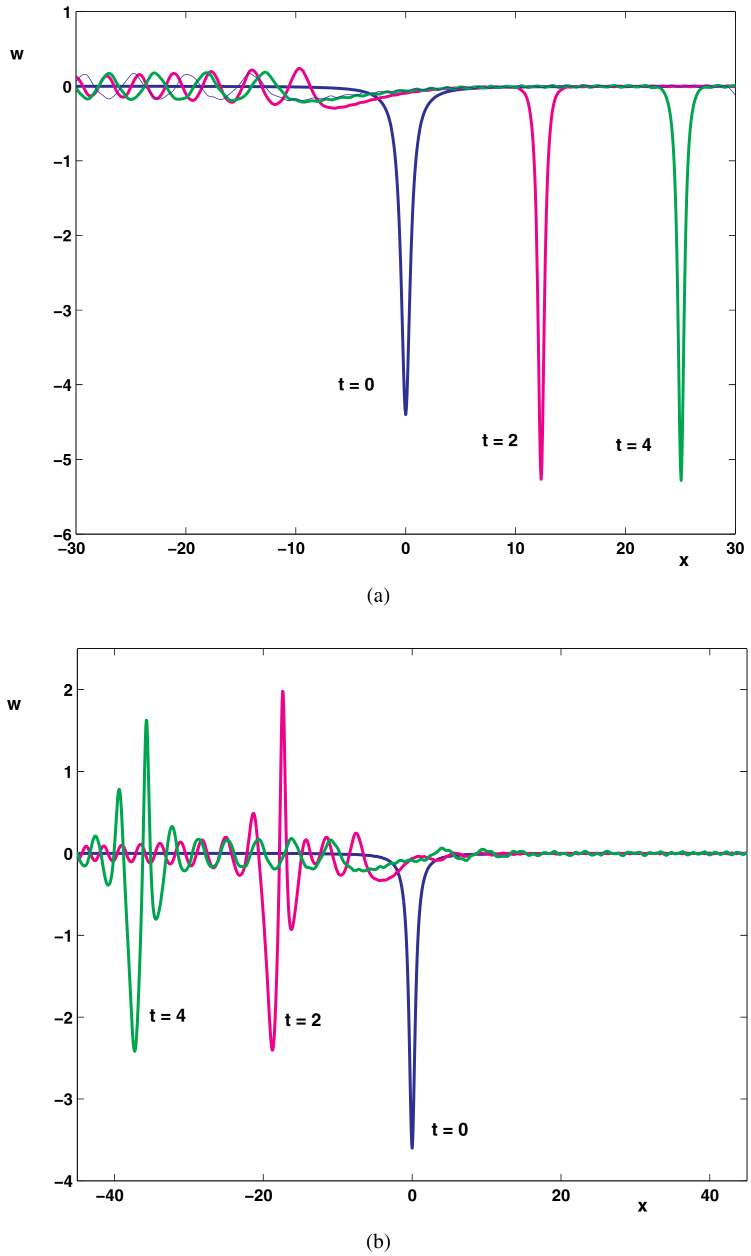

Fig. 2. Transformation of the perturbed algebraic soliton in the modified KdV equation (5.31) with (a) $\epsilon = 0.1$ and (b) $\epsilon = -0.1$.

When $\epsilon = 0$, the algebraic soliton (5.18) is an exact solution of (5.31). When $\epsilon > 0$, the perturbed algebraic soliton transforms to a travelling soliton that corresponds to a single real-valued eigenvalue $\lambda_b(\epsilon)$; see Fig. 2(a). When $\epsilon < 0$, the perturbed algebraic soliton transforms to a time-periodic breather that corresponds to a pair of complex-valued eigenvalues $\lambda_{c1,2}(\epsilon)$; see Fig. 2(b).

In applications to the focusing NLS equation (1.2), the potential $u(x)$ is complex-valued, and therefore $W_{\rm i}(x) \neq 0$. As a result, the eigenvalues $\lambda_p(\epsilon)$ and $\lambda_b(\epsilon)$ for the exponentially decaying potential (5.1) move according to the balance between $W_{\rm r}(x)$ and $W_{\rm i}(x)$. In general, the two eigenvalues become complex and do not coalesce for larger values of ϵ. While the recent works [AH90], [AHS96] addressed perturbations of time-homoclinic orbits, which are periodic functions of x, we have considered here the time-periodic orbits, which are exponentially decaying functions of x. Propositions 5.1 and 5.2 describe the transformation of the discrete spectrum of the AKNS spectral problem associated with the time-periodic, space-decaying solutions of the NLS equation.

For the algebraically decaying potential (5.18), the branch point bifurcation and the new eigenvalue $\lambda_b(\epsilon)$ depend again on the balance between $W_{\rm r}(x)$ and $W_{\rm i}(x)$. Either one eigenvalue $\lambda_b(\epsilon)$ or two eigenvalues $\lambda_{c1}(\epsilon)$, $\lambda_{c2}(\epsilon)$ generally bifurcate from the branch point $\lambda = 1$ under a complex-valued perturbation $\epsilon W(x)$. These perturbation results describe the transformation of the algebraic time-homoclinic orbit (1.6) to the time-periodic space-decaying solutions of the focusing NLS equation (1.2). The two different scenarios for this transformation generalize those for the time-homoclinic space-periodic orbits (see Fig. 2 in [AHS96]). According to [AHS96], the time-homoclinic orbits transform to time-periodic solutions of the focusing NLS equation, such that perturbations of opposite signs result in time-periodic solutions of different periods and different spatial symmetry.

The analysis of branch point bifurcations for exponentially and algebraically decaying potentials can be extended to the KN spectral problem (2.7). Two branch points of the continuous spectrum $\Sigma_{\rm con}$ exist at $\lambda = \pm i$. The discrete spectrum $\Sigma_{\rm dis}$ is located in the intervals $\mathrm{Re}(\lambda) = 0$ and $0 < |\mathrm{Im}(\lambda)| < 1$. The shift and bifurcations of eigenvalues of the discrete spectrum were studied in [HKM92a], [HKM92b], in the context of the derivative NLS equation (1.3). It was shown that two simple eigenvalues can collide to form a double eigenvalue that in turn splits into a pair of complex eigenvalues. Algebraic solitons of the KN spectral problem (2.7) were explicitly constructed in [KI78b], [M89]. The branch point bifurcations for algebraic solitons are expected to result in a similar cubic equation (5.29), since there exists an asymptotic correspondence between solutions of the modified KdV equation (1.1) and the derivative NLS equation (1.3) [HKM92b].

6. Evans Function for the KN Spectral Problem

We consider the KN spectral problem (2.7), when the potential $u(x)$ satisfies zero boundary conditions:

$$\lim_{x\to\pm\infty} u(x) = 0. \tag{6.1}$$

The KN spectral problem is rewritten explicitly as

$$\psi_1' = -\lambda u(x)\psi_2 - i\lambda^2\psi_1,$$
$$\psi_2' = \lambda\overline{u}(x)\psi_1 + i\lambda^2\psi_2. \quad (6.2)$$

In the case $u(x) \equiv 0$, two fundamental solutions of the problem (6.2) exist for $\lambda \in \mathbb{C}$ as follows:

$$\psi(x) = \mathbf{e}_\pm e^{\pm\kappa(\lambda)x}, \quad \mathbf{e}_+ = \begin{pmatrix}1\\0\end{pmatrix}, \quad \mathbf{e}_- = \begin{pmatrix}0\\1\end{pmatrix}, \quad \kappa(\lambda) = -i\lambda^2. \quad (6.3)$$

The complex plane is decomposed as $\mathbb{C} = \mathcal{D}_+ \cup \Sigma_{\text{con}} \cup \mathcal{D}_-$, where

$$\Sigma_{\text{con}} = \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\kappa(\lambda)) = 0\} = \{\lambda \in \mathbb{R}\} \cup \{\lambda \in i\mathbb{R}\}, \quad (6.4)$$

$$\mathcal{D}_+ = \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\kappa(\lambda)) > 0\} = \mathcal{D}_I \cup \mathcal{D}_{III}, \quad (6.5)$$

and

$$\mathcal{D}_- = \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\kappa(\lambda)) < 0\} = \mathcal{D}_{II} \cup \mathcal{D}_{IV}. \quad (6.6)$$

The decomposition of $\mathbb{C}$ into domains $\mathcal{D}_I$–$\mathcal{D}_{IV}$ is shown schematically in Figure 3. There exists only one crossing point $\lambda = \lambda_c = 0$, where $\kappa'(\lambda_c) = 0$. The stars show isolated eigenvalues of the discrete spectrum $\lambda = \pm\Lambda, \pm\bar{\Lambda} \in \Sigma_{\text{dis}}$. Due to the symmetry (3.7), isolated eigenvalues $\lambda \in \Sigma_{\text{dis}}$ are symmetric with respect to the line $\operatorname{Re}(\lambda) = 0$. Due to another symmetry,

$$\psi_1(x,\lambda) \mapsto \psi_1(x,-\lambda), \qquad \psi_2(x,\lambda) \mapsto -\psi_2(x,-\lambda), \quad (6.7)$$

the eigenvalues are also symmetric with respect to the origin $\lambda = 0$. We shall hence consider eigenvalues in the first quadrant $\lambda \in \mathcal{D}_I$ only.

We shall focus on the class of algebraically decaying potentials $u(x)$, such that

$$\lim_{x\to\pm\infty} |x|^p u(x) e^{-2i\delta^2 x} = b_\infty^\pm, \quad |b_\infty^+| = |b_\infty^-| = b_\infty, \quad (6.8)$$

where $0 < b_\infty < \infty$, $p \geq 1$, and $\delta > 0$. Algebraically decaying solitons were found in [KN78], [M78] for $p = 1$ and $b_\infty = 1/\delta$. We shall define *fundamental solutions* and *embedded eigenvalues* of the KN spectral problem (6.2). Then, we shall prove that embedded eigenvalues may exist only if the potential $u(x)$ has the decay rate (6.8) with $p = 1$.

Proposition 6.1. *Let $u(x)$ satisfy (6.8) with $p \geq 1$. For $p > 1$, there exist two sets of fundamental solutions $\phi^\pm(x;\lambda)$ and $\psi^\pm(x;\lambda)$ of the KN problem (6.2) with $\lambda \in \mathbb{C}$, such that*

$$\lim_{x\to-\infty} \phi^\pm(x;\lambda) e^{\mp\kappa(\lambda)x} = \mathbf{e}_\pm, \qquad \lim_{x\to+\infty} \psi^\pm(x;\lambda) e^{\mp\kappa(\lambda)x} = \mathbf{e}_\pm. \quad (6.9)$$

For $p = 1$, the two sets of fundamental solutions exist for any $\lambda \in \mathbb{C}\backslash\{\pm i\delta\}$.

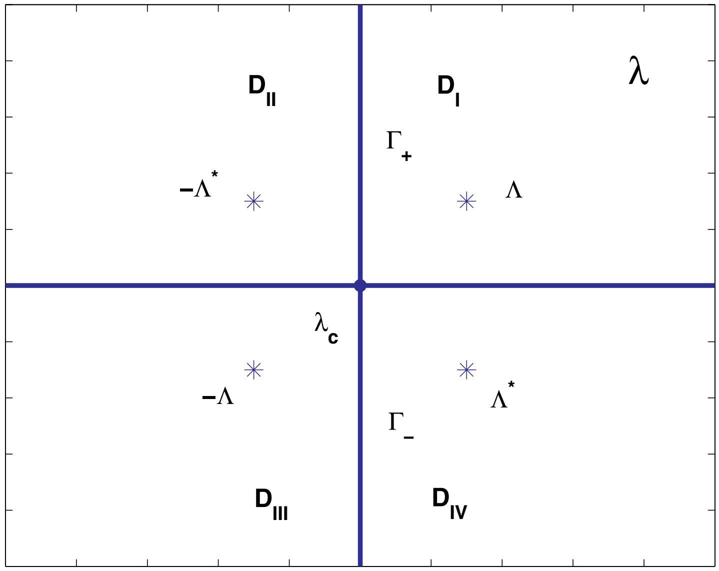

Fig. 3. Decomposition of $\mathbb{C}$ into domains $\mathcal{D}_{\mathrm{I}}$–$\mathcal{D}_{\mathrm{IV}}$ for the KN spectral problem (6.2).

Proof. By Lemma 3.1, fundamental solutions $\phi^{\pm}(x;\lambda)$ and $\psi^{\pm}(x;\lambda)$ of the KN problem (6.2) exist for $p > 1$, since eigenvalues of A are distinct for $\lambda \neq 0$ and $R(x) \in L^1(\mathbb{R})$ for $p > 1$. When $\lambda = 0$, the problem (6.2) has two linearly independent constant solutions $\mathbf{e}_{\pm}$. When $p = 1$, we substitute the following for $x \geq x_0 > 0$:

$$u(x) = w(x)e^{2i\delta^2 x}, \qquad \psi_1 = \varphi_1(x)e^{i\delta^2 x}, \qquad \psi_2 = \varphi_2(x)e^{-i\delta^2 x}, \tag{6.10}$$

such that the system (6.2) takes the form

$$\frac{d\varphi}{dx} = (A + V(x) + R(x))\,\varphi, \tag{6.11}$$

where

$$A = -i(\lambda^2 + \delta^2)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad V(x) = \frac{\lambda}{x}\begin{pmatrix}0 & -b_\infty^+\\ \bar{b}_\infty^+ & 0\end{pmatrix},$$

and

$$R(x) = \frac{\lambda}{x^2}\begin{pmatrix}0 & -x(xw(x) - b_\infty^+)\\ -x(\bar{b}_\infty^+ - x\bar{w}(x)) & 0\end{pmatrix},$$

such that $x\left(xw(x) - b_\infty^+\right) = \mathrm{o}(x)$ as $x \to +\infty$. When $\lambda \neq \pm i\delta$, the matrix A has distinct eigenvalues, $V(x) \to 0$, $V'(x) \in L^1(x \geq x_0)$, and $R(x) \in L^1(x \geq x_0)$. The existence of fundamental solutions (6.9) follows by Proposition 8.1 from [CL55, p. 92]. □

Definition 6.2. If there exists an L^2-eigenvector $\psi(x)$ of the KN problem (6.2) for $\lambda = \lambda_0 \in \Sigma_{\rm con}$, $\lambda_0 \neq 0$, the point $\lambda = \lambda_0$ is referred to as an embedded eigenvalue of the KN problem (6.2).

By Proposition 6.1, the nondecaying fundamental solutions exist for $p > 1$ and for $p = 1$ and $\lambda \neq \pm i\delta$, such that no *embedded eigenvalues* may exist for $p > 1$ and for $p = 1$ and $\lambda \in \Sigma_{\rm con}\backslash\{\pm i\delta\}$. On the other hand, Proposition 6.1 does not rule out *embedded eigenvalues* for $p = 1$ and $\lambda = \pm i\delta$. Due to the symmetries, we shall consider $\lambda = i\delta$ only.

Lemma 6.3. *Let $u(x)$ satisfy (6.8) with $p = 1$. The point $\lambda = i\delta$ can be an embedded eigenvalue of the KN problem (6.2) if $b_\infty > \frac{1}{2\delta}$.*

Proof. When $p = 1$ and $\lambda = i\delta$, we let $x \geq x_0 > 0$ and define

$$u(x) = \frac{b(x)}{x} e^{2i\delta^2 x}, \qquad \psi_1 = \frac{a_1(x)}{x^q} e^{i\delta^2 x}, \qquad \psi_2 = \frac{a_2(x)}{x^q} e^{-i\delta^2 x}, \tag{6.12}$$

where $q > 0$. The system (6.2) takes the form

$$x \frac{d\mathbf{a}}{dx} = (A + R(x))\,\mathbf{a}, \tag{6.13}$$

where

$$\begin{aligned} A &= \begin{pmatrix} q & -i\delta b_\infty^+ \\ i\delta \bar{b}_\infty^+ & q \end{pmatrix}, \\ R(x) &= \frac{i\delta}{x} \begin{pmatrix} 0 & -x(b(x) - b_\infty^+) \\ -x(\bar{b}_\infty^+ - \bar{b}(x)) & 0 \end{pmatrix}, \end{aligned} \tag{6.14}$$

such that $x\left(b(x) - b_\infty^+\right) = {\rm o}(x)$ as $x \to +\infty$. The eigenvalues of A are distinct for $b_\infty \neq 0$, while $\tilde{R}(z) = R(e^z)$ is in $L^1(z \geq z_0)$. By Lemma 3.1, there exist two solutions of the system (6.13), such that

$$\lim_{x\to\infty} \mathbf{a}_1 x^{-(q+\delta b_\infty)} = \begin{pmatrix} e^{i\theta_\infty} \\ i \end{pmatrix}, \qquad \lim_{x\to\infty} \mathbf{a}_2 x^{-(q-\delta b_\infty)} = \begin{pmatrix} e^{i\theta_\infty} \\ -i \end{pmatrix}, \tag{6.15}$$

where $\theta_\infty = \arg(b_\infty^+)$. These two solutions correspond to eigenvectors of the system (6.2) as $x \to +\infty$:

$$\boldsymbol{\psi}_1(x) \to \begin{pmatrix} e^{i\delta^2 x + i\theta_\infty} \\ i e^{-i\delta^2 x} \end{pmatrix} x^{\delta b_\infty}, \qquad \boldsymbol{\psi}_2(x) \to \begin{pmatrix} e^{i\delta^2 x + i\theta_\infty} \\ -i e^{-i\delta^2 x} \end{pmatrix} \frac{1}{x^{\delta b_\infty}}. \tag{6.16}$$

The second eigenvector, $\boldsymbol{\psi}_2(x)$, decays as $x \to +\infty$. When it exists, the decaying bound state $\boldsymbol{\psi}(x)$ belongs to $L^2(\mathbb{R})$, when $\delta b_\infty > \frac{1}{2}$. □

Lemma 6.4. *Let $u(x)$ satisfy (6.8) with $p = 1$ and $b_\infty > \frac{1}{2\delta}$. Let $\lambda = i\delta$ be an embedded eigenvalue of the KN problem (6.2). The geometric multiplicity of $\lambda = i\delta$ is one, and the algebraic multiplicity of $\lambda = i\delta$ is finite.*

Proof. We set $\lambda = i\delta$ and use the transformation (6.10). Only one decaying bound state $\varphi^{(0)}(x)$ may exist, since the other, linearly independent solution grows in x. If $\lambda = i\delta$ is a multiple eigenvalue, there exist bound states $\varphi^{(n)}(x)$ with $n > 0$ of the generalized problem

$$\begin{aligned}\varphi_1^{(n)\prime} &= -i\delta w(x)\varphi_2^{(n)} + 2\delta\varphi_1^{(n-1)}(x) - w(x)\varphi_2^{(n-1)}(x) - i\varphi_1^{(n-2)}(x),\\ \varphi_2^{(n)\prime} &= i\delta\bar{w}(x)\varphi_1^{(n)} - 2\delta\varphi_2^{(n-1)}(x) + \bar{w}(x)\varphi_1^{(n-1)}(x) + i\varphi_2^{(n-2)}(x),\end{aligned} \tag{6.17}$$

where $\boldsymbol{\varphi}^{(-1)} = \mathbf{0}$ and $n = 1, \ldots, N$. We show that $N < \infty$, similar to the proof of Lemma 4.4. It follows from the balance of algebraically decaying terms in the problem (6.17) that the nonhomogeneous solutions for $\boldsymbol{\varphi}^{(n)}(x)$ with $n > 0$ have the asymptotic form as $x \to +\infty$:

$$\boldsymbol{\varphi}^{(n)}(x) \to \frac{1}{x^{q-n}}\begin{pmatrix} c_n \\ d_n \end{pmatrix}, \quad q = \delta b_\infty, \tag{6.18}$$

where c_n and d_n satisfy the recurrence relation

$$\begin{aligned} c_n &= \frac{2\delta}{n(n-2q)}\left((n-q)c_{n-1} + i\delta b_\infty d_{n-1}\right),\\ d_n &= \frac{2\delta}{n(2q-n)}\left((n-q)d_{n-1} - i\delta\bar{b}_\infty c_{n-1}\right), \quad n \geq 1,\end{aligned} \tag{6.19}$$

and $c_0 = 1$, $d_0 = 1$. The generalized eigenfunctions $\boldsymbol{\varphi}^{(n)}(x)$ decay algebraically when $q > n$. Under this condition, the recurrence relations (6.19) are nonsingular for any n. Therefore, given $q < \infty$, there exists $N < \infty$, such that solutions of (6.17) with $n = N + 1$ become nondecaying as $x \to +\infty$. ☐

The discrete spectrum $\Sigma_{\rm dis}$ of the KN problem (6.2) in the domain $\lambda \in \mathcal{D}_I$ is defined by the zeros of $E(\lambda)$, *the Evans function* of the KN problem (6.2), given by

$$E(\lambda) = \det(\boldsymbol{\phi}^+(x;\lambda), \boldsymbol{\psi}^-(x;\lambda)), \quad \mathrm{Re}\,(\kappa(\lambda)) > 0. \tag{6.20}$$

By Proposition 6.1, $E(\lambda)$ is well-defined on $\lambda \in \mathcal{D}_I$ for the potential (6.8) with $p \geq 1$. By Lemma 6.3, the potentials $u(x)$ satisfying (6.8) with $p = 1$ may support an embedded eigenvalue at $\lambda = i\delta$. In order to study such an embedded eigenvalue, we define the semi-axis $\Gamma_+ = \{\lambda \in i\mathbb{R}_+\}$, which belongs to the continuous spectrum $\Sigma_{\rm con}$ (see Fig. 2). We use the parameter k for the the semi-axis Γ_+, such that $\lambda = i\sqrt{k}$, $\kappa(\lambda) = ik$, and $k \in \mathbb{R}_+$ on $\lambda \in \Gamma_+$. The function $G(k)$ is defined on $\lambda \in \Gamma_+$ from the matching condition,

$$G(k) = E(\lambda) = \det\left(\boldsymbol{\phi}^+(x;k), \boldsymbol{\psi}^-(x;k)\right), \quad \lambda = i\sqrt{k}, \quad k > 0. \tag{6.21}$$

The analysis of the singular behavior of $G(k)$ as $k \to \delta^2$ can be developed similarly to the proof of Lemma 4.8. We omit the details and only illustrate the regularization of the function $G(k)$ at $k \to \delta^2$ for an example of the algebraic soliton $u(x)$.

7. Examples of an Embedded Eigenvalue Bifurcation

We consider a family of exponentially decaying potentials,

$$u_\gamma(x) = 4\delta \sin\gamma\, e^{2\theta - 2i\varphi} \frac{e^{4\theta} + e^{i\gamma}}{(e^{4\theta} + e^{-i\gamma})^2}, \tag{7.1}$$

where $0 < \gamma < \pi$, $\delta > 0$, $\theta(x) = x\delta^2 \sin\gamma$, and $\varphi(x) = x\delta^2 \cos\gamma$. The family (7.1) corresponds to *the solitons* of the derivative NLS equation (1.3) [KN78] and of the MTM system (1.4) (where transformation (2.10) must be used) [KN77]. The fundamental solutions of the KN problem (6.2) with the potentials (7.1) take the explicit form (see [CY03])

$$\phi^+(x;\lambda) = e^{\kappa(\lambda)x} \begin{pmatrix} \dfrac{E(\lambda)e^{4\theta} + e^{-i\gamma}}{e^{4\theta} + e^{-i\gamma}} \\ -\dfrac{2i\delta\lambda \sin\gamma\, e^{2\theta + 2i\varphi - 2i\gamma}}{(\lambda^2 - \overline{\Lambda^2})(e^{4\theta} + e^{i\gamma})} \end{pmatrix}, \tag{7.2}$$

and

$$\psi^-(x;\lambda) = e^{-\kappa(\lambda)x} \begin{pmatrix} -\dfrac{2i\delta\lambda \sin\gamma\, e^{2\theta - 2i\varphi}}{(\lambda^2 - \overline{\Lambda^2})(e^{4\theta} + e^{-i\gamma})} \\ \dfrac{e^{4\theta} + E(\lambda)e^{i\gamma}}{e^{4\theta} + e^{i\gamma}} \end{pmatrix}, \tag{7.3}$$

where $\Lambda = \delta e^{i\gamma/2}$, $\kappa(\lambda) = -i\lambda^2$, and $E(\lambda)$ is the Evans function in the explicit form:

$$E(\lambda) = \frac{\lambda^2 - \Lambda^2}{\lambda^2 - \overline{\Lambda^2}} e^{-2i\gamma}. \tag{7.4}$$

The Evans function $E(\lambda)$ has a simple zero in $\mathcal{D}_I$ at $\lambda = \Lambda$, where a simple isolated eigenvalue of Σ_{dis} resides. The perturbation theory for the isolated eigenvalue was studied for the derivative NLS equation (1.3) in [WM84], [CY03] and for the MTM system (1.4) in [S02].

The family of exponentially decaying potentials (7.1) converges as $\gamma \to \pi$ to the algebraically decaying potential,

$$u_\pi(x) = 4\delta e^{2i\delta^2 x} \frac{4\delta^2 x + i}{(4\delta^2 x - i)^2}. \tag{7.5}$$

The special solution (7.5) corresponds to *the algebraic solitons* in the derivative NLS equation [KN78], [M78] and in the MTM system [KN77], [BPZ98]. The Evans function $E(\lambda)$ converges as $\gamma \to \pi$ to $E(\lambda) = 1$. The fundamental solutions of the KN problem follow from (7.2)–(7.3) in the limit $\gamma \to \pi$:

$$\phi^+(x;\lambda) = e^{\kappa(\lambda)x} \left[\mathbf{e}_+ + \frac{2i\lambda}{\lambda^2 + \delta^2} \begin{pmatrix} \frac{\lambda}{4\delta^2 x - i} \\ -\frac{\delta e^{-2i\delta^2 x}}{4\delta^2 x + i} \end{pmatrix} \right], \tag{7.6}$$

$$\psi^{-}(x;\lambda) = e^{-\kappa(\lambda)x}\left[\mathbf{e}_{-} - \frac{2i\lambda}{\lambda^2+\delta^2}\begin{pmatrix}\frac{\delta e^{2i\delta^2 x}}{4\delta^2 x - i}\\ \frac{\lambda}{4\delta^2 x + i}\end{pmatrix}\right]. \tag{7.7}$$

The fundamental solutions $\phi^{+}(x;\lambda)$ and $\psi^{-}(x;\lambda)$ have simple poles at $\lambda = i\delta$, which correspond to the embedded eigenvalue $\lambda = i\delta$, with the bound state $\psi^{(0)}(x)$ in the form

$$\psi^{(0)}(x) = \begin{pmatrix}\frac{i e^{i\delta^2 x}}{4\delta^2 x - i}\\ -\frac{e^{-i\delta^2 x}}{4\delta^2 x + i}\end{pmatrix}. \tag{7.8}$$

In agreement with Lemma 6.3, we have $q = \delta b_\infty = 1$. We regularize the Evans function $E(\lambda;\epsilon)$ by cancelling the pole singularities in the renormalized function:

$$\hat{E}(\lambda;\epsilon) = (\lambda^2+\delta^2)^2 E(\lambda;\epsilon). \tag{7.9}$$

The renormalized function $\hat{E}(\lambda;\epsilon)$ has a double zero at the point $\lambda = i\delta$ when $\epsilon = 0$. The double zero defines the algebraic structure of the embedded eigenvalue $\lambda = i\delta$. The changes in the discrete spectrum of the KN problem (6.2) as $\epsilon \neq 0$ are described in the following proposition.

Proposition 7.1. *Let $\hat{E}(\lambda;0) = \hat{E}(\lambda)$ be given by (7.9), such that $\hat{E}(i\delta) = 0$. Let $U(x)$ satisfy (6.8) with $p > 1$. There exist $\epsilon_0 > 0$ and $C > 0$ such that the function $\hat{E}(\lambda;\epsilon)$ has a simple zero at $\lambda = \lambda_\delta(\epsilon)$ in $\mathcal{D}_I$, where $0 < |\lambda_\delta(\epsilon) - i\delta| \leq C\epsilon_0^{1/2}$ for $0 < |\epsilon| \leq \epsilon_0$, if*

$$\int_{-\infty}^{\infty}\left(\frac{U(x)e^{-2i\delta^2 x}}{(4\delta^2 x + i)^2} - \frac{\overline{U(x)}e^{2i\delta^2 x}}{(4\delta^2 x - i)^2}\right)dx \neq 0. \tag{7.10}$$

Proof. Eigenvalues $\lambda \in \mathcal{D}_I$ corresponds to zeros of the Evans function $\hat{E}(\lambda;\epsilon)$ in $\mathcal{D}_I$, such that we can redefine the renormalized Evans function $\hat{E}(\lambda;\epsilon)$ in new variables κ and ϵ as follows:

$$\hat{G}(\kappa;\epsilon) = \hat{E}(i\sqrt{\delta^2-\kappa};\epsilon), \quad 0 < \arg(\kappa) < \pi. \tag{7.11}$$

At $\epsilon = 0$, there is a double zero at $\kappa = 0$, since

$$\hat{G}(\kappa,0) = \kappa^2. \tag{7.12}$$

It follows from the KN problem (6.2) that

$$\frac{\partial E}{\partial \epsilon}(\lambda;0) = -\lambda\int_{-\infty}^{\infty}\left[U(x)\phi_2^{+}(x;\lambda)\psi_2^{-}(x;\lambda) + \overline{U(x)}\phi_1^{+}(x;\lambda)\psi_1^{-}(x;\lambda)\right]dx, \tag{7.13}$$

such that

$$\frac{\partial \hat{G}}{\partial \epsilon}(0,0) = 4\delta^5\int_{-\infty}^{\infty}\left(\frac{U(x)e^{-2i\delta^2 x}}{(4\delta^2 x + i)^2} - \frac{\overline{U(x)}e^{2i\delta^2 x}}{(4\delta^2 x - i)^2}\right)dx. \tag{7.14}$$

By the Implicit Function Theorem, the double zero of $\hat{G}(\kappa; 0)$ splits according to the roots of the quadratic equation:

$$\kappa_\delta^2(\epsilon) + \frac{\partial \hat{G}}{\partial \epsilon}(0; 0)\epsilon = \mathrm{o}(\epsilon). \tag{7.15}$$

The leading-order behavior of $\kappa_\delta(\epsilon)$ is given by

$$\kappa_\delta^2(\epsilon) = -4\epsilon\delta^5 \int_{-\infty}^{\infty} \left(\frac{U(x)e^{-2i\delta^2 x}}{(4\delta^2 x + i)^2} - \frac{\overline{U(x)}e^{2i\delta^2 x}}{(4\delta^2 x - i)^2} \right) dx + \mathrm{o}(\epsilon), \tag{7.16}$$

$0 < \arg(\kappa_\delta(\epsilon)) < \pi$. The corresponding eigenvalues $\lambda_\delta(\epsilon)$ are found from $\lambda_\delta(\epsilon) = i\sqrt{\delta^2 - \kappa_\delta(\epsilon)}$. Under the constraint (7.10), we have $\kappa_\delta^2(\epsilon) \in i\mathbb{R}$, such that there exists exactly one zero $\kappa_\delta(\epsilon)$ in the domain (7.11). This zero corresponds to a simple complex eigenvalue $\lambda_\delta(\epsilon) \in \Sigma_{\mathrm{dis}}$ in the domain $\mathcal{D}_I$. □

In applications to the derivative NLS equation (1.3), Proposition 7.1 describes the transformation of the algebraic soliton (1.7) under a deformation of the initial data $\epsilon U(x)$. For a generic function $\epsilon U(x)$ that satisfies the constraint (7.10), the deformation shifts the embedded eigenvalue $\lambda = i\delta$ to the complex isolated eigenvalue $\lambda = \Lambda(\epsilon)$ in $\mathcal{D}_I$. Since $(\lambda_\delta(\epsilon) - i\delta) \sim e^{\pm\frac{i\pi}{4}}$, the change in the parameters $\delta(\epsilon)$ and $\gamma(\epsilon)$ of the shifted eigenvalue $\lambda_\delta(\epsilon) = \Lambda(\epsilon) = \delta(\epsilon)e^{i\gamma(\epsilon)/2}$ is of order $O(\epsilon)$. Since an isolated eigenvalue $\lambda_\delta(\epsilon)$ corresponds to the travelling and rotating soliton of the derivative NLS equation (1.3), we conclude that a generic deformation of the initial data smoothly transforms an algebraic soliton to an exponentially decaying soliton (7.1).

In applications to the MTM system (1.4), the same conclusion holds for the algebraic soliton (1.8) after the transformation (2.10). It was reported in [BPZ98], [KS02] that nonintegrable deformations of the MTM system result in spectral instability of the algebraic soliton (1.8). However, it follows from Proposition 7.1 that deformation of the initial data for the integrable MTM system (1.4) results not in spectral instabilities but rather in a smooth transformation of the algebraic soliton (1.8) into an exponentially decaying soliton (7.1). This property is related to the integrability of the MTM system (1.4) and to the existence of the linear (Lax) operator (2.1).

8. Summary

We have shown that the standard Evans function $E(\lambda; \epsilon)$ may have power and pole singularities at the continuous spectrum, if the AKNS and KN spectral problems have algebraically decaying potentials. Algebraic solitons of integrable nonlinear evolution equations appear to be typical examples of such potentials. The algebraic structure of pole singularities and related embedded eigenvalues is characterized by the renormalized Evans function $\hat{E}(\lambda; \epsilon)$. Branch point and embedded eigenvalue bifurcations occur when nongeneric algebraically decaying potentials (such as algebraic solitons) are perturbed by smooth generic perturbation functions. These bifurcations can be studied using the Implicit Function Theorem applied to the renormalized Evans function $\hat{E}(\lambda; \epsilon)$.

We have found two different types of transformations of algebraically decaying potentials. In the context of the AKNS spectral problem with nonzero boundary conditions, we showed that the branch point bifurcation results in either one or two new eigenvalues of the discrete spectrum which pop out from the branch point, depending on the sign of the perturbation function. The underlying algebraic structure of the branch point bifurcation is defined by solutions of a cubic equation along the corresponding sheet of the Riemann surface. In the context of the KN spectral problem with zero boundary conditions, we showed that the embedded eigenvalue bifurcation always results in one eigenvalue in the first quadrant of the complex plane. The underlying algebraic structure of the embedded eigenvalue bifurcation is defined by solutions of a quadratic equation. These results of the spectral theory of Lax operators are related to precise implications on the transformation of algebraic solitons in the modified KdV, focusing NLS, derivative NLS, and massive Thirring equations.

Appendix A Proofs of Lemmas 4.6–4.8

Here we shall prove Lemmas 4.6–4.8 of Section 4 by using the AKNS system (3.2).

Proof of Lemma 4.6. We introduce two linearly independent functions $\psi_{1,2}(x;k)$ that solve the AKNS system (3.2) with $\lambda = \sqrt{1-k^2}$, subject to the initial values

$$\psi_1(0;k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \psi_2(0;k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{A.1}$$

By Green's function methods, the functions $\psi_{1,2}(x;k)$ solve the Volterra integral equations

$$\psi_{1,2}(x;k) = \varphi_{1,2}(x;k) - \int_0^x K(x,s;k)\psi_{1,2}(s;k)\,ds, \tag{A.2}$$

where

$$\varphi_1(x;k) = \begin{bmatrix} \sqrt{1-k^2} \\ 1 \end{bmatrix} \frac{\sin kx}{k} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos kx,$$

and

$$\varphi_2(x;k) = -\begin{bmatrix} 1 \\ \sqrt{1-k^2} \end{bmatrix} \frac{\sin kx}{k} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos kx.$$

The function $G(k)$, given by (3.21), can be computed by setting $x = 0$. The leading-order singular behavior (4.20) in $G(k)$–$G(0)$ follows from the singular behavior of $\phi^{\pm}(0;k) - \phi^{\pm}(0;0)$ and $\psi^{\pm}(0;k) - \psi^{\pm}(0;0)$. Since the Wronskian determinant of two solutions of (3.2) is independent of x, then

$$\begin{aligned} \psi_1^-(0;k) &= \det(\psi^-(x;k), \psi_2(x;k)) \\ &= \lim_{x\to+\infty} \Big((\sqrt{1-k^2} - ik)\psi_{22}(x;k) - \psi_{21}(x;k) \Big) e^{-ikx} \end{aligned} \tag{A.3}$$

$$
\begin{aligned}
&= \sqrt{1-k^2} - ik \\
&\quad + \int_0^{+\infty} \Big(\bar{w}(s)\psi_{21}(s;k)(\sqrt{1-k^2} - ik) + w(s)\psi_{22}(s;k) \Big) e^{-iks}\, ds
\end{aligned}
$$

and

$$
\begin{aligned}
\psi_2^-(0;k) &= \det(\boldsymbol{\psi}_1(x;k), \boldsymbol{\psi}^-(x;k)) \\
&= \lim_{x\to+\infty} \Big(\psi_{11}(x;k) - (\sqrt{1-k^2} - ik)\psi_{12}(x;k) \Big) e^{-ikx} \qquad \text{(A.4)} \\
&= 1 - \int_0^{+\infty} \Big(\bar{w}(s)\psi_{11}(s;k)(\sqrt{1-k^2} - ik) + w(s)\psi_{12}(s;k) \Big) e^{-iks}\, ds.
\end{aligned}
$$

As a result, we have

$$
\begin{aligned}
\psi_1^-(0;k) - \psi_1^-(0;0) &= \sqrt{1-k^2} - 1 - ik \\
&\quad + \int_0^{+\infty} (\bar{w}(s)\psi_{21}(s;0) + w(s)\psi_{22}(s;0))\, (e^{-iks} - 1)\, ds \\
&\quad + \int_0^{+\infty} \Big(\bar{w}(s) \Big[\psi_{21}(s;k)(\sqrt{1-k^2} - ik) - \psi_{21}(s;0) \Big] \\
&\quad + w(s)\, [\psi_{22}(s;k) - \psi_{22}(s;0)] \Big)\, e^{-iks}\, ds \qquad \text{(A.5)}
\end{aligned}
$$

and

$$
\begin{aligned}
\psi_2^-(0;k) - \psi_2^-(0;0) &= -\int_0^{+\infty} (\bar{w}(s)\psi_{11}(s;0) + w(s)\psi_{12}(s;0))\, (e^{-iks} - 1)\, ds \\
&\quad - \int_0^{+\infty} \Big(\bar{w}(s) \Big[\psi_{11}(s;k)(\sqrt{1-k^2} - ik) - \psi_{11}(s;0) \Big] \\
&\quad + w(s)\, [\psi_{12}(s;k) - \psi_{12}(s;0)] \Big)\, e^{-iks}\, ds. \qquad \text{(A.6)}
\end{aligned}
$$

The integrals in (A.5) and (A.6) are of order $\mathrm{O}(k^{p-2})$ as $k \to 0$. In order to compute the leading-order behavior of (A.5) and (A.6) as $k \to 0$, we use the asymptotic representation as $x \to +\infty$:

$$
w(x) = \frac{b_\infty^+}{x^p} + \mathrm{o}\left(\frac{1}{x^p}\right), \qquad \boldsymbol{\psi}_{1,2}(x;0) = d_{1,2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \mathrm{o}(x), \qquad \text{(A.7)}
$$

where

$$
\begin{aligned}
d_1 &= 1 - \int_0^{\infty} (\bar{w}(s)\psi_{11}(s;0) + w(s)\psi_{12}(s;0))\; ds, \\
d_2 &= -1 - \int_0^{\infty} (\bar{w}(s)\psi_{21}(s;0) + w(s)\psi_{22}(s;0))\; ds.
\end{aligned}
$$

It follows from (A.3) and (A.4) that $\psi_1^-(0,0) = -d_2$ and $\psi_2^-(0,0) = d_1$. Following the proof of Theorem 3.1 in [K88a], we compute the singular terms in the first and second integrals in (A.5):

$$\int_0^{+\infty} (\bar{w}(s)\psi_{21}(s;0) + w(s)\psi_{22}(s;0))\,(e^{-iks} - 1)\,ds$$
$$= 2d_2 b_\infty k^{p-2} \int_0^\infty \frac{e^{-ix} - 1}{x^{p-1}}\,dx + \mathrm{o}(k^{p-2}), \tag{A.8}$$

and

$$\int_0^{+\infty} \Big(\bar{w}(s)\Big[\psi_{21}(s;k)(\sqrt{1-k^2} - ik) - \psi_{21}(s;0)\Big]$$
$$+\ w(s)\,[\psi_{22}(s;k) - \psi_{22}(s;0)]\Big)\,e^{-iks}\,ds$$
$$= 2d_2 b_\infty k^{p-2} \int_0^\infty \frac{e^{-ix}}{x^{p-1}} \left(\frac{\sin x}{x} - 1\right) dx + \mathrm{o}(k^{p-2}), \tag{A.9}$$

where $b_\infty = \mathrm{Re}(b_\infty^+)$ is used. The first and second integrals in (A.6) give exactly the same leading-order terms, with the replacement $-d_2 \mapsto d_1$. As a result, the leading-order behavior of $\boldsymbol{\psi}^-(0;k) - \boldsymbol{\psi}^-(0;0)$ is

$$\boldsymbol{\psi}^-(0;k) = \boldsymbol{\psi}^-(0;0)\left(1 + b_\infty 2^{p-1}\mathcal{I}_p k^{p-2}\right) + \mathrm{o}(k^{p-2}), \tag{A.10}$$

where

$$\mathcal{I}_p = -i\int_0^{+\infty} \frac{e^{-ix} - 1 + ix}{x^p}\,dx = \Gamma(1-p)e^{\frac{\pi i(p-2)}{2}}. \tag{A.11}$$

Similar computations for $\boldsymbol{\phi}^+(0;k)$ give

$$\boldsymbol{\phi}^+(0;k) = \boldsymbol{\phi}^+(0;0)\left(1 + b_\infty 2^{p-1}\mathcal{I}_p k^{p-2}\right) + \mathrm{o}(k^{p-2}). \tag{A.12}$$

Expanding $G(k)$ for small k, we obtain the leading-order behavior (4.20). □

Proof of Lemma 4.7. We consider the resonance case, when $G(0) = 0$ and $\boldsymbol{\psi}^-(x;0) = \gamma\boldsymbol{\phi}^+(x;0)$, $\gamma \neq 0$. A useful relation follows from the Volterra integral equations (3.19),

$$\boldsymbol{\psi}^-(x;0) = \begin{bmatrix}1\\1\end{bmatrix}\left(1 + \int_x^\infty (\bar{w}(s)\psi_1^-(s;0) + w(s)\psi_2^-(s;0))(x-s)\,ds\right)$$
$$+ \int_x^\infty \begin{bmatrix} w(s)\psi_2^-(s;0) \\ -\bar{w}(s)\psi_1^-(s;0)\end{bmatrix} ds, \tag{A.13}$$

such that

$$\psi_1^-(0;0) - \psi_2^-(0,0) = \int_0^\infty \left(\bar{w}(s)\psi_1^-(s;0) + w(s)\psi_2^-(s;0)\right) ds. \tag{A.14}$$

Since $\boldsymbol{\psi}^-(x;0)$ is bounded as $x \to -\infty$, we have additional relations,

$$\int_{-\infty}^\infty \left(\bar{w}(s)\psi_1^-(s;0) + w(s)\psi_2^-(s;0)\right) ds = 0, \tag{A.15}$$

and

$$\gamma = 1 - \int_{-\infty}^{\infty} s\left(\bar{w}(s)\psi_1^-(s;0) + w(s)\psi_2^-(s;0)\right) ds + \int_{-\infty}^{\infty} w(s)\psi_2^-(s)\, ds. \quad \text{(A.16)}$$

We introduce the function $\boldsymbol{\chi}(x;k)$ that solves the AKNS system (3.2) with $\lambda = \sqrt{1-k^2}$, subject to the initial value: $\boldsymbol{\chi}(0;k) = \boldsymbol{\psi}^-(0;0)$. Uniqueness implies that $\boldsymbol{\chi}(x;0) = \boldsymbol{\psi}^-(x;0)$. The function $\boldsymbol{\chi}(x;k)$ is useful in rewriting the Evans function $G(k)$ as

$$G(k) = \frac{\psi_1^-(0;k)G_1(k) + \phi_1^+(0;k)G_2(k)}{\psi_1^-(0;0)}, \quad \text{(A.17)}$$

where

$$\begin{aligned} G_1(k) &= \det(\boldsymbol{\phi}^+(0;k), \boldsymbol{\psi}^-(0;0)) = \det(\boldsymbol{\phi}^+(0;k), \boldsymbol{\chi}(0;k)), \\ G_2(k) &= \det(\boldsymbol{\psi}^-(0;0), \boldsymbol{\psi}^-(0;k)) = \det(\boldsymbol{\chi}(0;k), \boldsymbol{\psi}^-(0;k)), \end{aligned}$$

and we have assumed without loss of generality that $\psi_1^-(0;0) \neq 0$. It is clear that $\boldsymbol{\chi}(x;k) = \psi_1^-(0;0)\boldsymbol{\psi}_1(x;k) + \psi_2^-(0;0)\boldsymbol{\psi}_2(x;k)$, where $\boldsymbol{\psi}_{1,2}(x;k)$ solves the Volterra equations (A.2). Therefore, $\boldsymbol{\chi}(x;k)$ solves the Volterra equations:

$$\begin{aligned} \boldsymbol{\chi}(x;k) &= \left(\psi_1^-(0;0)\begin{bmatrix}\sqrt{1-k^2}\\ 1\end{bmatrix} - \psi_2^-(0;0)\begin{bmatrix}1\\ \sqrt{1-k^2}\end{bmatrix}\right)\frac{\sin kx}{k} \\ &\quad + \boldsymbol{\psi}^-(0;0)\cos kx - \int_0^x K(x,s;k)\boldsymbol{\chi}(s;k)\, ds. \end{aligned} \quad \text{(A.18)}$$

Since the Wronskian determinant of two solutions of (3.2) is constant in x, then

$$\begin{aligned} G_1(k) &= \lim_{x\to-\infty} \det(\boldsymbol{\phi}^+(x;k), \boldsymbol{\chi}(x;k)) \\ &= \lim_{x\to-\infty}\left((\sqrt{1-k^2}+ik)\chi_2(x;k) - \chi_1(x;k)\right)e^{ikx} \\ &= (\sqrt{1-k^2}+ik)\psi_2^-(0;0) - \psi_1^-(0;0) \\ &\quad - \int_{-\infty}^0 \left(\bar{w}(s)\chi_1(s;k)(\sqrt{1-k^2}+ik) + w(s)\chi_2(s;k)\right)e^{iks}\, ds \end{aligned} \quad \text{(A.19)}$$

and

$$\begin{aligned} G_2(k) &= \lim_{x\to+\infty} \det(\boldsymbol{\chi}(x;k), \boldsymbol{\psi}^-(x;k)) \\ &= \lim_{x\to+\infty}\left(\chi_1(x;k) - (\sqrt{1-k^2}-ik)\chi_2(x;k)\right)e^{-ikx} \\ &= \psi_1^-(0;0) - (\sqrt{1-k^2}-ik)\psi_2^-(0;0) \\ &\quad - \int_0^{+\infty}\left(\bar{w}(s)\chi_1(s;k)(\sqrt{1-k^2}-ik) + w(s)\chi_2(s;k)\right)e^{-iks}\, ds. \end{aligned} \quad \text{(A.20)}$$

We rewrite the integrals in $G_1(k)$ and $G_2(k)$ so that

$$\begin{aligned} G_1(k) \;=\; & (\sqrt{1-k^2}+ik)\psi_2^-(0;0) - \psi_1^-(0;0) \\ & - \int_{-\infty}^{0} \left(\bar{w}(s)\psi_1^-(s;0) + w(s)\psi_2^-(s;0)\right) e^{iks}\, ds \\ & - \int_{-\infty}^{0} \Big(\bar{w}(s)\left[\chi_1(s;k)(\sqrt{1-k^2}+ik) - \psi_1^-(s;0)\right] \\ & \qquad + w(s)\left[\chi_2(s;k) - \psi_2^-(s;0)\right]\Big) e^{iks}\, ds \end{aligned} \tag{A.21}$$

and

$$\begin{aligned} G_2(k) \;=\; & \psi_1^-(0;0) - (\sqrt{1-k^2}-ik)\psi_2^-(0;0) \\ & - \int_{0}^{+\infty} \left(\bar{w}(s)\psi_1^-(s;0) + w(s)\psi_2^-(s;0)\right) e^{-iks}\, ds \\ & - \int_{0}^{+\infty} \Big(\bar{w}(s)\left[\chi_1(s;k)(\sqrt{1-k^2}-ik) - \psi_1^-(s;0)\right] \\ & \qquad + w(s)\left[\chi_2(s;k) - \psi_2^-(s;0)\right]\Big) e^{-iks}\, ds. \end{aligned} \tag{A.22}$$

Since $\boldsymbol{\psi}^-(x;0)$ is bounded in $x \in \mathbb{R}$ and $w(x) \in L_1^1(\mathbb{R})$, the first integrals in (A.21) and (A.22) have the leading-order behavior

$$\begin{aligned} & - \int_{-\infty}^{0} \left(\bar{w}(s)\psi_1^-(s;0) + w(s)\psi_2^-(s;0)\right) ds \\ & \qquad - ik \int_{-\infty}^{0} s\left(\bar{w}(s)\psi_1^-(s;0) + w(s)\psi_2^-(s;0)\right) ds + \mathrm{o}(k), \end{aligned}$$

and

$$\begin{aligned} & - \int_{0}^{+\infty} \left(\bar{w}(s)\psi_1^-(s;0) + w(s)\psi_2^-(s;0)\right) ds \\ & \qquad + ik \int_{0}^{+\infty} s\left(\bar{w}(s)\psi_1^-(s;0) + w(s)\psi_2^-(s;0)\right) ds + \mathrm{o}(k). \end{aligned}$$

The solution $\boldsymbol{\chi}(x;k)$ satisfies the following upper bound for any $x \in \mathbb{R}$:

$$|\chi_j(x;k) - \chi_j(x;0)| \le C\left(\frac{kx}{1+kx}\right)^2, \quad j = 1, 2, \tag{A.23}$$

where $C > 0$ is some constant. The bound (A.23) is proved similarly to Eq. (2.5) in [K88b], where it was proved for the half-line $x \in \mathbb{R}_+$. The bound extends to the full line $x \in \mathbb{R}$, since $\chi(x;k)$ is computed for the resonance case. With the upper bound (A.23), we estimate the second integrals in (A.21) and (A.22) as

$$-ik \int_{-\infty}^{0} \bar{w}(s)\psi_1^-(s;0)\, ds + \mathrm{o}(k),$$

and

$$ik\int_0^{+\infty}\bar{w}(s)\psi_1^-(s;0)\,ds+\mathrm{o}(k).$$

Using (A.14), we conclude that

$$G_1(k) = ik\left(\psi_2^-(0;0)-\int_{-\infty}^0 s\left(\bar{w}(s)\psi_1^-(s;0)+w(s)\psi_2^-(s;0)\right)ds\right.$$
$$\left.-\int_0^{+\infty}\bar{w}(s)\psi_1^-(s;0)\,ds\right)+\mathrm{o}(k),$$

and

$$G_2(k) = ik\left(\psi_2^-(0;0)+\int_0^{+\infty} s\left(\bar{w}(s)\psi_1^-(s;0)+w(s)\psi_2^-(s;0)\right)ds\right.$$
$$\left.+\int_0^{+\infty}\bar{w}(s)\psi_1^-(s;0)\,ds\right)+\mathrm{o}(k).$$

Since $\lim_{x\to-\infty}\boldsymbol{\chi}(x;0)=\gamma(1,1)^T$ and $\lim_{x\to+\infty}\boldsymbol{\chi}(x;0)=(1,1)^T$, it follows from the Volterra integral equations (A.18) that

$$\psi_2^-(0;0)-\int_{-\infty}^0 s\left(\bar{w}(s)\psi_1^-(s;0)+w(s)\psi_2^-(s;0)\right)ds-\int_0^{+\infty}\bar{w}(s)\psi_1^-(s;0)\,ds=\gamma,$$

and

$$\psi_2^-(0;0)+\int_0^{+\infty} s\left(\bar{w}(s)\psi_1^-(s;0)+w(s)\psi_2^-(s;0)\right)ds+\int_0^{+\infty}\bar{w}(s)\psi_1^-(s;0)\,ds=1.$$

Thus, $G_1(k)=ik\gamma+\mathrm{o}(k)$ and $G_2(k)=ik+\mathrm{o}(k)$. Inserting these expressions into (A.17), we obtain the leading-order behavior (4.21). □

Proof of Lemma 4.8. Due to the Miura transformation [AC91], the AKNS system (3.2) with $\lambda=\sqrt{1-k^2}$ and real-valued $w(x)$ can be converted into the Schrödinger equation,

$$-\chi''+U(x)\chi=k^2\chi, \tag{A.24}$$

where $\chi=\psi_1+i\psi_2$ and $U(x)=-2w(x)-w^2(x)+iw'(x)$. We extract the long-range potential as

$$U(x)=\frac{q(q+1)}{x^2}+V(x),\quad |x|\geq x_0>0, \tag{A.25}$$

and

$$U(x)=V_0(x),\quad |x|\leq x_0, \tag{A.26}$$

where $V(x)\in L_1^1(|x|\geq x_0)$ and $V_0(x)\in L^1(|x|\leq x_0)$. We follow the analysis of [N86], [K88a] and introduce scalar Jost functions for the Schrödinger problem (A.24):

$$f^{\pm}(x;k)\to e^{\mp ikx},\quad x\to\pm\infty.$$

It is obvious from the system (3.2) and the relation $\chi = \psi_1 + i\psi_2$ that the Wronskian determinant of $f^{\pm}(x;k)$ is related to the Evans function $G(k)$, defined in (3.21):

$$W[f^-, f^+] = f^- f^{+\prime} - f^{-\prime} f^+ = -\frac{2i\sqrt{1-k^2}(\phi_{11}^-\phi_{22}^+ - \phi_{12}^-\phi_{21}^+)}{(\sqrt{1-k^2}+i)^2 + k^2} = -G(k). \quad \text{(A.27)}$$

The problem (A.24) with $U(x) = q(q+1)/x^2$ transforms to the Bessel equation with two linearly independent solutions $x^{1/2}J_\sigma(kx)$ and $x^{1/2}N_\sigma(kx)$, where $J_\sigma(z)$ and $N_\sigma(z)$ are Bessel and Neumann functions [AS74] and $\sigma = q + \frac{1}{2}$. We will use the linearly independent Hankel functions $H_\sigma^{(\pm)}(z) = J_\sigma(z) \pm i N_\sigma(z)$. Due to the asymptotic behavior of the Hankel functions at infinity [AS74], the Jost functions $f^{\pm}(x;k) = f_0^{\pm}(x;k)$, associated with the potential $U(x) = q(q+1)/x^2$, are given by

$$f_0^{\pm}(x;k) = \mp i\sqrt{\frac{\pi k x}{2}} H_{q+\frac{1}{2}}^{(\mp)}(kx) e^{\mp\frac{i\pi q}{2}}.$$

As follows from the Green's function, the Jost functions $f^{\pm}(x;k)$ satisfy the Volterra integral equations:

$$f^{\pm}(x;k) = f_0^{\pm}(x;k) + \int_x^{\pm\infty} g(x,s;k)V(s)f^{\pm}(s;k)\,ds, \qquad \text{(A.28)}$$

where

$$g(x,s;k) = \frac{\pi}{2}\sqrt{xs}\left(J_{q+\frac{1}{2}}(kx)N_{q+\frac{1}{2}}(ks) - J_{q+\frac{1}{2}}(ks)N_{q+\frac{1}{2}}(kx)\right).$$

It follows from the power series representation for Bessel functions [AS74] that there exists the limit

$$\lim_{k\to 0} k^q f^{\pm}(x;k) = F^{\pm}(x), \qquad \text{(A.29)}$$

where $F^{\pm}(x)$ solve the Volterra's integral equations:

$$F^{\pm}(x) = F_0^{\pm}(x) + \int_x^{\pm\infty} G(x,s)V(s)F^{\pm}(s)\,ds, \qquad \text{(A.30)}$$

such that

$$F_0^{\pm}(x) = \frac{2^q\Gamma\left(q+\frac{1}{2}\right)e^{\mp\frac{i\pi q}{2}}}{\sqrt{\pi}x^q},$$

and

$$G(x,s) = -\frac{\sqrt{xs}}{2q+1}\left(\left(\frac{x}{s}\right)^{q+\frac{1}{2}} - \left(\frac{s}{x}\right)^{q+\frac{1}{2}}\right).$$

It follows from (A.27) and (A.29) that

$$k^{2q}W[f^-, f^+] = W[F^-, F^+] + \mathrm{o}(1),$$

which proves the leading-order behavior (4.24) in the general case, when $F^+(x)$ and $F^-(x)$ are linearly independent. The functions $F^{\pm}(x)$ are solutions of (A.24) for $k = 0$, defined in (4.22)–(4.23). The correspondence between $\hat{\psi}^{\pm}_{1,2}(x)$ and the eigenfunctions $\phi^{+}(x;k)$ and $\psi^{-}(x;k)$ of the system (3.2) can be established via the large-x asymptotics.

Assume now that $F^{\pm}(x)$ are linearly dependent, such that $F^{-}(x) = \gamma F^{+}(x)$ and hence $W[F^{-}, F^{+}] = 0$. Since $F^{\pm}(x)$ decay as $|x| \to \infty$, a bound state $\psi^{(0)}(x)$ exists such that $\lambda = 1$ is a branch point eigenvalue. We introduce the function $\chi(x;k)$ that solves the scalar problem (A.24), subject to the initial values: $\chi(0;k) = F^{+}(0)$, $\chi'(0;k) = F^{+\prime}(0)$. Uniqueness implies that $\chi(x;0) = F^{+}(x)$. With the use of the function $\chi(x;k)$, we rewrite $W[f^{-}, f^{+}]$ as

$$W[f^{-}, f^{+}] = \frac{f^{-}(0;k)W_1(k) + f^{+}(0;k)W_2(k)}{F^{+}(0)}, \tag{A.31}$$

where

$$W_1(k) = W[F^{+}(0), f^{+}(0;k)] = W[\chi(x;k), f^{+}(x;k)], \tag{A.32}$$

$$W_2(k) = W[f^{-}(0;k), F^{+}(0)] = W[f^{-}(x;k), \chi(x;k)], \tag{A.33}$$

and $F^{+}(0) \neq 0$ is assumed without loss of generality. Using the Volterra integral equations, we express $\chi(x;k)$ for $x \geq x_0 > 0$ in the form

$$\chi(x;k) = \chi_0(x;k) - \int_{x_0}^{x} g(x,s;k)V(s)\chi(s,k)\,ds, \quad x \geq x_0 > 0, \tag{A.34}$$

where

$$\chi_0(x;k) = \alpha(k)\sqrt{x}J_{q+\frac{1}{2}}(kx) + \beta(k)\sqrt{x}N_{q+\frac{1}{2}}(kx).$$

The functions $\alpha(k)$ and $\beta(k)$ are found from the matching conditions at $x = x_0$ as

$$\alpha(k) = \frac{\pi}{2}W[\chi(x_0;k), \sqrt{x_0}N_{q+\frac{1}{2}}(kx_0)],$$

$$\beta(k) = \frac{\pi}{2}W[\sqrt{x_0}J_{q+\frac{1}{2}}(kx_0), \chi(x_0;k)], \tag{A.35}$$

where we have used the relation [AS74]

$$W[\sqrt{x}J_\sigma(kx), \sqrt{x}N_\sigma(kx)] = \frac{2}{\pi}, \quad \forall x > 0, \ \forall \sigma.$$

It follows from the integral equations (A.34) that the limit $k \to 0^{+}$ results in a nonsingular solution $\chi(x;0) = F^{+}(x)$, only if the following limits exist:

$$\alpha_0 = \lim_{k\to 0} k^{q+\frac{1}{2}}\alpha(k), \qquad \beta_0 = \lim_{k\to 0} k^{-q-\frac{1}{2}}\beta(k). \tag{A.36}$$

Using the power series representations for Bessel functions [AS74], we find that the Volterra's equation (A.34) is equivalent to the Volterra's equation (A.30) in the limit $k \to 0$, when

$$\alpha_0 = -2^{q-\frac{1}{2}}\Gamma\left(q+\frac{1}{2}\right)\int_{x_0}^{\infty} s^{-q}V(s)F^{+}(s)\,ds, \tag{A.37}$$

$$\beta_0 = -\sqrt{\frac{\pi}{2}}e^{-\frac{i\pi q}{2}} - \frac{\pi}{2^{q+\frac{3}{2}}\Gamma\left(q+\frac{3}{2}\right)}\int_{x_0}^{\infty} s^{q+1}V(s)F^{+}(s)\,ds. \tag{A.38}$$

We consider $W_1(k)$ in the limit $k \to 0$. It is clear from (A.32) and (A.34) that $W_1(k)$ can be computed in the limit $x \to +\infty$ as

$$W_1(k) = \sqrt{\frac{2k}{\pi}} e^{-\frac{i\pi q}{2}} \left(-\alpha(k) + i\beta(k) + \frac{i\pi}{2} \int_{x_0}^{\infty} \sqrt{s} H^{(-)}(ks) V(s) \chi(s;k)\, ds \right). \tag{A.39}$$

We use the following estimate for the solutions $\chi(x;k)$ (see (2.16) in [K88a]):

$$|\chi(x;k) - \chi(x;0)| \le Ck^2 \left(\frac{x}{1+kx} \right)^{q+1}, \quad x \ge x_0 > 0. \tag{A.40}$$

Using (A.40), we introduce a formal expansion,

$$\chi(x;k) = F^+(x) + k^2 \chi_2(x) + \mathrm{o}(k^2),$$

where $F^+(x)$ and $\chi_2(x)$ solve the problems

$$-F^{+\prime\prime} + \frac{q(q+1)}{x^2} F^+ + V(x) F^+ \;=\; 0, \quad x \ge x_0, \tag{A.41}$$

$$-\chi_2'' + \frac{q(q+1)}{x^2} \chi_2 + V(x)\chi_2 \;=\; F^+(x), \quad x \ge x_0. \tag{A.42}$$

We also expand the coefficients $\alpha(k)$ and $\beta(k)$ according to the limits (A.36):

$$k^{q+\frac{1}{2}} \alpha(k) = \alpha_0 + \alpha_2 k^2 + \mathrm{o}\left(k^2\right), \qquad \beta(k) = \beta_0 k^{q+\frac{1}{2}} + \mathrm{o}\left(k^{q+\frac{1}{2}}\right), \tag{A.43}$$

where

$$\alpha_2 = -2^{q-\frac{5}{2}} \Gamma\left(q - \frac{1}{2}\right) W[F^+(x_0), x_0^{-q+2}] - 2^{q-\frac{1}{2}} \Gamma\left(q + \frac{1}{2}\right) W[\chi_2(x_0), x_0^{-q}].$$

Expanding the integral in (A.39) in the limit $k \to 0$ and using the power series expansions above, we compute the leading-order terms as

$$k^q W_1(k) = \sqrt{\frac{2}{\pi}} e^{-\frac{i\pi q}{2}} \left[w_0 + k^2 w_2 + \mathrm{o}(k^2) \right], \tag{A.44}$$

where

$$\begin{aligned} w_0 \;&=\; -\alpha_0 - 2^{q-\frac{1}{2}} \Gamma\left(q + \frac{1}{2}\right) \int_{x_0}^{\infty} s^{-q} V(s) F^+(s)\, ds, \\ w_2 \;&=\; -\alpha_2 - 2^{q-\frac{5}{2}} \Gamma\left(q - \frac{1}{2}\right) \int_{x_0}^{\infty} s^{2-q} V(s) F^+(s)\, ds \\ &\quad - 2^{q-\frac{1}{2}} \Gamma\left(q + \frac{1}{2}\right) \int_{x_0}^{\infty} s^{-q} V(s) \chi_2(s)\, ds. \end{aligned}$$

Due to (A.37), we have $w_0 = 0$. Using (A.41)–(A.42) and integrating by parts, we have the relations

$$\int_{x_0}^{\infty} s^{2-q} V(s) F^+(s)\, ds \;=\; -W[x_0^{-q+2}, F^+(x_0)] + 2(1-2q) \int_{x_0}^{\infty} s^{-q} F^+(s)\, ds,$$

$$\int_{x_0}^{\infty} s^{-q} V(s) \chi_2(s)\, ds \;=\; \lim_{x\to\infty} W[x^{-q}, \chi_2(x)] - W[x_0^{-q}, \chi_2(x_0)] + \int_{x_0}^{\infty} s^{-q} F^+(s)\, ds.$$

By using these formulas in the expansion (A.44), we have

$$W_1(k) = k^{2-q} \lim_{x\to\infty} W[\chi_2(x), F^+(x)] + \mathrm{o}(k^{2-q}).$$

It follows from the problems (A.41)–(A.42), defined separately for $0 \le x \le x_0$ and $x \ge x_0$, that

$$\int_0^{\infty} \left(F^+(s)\right)^2 ds = \lim_{x\to\infty} W[\chi_2(x), F^+(x)],$$

where we have used the initial values $\chi_2(0) = 0$ and $\chi_2'(0) = 0$. As a result,

$$W_1(k) = k^{2-q} \int_0^{\infty} (F^+(s))^2\, ds + \mathrm{o}(k^{2-q}).$$

By similar arguments (replacing x by $-x$), we obtain

$$W_2(k) = k^{2-q} \int_{-\infty}^{0} (F^-(s))^2\, ds + \mathrm{o}(k^{2-q}).$$

Expanding the relation (A.31) as $k \to 0^+$ and using $F^-(x) = \gamma F^+(x)$, we obtain the leading-order behavior (4.25). ☐

Appendix B Regular Perturbation Theory for the AKNS Problem

Here we develop the regular perturbation theory for the AKNS system (3.2). These results are used in the proofs of Propositions 5.1–5.3 of Section 5.

Lemma B.1. *Let $w \in L^1(\mathbb{R})$. The Evans function $E(\lambda)$ of the AKNS problem (3.2) is continuously differentiable in $\lambda \in \mathcal{D}_+$, such that*

$$E'(\lambda) = \frac{\lambda E(\lambda)}{\lambda^2 - 1} + \int_{-\infty}^{\infty} \left(\phi_1^+(x;\lambda)\psi_2^-(x;\lambda) + \phi_2^+(x;\lambda)\psi_1^-(x;\lambda) - \frac{\lambda E(\lambda)}{\sqrt{\lambda^2 - 1}} \right) dx. \tag{B.1}$$

Proof. The fundamental solutions $\phi^+(x;\lambda)$ and $\psi^-(x;\lambda)$ are uniquely defined by the limits (3.11) for any $\lambda \in \mathcal{D}_+$ under the condition that $w(x) \in L^1(\mathbb{R})$. The fundamental solutions are also differentiable in $\lambda \in \mathcal{D}_+$. Using the AKNS problem (3.2), we obtain for any pair of solutions $\boldsymbol{\theta}(x;\lambda)$ and $\boldsymbol{\varphi}(x;\lambda)$,

$$\frac{d}{dx}\left(\varphi_2 \frac{\partial\theta_1}{\partial\lambda} - \varphi_1 \frac{\partial\theta_2}{\partial\lambda}\right) = \varphi_2\theta_1 + \varphi_1\theta_2. \tag{B.2}$$

We will apply this identity to compute $E'(\lambda)$, which we write as

$$E'(\lambda) = \det\left(\frac{\partial\boldsymbol{\phi}^+(x;\lambda)}{\partial\lambda}, \boldsymbol{\psi}^-(x;\lambda)\right) + \det\left(\boldsymbol{\phi}^+(x;\lambda), \frac{\partial\boldsymbol{\psi}^-(x;\lambda)}{\partial\lambda}\right). \tag{B.3}$$

We will evaluate the right-hand side at $x = 0$. Using (3.12), (3.14), and (3.16), we derive

$$\lim_{x\to\pm\infty}\left(\phi_1^+(x;\lambda)\psi_2^-(x;\lambda) + \phi_2^+(x;\lambda)\psi_1^-(x;\lambda)\right) = 2a(\lambda)\lambda = \frac{\lambda E(\lambda)}{\sqrt{\lambda^2-1}}. \tag{B.4}$$

Let $x < 0$. Integrating (B.2) with $\boldsymbol{\varphi} = \boldsymbol{\psi}^-(x;\lambda)$ and $\boldsymbol{\theta} = \boldsymbol{\phi}^+(x;\lambda)$ over $[x, 0]$) gives

$$\begin{aligned}
&\frac{\partial\phi_1^+(0;\lambda)}{\partial\lambda}\psi_2^-(0;\lambda) - \frac{\partial\phi_2^+(0;\lambda)}{\partial\lambda}\psi_1^-(0;\lambda)\\
&\quad= \frac{\partial\phi_1^+(x;\lambda)}{\partial\lambda}\psi_2^-(x;\lambda) - \frac{\partial\phi_2^+(x;\lambda)}{\partial\lambda}\psi_1^-(x;\lambda)\\
&\qquad - \frac{\lambda E(\lambda)x}{\sqrt{\lambda^2-1}} + \int_x^0\left(\phi_1^+(x;\lambda)\psi_2^-(x;\lambda) + \phi_2^+(x;\lambda)\psi_1^-(x;\lambda) - \frac{\lambda E(\lambda)}{\sqrt{\lambda^2-1}}\right)dx.
\end{aligned} \tag{B.5}$$

By (3.11) and (3.14), we obtain

$$\begin{aligned}
&\lim_{x\to-\infty}\left(\frac{\partial\phi_1^+(x;\lambda)}{\partial\lambda}\psi_2^-(x;\lambda) - \frac{\partial\phi_2^+(x;\lambda)}{\partial\lambda}\psi_1^-(x;\lambda) - \frac{\lambda E(\lambda)x}{\sqrt{\lambda^2-1}}\right)\\
&\quad= a(\lambda)\left(1 - \frac{i\lambda}{\sqrt{\lambda^2-1}}\right),
\end{aligned} \tag{B.6}$$

and thus

$$\begin{aligned}
\frac{\partial\phi_1^+(0;\lambda)}{\partial\lambda}\psi_2^-(0;\lambda) - \frac{\partial\phi_2^+(0;\lambda)}{\partial\lambda}\psi_1^-(0;\lambda) &= a(\lambda)\left(1 - \frac{i\lambda}{\sqrt{\lambda^2-1}}\right)\\
&+ \int_{-\infty}^0\left(\phi_1^+(x;\lambda)\psi_2^-(x;\lambda) + \phi_2^+(x;\lambda)\psi_1^-(x;\lambda) - \frac{\lambda E(\lambda)}{\sqrt{\lambda^2-1}}\right)dx.
\end{aligned} \tag{B.7}$$

Similarly, choosing $x > 0$, integrating (B.2) over $[0, x]$, and then taking $x \to +\infty$, we obtain

$$\begin{aligned}
\phi_1^+(0;\lambda)\frac{\partial\psi_2^-(0;\lambda)}{\partial\lambda} - \phi_2^+(0;\lambda)\frac{\partial\psi_1^-(0;\lambda)}{\partial\lambda} &= -a(\lambda)\left(1 + \frac{i\lambda}{\sqrt{\lambda^2-1}}\right)\\
&+ \int_0^\infty \left(\phi_1^+(x;\lambda)\psi_2^-(x;\lambda) + \phi_2^+(x;\lambda)\psi_1^-(x;\lambda)\right.\\
&\left. - \frac{\lambda E(\lambda)}{\sqrt{\lambda^2-1}}\right)dx.
\end{aligned} \tag{B.8}$$

Combining (B.3)–(B.8), we obtain (B.1). □

Corollary B.2. *If $\lambda = \lambda_0$ is an isolated eigenvalue of the discrete spectrum $\lambda \in \Sigma_{\text{dis}}$, such that $E(\lambda_0) = 0$ and $\boldsymbol{\phi}^+(x;\lambda_0) = b_0 \boldsymbol{\psi}^-(x;\lambda_0)$, then*

$$E'(\lambda_0) = 2b_0 \int_{-\infty}^{\infty} \psi_1^-(x;\lambda_0)\psi_2^-(x;\lambda_0)\,dx. \tag{B.9}$$

Lemma B.3. *Let $w_\epsilon(x) = w(x) + \epsilon W(x) \in L^1(\mathbb{R})$ and $E(\lambda;\epsilon)$ be defined by (3.16) for $w_\epsilon(x)$. For any $\lambda \in \mathcal{D}_+ \cup \Gamma_+ \setminus \{1\}$, the function $E(\lambda;\epsilon)$ is an entire function of ϵ and*

$$\frac{\partial E}{\partial \epsilon}(\lambda;0) = -\int_{-\infty}^{\infty} \left(W(x)\phi_2^+(x;\lambda)\psi_2^-(x;\lambda) + \overline{W}(x)\phi_1^+(x;\lambda)\psi_1^-(x;\lambda) \right) dx. \tag{B.10}$$

Proof. The proof is similar to that of Lemma B.1. It is based on the following relation between any two solutions $\boldsymbol{\varphi}(x;\lambda,\epsilon)$ and $\boldsymbol{\theta}(x;\lambda,\epsilon)$ of the AKNS problem (3.2) with $w_\epsilon(x) = w(x) + \epsilon W(x)$:

$$\frac{d}{dx}\left(\varphi_2 \frac{\partial \theta_1}{\partial \epsilon} - \varphi_1 \frac{\partial \theta_2}{\partial \epsilon} \right) = -W(x)\varphi_2\theta_2 - \overline{W(x)}\varphi_1\theta_1. \tag{B.11}$$

The analyticity of $\boldsymbol{\phi}^+(x;\lambda,\epsilon)$ and $\boldsymbol{\psi}^-(x;\lambda,\epsilon)$ in ϵ at every ϵ follows from the integral equations (3.19) by iteration and from the definition (3.16). Note that after we have replaced $w(s)$ by $w(s) + \epsilon W(s)$ and $\bar{w}(s)$ by $\bar{w}(s) + \epsilon \bar{W}(s)$ in (3.20), we can view ϵ as a complex parameter. Then, iteration of (3.19) yields an entire function of ϵ. Moreover, it follows from the integral equations (3.19) that

$$\left| \frac{\partial \phi_{1,2}^+(x;\lambda,\epsilon)}{\partial \epsilon} \right| \le C_\epsilon^+ e^{\text{Im}(k)x} \int_{-\infty}^{x} (|w(s)| + |\epsilon||W(s)|)\,ds, \tag{B.12}$$

$$\left| \frac{\partial \psi_{1,2}^-(x;\lambda,\epsilon)}{\partial \epsilon} \right| \le C_\epsilon^- e^{-\text{Im}(k)x} \int_{x}^{\infty} (|w(s)| + |\epsilon||W(s)|)\,ds, \tag{B.13}$$

where $\lambda = \sqrt{1-k^2}$, $\text{Im}(k) \le 0$ on $\lambda \in \mathcal{D}_+ \cup \Gamma_+ \setminus \{1\}$, and $C_\epsilon^\pm$ are suitable constants. The bound (B.12) implies that $\frac{\partial \phi_1^+}{\partial \epsilon}\psi_2^- - \frac{\partial \phi_2^+}{\partial \epsilon}\psi_1^-$ tends to zero as $x \to -\infty$, while the bound (B.13) implies that $\phi_1^+ \frac{\partial \psi_1^-}{\partial \epsilon} - \phi_2^+ \frac{\partial \psi_1^-}{\partial \epsilon}$ tends to zero as $x \to +\infty$. Integrating (B.11) under these conditions, as in the proof of Lemma B.1, we obtain (B.10). □

Using the same methods, we obtain the following corollary.

Corollary B.4. *Let $w_\epsilon \in L_1^1(\mathbb{R})$ and $G(k;\epsilon)$ be defined by (3.21) for $w_\epsilon(x)$. Then $G(k;\epsilon)$ is an entire function of ϵ for every k in the closed lower half-plane.*

Lemma B.5. *Let $w(x)$ be real, satisfy (4.1) with $p = 2$ and $b_\infty < -\frac{3}{8}$, and $w'(x) \in L_1^1(\mathbb{R})$. Let $W(x)$ be real, satisfy (4.1) with $p > 2$, and $W'(x) \in L_1^1(\mathbb{R})$. Let $\hat{G}(k;\epsilon)$ be defined similarly to $\hat{G}(k)$ for $w_\epsilon = w(x) + \epsilon W(x)$. Then $\hat{G}(k;\epsilon)$ is an entire function of ϵ for every k in the closed lower half-plane.*

Proof. The potential $V(x)$ in the decomposition (A.25) becomes

$$V_\epsilon(x) = V(x) - 2\epsilon W(x) - 2\epsilon w(x)W(x) + i\epsilon W'(x) - \epsilon^2 W^2(x).$$

From the assumptions of Lemma B.5, it follows that $V_\epsilon(x) \in L_1^1(\mathbb{R})$. Replacing $V(s)$ by $V_\epsilon(s)$ in (A.28), we obtain integral equations for $f_\epsilon^\pm(x;k)$, where the terms $f_0^\pm(x;k)$ are independent of ϵ. Considering the renormalization $\hat{f}_\epsilon^\pm(x;k) = k^q f_\epsilon^\pm(x;k)$, we prove the existence of the limit as $k \to 0$. It also follows by iteration (as in the proof of Lemma B.3) that the functions $\hat{f}_\epsilon^\pm(x;k)$ are continuously differentiable with respect to ϵ for $0 \leq k < 1$. Therefore, the Wronskian relation (A.27) is generalized for $\epsilon \neq 0$ and the assertion of the lemma follows. □

Lemma B.6. *Suppose that $w_0(x)$ is given by (5.18), $W(x)$ is real, and $W(x)$, $W'(x) \in L_2^1(\mathbb{R})$. Let $\hat{G}(k;\epsilon)$ be as in Lemma B.5. Then for $\kappa \in \mathcal{D}_+$, $\hat{G}(k;\epsilon)$ is continuously differentiable with respect to κ and $\frac{\partial \hat{G}}{\partial \kappa}(\kappa;\epsilon) = o(1/\kappa)$ as $\kappa \to 0$ uniformly in ϵ on any interval $[0,\epsilon_0]$, $\epsilon_0 > 0$. Furthermore, $\frac{\partial^2 \hat{G}}{\partial \kappa \partial \epsilon}(\kappa;\epsilon) = o(1/\kappa)$ uniformly in $|\epsilon| < \epsilon_0$.*

Proof. Let us fix $\epsilon_0 > 0$. We use (A.24), but with potential

$$U_\epsilon(x) = U_0(x) + \tilde{U}_\epsilon(x),$$

where

$$U_0(x) = -2w_0(x) - w_0^2(x) + iw_0'(x),$$

$$\tilde{U}_\epsilon(x) = -2\epsilon W(x) - 2\epsilon w_0(x)W(x) + i\epsilon W'(x) - \epsilon^2 W^2(x).$$

It suffices to choose an $x_0 > 0$ and to consider (A.24) on $[x_0, \infty)$. We think of $\tilde{U}_\epsilon(x)$ as a perturbation of $U_0(x)$ and use a modified form of the integral equation (A.28). In contrast to the proof of Lemma B.5, we do not include $w_0'(x)$ in the perturbation, since it falls off like $x^{-3}$and therefore misses being in $L_2^1(\mathbb{R})$. Including $w_0'(x)$ in the perturbation would lead to complications when we differentiate (A.28) with respect to k. We let

$$\chi^+(x;k) = \phi_1^+(x;k) + i\phi_2^+(x;k), \qquad \chi^-(x;k) = \psi_1^-(x;k) + i\psi_2^-(x;k),$$

where ϕ_k^+ and ψ_k^- $(k = 1,2)$ are the components of the solutions given in (5.19) and (5.20), respectively. Then the function

$$f_0^+(x;k) = [\sqrt{1-k^2} + i(1-k)]^{-1}\chi^-(x;k) \to e^{-ikx}, \quad x \to +\infty,$$

is the Jost solution for the unperturbed problem with potential $U_0(x)$. The Jost solution of the perturbed problem will be denoted by $f^+(x;k)$; the ϵ dependence is suppressed. Writing the integral equation in terms of (note that $q = 1$)

$$h^+(x;k) = ke^{ikx}f^+(x;k), \qquad h_0^+(x;k) = ke^{ikx}f_0^+(x;k),$$

we obtain

$$h^+(x;k) = h_0^+(x;k) + \int_x^\infty e(x,s;k)\tilde{U}_\epsilon(s)h^+(s;k)\,ds, \tag{B.14}$$

where

$$e(x,s;k) = (-4k\sqrt{1-k^2})^{-1} e^{ik(x-s)} (\chi^-(x;k)\chi^+(s;k) - \chi^-(s;k)\chi^+(x;k)), \quad \text{(B.15)}$$

$$h_0^+(x;k) = \frac{2i}{i-2x} + k.$$

Let

$$y_m(x,s;k) = e^{2ik(x-s)} - \sum_{n=0}^{m} \frac{[2ik(x-s)]^n}{n!}, \quad m = 0, 1, 2, 3,$$

be the remainders of successive Taylor polynomials approximating $e^{2ik(x-s)}$. The kernel $e(x,s;k)$ in (B.15) is given by

$$\begin{aligned} &2(-i+2x)(-i+2s)e(x,s;k) \\ &\quad - \frac{1}{k}(i + s(-2-4ix) - 2x)y_0 + \frac{4}{k^2}(s-x)y_1 - \frac{4i}{k^3}y_2. \end{aligned} \quad \text{(B.16)}$$

Using the estimates

$$|y_m(x,s;k)| \le C\frac{[k(s-x)]^{m+1}}{1+k(s-x)} \le C\frac{(ks)^{m+1}}{1+ks}, \quad s \ge x \ge x_0, \quad \text{(B.17)}$$

we obtain

$$|e(x,s;k)| \le \frac{Cs^2}{x}, \quad s \ge x \ge x_0. \quad \text{(B.18)}$$

Using this bound in (B.14), together with Gronwall's inequality, we conclude that $|h^+(x;k)| \le Ck$, where C is independent of x, k, and ϵ. In order to get a handle on the k derivative of $h^+(x;k)$, we differentiate (B.14) with respect to k (denoting the derivative by a dot), which yields

$$\begin{aligned} \dot{h}^+(x;k) \;&=\; 1 + \int_x^\infty \dot{e}(x,s;k)\tilde{U}_\epsilon(s)h^+(s;k)\,ds \\ &\quad + \int_x^\infty e(x,s;k)\tilde{U}_\epsilon(s)\dot{h}^+(s;k)\,ds. \end{aligned} \quad \text{(B.19)}$$

The expression for $\dot{e}(x,s;k)$ is lengthy and so will not be stated explicitly. Using (B.16) and (B.17), we estimate

$$|\dot{e}(x,s;k)| \le \frac{C}{k}\left(\frac{ks}{1+ks}\right)\frac{s^2}{x}, \quad s \ge x \ge x_0. \quad \text{(B.20)}$$

We briefly describe the strategy used in deriving (B.20) by considering a typical term. First note that $\dot{e}(x,s;k)$ is continuous at $k=0$ and so all the terms involving inverse powers of k must cancel out. When we differentiate the right-hand side of (B.16) with respect to k, the most singular term involves the fraction $e^{2ik(x-s)}/k^4$. We write it as

$$\frac{e^{2ik(x-s)}}{k^4} = \frac{1}{k^4} + \frac{2i(x-s)}{k^3} - \frac{2(x-s)^2}{k^2} - i\frac{4(x-s)^3}{3k} + \frac{y_3(x,s;k)}{k^4}.$$

As already mentioned, the first four terms will cancel out exactly when combined with other similar terms from the k-derivative of (B.16). The fifth term is estimated as follows:

$$\frac{|y_3(x,s;k)|}{k^4} \le \frac{C}{k}\left(\frac{ks}{1+ks}\right)s^3. \tag{B.21}$$

In view of the factors on the left-hand side of (B.16), the expression (B.21) gives a contribution to $|\dot{e}(x,s;k)|$ of the form of the right-hand side of (B.20). Using (B.20) and applying the dominated convergence theorem, we see that

$$\left|\int_x^\infty \dot{e}(x,s;k)\tilde{U}_\epsilon(s)h^+(s;k)\,ds\right| \le \frac{C}{kx}\int_x^\infty \left(\frac{ks}{1+ks}\right)s^2|\tilde{U}_\epsilon(s)|\,ds = o(1/k).$$

Hence, by using Gronwall's inequality in (B.19), it follows that $|\dot{h}^+(x;k)| = o(1/k)$ and thus $f^+(x;k) = o(1/k)$ (uniformly for $x \ge x_0$ and $0 \le \epsilon \le \epsilon_0$). Similarly, by differentiating (B.19) with respect to x, we find that $f^{+\prime}(x;k) = o(1/k)$. Moreover, by completely analogous arguments we see that $f^-(x;k) = o(1/k)$ and $f^{-\prime}(x;k) = o(1/k)$, on $x \le -x_0$. Since on $[-x_0, x_0]$ the potential $U_\epsilon(x)$ is integrable, this estimate can be transferred to the point x_0. Evaluating the Wronskian in (A.27) at x_0 and differentiating it with respect to k, we obtain the first assertion of the lemma. The second assertion follows from writing $\partial^2\hat{G}(\kappa;\epsilon)/\partial\kappa\partial\epsilon$ as a Cauchy integral involving $\partial\hat{G}(\kappa;\epsilon)/\partial\kappa$ and estimating the integral. For the contour of integration we choose a circle of radius $2\epsilon_0$ about zero in the ϵ-plane and consider $|\epsilon| \le \epsilon_0$. Then $|\partial^2\hat{G}(\kappa;\epsilon)/\partial\kappa\partial\epsilon| \le 2M(\kappa)/\epsilon_0$, where $M(\kappa) = \sup_{|\epsilon|=2\epsilon_0}|\partial\hat{G}(\kappa;\epsilon)/\partial\kappa|$. Since $M(\kappa) = o(1/\kappa)$ and ϵ_0 is arbitrary, the result follows. □

Acknowledgments

This work was supported by the EPSRC Visiting Fellowship GR/R88885/01, EPSRC Research grant GR/R02702/01, and NSERC Research grant No. 5-36694.

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