Traveling Waves in Fractional Models

Dmitry E. Pelinovsky

Abstract Fractional models of the Korteweg-de Vries (KdV) type are discussed in the context of propagation of one-dimensional traveling waves in nonlocal nonlinear dispersive systems. Spatially periodic waves can be constructed by using small-amplitude expansions, fixed-point methods, and calculus of variations. The existence theory is closely related to the stability theory, both of which provide the first step towards understanding of the nonlinear dynamics of traveling periodic waves in such nonlocal systems. Recent existence and stability results on the traveling periodic waves are reviewed for the fractional KdV models with quadratic and cubic nonlinearities.

1 Introduction

Nonlinear dispersive systems can be analyzed by using the power expansion of the dispersion relation near a selected wave number and the asymptotic small-amplitude slowly varying expansion of the wave profile near a constant equilibrium. It is well-known, see, e.g. [1], that the universal long-scale model for unidirectional wave propagation which appears in many physical contexts is the Korteweg–de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0, \tag{1}$$

where $u = u(x,t)$ is a scalar function in spatial coordinate x and time variable t and all variables have been normalized for convenience of notation. In the cases when the coefficient of the quadratic nonlinearity vanishes (which may happen due to the symmetry with respect to the sign change in the wave profile), the main model is written with the cubic nonlinearity and is usually referred to as the modified KdV (mKdV) equation

$$u_t + u^2 u_x + u_{xxx} = 0. \tag{2}$$

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Both models correspond to the expansion of the dispersion relation for the wave frequency ω with the wave number k up to the cubic powers via

$$\omega(k) = c_1 k + c_3 k^3 + \mathcal{O}(k^5) \quad \text{as} \quad k \to 0,$$

where c_1, c_3 are coefficients of the expansion. The nonlocal nonlinear dispersive systems of the KdV type were introduced in [2] as the way to generalize the dispersion relation to a non-cubic power expansion, e.g.

$$\omega(k) = c_1 k + c_{1+\alpha}|k|^\alpha k + \mathcal{O}(|k|^{2\alpha} k) \quad \text{as} \quad k \to 0,$$

where $\alpha \in (0, 2)$ is a parameter which models the fractional power of the wave dispersion. This expansion defines the fractional KdV equation

$$u_t + u u_x = (-\Delta)^{\alpha/2} u_x \tag{3}$$

and the fractional mKdV equation

$$u_t + u^2 u_x = (-\Delta)^{\alpha/2} u_x, \tag{4}$$

which are the main nonlocal nonlinear dispersive models considered here. When $\alpha = 2$, these equations reduce to the KdV equations (1) and (2). When $\alpha = 1$, these equations are referred to as the Benjamin–Ono (BO) equations, for which $(-\Delta)^{1/2} u_x = -\mathbf{H} u_{xx}$ is expressed by using the Hilbert transform $\mathbf{H}$.

The best way to understand the nonlocal operator $(-\Delta)^{\alpha/2}$ called *the fractional Laplacian* is to use the Fourier methods which depend on the spatial domain of the wave profiles. Spatially decaying (solitary) waves are posed on the real line $\mathbb{R}$, for which we can work with the Fourier transform. Spatially periodic waves are analyzed by using the Fourier series for periodic functions. For convenience of notation, we will always work in the normalized periodic domain $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, for which the Fourier series and the fractional Laplacian are defined by

$$f(x) = \sum_{n \in \mathbb{Z}} f_n e^{inx}, \quad (-\Delta)^{\alpha/2} f(x) = \sum_{n \in \mathbb{Z}} |n|^\alpha f_n e^{inx}. \tag{5}$$

Since the Fourier transform is an isomorphism in the space of squared integrable functions, the fractional models (3) and (4) are usually considered in a subset of squared integrable functions for which some integral quantities are conserved in the time evolution. The two basic conserved quantities have the meaning of energy

$$E(u) = \frac{1}{2} \int_{\mathbb{T}} ((-\Delta)^{\alpha/4} u)^2 dx - \frac{1}{k(k+1)} \int_{\mathbb{T}} u^{k+1} dx, \tag{6}$$

and momentum

$$F(u) = \frac{1}{2} \int_{\mathbb{T}} u^2 dx, \tag{7}$$

where $k = 2$ for (3) and $k = 3$ for (4). Another conserved quantity having the meaning of mass also makes sense on the periodic domain:

$$M(u) = \int_{\mathbb{T}} u \, dx. \tag{8}$$

The momentum and mass in (7) and (8) are defined in the space $L^2(\mathbb{T})$ of squared integrable functions on $\mathbb{T}$ thanks to the Cauchy–Schwarz inequality:

$$\|f\|_{L^1} \leq \sqrt{2\pi} \|f\|_{L^2}, \quad \forall f \in L^2(\mathbb{T}).$$

The first term of energy in (6) is defined in a subset of $L^2(\mathbb{T})$ for which $(-\Delta)^{\alpha/4} u \in L^2(\mathbb{T})$, which coincides with the Sobolev space $H^{\alpha/2}(\mathbb{T})$. Functions in $H^{\alpha/2}(\mathbb{T})$ are continuous and bounded if $\alpha > 1$, for which the periodic boundary conditions $u(-\pi) = u(\pi)$ are satisfied at the end points of the spatial domain. The second term of energy in (6) is bounded if $\alpha > (k-1)/(k+1)$ due to the Sobolev inequality [3],

$$\|f\|_{L^{k+1}} \leq C \|f\|_{H^{\alpha/2}}, \quad \forall f \in H^{\alpha/2}(\mathbb{T}),$$

where $k + 1 < 2/(1 - \alpha)$ and $C > 0$ is independent of $f \in H^{\alpha/2}(\mathbb{T})$. For $k = 2$ in (3), this yields the restriction $\alpha > 1/3$, and for $k = 3$ in (4), this yields the restriction $\alpha > 1/2$. The fractional KdV equations (3) and (4) are hence defined in the energy space $H^{\alpha/2}(\mathbb{T})$ with $\alpha > 1/3$ and $\alpha > 1/2$ respectively.

The initial-value problem for the fractional KdV equations must have a unique local solution in $H^{\alpha/2}(\mathbb{T})$ which is continuous with respect to the time t and to the initial data if we want the evolution model to be physically relevant and the conserved quantities (6), (7), and (8) to be useful. This leads to the concept of well-posedness which we will review on the infinite line $\mathbb{R}$.

We say that the initial-value problem for the fractional KdV equations is locally well-posed in $H^{\alpha/2}(\mathbb{R})$ if for every $u_0 \in H^{\alpha/2}(\mathbb{R})$, there is a maximal existence time $\tau_0 > 0$ and a solution $u \in C^0([0, \tau_0), H^{\alpha/2}(\mathbb{R}))$ such that $u(0, \cdot) = u_0$ and for every $\tau \in (0, \tau_0)$, the solution $u \in C^0([0, \tau], H^{\alpha/2}(\mathbb{R}))$ depends continuously on the initial data in a neighborhood of $u_0 \in H^{\alpha/2}(\mathbb{R})$. If $\tau_0 = \infty$, we say that the initial-value problem is globally well-posed.

Local well-posedness for (3) in $H^s(\mathbb{R})$ was obtained for $\alpha \in [1, 2]$ in [4] with $s > (9 - 3\alpha)/4$. This yields the global well-posedness in the energy space $H^{\alpha/2}(\mathbb{R})$ for $\alpha \geq 9/5$. Global well-posedness in $L^2(\mathbb{R})$ was shown in [5] for $\alpha \in (1, 2]$ and in [6] for $\alpha = 1$. Local well-posedness in $H^s(\mathbb{R})$ was obtained for $\alpha \in (0, 1)$ in [7] for $s > 3(4 - \alpha)/8$ and in [8] for $s > (6 - 5\alpha)/4$. As a result, the global well-posedness in the energy space $H^{\alpha/2}(\mathbb{R})$ was obtained for $\alpha \in (6/7, 1)$ and it was conjectured in [7] that the global well-posedness holds for all $\alpha \in (1/2, 1)$.

Local well-posedness for (4) in $H^s(\mathbb{R})$ was shown for $\alpha \in (1, 2]$ in [9] for $s \geq (7 - 3\alpha)/4$ and in [10] for $s \geq (3 - \alpha)/4$. Global well-posedness hold in the energy space $H^{\alpha/2}(\mathbb{R})$ for every $\alpha \in (1, 2]$. Global existence for small initial data in the energy space was shown for $\alpha = 1$ in [11]. A finite-time blow up for the critical value of momentum $F(u) = F_0$ was obtained in [12], where the critical value of momentum F_0 is the same for the entire family of the traveling solitary wave solutions of (4) with $\alpha = 1$.

Although the initial-value problem for (3) and (4) was considered on $\mathbb{R}$, similar (but appropriately modified) results are also relevant on $\mathbb{T}$.

The main focus of this chapter is on the existence and stability of traveling periodic waves in the fractional KdV equations. Compared to the local KdV equations, the existence and stability results cannot be concluded with the ODE methods. We have to deal with small-amplitude expansions, fixed-point methods, and calculus of variations to characterize the existence and stability of the traveling periodic waves.

The existence and stability results have already been obtained for traveling solitary waves on the infinite line in the energy space $H^{\alpha/2}(\mathbb{R})$. Solitary waves for (3) were characterized as minimizers of energy subject to the fixed momentum in [13] for $\alpha \geq 1$ and in [14] for $\alpha > 1/2$. Existence and uniqueness (modulo translations) of solitary waves was shown in [15] for (3) with $\alpha > 1/3$ and for (4) with $\alpha > 1/2$ by using a different variational characterization of solitary waves as minimizers of the Gagliardo–Nirenberg inequality. Existence results from [13, 14] combined with the local well-posedness in the energy space from [5–8] imply the orbital stability of solitary waves in (3) for $\alpha > 1/2$ in the following sense.

Let $\phi_c \in H^{\alpha/2}(\mathbb{R})$ be the profile of the traveling solitary wave

$$u(x, t) = \phi_c(x - ct)$$

with some speed $c \in \mathbb{R}$. We say that the traveling solitary wave is orbitally stable in $H^{\alpha/2}(\mathbb{R})$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if the initial data $u_0 \in H^{\alpha/2}(\mathbb{R})$ is close to ϕ_c as

$$\|u_0 - \phi_c\|_{H^{\alpha/2}} < \delta,$$

then the unique global solution $u \in C^0([0, \infty), H^{\alpha/2}(\mathbb{R}))$ to the fractional KdV equation with $u(\cdot, 0) = u_0$ stays close to the orbit $\{\phi_c(\cdot - s)\}_{s\in\mathbb{R}}$ as

$$\inf_{s\in\mathbb{R}} \|u(\cdot, t) - \phi_c(\cdot - s)\|_{H^{\alpha/2}} < \epsilon, \quad t \geq 0.$$

The solitary waves in (3) were shown in [16] to be orbitally stable for $\alpha > 1/2$ and unstable for $1/3 < \alpha < 1/2$. The border case of $\alpha = 1/2$ was left open. Similarly, the solitary waves in (4) were shown to be orbitally stable for $\alpha > 1$ and unstable for $1/2 < \alpha < 1$. Instability of solitary waves in (4) for the border case $\alpha = 1$ was shown in [12]. These results were extended to solitary waves in all dimensions in [17]. Transverse stability of solitary waves in the fractional models was addressed in [18].

In regards to the traveling periodic waves, we should agree to only consider the periodic solutions with a *single-lobe* profile for which there exists only one maximum (and minimum) point on $\mathbb{T}$. Periodic solutions with a multiple-lobe profile can be constructed by a scaling transformation from periodic solutions with a single-lobe profile.

Existence and stability of such traveling periodic waves were analyzed for (3) by using small-amplitude expansions [19] and fixed-point methods [20]. A variational characterization of periodic waves as constrained minimizers of energy E subject to fixed momentum F and mass M was developed in [21, 22] for (3) with $\alpha > 1/3$ under some non-degeneracy condition on the kernel of the linearized operator. Positivity of profiles of the periodic waves was proven in [23] based on the fixed-point methods under the same non-degeneracy condition. Positive waves were also characterized in [24] for $\alpha > 1/2$ as constrained minimizers of energy E subject to fixed momentum F without constraint on the mass M. Minimizers of energy exist for every positive momentum and each such minimizer is degenerate only up to the translation symmetry.

A variational characterization of periodic waves similar to the minimization of the Gagliardo–Nirenberg inequality was developed in [25] for (3) and in [26] for (4). The periodic waves were characterized as constrained minimizers of the quadratic part of energy E and the quadratic momentum F subject to the fixed non-quadratic part of energy E and mass M. It was shown in [25] for (3) that the zero-mass constraint $M(u) = 0$ can be set without loss of generality to obtain *all* traveling periodic waves for $\alpha > 1/3$ including those for which the non-degeneracy condition from [22] fails and those which are not constrained minimizers of energy for fixed momentum in [24]. On the other hand, a more complicated structure of traveling periodic waves arises for (4) with three distinct branches obtained in [26]. A complete characterization of all branches was developed in [27] but only in the local case $\alpha = 2$, for which the fractional mKdV equation (4) becomes the mKdV equation (2). Further results on existence and stability of traveling periodic waves in other fractional models were obtained in [28, 29].

The main purpose of this chapter is to review the recent results on the existence and stability of the traveling periodic waves in (3) and (4). In the sequential order of appearance, we will explain the small-amplitude expansion in Sect. 2, the fixed-point methods in Sect. 3, and the variational characterization of periodic waves in Sect. 4. Section 5 specifies the criteria for spectral stability of the traveling periodic waves, which also yield the orbital stability in cases when the time evolution of the initial-value problem is locally well-posed in the energy space $H^{\alpha/2}(\mathbb{T})$. Section 6 gives numerical examples of the traveling periodic waves with the single-lobe profiles. Section 7 concludes the chapter with a list of open questions for future studies.

2 Small-Amplitude Expansions

A traveling wave in the fractional KdV equations is given by a solution of the form $u(x,t) = \phi_c(x - ct)$, where c is the wave speed and ϕ_c is the wave profile defined on $\mathbb{T}$. Traveling wave solutions satisfy the stationary equation:

$$(-\Delta)^{\alpha/2}\phi_c + c\phi_c + b = \frac{1}{k}\phi_c^k, \tag{9}$$

where b is another constant obtained after integrating (3) and (4) in x. As before, we use $k = 2$ for (3) and $k = 3$ for (4). It follows from (9) that the profile ϕ_c actually depends on two parameters c and b, so that we should write $\phi_c \equiv \phi_{c,b}$ in a general case. This is one complication in the study of the traveling periodic waves compared to the traveling solitary waves for which $b = 0$ follows from the decay of ϕ_c to zero at infinity.

For the fractional KdV equation with quadratic nonlinearity, $k = 2$, the problem with appearance of the new parameter b can be resolved with a transformation, which is usually referred as *the Galilean transformation*.

If $\phi_{c,b}$ is a suitable solution to (9) with $k = 2$ for some (c, b) satisfying $c^2 + 2b \geq 0$, then

$$\varphi_\omega := \phi_{c,b} - c - \sqrt{c^2 + 2b}, \quad \omega := \sqrt{c^2 + 2b} \tag{10}$$

is a solution of

$$(-\Delta)^{\alpha/2}\varphi_\omega - \omega\varphi_\omega = \frac{1}{2}\varphi_\omega^2. \tag{11}$$

Every periodic solution φ_ω of (11) for some fixed $\omega \geq 0$ generates a curve of periodic solutions of (9) with $k = 2$ in the parameter plane (c, b) in the region $c^2 + 2b \geq 0$. Moreover, if φ_ω is a solution of (11) for $\omega > 0$, then $\varphi_{-\omega} := 2\omega + \varphi_\omega$ is a solution of (11) with ω replaced by $-\omega$.

Since $b + c\phi - \phi^2/2$ is strictly negative for $c^2 + 2b < 0$, there exist no periodic solutions in the region $c^2 + 2b < 0$ with the contradiction appearing for every $\phi_c \in H^{\alpha/2}(\mathbb{T})$:

$$0 = \int_{\mathbb{T}} (-\Delta)^{\alpha/2}\phi_c dx = -\int_{\mathbb{T}} (b + c\phi_c - \frac{1}{2}\phi_c^2) dx > 0.$$

Hence, the Galilean transformation transforms all periodic solutions of (9) with $k = 2$ to the stationary equation (11) with the only parameter ω. Figure 1 shows the existence region of the single-lobe periodic solutions of (11) with $\omega \geq 1$ on the (c, b) parameter plane obtained by means of (10).

A robust method to understand the existence properties of periodic solutions of (11) is based on the perturbative method which is referred to as *the small-amplitude expansions*. This method describes the bifurcation of a family of periodic solutions

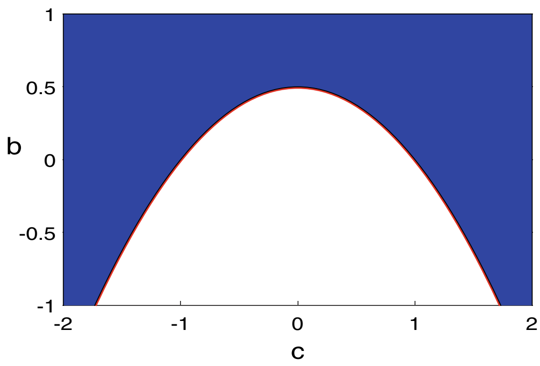

Fig. 1 The existence region on the parameter plane (c, b) for the single-lobe periodic solutions of the stationary equation (9) obtained with (10) and (11) for $\omega \geq 1$

from the constant solution. The linear operator $\mathcal{L}_\omega := (-\Delta)^{\alpha/2} - \omega$ in $L^2(\mathbb{T})$ has a purely discrete spectrum of eigenvalues. It follows from the Fourier series (5) that eigenvalues of $\mathcal{L}_\omega$ are located at the points $\{|n|^\alpha - \omega, \;\; n \in \mathbb{Z}\}$. Periodic solutions with the single-lobe profile (defined in Sect. 1) correspond to the first Fourier modes with $n = \pm 1$ for which the operator $\mathcal{L}_\omega$ has zero eigenvalue for $\omega = \omega_0 := 1$. Bifurcation of the periodic solutions with the single-lobe profile is expected for $\omega \neq \omega_0$ in either $\omega > \omega_0$ or $\omega < \omega_0$.

Computations of the small-amplitude expansions are fully algorithmic. We substitute the following expansions

$$\varphi_\omega(x) = a\varphi_1(x) + a^2\varphi_2(x) + a^3\varphi_3(x) + O(a^4),$$
$$\omega = \omega_0 + a^2\omega_2 + O(a^4)$$

into (11) and collect equations in powers of the formal small parameter a:

$$\begin{aligned} O(a) &: \quad ((-\Delta)^{\alpha/2} - \omega_0)\varphi_1 = 0, \\ O(a^2) &: \quad ((-\Delta)^{\alpha/2} - \omega_0)\varphi_2 = \tfrac{1}{2}\varphi_1^2, \\ O(a^3) &: \quad ((-\Delta)^{\alpha/2} - \omega_0)\varphi_3 - \omega_2\varphi_1 = \varphi_1\varphi_2. \end{aligned}$$

Since $(-\Delta)^{\alpha/2}$ has the translation symmetry and preserves even parity, we can consider the wave profile φ_ω to be even in x in order to define the translational parameter uniquely. Since a is the free amplitude parameter (positive or negative), we normalize the leading-order solution as

$$\varphi_1(x) = \cos(x)$$

and require that all other corrections of the expansion of φ_ω to be even and orthogonal to φ_1. Solving the inhomogeneous equation at $O(a^2)$ yields the exact periodic solution as

$$\varphi_2(x) = -\frac{1}{4} + \frac{1}{4(2^\alpha - 1)}\cos(2x).$$

The inhomogeneous equation at $\mathcal{O}(a^3)$ admits non-periodic solutions of the type $x\sin(x)$ unless ω_2 is uniquely defined by removing the non-periodic term. This procedure yields the exact value:

$$\omega_2 = \frac{1}{4} - \frac{1}{8(2^\alpha - 1)}.$$

With this value of ω_2, the inhomogeneous equation at $\mathcal{O}(a^3)$ yields the exact periodic solution:

$$\varphi_3(x) = \frac{1}{8(2^\alpha - 1)(3^\alpha - 1)} \cos(3x).$$

This completes the small-amplitude expansions up to the $\mathcal{O}(a^4)$ order.

The method of *Lyapunov–Schmidt reductions* can be used to justify the small-amplitude expansions. This justification was developed in Lemma 2.1 and Theorem A.1 of [19] for $\alpha > 1/2$, which is the restriction to ensure smoothness of the nonlinear term in the domain space $H^\alpha(\mathbb{T})$ of operator $\mathcal{L}_\omega$. The result can be extended to every $\alpha > 1/3$ by working in the form domain $H^{\alpha/2}(\mathbb{T})$ of $\mathcal{L}_\omega$. The following theorem presents the small-amplitude expansion of the periodic function with the profile ϕ_c which solves (9) with $k = 2$ and $b = 0$.

Theorem 1 *For every $\alpha > 1/3$, there exists $a_0 > 0$ such that for every $a \in (-a_0, a_0)$, there exists a locally unique, even, single-lobe solution (c, ϕ_c) of (9) with $k = 2$ and $b = 0$, which is expressed by the expansion*

$$\phi_c(x) = 2c + a\cos(x) - \frac{1}{4}a^2 + \frac{1}{4(2^\alpha - 1)}a^2\cos(2x) + \frac{1}{8(2^\alpha - 1)(3^\alpha - 1)}a^3\cos(3x) + \mathcal{O}(a^4), \tag{12}$$

and

$$c = 1 + \frac{2^{\alpha+1} - 3}{8(2^\alpha - 1)}a^2 + \mathcal{O}(a^4). \tag{13}$$

The mapping $a \mapsto (c, \phi_c)$ is smooth near $a = 0$.

One striking feature follows from (13) that the single-lobe periodic solution bifurcates to $c > 1$ if $\alpha > \alpha_0$ and to $c < 1$ if $\alpha < \alpha_0$, where

$$\alpha_0 := \frac{\log 3}{\log 2} - 1 \approx 0.585. \tag{14}$$

This property was discovered in [23]. The presence of α_0 implies a non-trivial fold bifurcation of the periodic waves in the fractional KdV equation (3) with $\alpha < \alpha_0$ explored in [25]. We will elaborate more on the fold bifurcation of the traveling periodic waves for $\alpha < \alpha_0$ in Sect. 4. Solutions of Theorem 1 correspond to the line $\{(c, b) : \; c \geq 1, \; b = 0\}$ on Fig. 1 for $\alpha > \alpha_0$.

Small-amplitude expansions are also important for the fractional mKdV equation (4), see Sect. 3.1, 5.1 and Appendix A in [26]. However, the construction of traveling periodic waves is more complicated in the cubic case because the additional parameter b in the stationary equation (9) with $k = 3$ cannot be eliminated by the Galilean transformation. As a result, three branches of solutions coexist for $\alpha \in (\frac{1}{2}, 2]$, see [26, 27].

The small-amplitude expansions of one of the three branches of the periodic solutions can be computed for (9) with $k = 3$ and $b = 0$. The periodic solutions with the single-lobe profile bifurcate from the constant solution $\phi_c = \sqrt{3c}$ at $c = 1/2$, see Proposition 5.1 in [26]. It has been discovered that the branch of the periodic solutions bifurcates to $c > 1/2$ if $\alpha > \alpha_0$ and to $c < 1/2$ if $\alpha < \alpha_0$, where the threshold value of $\alpha = \alpha_0$ is now given by

$$\alpha_0 := \frac{\log 8 - \log 5}{\log 2} - 1 \approx 0.678. \tag{15}$$

A fold bifurcation exists for the periodic waves in the fractional mKdV equation (4) with $\alpha < \alpha_0$ in addition to the pitchfork bifurcation of the three solution branches which exists for every $\alpha \in (1/2, 2]$, see [26, 27].

3 Fixed-Point Methods

Fixed-point methods allow us to obtain information on the periodic solutions of the stationary equation (9) with (c, b) away from the bifurcation point where the periodic solutions bifurcate from the constant solutions. Positivity of the single-lobe profile $\phi_{c,b}$ can be proven for a subset of parameters (c, b).

In order to rewrite the stationary equation (9) as the fixed-point problem, we set $b = 0$ and consider $c > 0$, then the linear operator $\mathcal{L}_c := c + (-\Delta)^{\alpha/2}$ in $L^2(\mathbb{T})$ is strictly positive and invertible with a bounded inverse. By using the Fourier series (5), we can write the inverse operator $\mathcal{L}_c^{-1}$ in the form

$$\mathcal{L}_c^{-1} f = \sum_{n \in \mathbb{Z}} \frac{f_n}{c + |n|^\alpha} e^{inx}.$$

It can also be written in the convolution form

$$[\mathcal{L}_c^{-1} f](x) = \int_{\mathbb{T}} G_c(x - y) f(y) dy,$$

where G_c is the Green function defined by the Fourier series

$$G_c(x) := \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{e^{inx}}{c + |n|^\alpha}. \tag{16}$$

Let us use the Green function G_c in order to rewrite the stationary equation (9) with $k = 2$, $b = 0$, and $c > 0$ as the fixed-point equation $\phi_c = \mathcal{A}_c(\phi_c)$ for the following nonlinear operator

$$[\mathcal{A}_c(\phi)](x) := \frac{1}{2}\int_{\mathbb{T}} G_c(x-y)\phi(y)^2 dy. \tag{17}$$

Existence and positivity of the single-lobe profile ϕ_c as a fixed point of $\mathcal{A}_c$ in $H^{\alpha/2}(\mathbb{T})$ follows from applications of *the Krasnoselskii theorem*, see Corollary 20.1 in [30]. This was shown in Theorem 2.2 of [23] for every $c > 1$ under a non-degeneracy assumption on the linearized operator

$$\mathcal{M}_c := c + (-\Delta)^{\alpha/2} - \phi_c.$$

Due to the translation invariance of the stationary equation (9) and smoothness of the wave profile $\phi_c \in H^\infty(\mathbb{T})$, we have

$$\mathcal{M}_c \partial_x \phi_c = 0$$

with $\partial_x \phi_c \in \mathrm{Ker}(\mathcal{M}_c)$. The non-degeneracy assumption

$$\mathrm{Ker}(\mathcal{M}_c) = \mathrm{span}(\partial_x \phi_c)$$

implies that the zero eigenvalue of $\mathcal{M}_c$ is simple in $L^2(\mathbb{T})$. It was shown in [25] that the non-degeneracy assumption can only be satisfied if $\alpha > \alpha_0$, where α_0 is given by (14), and is violated for some periodic solutions with the single-lobe profile if $\alpha < \alpha_0$. The following theorem states positivity of the single-lobe profile ϕ_c as a fixed point of $\mathcal{A}_c$ in $H^{\alpha/2}(\mathbb{T})$ for $c > 1$ if $\alpha > \alpha_0$.

Theorem 2 *Fix $\alpha \in (\alpha_0, 2]$. For every $c > 1$, there exists a fixed point $\phi_c \in H^{\alpha/2}(\mathbb{T})$ of the nonlinear operator $\mathcal{A}_c$ in (17) such that $\min_{x\in\mathbb{T}} \phi_c(x) > 0$ provided that* $\mathrm{Ker}(\mathcal{M}_c) = \mathrm{span}(\partial_x \phi_c)$ *for every $c > 1$.*

We illustrate the proof of Theorem 2 based on the following steps.

1. *For every $c > 0$ and every $\alpha \in (0, 2]$, the Green function G_c in (16) is even, strictly positive on $\mathbb{T}$, and monotonically decreasing away from $x = 0$.*

   This result on $\mathbb{R}$ was shown in Lemma A.4 in [15]. The proof on $\mathbb{T}$ was developed in Theorem 1.1 in [31] by exploring relations of the Green function G_c to the Mittag–Leffler functions. A simpler proof exploring a probabilistic representation of the Green function G_c was given in [32]. For illustration, we show profiles of the Green function G_c for specific values of c in Fig. 2 for $\alpha = 0.5$ (left) and $\alpha = 1.5$ (right). Note that $G_c(0) = \infty$ for $\alpha \leq 1$ and $G_c(0) < \infty$ for $\alpha > 1$.

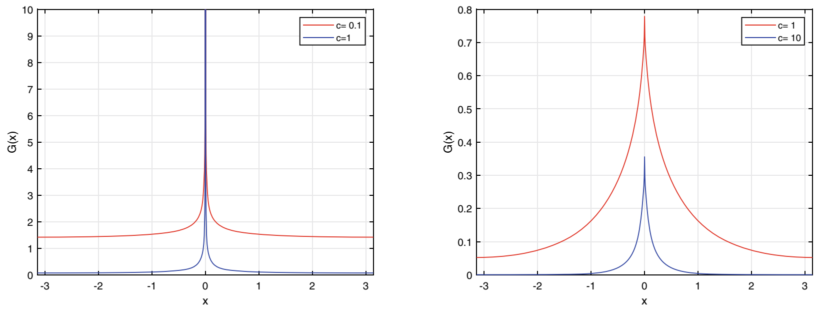

Fig. 2 Profiles of G_c for specific values of c are plotted versus x on $\mathbb{T}$ for $\alpha = 0.5$ (left) and $\alpha = 1.5$ (right). Reproduced from [31]. ©2021 by Springer Nature

2. *For every* $c > 0$ *and* $\alpha \in (1/2, 2]$, *there exists a positive set* $P_c \subset L^2(\mathbb{T})$ *such that the nonlinear operator* $\mathcal{A}_c : P_c \mapsto P_c$ *is compact.*

   We can give a precise definition of the positive set $P_c \subset L^2(\mathbb{T})$. From Step 1, it follows that there is $m_c > 0$ such that $G_c(x) \geq m_c$ for every $x \in \mathbb{T}$. On the other hand, it follows from (16) that $G_c \in L^2(\mathbb{T})$ for $\alpha > 1/2$, hence $M_c := \|G_c\|_{L^2}$. The positive set is then given by

$$P_c := \left\{ \phi \in L^2(\mathbb{T}) : \quad \phi(x) \geq \frac{m_c}{M_c} \|\phi\|_{L^2} \right\}.$$

   The operator $\mathcal{A}_c$ is bounded and continuous on $L^2(\mathbb{T})$ due to Young's convolution inequality:

$$\|\mathcal{A}_c(\phi)\|_{L^2} \leq \frac{1}{2} \|G_c\|_{L^2} \|\phi^2\|_{L^1} = \frac{1}{2} M_c \|\phi\|_{L^2}^2.$$

   Moreover, it is the limit of compact operators defined by the partial sum of the Fourier series, hence it is compact. Finally, $\mathcal{A}_c$ is invariant in P_c due to the following bound:

$$[\mathcal{A}_c(\phi)](x) \geq \frac{1}{2} m_c \|\phi\|_{L^2}^2 \geq \frac{m_c}{M_c} \|\mathcal{A}_c(\phi)\|_{L^2}.$$

3. *There are* $0 < r_- < r_+ < \infty$ *such that there exists a fixed point of* $\mathcal{A}_c$ *in* $P_c \cap (\overline{B}_{r_+} \backslash B_{r_-})$, *where* B_r ($\overline{B}_r$) *is an open (closed) ball of radius* r *in* $L^2(\mathbb{T})$.

   If

$$r_- < \frac{2}{M_c} \leq \frac{2}{\sqrt{2\pi} m_c} < r_+,$$

   then bounds in Step 2 imply that

$$\|\mathcal{A}_c(\phi)\|_{L^2} \le \frac{1}{2} M_c r_- \|\phi\|_{L^2} < \|\phi\|_{L^2}, \quad \phi \in P_c \cap \partial B_{r_-}$$

and

$$\|\mathcal{A}_c(\phi)\|_{L^2} \ge \frac{1}{2}\sqrt{2\pi} m_c r_+ \|\phi\|_{L^2} > \|\phi\|_{L^2}, \quad \phi \in P_c \cap \partial B_{r_+},$$

where ∂B_r is the boundary of $\overline{B}_r$. Note that $G_c(x) \ge m_c$ implies $M_c \ge \sqrt{2\pi} m_c$ which ensures that $r_+ > r_-$. The existence of a fixed point of $\mathcal{A}_c$ in $P_c \cap (\overline{B}_{r_+} \backslash B_{r_-})$ follows from the Krasnoselskii fixed-point theorem.

4. *If* $\phi_c \in L^2(\mathbb{T})$ *is a fixed point for* $\mathcal{A}_c$*, then* $\phi_c \in H^\infty(\mathbb{T})$.

   This result has been proven independently by several authors, see Proposition 2.1 in [22], Theorem 2.2 in [23], and Proposition 2.4 in [25].

5. *If* $\alpha \in (\alpha_0, 2]$*, the fixed point* ϕ_c *of* $\mathcal{A}_c$ *has the positive single-lobe profile for every* $c > 1$ *provided that* $\mathrm{Ker}(\mathcal{M}_c) = \mathrm{span}(\partial_x \phi_c)$ *for every* $c > 1$.

   The constraint $c > 1$ appears from the fact that the constant solution $\phi = 2c$ is always a fixed point of $\mathcal{A}_c$ in $P_c \cap (\overline{B}_{r_+} \backslash B_{r_-})$. However, the associated linearized operator $\mathcal{A}_c'(2c) = 2c\mathcal{L}_c^{-1}$ in $L^2(\mathbb{T})$ admits eigenvalues

$$\{2c(c + |n|^\alpha)^{-1}, \; n \in \mathbb{Z}\}.$$

   Recall that the fixed-point operator $\mathcal{A}_c(\phi_c)$ is a contraction at a fixed point $\phi_c \in H^{\alpha/2}(\mathbb{T})$ if all eigenvalues of $\mathcal{A}_c'(\phi_c)$ in $L^2(\mathbb{T})$ are inside the unit disk. For $c \in (0, 1)$ there is only one eigenvalue of $\mathcal{A}_c'(2c)$ outside the unit disk, it corresponds to $n = 0$. At $c = 1$, a pair of eigenvalues of $\mathcal{A}_c'(2c)$ for $n = \pm 1$ crosses the unit circle and this corresponds to the bifurcation studied in Theorem 1. For $c > 1$, three or more eigenvalues of $\mathcal{A}_c'(2c)$ are outside the unit circle and, if $\alpha > \alpha_0 \approx 0.585$, the single-lobe positive solution $\phi = \phi_c$ bifurcates from the constant solution $\phi = 2c$ for $c > 1$ and exists as the small-amplitude expansion (12)–(13) in Theorem 1. Since the kernel of $\mathcal{M}_c$ is assumed to be non-degenerate, the number of negative eigenvalues of $\mathcal{M}_c$ persists for $c > 1$. By Lemma 2.2 and 2.4 in [23], there is only one negative eigenvalue of $\mathcal{M}_c$ for every $c > 1$ and for $\alpha > \alpha_0$. Hence, there is only one eigenvalue outside the unit disk for the associated linearized operator

$$\mathcal{A}_c'(\phi_c) = \mathcal{L}_c^{-1}\phi_c = \mathrm{Id} - \mathcal{L}_c^{-1}\mathcal{M}_c.$$

   As a result, the branch of the fixed points of $\mathcal{A}_c$ with a positive single-lobe profile $\phi_c \in H^{\alpha/2}(\mathbb{T})$ in $P_c \cap (\overline{B}_{r_+} \backslash B_{r_-})$ has only one eigenvalue of $\mathcal{A}_c'(\phi_c)$ outside the unit disk for every $c > 1$, whereas any other branch of fixed points of $\mathcal{A}_c$ with constant or multiple-lobe periodic profiles has three or more eigenvalues. Hence, the branch with the positive single-lobe profile ϕ_c does not coalesce with any other branches of fixed points of $\mathcal{A}_c$ and persists for every $c > 1$.

The result of Theorem 2 is very useful in the numerical approximations of the periodic solutions of the stationary equation (9) with $k = 2$ and $b = 0$ by using Petviashvili's iteration method [23]. In fact, Petviashvili's method implements iterations of the operator $\mathcal{A}_c$ after adding a constraint which removes the only eigenvalue of $\mathcal{A}_c'(\phi_c)$ outside the unit disk. Although it might not be obvious why the positive solution of Theorem 2 should have the single-lobe profile for every $c > 1$ far away from the bifurcation point, this follows from Sturm's oscillation theory for the fractional operator $\mathcal{M}_c$ developed in [22] since $\mathcal{M}_c$ admits only one negative eigenvalue in $L^2(\mathbb{T})$ under the non-degeneracy assumption $\text{Ker}(\mathcal{M}_c) = \text{Ker}(\partial_x \phi_c)$.

Theorem 4 below suggests that $\mathcal{M}_c$ has exactly one negative eigenvalue and a simple zero eigenvalue for c being close to 1 if $\alpha > \alpha_0$, see (32). Hence, the non-degeneracy assumption of Theorem 2 is satisfied for $\alpha > \alpha_0$ at least for c close to 1. On the other hand, $\mathcal{M}_c$ has two negative eigenvalues and a simple zero eigenvalue for c being close to 1 if $\alpha < \alpha_0$. Consequently, we are not able to use the continuation argument in Step 5 to guarantee positivity of the single-lobe profile ϕ_c as the fixed point of $\mathcal{A}_c$ for $\alpha < \alpha_0$.

If the kernel of $\mathcal{M}_c$ remains non-degenerate, the periodic solution with the single-lobe profile ϕ_c remains positive by Theorem 2. It follows from Theorem 3.5 in [20] that the Fourier coefficients of this periodic solution are also positive. The converse statement was obtained in Theorem 4.1 in [33]: if the profile of the periodic solution is positive and the Fourier coefficients are positive, then $\mathcal{M}_c$ has only a simple negative eigenvalue and a simple zero eigenvalue while the rest of its spectrum is strictly positive.

It is not so easy to extend the result of Theorem 2 for $k = 3$. The nonlinear operator

$$[\mathcal{A}_c(\phi)](x) := \frac{1}{3} \int_{\mathbb{T}} G_c(x - y)\phi(y)^3 dy$$

can be closed either in $H^{\alpha/2}(\mathbb{T})$ or in $L^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$ but the positive set P_c does not follow from the lower bound on $[\mathcal{A}_c(\phi)](x)$ because the cubic term ϕ^3 can be negative. Although Petviashvili's method approximates well the periodic solutions with the positive single-lobe profile ϕ_c for (9) with $k = 3$ and $b = 0$, ***the analytical proof of the existence of positive fixed points of*** $\mathcal{A}_c$ ***for*** $k = 3$ ***is an open question for further studies.***

4 Variational Characterizations

Let us summarize on the existence of periodic solutions with the single-lobe profile ϕ_c in the stationary equation (9) with $k = 2$ and $b = 0$ given by Theorems 1 and 2. Two small-amplitude periodic solutions exist and can be constructed by using the periodic solution with the single-lobe profile φ_ω defined by the periodic solutions of (11) with $\omega > 0$. One solution has the positive single-lobe profile $\phi_c = 2c + \varphi_c$ and exists for $c > 1$ if $\alpha \in (\alpha_0, 2]$. Another solution has the sign-indefinite single-lobe profile $\phi_c = \varphi_{-c}$ and exists for $c < -1$ if $\alpha \in (\alpha_0, 2]$. The two solutions correspond to the following two semi-infinite lines in Fig. 1:

$$\{(c,b): \ c \geq 1, \ b = 0\} \quad \text{and} \quad \{(c,b): \ c \leq -1, \ b = 0\}.$$

We can ask if we can characterize *all* periodic solutions with the single-lobe profile $\phi_{c,b}$ in the stationary equation (9) with $k = 2$, $c^2 + 2b \geq 0$, and $\alpha \in (1/3, 2]$. This task can be accomplished with *a variational formulation* of the periodic solutions.

A straightforward variational formulation follows from the fact that the stationary equation (9) is the Euler–Lagrange equation for the augmented energy $G_{c,b}(u) := E(u) + cF(u) + bM(u)$, where E, F, and M are the conserved energy, momentum, and mass in (6), (7), and (8). This suggests that the periodic solutions of (9) can be characterized from the following constrained minimization problem:

$$\inf_{u \in H^{\alpha/2}(\mathbb{T})} \{E(u): \quad F(u) = F_0, \quad M(u) = M_0\}. \tag{18}$$

By Proposition 2.1 in [22], a local minimizer of (18) exists for $k = 2$ and every $\alpha \in (1/3, 2]$ and it is a critical point of $G_{c,b}$ for some (c, b). However, some questions are not answered by the variational formulation (18):

1. What are admissible values of F_0 and M_0 in (18) and of c and b in (9)?
2. Are the local minimizers C^1 smooth with respect to parameters (c, b)?
3. Do the local minimizers have the single-lobe profile $\phi_{c,b}$?

The first question is important for obtaining *all* periodic solutions of (9) but the variational formulation (18) does not give a precise relation between (c, b) and (F_0, M_0) as well as their admissible values.

The second question is important for characterizing the non-degeneracy of the linearized operator

$$\mathcal{M}_{c,b} : H^\alpha(\mathbb{T}) \subset L^2(\mathbb{T}) \to L^2(\mathbb{T}), \quad \mathcal{M}_{c,b} := (-\Delta)^{\alpha/2} + c - \phi_{c,b}. \tag{19}$$

If $\phi_{c,b}$ is C^1 smooth with respect to (c, b), then we have

$$\mathcal{M}_{c,b}\partial_c\phi_{c,b} = -\phi_{c,b} \quad \text{and} \quad \mathcal{M}_{c,b}\partial_b\phi_{c,b} = -1, \tag{20}$$

so that $\{1, \phi_{c,b}\} \in \text{Range}(\mathcal{M}_{c,b})$. Since we also have

$$\mathcal{M}_{c,b}\phi_{c,b} = -b - \frac{1}{2}\phi_{c,b}^2, \tag{21}$$

this would imply that $\{1, \phi_{c,b}, \phi_{c,b}^2\} \in \text{Range}(\mathcal{M}_{c,b})$, in which case the result of Lemma 3.3 in [22] suggests the non-degeneracy

$$\text{Ker}(\mathcal{M}_{c,b}) = \text{span}(\partial_x \phi_{c,b}). \tag{22}$$

Unfortunately, C^1 smoothness of local minimizers of (18) cannot be taken for granted and it fails for $\alpha \in (1/3, \alpha_0)$ due to the fold bifurcation of the local minimizers of (18), see [23, 25]. The fold bifurcation corresponds to the values of (c, b) for which $\text{Ker}(\mathcal{M}_{c,b})$ is spanned by $\partial_x \phi_{c,b}$ and another eigenfunction.

The third question is important to distinguish the periodic functions with the single-lobe profile $\phi_{c,b}$ from the constant functions. The constant functions can also be the local minimizers of (18) and they may exist for values of (F_0, M_0) for which the single-lobe minimizers of (18) do not exist, see [23, 25].

To answer some of these questions, a modified variational formulation was proposed in [24]. The periodic solutions of (9) are characterized from the following constrained minimization problem:

$$\inf_{u \in H^{\alpha/2}(\mathbb{T})} \{E(u) + bM(u) : \quad F(u) = F_0\}. \tag{23}$$

By Proposition 3.1 in [24], a local minimizer of (23) exists for every $\alpha \in (1/2, 2]$, $b \in \mathbb{R}$, and $F_0 > 0$, and it is a critical point of $G_{c,b}$ for some $c \neq 0$. The restriction of $\alpha > 1/2$ is similar to the stability constraint for the solitary waves of the same model [16]. By Lemma 4.1 in [24], the local minimizer of (23) is non-degenerate in the sense of (22), hence it is C^1 smooth with respect to parameter c for fixed $b \in \mathbb{R}$. The second question is thus answered adequately, but the first and third questions are still open as the local minimizers of (23) do not include *all* periodic solutions of (9) even if $\alpha \in (1/2, 2]$ and the local minimizers of (23) could be attained at the constant functions rather than at the periodic functions with the single-lobe profile $\phi_{c,b}$.

In order to give a superior characterization of the periodic solutions of (9) with $k = 2$, the following constrained minimization problem was considered in [25]:

$$\inf_{u \in H^{\alpha/2}(\mathbb{T})} \left\{ \int_{\mathbb{T}} [((-\Delta)^{\alpha/4} u)^2 + cu^2] dx : \quad \int_{\mathbb{T}} u^3 dx = 1, \quad \int_{\mathbb{T}} u dx = 0 \right\}. \tag{24}$$

The quadratic part of energy E is minimized in (24) together with the momentum F in the same linear combination as in the augmented energy $G_{c,b}(u) = E(u) + cF(u) + bM(u)$ to fix the parameter c of the stationary equation (9). On the other hand, the cubic part of energy E is normalized to unity and the zero-mean constraint is used to set $M(u) = 0$.

The zero-mean constraint $M(u) = 0$ does not mean that the parameter b in (9) is fixed to zero. On the contrary, the parameter b in (9) is uniquely determined by the zero-mean constraint in the form:

$$b = b(c) := \frac{1}{4\pi} \int_{\mathbb{T}} \phi_c^2 dx = \frac{1}{2\pi} F(\phi_c) > 0. \tag{25}$$

This unique definition of the parameter $b = b(c)$ does not limit the generality of the periodic solutions of the stationary equation (9) with $k = 2$ due to the Galilean transformation (10).

The constraint $\int_{\mathbb{T}} u^3 dx = 1$ can be normalized for every $c > -1$ by using the scaling transformation since positivity of $\mathcal{L}_c := (-\Delta)^{\alpha/2} + c$ is required for minimization of the quadratic form in (24), which suggests that

$$\frac{1}{2} \int_{\mathbb{T}} \phi_c^3 dx = \langle ((-\Delta)^{\alpha/2} + c)\phi_c, \phi_c \rangle > 0. \tag{26}$$

Minimizers of (24) satisfy the Euler–Lagrange equation

$$(-\Delta)^{\alpha/2} \chi_c + c\chi_c = C_1 \chi_c^2 + C_2,$$

where the Lagrange multipliers C_1 and C_2 are uniquely determined by

$$C_1 = \langle ((-\Delta)^{\alpha/2} + c)\chi_c, \chi_c \rangle > 0, \quad C_2 = -\frac{1}{2\pi} \left(\int_{\mathbb{T}} \chi_c^2 dx \right) C_1.$$

The Euler–Lagrange equation for χ_c is mapped to the stationary equation (9) with $k = 2$ by the invertible transformation $\phi_c = 2C_1\chi_c$ with

$$b = -2C_1C_2 = b(c).$$

Using positivity of $\mathcal{L}_c$ on the subspace of periodic functions in $L^2(\mathbb{T})$ with the zero mean for $c > -1$ and a compact embedding of $H^{\alpha/2}(\mathbb{T})$ into $L^3(\mathbb{T})$ for $\alpha > 1/3$, the following theorem was proven, see Theorem 2.1 in [25].

Theorem 3 *For every $\alpha \in (1/3, 2]$ and $c \in (-1, \infty)$, there exists a global minimizer of (24) with the single-lobe profile $\chi_c \in H^{\alpha/2}(\mathbb{T})$.*

The single-lobe property of the profile χ_c follows from the fractional Polya–Szegö inequality for $\alpha \leq 2$, see [34]. Compared to the other two variational characterizations (18) and (23), the single-lobe periodic functions cannot be mixed up with the constant functions due to the zero-mean constraint in (24).

By using the Galilean transformation (10) and the small-amplitude expansions (12) and (13) of Theorem 1, we can write the small-amplitude expansions of the periodic solutions with the single-lobe profile ϕ_c obtained from the minimizers of (24) in Theorem 3. Inverting (10) for the zero-mean periodic function ϕ_c yields

$$\phi_c = \varphi_\omega - \frac{1}{2\pi} \int_{\mathbb{T}} \varphi_\omega dx, \quad c = -\omega - \frac{1}{2\pi} \int_{\mathbb{T}} \varphi_\omega dx, \quad b(c) = \frac{1}{2}(\omega^2 - c^2). \tag{27}$$

from which we find

$$\phi_c(x) = a\cos(x) + \frac{1}{4(2^\alpha - 1)}a^2\cos(2x) + \frac{1}{8(2^\alpha - 1)(3^\alpha - 1)}a^3\cos(3x) + O(a^4),$$

and

$$c = -1 + \frac{1}{8(2^\alpha - 1)}a^2 + O(a^4), \quad b(c) = \frac{1}{4}a^2 + O(a^4). \tag{28}$$

The threshold value $\alpha = \alpha_0$ seen in (13) does not appear in the expansion (28). As a result, the solution branch for (c, ϕ_c) obtained from the minimizers of (24) is C^1 smooth with respect to c near $c = -1$. It was shown in [25] that the unique C^1 continuation of minimizers of (24) in c holds for every $c > -1$ under the non-degeneracy assumption

$$\mathrm{Ker}(\mathcal{M}_c|_{\{1\}^\perp}) = \mathrm{span}(\partial_x \phi_c), \tag{29}$$

where $\mathcal{M}_c$ is the linearized operator $\mathcal{M}_{c,b}$ with $\phi_{c,b} \equiv \phi_c$ and $b = b(c)$ and $\mathcal{M}_c|_{\{1\}^\perp}$ is its restriction to the subspace of L^2 functions with zero mean defined as

$$\mathcal{M}_c|_{\{1\}^\perp} f := \mathcal{M}_c f + \frac{1}{2\pi}\int_{\mathbb{T}} \phi_c f dx.$$

No numerical examples were found in [25] which would violate the non-degeneracy condition (29), which suggests that the minimizers of (24) are C^1 smooth in c for every $c > -1$. ***It is still open to prove that all minimizers of (24) satisfy (29) for every*** $c \in (-1, \infty)$ ***and*** $\alpha \in (1/3, 2]$**.**

Under the non-degeneracy assumption (29), we can fully characterize the number of negative eigenvalues of $\mathcal{M}_c$ and the non-degeneracy condition (22). This is formulated in the following theorem, and since it is important for practical applications, we give computational details of its proof.

Theorem 4 *Assume that $\phi_c \in H^{\alpha/2}(\mathbb{T})$ obtained from Theorem 3 satisfies (29). Then, we have the following:*

- *if $c + b'(c) > 0$, then $\mathcal{M}_c$ has one simple negative eigenvalue and a simple zero eigenvalue,*
- *if $c + b'(c) = 0$, then $\mathcal{M}_c$ has one simple negative eigenvalue and a double zero eigenvalue,*
- *if $c + b'(c) < 0$, then $\mathcal{M}_c$ has two simple negative eigenvalues (or a double negative eigenvalue) and a simple zero eigenvalue.*

All other eigenvalues of $\mathcal{M}_c$ are strictly positive.

For the proof of Theorem 4, we use the C^1 smoothness of the solution branch (c, ϕ_c) in c and deduce the following relations:

$$\mathcal{M}_c 1 = c - \phi_c, \quad \mathcal{M}_c\phi_c = -b(c) - \frac{1}{2}\phi_c^2, \quad \mathcal{M}_c\partial_c\phi_c = -b'(c) - \phi_c,$$

from which we derive

$$\mathcal{M}_c(1 - \partial_c \phi_c) = c + b'(c). \tag{30}$$

If $c + b'(c) \neq 0$, then $\{1, \phi_c, \phi_c^2\} \in \text{Range}(\mathcal{M}_c)$ and the result of Lemma 3.3 in [22] implies (22), in which case the zero eigenvalue of $\mathcal{M}_c$ is simple. If $c + b'(c) = 0$, then

$$\text{Ker}(\mathcal{M}_c) = \text{span}(\partial_x \phi_c, 1 - \partial_c \phi_c),$$

and the zero eigenvalue of $\mathcal{M}_c$ is double.

Denote the number of negative eigenvalues of $\mathcal{M}_c$ by $n(\mathcal{M}_c)$ and the multiplicity of the zero eigenvalue of $\mathcal{M}_c$ by $z(\mathcal{M}_c)$. Since the variational problem (24) involves two constraints and ϕ_c is obtained from a minimizer of (24) satisfying (29), it follows from Theorem 4.1 in [35] that

$$\begin{cases} 0 = n(\mathcal{M}_c) - n_0 - z_0, \\ 1 = z(\mathcal{M}_c) + z_0 - z_\infty, \end{cases} \tag{31}$$

where n_0, z_0 are the number of negative and zero eigenvalues of the limit $\lambda \to 0$ of the following symmetric 2-by-2 matrix

$$P(\lambda) := \begin{bmatrix} \langle (\mathcal{M}_c - \lambda I)^{-1} \phi_c^2, \phi_c^2 \rangle & \langle (\mathcal{M}_c - \lambda I)^{-1} \phi_c^2, 1 \rangle \\ \langle (\mathcal{M}_c - \lambda I)^{-1} 1, \phi_c^2 \rangle & \langle (\mathcal{M}_c - \lambda I)^{-1} 1, 1 \rangle \end{bmatrix},$$

and z_∞ is the number of eigenvalues of $P(\lambda)$ diverging to infinity as $\lambda \to 0$. If $c + b'(c) \neq 0$, then $z_\infty = 0$ and we compute from (30) that

$$\begin{aligned} \langle \mathcal{M}_c^{-1} 1, 1 \rangle &= \frac{2\pi}{c + b'(c)}, \\ \langle \mathcal{M}_c^{-1} 1, \phi_c^2 \rangle &= \langle \mathcal{M}_c^{-1} \phi_c^2, 1 \rangle = -2b(c) \langle \mathcal{M}_c^{-1} 1, 1 \rangle, \\ \langle \mathcal{M}_c^{-1} \phi_c^2, \phi_c^2 \rangle &= -2 \langle \phi_c, \phi_c^2 \rangle - 2b(c) \langle \mathcal{M}_c^{-1} 1, \phi_c^2 \rangle, \end{aligned}$$

so that

$$\lim_{\lambda \to 0} \det P(\lambda) = -\frac{4\pi}{c + b'(c)} \int_{\mathbb{T}} \phi_c^3 dx,$$

where $\int_{\mathbb{T}} \phi_c^3 dx > 0$ due to (26). Hence, we have $n(\mathcal{M}_c) = 1$ if $c + b'(c) > 0$ since $n_0 = 1$ and $z_0 = 0$ and we have $n(\mathcal{M}_c) = 2$ if $c + b'(c) < 0$ since $n_0 = 2$ and $z_0 = 0$. In both cases, $z(\mathcal{M}_c) = 1$. In the degenerate case $c + b'(c) = 0$, we have $n(\mathcal{M}_c) = 1$ and $z(\mathcal{M}_c) = 2$ since $n_0 = 1$, $z_0 = 0$, and $z_\infty = 1$.

We note that the small-amplitude expansion (28) implies that

$$c + b'(c) = -1 + 2(2^\alpha - 1) + O(a^2) = 2^{\alpha+1} - 3 + O(a^2), \tag{32}$$

which suggests that $\mathcal{M}_c$ has only one negative eigenvalue for small a if $\alpha > \alpha_0$ and two negative eigenvalues for small a if $\alpha < \alpha_0$, where α_0 is given by (14). The non-degeneracy condition (22) is violated in the small-amplitude limit $a \to 0$ for $\alpha = \alpha_0$.

For the fractional mKdV equation (4), the traveling periodic solutions with the single-lobe profile $\phi_{c,b}$ are found from the stationary equation (9) with $k = 3$, for which no Galilean transformation is available. A general variational characterization for such periodic solutions was proposed in [26, 27]:

$$\inf_{u \in H^{\alpha/2}(\mathbb{T})} \left\{ \int_{\mathbb{T}} [((-\Delta)^{\alpha/4} u)^2 + c u^2] dx : \int_{\mathbb{T}} u^4 dx = 1, \; \frac{1}{2\pi} \int_{\mathbb{T}} u dx = m \right\}, \quad (33)$$

where c is the same speed parameter but m is a new parameter of the mean value of the periodic solutions.

It was shown in Appendix B in [26] that a global minimizer $\chi_{c,m} \in H^{\alpha/2}(\mathbb{T})$ of (33) exists for every $\alpha \in (1/2, 2]$, $c \in (-1, \infty)$, and $m \in [-m_0, m_0]$, where $m_0 := (2\pi)^{-1/4}$. The minimizer has the single-lobe profile if $m \in (-m_0, m_0)$ and the constant profile if $m = \pm m_0$, which can be checked from the Hölder inequality:

$$\left| \int_{\mathbb{T}} u dx \right| \leq \left(\int_{\mathbb{T}} 1^{\frac{4}{3}} dx \right)^{\frac{3}{4}} \left(\int_{\mathbb{T}} u^4 dx \right)^{\frac{1}{4}} = (2\pi)^{\frac{3}{4}}.$$

Minimizers of (33) satisfy the Euler–Lagrange equation

$$(-\Delta)^{\alpha/2} \chi + c\chi = C_1 \chi^3 + C_2,$$

where C_1, C_2 are Lagrange multipliers found from the following system of two equations:

$$\begin{aligned} C_1 + 2\pi m C_2 &= \langle ((-\Delta)^{\alpha/2} + c)\chi, \chi \rangle, \\ \int_{\mathbb{T}} \chi^3 dx C_1 + 2\pi C_2 &= 2\pi c m. \end{aligned}$$

By eliminating C_2 from the second equation and substituting it into the first equation, we obtain

$$\left[1 - m \int_{\mathbb{T}} \chi^3 dx \right] C_1 = \langle ((-\Delta)^{\alpha/2} + c)\chi, \chi \rangle - 2\pi c m^2.$$

The value of C_1 is uniquely defined for $c \in (-1, \infty)$ and $m \in (-m_0, m_0)$ because both the left-hand side and the right-hand side are strictly positive:

$$\begin{aligned} 1 - m \int_{\mathbb{T}} \chi^3 dx &= \frac{1}{2\pi} \left[\left(\int_{\mathbb{T}} dx \right) \left(\int_{\mathbb{T}} \chi^4 dx \right) - \left(\int_{\mathbb{T}} \chi dx \right) \left(\int_{\mathbb{T}} \chi^3 dx \right) \right] \\ &= \frac{1}{16\pi} \int_{\mathbb{T}} \int_{\mathbb{T}} \left([\chi(x) - \chi(y)]^4 + 3 \left[\chi^2(x) - \chi^2(y) \right]^2 \right) dx dy \end{aligned}$$

and

$$\langle((-\Delta)^{\alpha/2}+c)\chi,\chi\rangle - 2\pi c m^2 = \|(-\Delta)^{\alpha/4}\chi\|_{L^2}^2 + c\|\chi - m\|_{L^2}^2 \\ \geq (1+c)\|\chi - m\|_{L^2}^2,$$

since $1+c>0$ and χ is non-constant if $m \in (-m_0, m_0)$. After the unique (positive) value of C_1 is found, any of the two equations of the system determines the unique value of C_2 and $\phi_{c,m} := \sqrt{3C_1}\chi_{c,m}$ becomes a solution of the stationary equation (9) with $k=3$ and

$$b = -\sqrt{3C_1}C_2 = b(c,m).$$

If $m=0$, then the minimizer $\chi_{c,m=0} \in H^{\alpha/2}(\mathbb{T})$ of (33) generates a solution $\phi_c \in H^{\alpha/2}(\mathbb{T})$ of (9) with $k=3$ and

$$b = b(c) = \frac{1}{6\pi}\int_{\mathbb{T}} \phi_c^3 dx. \tag{34}$$

A pitchfork bifurcation exists for some $c_0 \in (-1,\infty)$ such that the 2π-periodic zero-mean function ϕ_c obtained from the minimizer $\chi_{c,m=0}$ of (33) satisfies the half-period symmetry

$$\phi_c(\pi - x) = -\phi_c(x), \quad x \in \mathbb{T}$$

for $c \in (-1, c_0)$ and violates the half-period symmetry for $c \in (c_0, \infty)$, see [26]. This pitchfork bifurcation was earlier discovered for $\alpha = 2$ in [36–38].

For $m \neq 0$, the pitchfork bifurcation breaks into a fold bifurcation of two branches of the periodic solutions and a global continuation of the third branch. The latter branch coincides with the global minimizers of (33), see [27] for $\alpha = 2$. ***The existence and multiplicity of critical points of (33) in the fractional case $\alpha \in (1/2, 2)$ is an open question for further studies.***

5 Stability of the Traveling Periodic Waves

Stability of the traveling periodic waves of the form $u(x,t) = \phi_{c,b}(x-ct)$ in the time evolution of the fractional models (3) and (4) can be studied from the linearization of these models. Substituting a sum of the traveling wave $\phi_{c,b}(x-ct)$ and a perturbation $v(x-ct,t)$ into the nonlinear systems and truncating at the linear terms in v, we obtain a linearized equation of motion in the form

$$v_t = \partial_x \mathcal{M}_{c,b} v, \tag{35}$$

where $\mathcal{M}_{c,b}$ is the linearized operator given by (19).

We say that the traveling periodic wave with the profile $\phi_{c,b} \in H^\infty(\mathbb{T})$ is spectrally unstable if there exists an eigenvalue λ_0 of the linear operator

$$\partial_x \mathcal{M}_{c,b} : H^{\alpha+1}(\mathbb{T}) \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})$$

such that $\mathrm{Re}(\lambda_0) > 0$. Otherwise, the wave is spectrally stable.

One possible approach to the proof of spectral stability is to use the variational characterization (18) and to show that the traveling periodic wave with the profile $\phi_{c,b}$ is related to a local constrained minimizer of the energy E subject to fixed momentum F and mass M. The orthogonal complement to the tangent plane of the two constraints in (18) is given by

$$L^2|_{\{1,\phi_{c,b}\}^\perp} := \left\{ v \in L^2(\mathbb{T}) : \quad \langle 1, v \rangle = 0, \quad \langle \phi_{c,b}, v \rangle = 0 \right\}. \tag{36}$$

It has been well established in the literature, see [36, 37], that if the non-degeneracy condition (22) is satisfied for the linearized operator $\mathcal{M}_{c,b}$, then the local constrained minimizers of (18) are linearly stable according to the following definition.

We say that the traveling periodic wave with the profile $\phi_{c,b} \in H^\infty(\mathbb{T})$ is linearly stable in $H^{\alpha/2}(\mathbb{T})$ if for every $v_0 \in H^{\alpha/2}(\mathbb{T}) \cap L^2|_{\{1,\phi_{c,b}\}^\perp}$, there exists the unique solution $v \in C^0([0,\infty), H^{\alpha/2}(\mathbb{T}))$ to the linearized equation (35) and a function $a \in C^0([0,\infty))$ such that

$$\|v(t,\cdot) - a(t)\partial_x \phi_{c,b}\|_{H^{\alpha/2}} \leq C \|v_0\|_{H^{\alpha/2}}, \quad t \geq 0,$$

for some v_0-independent constant $C > 0$.

Linear stability implies spectral stability but the spectral stability does not imply the linear stability because multiple eigenvalues of $\partial_x \mathcal{M}_{c,b}$ on the imaginary axis may lead to the polynomial growth of solutions of the linearized equation (35). The reason why the minimization property in (18) and the non-degeneracy (22) are sufficient for the spectral and linear stability is the strict positivity of the operator $\mathcal{M}_{c,b}$ under the three orthogonality conditions

$$\langle 1, v \rangle = 0, \quad \langle \phi_{c,b}, v \rangle = 0, \quad \langle \partial_x \phi_{c,b}, v \rangle = 0. \tag{37}$$

It follows from the linearized equation (35) that the first two orthogonality conditions are preserved in time: if they are satisfied for $t = 0$, they are satisfied for $t \geq 0$. For the last orthogonality condition, we should use the decomposition

$$v(t,\cdot) = a(t)\partial_x \phi_{c,b} + w(t,\cdot), \quad \langle \partial_x \phi_{c,b}, w(t,\cdot) \rangle = 0, \tag{38}$$

so that $w(t,\cdot)$ satisfies all three orthogonality conditions in (37) if v_0 satisfies the two orthogonality conditions:

$$\langle 1, v_0 \rangle = 0 \quad \text{and} \quad \langle \phi_{c,b}, v_0 \rangle = 0.$$

The Lyapunov functional $\langle \mathcal{M}_{c,b} w(t,\cdot), w(t,\cdot) \rangle$ is a constant of motion for the linearized equation (35) and the linear stability follows from the conservation and coercivity of the Lyapunov functional. Spectral stability follows from the linear stability.

The minimization property in (18) and the non-degeneracy (22) also implies the nonlinear orbital stability of the traveling periodic waves if the local well-posedness of the nonlinear fractional models can be established in the energy space $H^{\alpha/2}(\mathbb{T})$, see [22, 39]. As we discussed in the introduction, the local well-posedness can only be obtained for sufficiently large α and hence we will use the concept of linear stability instead of the nonlinear orbital stability to avoid subtle issues related to the local well-posedness of the nonlinear fractional models.

The following theorem presents the state of the art in the proof of linear stability of the traveling periodic waves in the fractional KdV models. It is deduced from the constrained minimization problem (18), see Theorem 2.3 in [36] and Theorem 1 in [37].

Theorem 5 *Let $\phi_{c,b} \in H^\infty(\mathbb{T})$ be a solution of the stationary equation (9). Assume that $\phi_{c,b}$ is a C^1 function of (c, b). Then the non-degeneracy condition (22) is satisfied and the traveling periodic wave is linearly stable if the Jacobian of the transformation*

$$(c, b) \mapsto (F(\phi_{c,b}), M(\phi_{c,b}))$$

admits no zero eigenvalue and the number of its positive eigenvalues is equal to the number of negative eigenvalues of $\mathcal{M}_{c,b}$ in $L^2(\mathbb{T})$.

For the proof of non-degeneracy in Theorem 5, we again use the relations (20) which are deduced from the assumption that $\phi_{c,b}$ is a C^1 function of (c, b). Together with (21) they ensure validity of the non-degeneracy condition (22) by the theory in [22] since $\{1, \phi_{c,b}, \phi_{c,b}^2\} \in \text{Range}(\mathcal{M}_{c,b})$.

For the proof of linear stability in Theorem 5, we need to show that $\phi_{c,b}$ is related to a local constrained minimizer of the variational problem (18). For this, we need to show validity of the relations (31), where n_0, z_0, z_∞ are now related to the number of negative, zero, and diverging eigenvalues of the limit $\lambda \to 0$ of the following symmetric 2-by-2 matrix

$$D(\lambda) := \begin{bmatrix} \langle (\mathcal{M}_{c,b} - \lambda I)^{-1} \phi_{c,b}, \phi_{c,b} \rangle & \langle (\mathcal{M}_{c,b} - \lambda I)^{-1} \phi_{c,b}, 1 \rangle \\ \langle (\mathcal{M}_{c,b} - \lambda I)^{-1} 1, \phi_{c,b} \rangle & \langle (\mathcal{M}_{c,b} - \lambda I)^{-1} 1, 1 \rangle \end{bmatrix}.$$

While $P(\lambda)$ is constructed from the orthogonal complement of the tangent plane of the two constraints in (24), the matrix $D(\lambda)$ is constructed for the constrained subspace (36) related to the two constraints in (18). By using (20), this yields

$$\lim_{\lambda\to 0} D(\lambda) = \begin{bmatrix} \langle \mathcal{M}_{c,b}^{-1}\phi_{c,b}, \phi_{c,b}\rangle & \langle \mathcal{M}_{c,b}^{-1}\phi_{c,b}, 1\rangle \\ \langle \mathcal{M}_{c,b}^{-1}1, \phi_{c,b}\rangle & \langle \mathcal{M}_{c,b}^{-1}1, 1\rangle \end{bmatrix}$$
$$= -\begin{bmatrix} \langle \partial_c\phi_{c,b}, \phi_{c,b}\rangle & \langle \partial_c\phi_{c,b}, 1\rangle \\ \langle \partial_b\phi_{c,b}, \phi_c\rangle & \langle \partial_b\phi_{c,b}1, 1\rangle \end{bmatrix}$$
$$= -\begin{bmatrix} \partial_c F(\phi_{c,b}) & \partial_c M(\phi_{c,b}) \\ \partial_b F(\phi_{c,b}) & \partial_b M(\phi_{c,b}) \end{bmatrix},$$

where F and M are the conserved momentum and mass in (7) and (8). Since $\phi_{c,b} \in H^\infty(\mathbb{T})$ is a C^1 function of (c, b), we have $z_\infty = 0$, whereas n_0 and z_0 coincide with the number of positive and zero eigenvalues of the Jacobian of the transformation

$$(c, b) \mapsto (F(\phi_{c,b}), M(\phi_{c,b})).$$

If $n_0 = n(\mathcal{M}_{c,b})$ and $z_0 = 0$, then the relations (31) are satisfied and $\phi_{c,b}$ is related to a local constrained minimizer of the variational problem (18). Hence, $\langle M_{c,b}v, v\rangle$ is a Lyapunov functional for the linearized equation (35) and $\phi_{c,b}$ is linearly stable.

We have already discussed that the assumption of Theorem 5 on the C^1 smoothness of $\phi_{c,b} \in H^\infty(\mathbb{T})$ in (c, b) may fail at the fold bifurcation. For $k = 2$, the variational characterization of the profile $\phi_{c,b} \in H^\infty(\mathbb{T})$ is alternatively developed by using the constrained minimization problem (24), for which $b = b(c)$ is related to $F(\phi_c)$ by (25). In this case, the linear stability of the local constrained minimizer of (24) is determined by a scalar function $b = b(c)$ as in the following theorem.

Theorem 6 *Let $\phi_c \in H^\infty(\mathbb{T})$ be a solution of the stationary equation (9) obtained from a local constrained minimizer of (24) with $b = b(c)$ given by (25). Assume the non-degeneracy condition (29). Then the traveling periodic wave is linearly stable if $b'(c) > 0$ and is linearly unstable if $b'(c) \le 0$.*

The non-degeneracy condition (29) is again the key to the C^1 smoothness of $\phi_c \in H^\infty(\mathbb{T})$ in c since it yields (30). If $c + b'(c) \neq 0$, then $z_\infty = 0$ in (31) and

$$\begin{aligned} \langle \mathcal{M}_c^{-1}1, 1\rangle &= \frac{2\pi}{c + b'(c)}, \\ \langle \mathcal{M}_c^{-1}1, \phi_c\rangle &= \langle \mathcal{M}_c^{-1}\phi_c, 1\rangle = -\frac{2\pi b'(c)}{c + b'(c)}, \\ \langle \mathcal{M}_c^{-1}\phi_c, \phi_c\rangle &= -2\pi b'(c) - \frac{2\pi (b'(c))^2}{c + b'(c)}, \end{aligned}$$

so that

$$\lim_{\lambda\to 0} \det D(\lambda) = -\frac{4\pi^2 b'(c)}{c + b'(c)}.$$

We have $n(\mathcal{M}_c) = 1$ if $c + b'(c) > 0$ and $n(\mathcal{M}_c) = 2$ if $c + b'(c) < 0$ by Theorem 4. In the former case, det $D(\lambda)$ as $\lambda \to 0$ shows that $n_0 = 1$, $z_0 = 0$ if $b'(c) > 0$ [in which case, the periodic wave is related to a local constrained minimizer of (18)] and $n_0 = 0$, $z_0 = 1$ if $b'(c) < 0$ [in which case, the periodic wave is related to a local constrained saddle point of (18)]. This implies the conclusion on linear stability if $b'(c) > 0$ and linear instability if $b'(c) < 0$ by Theorem 2.3 in [36] and Theorem 1 in [37]. In the latter case, det $D(\lambda)$ as $\lambda \to 0$ shows that $n_0 = 2$, $z_0 = 0$ if $b'(c) > 0$ and $n_0 = 1$, $z_0 = 0$ if $b'(c) < 0$. This yields the same stability conclusion. In the degenerate case $c + b'(c) = 0$, we have $n(\mathcal{M}_c) = 1$ and $z(\mathcal{M}_c) = 2$ with $z_\infty = 1$. We obtain $n_0 = 1$, $z_0 = 0$ if $b'(c) > 0$ and $n_0 = 0$, $z_0 = 0$ if $b'(c) < 0$, which yields the same stability conclusion.

The stability threshold case is $b'(c) = 0$, for which the linearized equation (35) admits the growing solution

$$v = \frac{1}{2} t^2 \partial_x \phi_c - t \partial_c \phi_c - \mathcal{M}_c^{-1} \partial_x^{-1} \partial_c \phi_c,$$

where ∂_x^{-1} is the anti-derivative of the zero-mean periodic functions with the zero mean and $\mathcal{M}_c^{-1} \partial_x^{-1} \partial_c \phi_c$ is well-defined because $\langle 1, \partial_c \phi_c \rangle = 0$ and $\langle \phi_c, \partial_c \phi_c \rangle = 0$. The growing solution $v(t, \cdot)$ satisfies both the constraints $\langle 1, v(t, \cdot) \rangle = 0$ and $\langle \phi_c, v(t, \cdot) \rangle = 0$ and $\| v(t, \cdot) - \frac{1}{2} t^2 \partial_c \phi_c \|_{L^2}$ grows linearly in t. Hence, the threshold case $b'(c) = 0$ is also linearly unstable with the linear (rather than the exponential) rate of instability.

Note that $b'(c) = 2(2^\alpha - 1) + \mathcal{O}(a^2)$ in the small-amplitude limit according to the expansions (28). Hence, the small-amplitude periodic wave is linearly stable for every $\alpha \in (1/3, 2]$. Also note that

$$\begin{aligned} \partial_c F(\phi_c) &= \langle \phi_c, \partial_c \phi_c \rangle = 2\pi b'(c), \\ \partial_c M(\phi_c) &= \langle 1, \partial_c \phi_c \rangle = 0, \end{aligned}$$

and

$$\langle 1, \mathcal{M}_c^{-1} 1 \rangle = \frac{2\pi}{c + b'(c)}.$$

Hence, the stability criterion in Theorem 6 can be deduced from the stability criterion in Theorem 5 in the particular case $b = b(c)$ for the zero-mean periodic solutions. It is remarkable that the stability criterion is now defined by the slope of the mapping $c \mapsto b(c)$ given by (25). This is the same stability condition as in the case of solitary waves for (3), see [16].

For $k = 3$, the variational characterization of the traveling periodic waves is developed by using the constrained minimization problem (33) with two parameters (c, m) and two constraints. The function $b = b(c, m)$ can be constructed from the stationary equation (9) with $k = 3$ in the form

$$b = b(c, m) = \frac{1}{6\pi} \int_{\mathbb{T}} \phi_{c,m}^3 dx - cm \left(\int_{\mathbb{T}} \phi_{c,m}^4 dx \right)^{1/4}, \tag{39}$$

which generalizes (34) for $m \neq 0$. However, this function is not related to the conserved quantities $F(\phi_{c,m})$ and $M(\phi_{c,m})$, hence it does not determine the stability criterion of the traveling periodic waves in the fractional mKdV equation (4). Moreover, there is a pitchfork bifurcation at $c_0 \in (-1, \infty)$ in the case of $m = 0$ such that three branches of periodic solutions with the single-lobe profile coexist for $c \in (c_0, \infty)$, two branches are minimizers of the constrained variational problem (33) and one is the saddle point, see [26, 27]. The only branch of periodic solutions for $c \in (-1, c_0)$ is the minimizer and satisfies the half-period symmetry: $\phi_c(\pi - x) = -\phi_c(x)$. The pitchfork bifurcation is broken for fixed $m \neq 0$ with one branch being a global minimizer for $c \in (-1, \infty)$ and the other two branches existing for a subset (c_0, ∞) with a fold bifurcation at $c_0 \in (-1, \infty)$, see [27].

The linear stability criterion was obtained for all three branches of the periodic solutions in [27] and was illustrated numerically with examples in the local case $\alpha = 2$. Assuming that

$$\det \begin{vmatrix} \partial_c F(\phi_{c,m}) & \partial_c M(\phi_{c,m}) \\ \partial_m F(\phi_{c,m}) & \partial_m M(\phi_{c,m}) \end{vmatrix} > 0, \tag{40}$$

it was proven in [27] that every local minimizer of the variational problem (33) satisfying the non-degeneracy condition

$$\mathrm{Ker}(\mathcal{M}_{c,m}|_{\{1,\phi_{c,m}^3\}^\perp}) = \mathrm{span}(\partial_x \phi_{c,m}) \tag{41}$$

is linearly stable similarly to Theorem 6. Moreover, the slope of $\partial_m b(c, m)$ gives the criterion for $\mathcal{M}_{c,m}$ to have a simple negative eigenvalue if $\partial_m b < 0$, a double zero eigenvalue if $\partial_m b = 0$, and two simple negative eigenvalues (or a double negative eigenvalue) if $\partial_m b > 0$, similarly to Theorem 4. It was also proven in [27] that every saddle point of the variational problem (33) with (40), (41), $\partial_m b < 0$, and $\mathcal{M}_{c,m}|_{\{1,\phi_{c,m}^3\}^\perp}$ admitting only one simple negative eigenvalue is lineary unstable. These criteria generalize the stability analysis in Theorem 1.3 of [26], where the case $m = 0$ was studied for every $\alpha \in (1/2, 2]$.

6 Examples of Traveling Periodic Waves

For the integrable BO equation (3) with $\alpha = 1$, the single-lobe periodic solution to the stationary equation (11) is known in the exact form:

$$\omega = \coth \gamma, \quad \varphi_\omega(x) = \frac{2 \sinh \gamma}{\cosh \gamma - \cos x} - 2\omega, \tag{42}$$

where $\gamma \in (0, \infty)$ is a free parameter of the solution and the small-amplitude limit of Theorem 1 corresponds to the limit $\gamma \to \infty$. Since

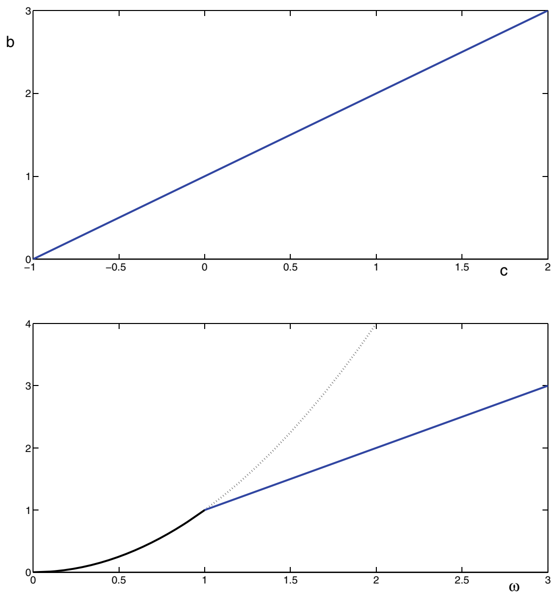

Fig. 3 The dependence of b versus c (top) and μ versus $\omega = c$ (bottom) for $\alpha = 1$. The dotted (solid) curves correspond to constant (single-lobe) solutions. Reproduced from [25]. ©2022 by IOP Publishing

$$\int_{-\pi}^{\pi} \varphi_\omega(x)dx = 4\pi(1-\omega),$$

we compute explicitly from (27) that

$$\phi_c(x) = \frac{2\sinh\gamma}{\cosh\gamma - \cos x} - 2, \quad c = \omega - 2, \quad b(c) = 2(\omega - 1),$$

which characterize periodic solutions of the variational problem (24). Eliminating $\omega \in (1, \infty)$ yields $b(c) = 2(c+1)$ for $c \in (-1, \infty)$ shown on Fig. 3 (top). For comparison, Fig. 3 (bottom) characterizes periodic solutions of the variational problem (23) with $b = 0$ and $F_0 \equiv \mu$. The black (dotted) curve shows the dependence of μ from c for the constant solution $\phi = 2c$ of the stationary equation (9) with $k = 2$ and $b = 0$. The blue (solid) line corresponds to the periodic solution with the single-lobe profile obtained from (42) by taking

$$\phi_c = 2c + \varphi_c = \frac{2\sinh\gamma}{\cosh\gamma - \cos x} \tag{43}$$

with $c = \coth\gamma$. For $c > 1$ and $\mu > 4\pi c^2$, the constant solution is no longer a minimizer of (23) and the periodic solution (43) with the positive single-lobe profile ϕ_c becomes a minimizer of (23).

Figure 4 (left) illustrates how the periodic solutions with the zero-mean single-lobe profile ϕ_c are defined by the variational problem (24). The unique solution exists for

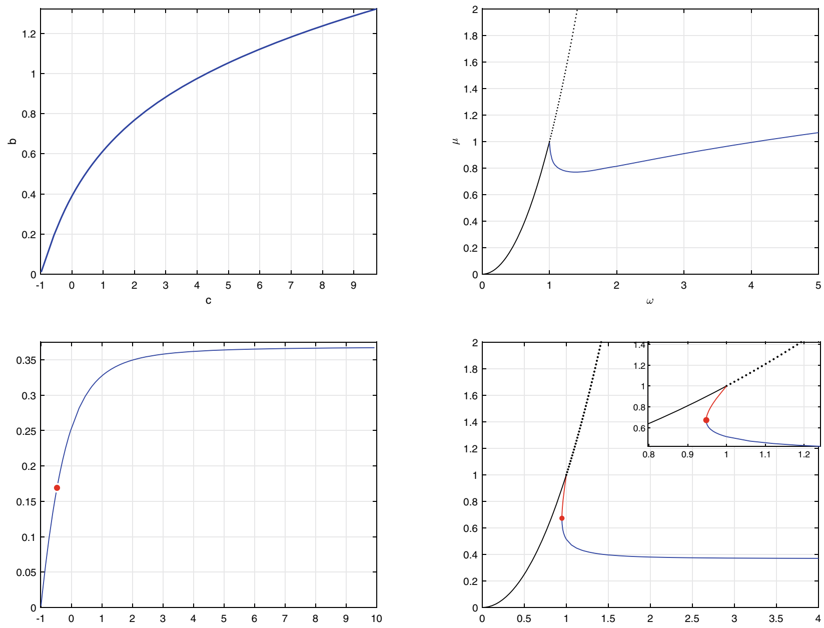

Fig. 4 Left: the dependence of b versus c for $\alpha = 0.6$ (top) and $\alpha = 0.5$ (bottom). Right: the dependence of μ versus $\omega = c$ for $\alpha = 0.6$ (top) and $\alpha = 0.5$ (bottom). Reproduced from [25]. ©2022 by IOP Publishing

every $c \in (-1, \infty)$ and the dependence of b versus c is monotonically increasing for $\alpha \in [1/2, 2]$. By Theorem 6, this implies the linear stability of the periodic waves. For $\alpha \in (1/3, 1/2)$, the dependence of b versus c is not monotone. In agreement with the instability criterion for the solitary waves [16], which correspond to the limit $c \to \infty$, we have found numerically for $\alpha \in (1/3, 1/2)$ that there is a single point $c_* \in (-1, \infty)$ such that the dependence of b versus c is increasing for $c \in (-1, c_*)$ and decreasing for $c \in (c_*, \infty)$, see [25].

Figure 4 (right) illustrates properties of the periodic solutions obtained from the variational problem (23) with $b = 0$ and $F_0 \equiv \mu$. Besides the fact that the constant solutions are global minimizers of (23) for $c \in (0, 1)$, the dependence of μ versus c for the periodic solutions with the single-lobe profile bifurcates below $\mu_0 = 4\pi c^2$ for $\alpha < 0.737$ and to the left of $c_0 = 1$ for $\alpha < \alpha_0 \approx 0.585$. The fold bifurcation at the bottom right panel corresponds to the point where two branches of periodic solutions meet: one branch (red line) has two negative eigenvalues of $\mathcal{M}_{c,b=0}$ and the other branch (blue line) has only one simple negative eigenvalue of $\mathcal{M}_{c,b=0}$. The branch with two negative eigenvalues is a saddle point of (23) and a constrained minimizer of (18). Not only the folded picture is unfolded in (24) but also the stability criterion is clearly given by the monotonicity of the mapping $c \mapsto b(c)$.

The periodic solutions of the fractional mKdV equation (4) with the single-lobe profile are constructed from the variational problem (33) with two constraints. If

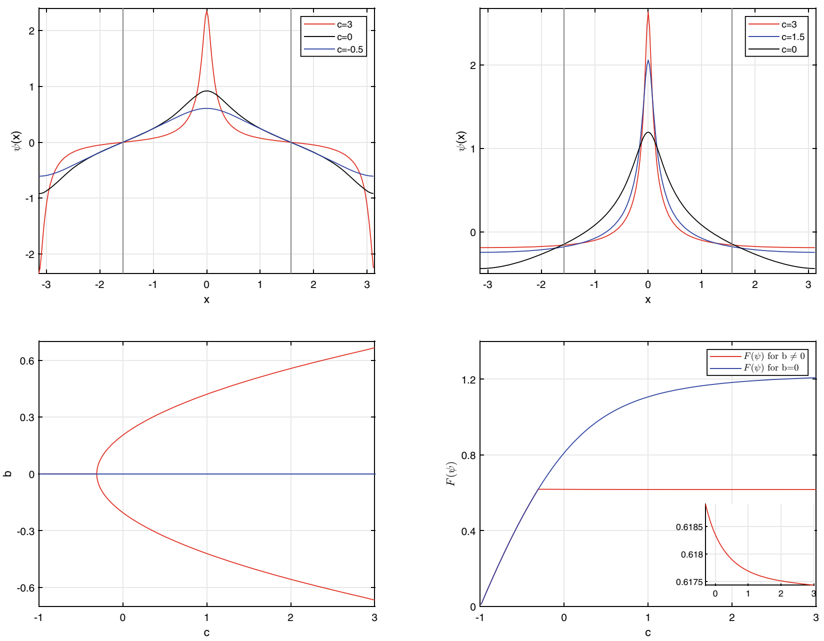

Fig. 5 Periodic solutions of the variational problem (33) with $m = 0$ for $\alpha = 1$. Top: Profiles of ϕ_c with $b = 0$ (left) and with $b = b(c) \neq 0$ (right) for three different values of c. Bottom left: Dependence of b versus c showing the pitchfork bifurcation point c_0. Bottom right: Dependence of the momentum $\mu = F(\phi_c)$ versus c. Reproduced from [26]. ©2022 by Springer Nature

$m = 0$, the branch of periodic solutions with the half-period symmetry $\phi_c(\pi - x) = -\phi_c(x)$ may coexist with the two symmetric branches of periodic solutions which violate the symmetry. Figure 5 illustrates the three branches in the case of the modified BO equation with $\alpha = 1$. The results were obtained numerically by using Newton's method in [26].

The top panels show the single-lobe profiles ϕ_c for three different values of parameter c: solutions with the half-period symmetry are on the left and solutions without the half-period symmetry are on the right. The former solutions have $b = 0$ in the stationary equation (9) with $k = 3$ and the latter solutions have $b = b(c) \neq 0$ given by (34). The bottom panels show the pitchfork bifurcation at $c = c_0$ on the (c, b) plane (left) and the dependence of $\mu := F(\phi_c)$ versus c for the three branches (right). The mapping $c \mapsto \mu$ is monotonically increasing for the former solutions with $b = 0$, however, the periodic solutions are linearly stable only before the pitchfork bifurcation and linearly unstable after the bifurcation, as shown in Theorem 2.14 of [26]. The mapping $c \mapsto \mu$ is monotonically decreasing for the latter solutions with $b = b(c) \neq 0$ and this implies that the periodic solutions are linearly unstable. It was also found in [26] that the periodic solutions with $b = b(c) \neq 0$ pass through the point $c_* \in (c_0, \infty)$, where the number of negative eigenvalues of the linearized operator $\mathcal{M}_{c,b(c)}$ changes from 2 for $c \in (c_0, c_*)$ to 1 for $c \in (c_*, \infty)$. However, the periodic

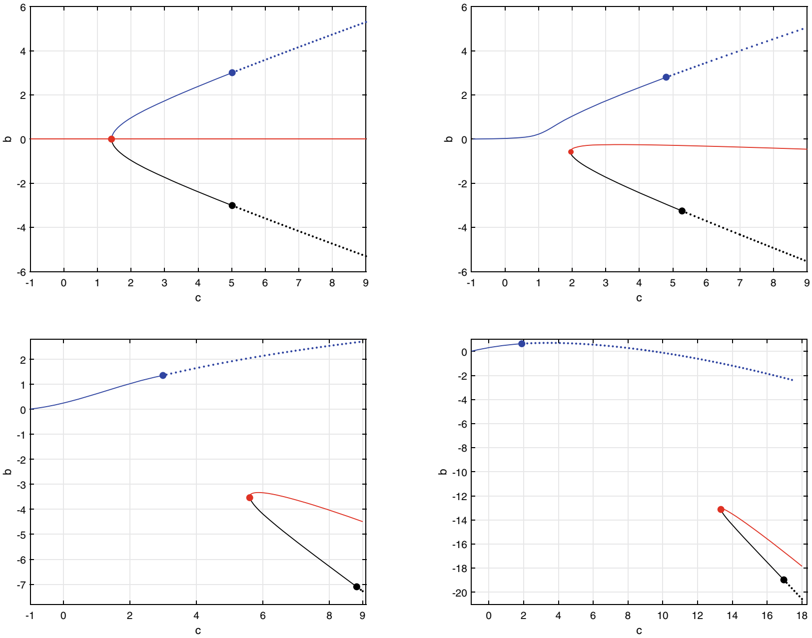

Fig. 6 Three solution families on the (b, c) diagram for fixed values of m: $m = 0$ (top left), $m = 0.01$ (top right), $m = 0.1$ (bottom left), and $m = 0.2$ (bottom right). Reproduced from [27]. ©2022 by the Society for Industrial and Applied Mathematics

solution with the profile ϕ_c is smoothly continued with respect to the parameter c in the framework of the variational problem (33) with $m = 0$. This shows that the fold bifurcation of the periodic solutions is unfolded in the variational problem (33) with $m = 0$.

Further studies of the periodic solutions of the variational problem (33) with $m \neq 0$ were performed in [27] in the local case $\alpha = 2$ for which the exact periodic solutions are available in the closed form [40]. The pitchfork bifurcation for $m = 0$ is very similar between the case $\alpha = 1$ in Fig. 5 and the case $\alpha = 2$ in the top left panel of Fig. 6. The dots show the border between two different solution forms, see [27, 40].

Other panels of Fig. 6 show the broken pitchfork bifurcation for $m \neq 0$, for which one branch is detached from the other two branches which coalesce in the fold bifurcation. It was confirmed numerically in [27] that the isolated branch (shown by blue curve) is related to the global minimizer of (33), whereas the top branch to the right of the fold bifurcation point (shown by red curve) is a saddle point of (33) and the bottom branch to the right of the fold bifurcation point (shown by black curve) is a local minimizer of (33). Computations of the solution surface $b = b(c, m)$ given by (39) have confirmed that the two minimizers of (33) are linearly stable everywhere in the (c, m) plane and the saddle point of (33) is linearly unstable, see [27]. These results correspond to the case $\alpha = 2$ for which the solitary waves are stable, see [16]. Stability of the minimizers of (33) for $\alpha = 1$ is clearly different and is affected by the instability of the solitary waves, see [12], which arises in the limit $c \to \infty$, as is seen in Fig. 5. ***Approximations of minimizers and saddle points of (33) with $\alpha \in (1/2, 2)$ and $m \neq 0$ are open questions for further studies.***

7 Conclusion

We have reviewed three methods to study existence and stability of traveling periodic waves in the fractional KdV models.

- The first method of the small-amplitude expansions is restricted to the periodic functions with the single-lobe profile bifurcating from the constant functions. It allows us to obtain some explicit approximations to the dependence of the wave speed from the wave amplitude, the degeneracy of the linearized operator, and the linear stability of the traveling wave.
- The second method is based on the formulation of the existence problem as the fixed-point iteration method. This technique allows us to control positivity of the periodic functions with the single-lobe profile as long as the linearized operator remains non-degenerate along the solution branch.
- The third method is based on the variational formulation, the Euler–Lagrange equations of which give the same equation for the traveling periodic wave after some suitable scaling transformation. Three variational formulations are possible: finding a minimum of energy subject to constrained mass and momentum, finding a minimum of energy with prescribed mass subject to the constrained momentum, and finding a minimum of the quadratic parts of the energy and momentum subject to the constrained higher power of the energy and the mass. The third method has showed superior performance both in quadratic and cubic case as it allows us to characterize the fold and pitchfork bifurcations of the solution branches, the degeneracy condition of the linearized operator, and the linear stability of the traveling waves.

Numerical methods are also available to complement the analytical results with precise approximation away from the bifurcation points. Newton's method remains the best method to approximate all branches of the periodic solutions with the single-lobe profiles [25, 26], although the solution branches with the positive single-lobe profiles can also be approximated by Petviashvili's fixed-point iteration method [23, 41]. In the integrable cases of the KdV and mKdV equations, one can also use the explicit solutions for the traveling periodic waves expressed by the Jacobian elliptic functions, see [25] and [27] respectively.

More questions are open in regards to the traveling periodic waves in the fractional KdV models. Here is a rather incomplete list of further open questions which complements the open questions posed in the previous sections (as the bold-face italic font).

- Numerical results strongly suggest that minimizers of the variational problems (24) and (33) are non-degenerate in the sense of (29) and (41), with the exception of the pitchfork bifurcation in (33) for $m = 0$ which has been well understood. The non-degeneracy may be a fundamental result which still needs further study.
- Uniqueness of minimizers in the variational problems (24) and (33) might be related to their non-degeneracy. Since uniqueness for the solitary wave solutions has been shown in the fractional models on the line [15], it is intriguing to show uniqueness of the periodic solutions both in the quadratic and cubic models.
- Some traveling periodic waves are linearly unstable for sufficiently small α both in (3) and (4) but well-posedness theory is not yet available in the energy space of the fractional models. It is not apriori clear if this instability can be observed in the suitable function spaces for which the initial-value problem is well-posed.
- Although similar existence and stability problems have been studied in other fractional models with quadratic and cubic nonlinearities, see [24, 28, 29], it is rather interesting to see if the same questions can be analyzed for higher-order (quartic, quintic, etc) power nonlinearities and for nonlinearities with several powers.
- Questions of modulational stability of the traveling periodic waves are widely open in the context of fractional models, see [22] for some previous results. Similarly, transverse stability of the traveling periodic waves in multi-dimensional generalizations of the fractional models has not been studied, see [18] for study of transverse stability of the traveling solitary waves.

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