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Positive solutions of the Gross–Pitaevskii equation for energy critical and supercritical nonlinearities

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Abstract

We consider positive and spatially decaying solutions to the following Gross–Pitaevskii equation with a harmonic potential:

$$-\Delta u + |x|^2 u = \omega u + |u|^{p-2} u \quad \text{in } \mathbb{R}^d,$$

where $d \geqslant 3$, $p > 2$ and $\omega > 0$. For $p = \frac{2d}{d-2}$ (energy-critical case) there exists a ground state u_ω if and only if $\omega \in (\omega_*, d)$, where $\omega_* = 1$ for $d = 3$ and $\omega_* = 0$ for $d \geqslant 4$. We give a precise description on asymptotic behaviours of u_ω as $\omega \to \omega_*$ up to the leading order term for different values of $d \geqslant 3$. When $p > \frac{2d}{d-2}$ (energy-supercritical case) there exists a singular solution u_∞ for some $\omega \in (0, d)$. We compute the Morse index of u_∞ in the class of radial functions and show that the Morse index of u_∞ is infinite in the oscillatory case, is equal to 1 or 2 in the monotone case for p not large enough and is equal to 1 in the monotone case for p sufficiently large.

Keywords: Gross–Pitaevskii equation, critical and supercritical nonlinearity, positive solutions, asymptotic behaviour, Morse index

Mathematics Subject Classification numbers: 35B09, 35B33, 35B40, 35J20

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1. Introduction

1.1. Background

We consider positive and spatially decaying solutions to the following stationary Gross–Pitaevskii equation with a harmonic potential:

$$-\Delta u+|x|^2u=\omega u+|u|^{p-2}u \quad \text{in } \mathbb{R}^d, \tag{1.1}$$

where $d\geqslant 3$, $p>2$ and $\omega>0$.

The stationary equation (1.1) is a classical model to describe the Bose–Einstein condensate with attractive inter-particle interactions under magnetic trap (see [41]) if $d=1,2,3$ and $p=4$ (the cubic case) or $p=6$ (the quintic case). In this context, $\psi(t,x)=\mathrm{e}^{-\mathrm{i}\omega t}u(x)$ is a standing wave solution of the time-dependent Gross–Pitaevskii equation

$$\mathrm{i}\,\partial_t\psi=-\Delta\psi+|x|^2\psi-|\psi|^{p-2}\psi \quad \text{in } \mathbb{R}^d, \tag{1.2}$$

where ψ stands for the macroscopic wave function, $|x|^2$ is an isotropic trapping potential that confines the Bose–Einstein condensate, and the nonlinear term corresponds to attractive inter-atomic interactions. Positive and spatially decaying solutions are called the bright solitons in the physics literature. We refer readers to [9] for the physical backgrounds of the Gross–Pitaevskii equation (1.2).

Since the operator $-\Delta+|x|^2$ is compact in $L^2(\mathbb{R}^d)$, the energy-subcritical case $2<p<\frac{2d}{d-2}$ can be studied by classical variational methods or bifurcation methods (see [18, 26, 36]). On the other hand, energy-critical $p=\frac{2d}{d-2}$ and energy-supercritical $p>\frac{2d}{d-2}$ cases with $d\geqslant 3$ were less investigated in the literature. In the energy-critical case, based on the well-known Gidas–Ni–Nirenberg theorem (see [19]), the existence of positive and spatially decaying solutions of the stationary equation (1.1) has been shown in [36, 37, 39] for $\omega\in(\omega_*,d)$ by variational methods, where

$$\omega_*=\begin{cases}1, & d=3,\\ 0, & d\geqslant 4.\end{cases} \tag{1.3}$$

In the energy-supercritical case, the existence and uniqueness of spatially decaying solutions of the stationary equation (1.1) is out of reach from the point of variational methods. Nevertheless, some results were obtained in [4, 15, 38] by using shooting methods since positive and radially symmetric solutions satisfy an ordinary differential equation.

Besides the existence and nonexistence of solutions, an interesting problem for critical elliptic equations is to study the concentration phenomenon and the asymptotic behaviour of solutions for the parameters close to the boundary of the existence interval. It has been proven in [37, theorem 5], by the method of Lyapunov–Schmidt reductions, that if $u_b\sim bu_0$, where u_0 is the normalised ground state of $-\Delta+|x|^2$, $b>0$ is a small parameter, and u_b is the positive solution of (1.1), then $\omega\sim d-\omega_2b^2$ with $\omega_2>0$ defined uniquely from the Lyapunov–Schmidt projections. A more interesting asymptotic behaviour of solutions of (1.1) appear in the limit $\omega\to\omega_*$. Such studies were initialed by Brezis *et al* [5–7] in the context of the following Dirichlet problem

$$\begin{cases}-\Delta u+a(x)u=\omega u+|u|^{\frac{4}{d-2}}u & \text{in } \Omega,\\ u(x)=0, & \text{on } \partial\Omega,\end{cases} \tag{1.4}$$

where $\Omega\subset\mathbb{R}^d$ $(d\geqslant 3)$ is a bounded domain with smooth boundary and $a(x)$ is a smooth weight (see [1, 8, 12–14, 16, 17, 21–25, 28–30, 33–35]). The concentration phenomenon of solutions of the Dirichlet problem (1.4) depends on the geometry of the domain Ω. More precisely, solutions concentrate around the critical points of the Robin function of the domain Ω. To our best knowledge, the concentration phenomenon and the asymptotic behavior of positive and spatially decaying solutions of the stationary equation (1.1) in the energy-critical case $p = \frac{2d}{d-2}$ have not been studied yet. *Thus, the first purpose of this paper is to give a precise description of the latter problems in the energy-critical case.* Together with [37, theorem 5] as $\omega \to d^-$, this result suggests how the ground state solutions of (1.1) change as ω increases from ω_* to d. Our results are valid for $p = 6$ (quintic case) and $d = 3$ (three dimensions) where they have physical applications (see also [11]).

While the existence results of the spatially decaying solutions of the stationary equation (1.1) are available for the energy-critical and energy-supercritical cases, their stability in the time-dependent equation (1.2) is determined by the Morse index, which is the number of negative eigenvalues of the associated linearisation operator. In the energy-critical case, the solutions of (1.1) constructed in [36, 37] by variational methods are the ground state solutions (in the sense of definition 1.1). It is standard to show that their Morse indices are equal to 1. However, in the energy-supercritical case, the solutions of (1.1) constructed in [4] are obtained by using shooting methods, thus, no variational formulation can be used to compute their Morse indices. *Hence, the second purpose of this paper is to estimate the Morse index of solutions of (1.1) for the entire range of energy-supercritical cases.*

1.2. Main results

We shall first introduce some notations and definitions to state our main results. Let $X \subset L^2(\mathbb{R}^d)$ be the form domain of the operator $-\Delta + |x|^2$ equipped with the norm

$$\|u\|_X := \left(\int_{\mathbb{R}^d} \left(|\nabla u|^2 + |x|^2 |u|^2 \right) \mathrm{d}x \right)^{\frac{1}{2}},$$

In the energy-critical case with $p = \frac{2d}{d-2}$, we introduce the energy space

$$\Sigma := X \cap L^{\frac{2d}{d-2}}(\mathbb{R}^d). \tag{1.5}$$

For fixed $\omega \in (\omega_*, d)$, we define

$$\mathcal{I}_\omega = \inf_{v \in \Sigma} \left\{ I_\omega(v) : \quad \|v\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} = 1 \right\}, \quad I_\omega(v) := \|v\|_X^2 - \omega \|v\|_{L^2(\mathbb{R}^d)}^2. \tag{1.6}$$

By the method of Lagrange's multipliers and the scaling transformation, $u = (\mathcal{I}_\omega)^{\frac{d-2}{4}} v$ is a nontrivial solution of the stationary equation (1.1) if v is a minimiser of the variational problem (1.6). Based on the above observations, we can introduce the following definition.

Definition 1.1. We say that u_ω is a ground state of the stationary equation (1.1) if $v_\omega \in \Sigma$ is a minimiser of the variational problem (1.6) such that $I_\omega(v_\omega) = \mathcal{I}_\omega$ and $u_\omega := (\mathcal{I}_\omega)^{\frac{d-2}{4}} v_\omega$.

Let

$$U_\varepsilon(x) = \varepsilon^{\frac{d-2}{2}} [d(d-2)]^{\frac{d-2}{4}} \left(\frac{1}{\varepsilon^2 + |x|^2} \right)^{\frac{d-2}{2}}, \quad \varepsilon > 0 \tag{1.7}$$

be a family of the algebraic solitons (also called the Aubin–Talanti bubbles [2, 40]) which satisfy the elliptic problem

$$-\Delta u = u^{\frac{d+2}{d-2}}, \quad u \in D^{1,2}(\mathbb{R}^d), \tag{1.8}$$

where $D^{1,2}(\mathbb{R}^d)$ denotes the space of closure of $C_0^\infty(\mathbb{R}^d)$ under the norm $\|\nabla \cdot\|_{L^2(\mathbb{R}^d)}$.

For the sake of simplicity, we also denote $U_{\varepsilon=1}$ by U. It is well known (see [2, 40]) that U_ε for every $\varepsilon > 0$ attains the best constant of the Sobolev embedding

$$\|u\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \leqslant \mathcal{S}^{-\frac{1}{2}} \|\nabla u\|_{L^2(\mathbb{R}^d)},$$

where $\mathcal{S}$ is given by

$$\mathcal{S} = \inf_{v \in D^{2,1}(\mathbb{R}^d)} \left\{ \|\nabla v\|^2_{L^2(\mathbb{R}^d)} : \quad \|v\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} = 1 \right\}. \tag{1.9}$$

By the scaling transformation, if v is a minimiser of the variational problem (1.9), then $u := (\mathcal{S})^{\frac{d-2}{4}} v$ is a solution of the elliptic problem (1.8) given by the family of algebraic solutions (1.7) up to spatial translations in $\mathbb{R}^d$.

Since the operator $-\Delta + |x|^2 - \omega_*$ is positive in X by (1.3), we can define the unique solution of the following inhomogeneous equation

$$-\Delta u + (|x|^2 - \omega_*) u = U_\varepsilon^{\frac{d+2}{d-2}}, \quad u \in X, \tag{1.10}$$

denoted by PU_ε. Moreover, since $U_\varepsilon > 0$, by the positivity of the operator $-\Delta + |x|^2 - \omega_*$ and the maximum principle, we know that $PU_\varepsilon > 0$ in $\mathbb{R}^d$.

Let G be the Green function of the positive operator $-\Delta + |x|^2 - \omega_*$,

$$\begin{cases} -\Delta G + (|x|^2 - \omega_*) G = (d-2)|\mathbb{S}^{d-1}|\delta_0 & \text{in } \mathbb{R}^d, \\ G(x) \to 0 & \text{as } |x| \to +\infty, \end{cases} \tag{1.11}$$

where δ_0 is the Dirac measure supported at $x = 0$ and $|\mathbb{S}^{d-1}|$ is the Lebesgue measure of the unit sphere in $\mathbb{R}^d$. This gives the unique normalisation of the Green function such that $G = |x|^{2-d} - H$, where H is a regular part of G satisfying the following equation

$$\begin{cases} -\Delta H + (|x|^2 - \omega_*) H = (|x|^2 - \omega_*)|x|^{2-d} & \text{in } \mathbb{R}^d, \\ H(x) \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{1.12}$$

By uniqueness of solutions to the elliptic problems (1.11) and (1.12), G and H are radially symmetric. Our main results in the energy-critical case $p = \frac{2d}{d-2}$ can be stated as follows.

Theorem 1.1. *Let $d \geqslant 3$, $p = \frac{2d}{d-2}$, and u_ω be the ground state solution of the stationary equation (1.1) for $\omega \in (\omega_*, d)$, where ω_* is given by (1.3). There exists $\varepsilon_\omega > 0$ such that*

- $u_\omega = PU_{\varepsilon_\omega} + \hat{u}_\omega$ *for* $3 \leqslant d \leqslant 6$
- $u_\omega = U_{\varepsilon_\omega} + \hat{u}_\omega$ *for* $d \geqslant 7$,

with $\varepsilon_\omega \to 0$ and $\|\hat{u}_\omega\|_X \to 0$ as $\omega \to \omega_*^+$. Moreover, $\mathcal{I}_\omega < \mathcal{S}$ for $\omega \in (\omega_*, d)$, $\mathcal{I}_\omega \to \mathcal{S}$ as $\omega \to \omega_*^+$, and there exist positive constants a_d, b_d and c_d which only depend on the dimension d, *such that the concentration rate ε_ω and the ground state energy $\mathcal{I}_\omega$ satisfy*

- *for* $\mathrm{d} = 3$

$$a_d = \lim_{\omega \to 1^+} \frac{\varepsilon_\omega}{(\omega - 1)\|G\|^2_{L^2(\mathbb{R}^3)}}, \qquad b_d = \lim_{\omega \to 1^+} \frac{\mathcal{S} - \mathcal{I}_\omega}{((\omega - 1)\|G\|^2_{L^2(\mathbb{R}^3)})^2},$$

- *for* d = 4,

$$a_d = \lim_{\omega\to 0+} \frac{\omega|\log\varepsilon_\omega|}{H(0)\|U\|^3_{L^3(\mathbb{R}^4)}}, \qquad b_d = \lim_{\omega\to 0+} \frac{\omega\left|\log(\mathcal{S}-\mathcal{I}_\omega) - \log\left(c_d H(0)\|U\|^3_{L^3(\mathbb{R}^4)}\right)\right|}{H(0)\|U\|^3_{L^3(\mathbb{R}^4)}},$$

- *for* d = 5,

$$a_d = \lim_{\omega\to 0+} \frac{H(0)\|U\|^{\frac{7}{3}}_{L^{\frac{7}{3}}(\mathbb{R}^d)}\varepsilon_\omega}{\|U\|^2_{L^2(\mathbb{R}^5)}\omega}, \qquad b_d = \lim_{\omega\to 0+} \frac{\left(H(0)\|U\|^{\frac{7}{3}}_{L^{\frac{7}{3}}(\mathbb{R}^d)}\right)^2(\mathcal{S}-\mathcal{I}_\omega)}{\|U\|^6_{L^2(\mathbb{R}^5)}\omega^3},$$

- *for* d = 6,

$$a_d = \lim_{\omega\to 0+} \frac{|\log\omega|\varepsilon^2_\omega}{\|U\|^2_{L^2(\mathbb{R}^6)}\omega}, \qquad b_d = \lim_{\omega\to 0+} \frac{|\log\omega|(\mathcal{S}-\mathcal{I}_\omega)}{\|U\|^4_{L^2(\mathbb{R}^6)}\omega^2},$$

- *for* $d \geqslant 7$,

$$\frac{1}{2} = \lim_{\omega\to 0+} \frac{\|xU\|^2_{L^2(\mathbb{R}^d)}\varepsilon^2_\omega}{\|U\|^2_{L^2(\mathbb{R}^d)}\omega}, \qquad b_d = \lim_{\omega\to 0+} \frac{\|xU\|^2_{L^2(\mathbb{R}^d)}(\mathcal{S}-\mathcal{I}_\omega)}{\|U\|^4_{L^2(\mathbb{R}^d)}\omega^2}.$$

Remark 1.1. Theorem 1.1 is the first result on the concentration phenomena and the asymptotic behaviour of solutions of the stationary equation (1.1) in the energy-critical case $p = \frac{2d}{d-2}$. It is worth pointing out that a formal and brief calculation on the upper bounds of $\mathcal{I}_\omega$ is obtained in [36, section 5] to ensure the existence of minimisers of $\mathcal{I}_\omega$. These upper bounds of $\mathcal{I}_\omega$ are calculated in a standard way by choosing the Aubin–Talanti bubbles as test functions of $\mathcal{I}_\omega$, as that in [6]. However, the main difficulty in proving theorem 1.1 is to obtain a good lower bound of $\mathcal{I}_\omega$ which will match the upper bound generated by the Aubin–Talanti bubbles up to the leading order terms. To achieve this, we need to further employ the ideas in literature [7, 13, 14, 16, 17, 21, 22, 34, 35], that is, splitting of u_ω into two parts in X and estimating of these two parts precisely up to the leading order term. We remark that, due to the growth of the harmonic potential at infinity and the unboundedness of $\mathbb{R}^d$, the regular part of the Green function of the operator $-\Delta + |x|^2 - \omega_*$ is no longer bounded for all $d \geqslant 3$. Thus, we need to modify the arguments of the proofs in a nontrivial way to capture the leading order terms of ε_ω and $\mathcal{I}_\omega$ for all $d \geqslant 3$, which also makes the concentration phenomena of positive solutions of the stationary equation (1.1) to be more complicated than that of the Dirichlet problem (1.4).

Remark 1.2. One can use parameter ε in the family of algebraic solitons (1.7) to parameterise the family of the ground states (ω, u_ω) of the stationary equation (1.1). It follows from theorem 1.1 that the asymptotic behaviour of the mapping $\varepsilon \mapsto \omega$ as $\varepsilon \to 0$ depends on the dimension $d \geqslant 3$ and satisfies

$$\omega - \omega_* \sim \begin{cases} \varepsilon & \text{for } d = 3, \\ |\log\varepsilon|^{-1} & \text{for } d = 4, \\ \varepsilon & \text{for } d = 5, \\ \varepsilon^2|\log\varepsilon| & \text{for } d = 6, \\ \varepsilon^2 & \text{for } d \geqslant 7. \end{cases}$$

This asymptotic dependence for $d \geqslant 5$ was recently confirmed by the outcomes of the shooting method in [32]. However, the asymptotic expressions for $d = 3, 4$ were not recovered with the shooting method in [32].

Proceeding now with the energy-supercritical case, we will fix $p = 4$ to simplify the computations similarly to what was adopted in [4, 31] since this case has more physical applications (see [9]) and solutions in the energy-supercritical case are less sensitive to the nonlinearity power p compared to the energy-critical case. The energy-supercritical case for $p = 4$ corresponds to $d \geqslant 5$ and the stationary equation (1.1) is reduced to

$$-\Delta u + |x|^2 u = \omega u + u^3, \quad \text{in } \mathbb{R}^d. \tag{1.13}$$

It has been proved in [38] (see also [4] for a different proof), that there exists a singular radial solution u_∞ of the stationary equation (1.13) for some $\omega_\infty \in (d-4, d)$ satisfying

$$u_\infty(x) = \frac{\sqrt{d-3}}{|x|}\left[1 + \mathcal{O}(|x|^2)\right] \quad \text{as } |x| \to 0. \tag{1.14}$$

Moreover, by [4, theorem 1.1], for every $b > 0$, there exists a positive radial solution u_b of the stationary equation (1.13) for some $\omega_b \in (d-4, d)$ satisfying $u_b(0) = b$. By [38, theorem 1.2], it is known that $u_b \to u_\infty$ strongly in Σ and $\omega_b \to \omega_\infty$ as $b \to +\infty$, where Σ is given by (1.5). The precise asymptotic behaviour of ω_b as $b \to +\infty$ is obtained in [4, theorem 1.3] under some nondegeneracy assumptions. By [4, theorem 1.3], ω_b is oscillatory around ω_∞ as $b \to +\infty$ for $5 \leqslant d \leqslant 12$ and ω_b converges to ω_∞ monotonically as $b \to +\infty$ for $d \geqslant 13$. Moreover, it was proven in [31, theorem 1.2] that the Morse index of u_b in the class of radial functions is equal for large b to the Morse index of u_∞ in the monotone case $d \geqslant 13$. It was also conjectured in [31] based on numerical evidences that it is equal to 1 for the monotone case $d \geqslant 13$, where the definition for the Morse index of the singular solution u_∞ in the class of radial functions is the following.

Definition 1.2. Let u_∞ be the singular radial solution of the stationary equation (1.13) for some $\omega_\infty \in (d-4, d)$ satisfying (1.14) and consider the linearised operator

$$L_\infty := -\Delta + |x|^2 - \omega_\infty - 3u_\infty^2$$

in $X_{\text{rad}} := \{f \in X : f \text{ is radial}\}$. The Morse index of u_∞ denoted by $\mathfrak{m}(u_\infty)$ is the number of negative eigenvalues of L_∞ in X_{rad}.

The following theorem states that the Morse index of u_∞ in the class of radial functions is infinite for the oscillatory behavior with $5 \leqslant d \leqslant 12$ and finite for the monotone behaviour with $d \geqslant 13$. In the latter case, we give a precise estimation of $\mathfrak{m}(u_\infty)$.

Theorem 1.2. *Let $p = 4$, $d \geqslant 5$ and u_∞ be the singular radial solution of the stationary equation (1.13) for some $\omega_\infty \in (d-4, d)$ satisfying (1.14). Then*

$$\mathfrak{m}(u_\infty) = \begin{cases} \infty, & 5 \leqslant d \leqslant 12, \\ 1 \textit{ or } 2, & 13 \leqslant d \leqslant 15, \\ 1, & d \geqslant 16. \end{cases}$$

Remark 1.3. To prove theorem 1.2 for $5 \leqslant d \leqslant 12$, we shall mainly follow the ideas in [20]. The oscillation of ω_b around ω_∞ as $b \to +\infty$ is obtained in [4, theorem 1.3] under some nondegeneracy assumptions, which are hard to verify. In order to avoid making these nondegeneracy assumptions, we need to modify the arguments in [20].

Remark 1.4. In proving theorem 1.2 for $d \geqslant 13$, we consider the limiting spectral problem

$$-\Delta u + |x|^2 u - \frac{3(d-3)}{|x|^2} u = \sigma u, \quad u \in X_{\mathrm{rad}}, \tag{1.15}$$

whose eigenvalues $\{\sigma_n\}_{n\in\mathbb{N}}$ are completely known in the literature from the confluent hypergeometric equation [42]. We compare $\omega_\infty + 3u_\infty^2$ and $\sigma_3 + \frac{3(d-3)}{r^2}$ to control $\mathfrak{m}(u_\infty)$ by the Morse index of the radial eigenfunctions of the spectral problem (1.15). As a by-product, we also prove that u_∞ is nondegenerate for $d \geqslant 16$, this avoids the nondegeneracy assumptions of [31]. See remark 3.3 for more details.

Remark 1.5. As pointed out in [31], if Morse index of the solution u_∞ is equal to 1, then the Vakhitov–Kolokolov stability criterion can be used to show orbital stability of u_∞ in the time evolution of the Gross–Pitaevskii equation (1.2). By theorem 1.2, $\mathfrak{m}(u_\infty) = 1$ in X_{rad} for $d \geqslant 16$. However, the Morse index of u_∞ in the general case of non-radial functions in X is still an open problem. We conjecture that in the monotone case with $d \geqslant 13$, there are no negative eigenvalues of L_∞ for non-radial functions so that the Morse index of u_∞ in X is equal to $\mathfrak{m}(u_\infty)$ in X_{rad}.

Remark 1.6. By [38, theorem 1.1], the positive radial singular solution u_∞ for general p behaviours like $|x|^{-\frac{2}{p-1}}$ near $|x| = 0$. Our method in estimating Morse index of the positive radial singular solution also works for general p and the result can be stated as follows. For fixed d, there exists $p_d > p_*$ in the case of $d \geqslant 11$, where $p_* := 2/(d-4-2\sqrt{d-1})$ was introduced in [27], such that $\mathfrak{m}(u_\infty) = \infty$ for $3 \leqslant d \leqslant 10$ and

$$\mathfrak{m}(u_\infty) = \begin{cases} \infty, & \frac{2d}{d-2} < p < p_*, \\ 1 \text{ or } 2, & p_* \leqslant p \leqslant p_d, \\ 1, & p_d < p \end{cases}$$

for $d \geqslant 11$.

1.3. Notations

Throughout this paper, C and $C^{'}$ are indiscriminately used to denote various positive constants. Notation $a \lesssim b$ means that there exists $C > 0$ such that $a \leqslant Cb$. Notation $a = \mathcal{O}(b)$ means that there exist $C, C' > 0$ such that $C'b \leqslant a \leqslant Cb$. Notation $a = o(b)$ means that $\lim\limits_{b\to 0} a/b = 0$. Notation $a \sim b$ as $b \to 0$ means that $\lim\limits_{b\to 0} a/b = 1$ (the same convention is used if $b \to \infty$).

2. The energy-critical case

2.1. Preliminaries

It has been proved in [36, section 5], without the statement of theorems, that $\mathcal{I}_\omega$ is attained for $\omega \in (\omega_*, d)$. On the other hand, by the Pohozaev identity, see, e.g. [4, proposition 2.2], we know that the stationary equation (1.1) has no solutions in Σ for $\omega \leqslant 0$ which implies that $\mathcal{I}_\omega$ can not be attained for $\omega \leqslant 0$. Moreover, since d is the first eigenvalue of $-\Delta + |x|^2$ in X, by multiplying (1.1) with the first eigenfunction of the operator $-\Delta + |x|^2$ on both sides and integrating by parts, see, e.g. [4, proposition 2.1], we know that the stationary equation (1.1) has no positive solutions for $\omega \geqslant d$. This implies that $\mathcal{I}_\omega$ can not be attained for $\omega \geqslant d$ either since minimisers of the variational problem (1.6) are positive and radially symmetric. In addition, by

[37, theorem 3] or [39, theorem 7], the stationary equation (1.1) also has no positive solutions for $\omega \leqslant 1$ in the case of $d = 3$. Thus, we know that $\mathcal{I}_\omega$ is attained if and only if $\omega \in (\omega_*, d)$.

Since $\mathcal{I}_\omega$ is attained for $\omega \in (\omega_*, d)$, it can be proven in a standard way that $\mathcal{I}_\omega$ is strictly decreasing for $\omega \in [\omega_*, d]$ with $\mathcal{I}_{\omega=\omega_*} = \mathcal{S}$ and $\mathcal{I}_{\omega=d} = 0$, where $\mathcal{S}$ is the best constant of the Sobolev embedding given by the variational problem (1.9). The monotone property was first pointed out by Brezis and Nirenberg in [6, remark 1.5]. The detailed proofs were recently given in [10, lemma 2.1] and [44, lemma 3.3]. Hence, we have

$$0 < \mathcal{I}_\omega < \mathcal{S} = \mathcal{I}_{\omega_*} \quad \text{for all } \omega \in (\omega_*, d). \tag{2.1}$$

Let v_ω be the minimiser of the variational problem (1.6) for $\omega \in (\omega_*, d)$. Then, $u_\omega := (\mathcal{I}_\omega)^{\frac{d-2}{4}} v_\omega$ is the ground state solution of the stationary equation (1.1). Since we are interested in $\omega \to \omega_*$ with $\omega_* < d$, it is standard to show that $\{u_\omega\}$ is bounded in X. By the compactness of the embedding from X to $L^2(\mathbb{R}^d)$ due to the harmonic potential $|x|^2$, we may assume that there exists $u_* \in \Sigma$ such that $u_\omega \rightharpoonup u_*$ weakly in Σ and $u_\omega \to u_*$ strongly in $L^2(\mathbb{R}^d)$ as $\omega \to \omega_*^+$. We claim that $u_* = 0$. Indeed, if $u_* \neq 0$, then $u_* \in \Sigma$ satisfies

$$-\Delta u_* + |x|^2 u_* = \omega_* u_* + |u_*|^{\frac{4}{d-2}} u_* \tag{2.2}$$

in the weak sense, which, together with (2.1), implies

$$I_{\omega_*}(u_*) = \|u_*\|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \leqslant \|u_\omega\|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} + o(1) = I_\omega(u_\omega) + o(1) \leqslant I_{\omega_*}(u_*) + o(1). \tag{2.3}$$

Thus, u_* corresponds to the minimiser v_* with $I_{\omega_*}(v_*) = \mathcal{I}_{\omega_*}$ by $u_* := (\mathcal{I}_{\omega_*})^{\frac{d-2}{4}} v_*$ so that u_* is positive and radially symmetric. This contradicts the previously reviewed results, from which no positive and radially symmetric solution of the stationary equation (2.2) exists in Σ with ω_* given by (1.3). Therefore, we must have $u_* = 0$ and $u_\omega \rightharpoonup 0$ weakly in X and $u_\omega \to 0$ strongly in $L^2(\mathbb{R}^d)$ as $\omega \to \omega_*^+$. Moreover, since v_ω is the minimiser of the variational problem (1.6) and $u_\omega = (\mathcal{I}_\omega)^{\frac{d-2}{4}} v_\omega$, by (2.1) and (2.3),

$$\|u_\omega\|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} = (\mathcal{I}_\omega)^{\frac{d}{2}} \|v_\omega\|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} = \mathcal{S}^{\frac{d}{2}} + o(1) \quad \text{as} \quad \omega \to \omega_*^+. \tag{2.4}$$

Since u_ω is also the ground state solution of the stationary equation (1.1), by multiplying (1.1) with u_ω on both sides and integrating by parts, we also have

$$\|u_\omega\|_X^2 = \omega \|u_\omega\|^2_{L^2(\mathbb{R}^d)} + \|u_\omega\|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} = \mathcal{S}^{\frac{d}{2}} + o(1) \quad \text{as} \quad \omega \to \omega_*^+ \tag{2.5}$$

since $u_\omega \to 0$ strongly in $L^2(\mathbb{R}^d)$ as $\omega \to \omega_*^+$. On the other hand, it follows from(1.9), (2.4), and (2.5) that

$$\|\nabla u_\omega\|^2_{L^2(\mathbb{R}^d)} \geqslant \mathcal{S} \|u_\omega\|^2_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} = \mathcal{S}^{1+\frac{d-2}{2}} + o(1) = \mathcal{S}^{\frac{d}{2}} + o(1) \quad \text{as} \quad \omega \to \omega_*^+,$$

which implies that

$$\|x u_\omega\|^2_{L^2(\mathbb{R}^d)} = o(1) \quad \text{as} \quad \omega \to \omega_*^+. \tag{2.6}$$

2.2. *Expansions of* u_ω

Since u_ω is a ground state solution of the stationary equation (1.1) related to a minimiser of the variational problem (1.6), the moving-plane method (see [19]) or the Schwarz symmetrisation (see [45]) imply that u_ω is radial, positive and strictly decreasing in $r = |x|$. The following lemma clarifies the construction of PU_ε from solutions of the inhomogeneous equation (1.10).

Lemma 2.1. *Let* $3 \leqslant d \leqslant 6$*, then*

$$PU_\varepsilon = U_\varepsilon - \varepsilon^{\frac{d-2}{2}}[d(d-2)]^{\frac{d-2}{4}} H - \eta_\varepsilon, \quad |x| \lesssim 1 \tag{2.7}$$

and

$$PU_\varepsilon(x) \lesssim \varepsilon^{\frac{d+2}{2}} |x|^{-(4+d)} \quad \textit{for } |x| \gtrsim 1, \tag{2.8}$$

where H *is defined by (1.12) and the correction term* η_ε *satisfies*

$$\|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \lesssim \varepsilon^{\frac{d+2}{2}} \quad \text{for } 3 \leqslant d \leqslant 5, \tag{2.9}$$

and

$$\|\eta_\varepsilon\|_{W^{2,\frac{3}{2}}(\mathbb{R}^6)} \lesssim \varepsilon^4, \quad \text{for } d = 6. \tag{2.10}$$

Moreover,

$$H(x) = \begin{cases} H(0) + \dfrac{1}{2}|x| + \mathcal{O}(|x|^2), & d = 3, \\ H(0) + \mathcal{O}(|x|^\alpha), & d = 4, 5, \\ -\dfrac{1}{4}\log|x| + \mathcal{O}(1), & d = 6 \end{cases}$$

near $|x| = 0$ *and*

$$\|PU_{\varepsilon_\omega}\|^2_{L^2(\mathbb{R}^d)} = \begin{cases} \varepsilon_\omega 3^{\frac{1}{2}} \displaystyle\int_{\mathbb{R}^3} G^2 \mathrm{d}x + \mathcal{O}(\varepsilon_\omega^{2-\sigma}), & d = 3, \\ 8|\mathbb{S}^3| \varepsilon_\omega^2 |\log \varepsilon_\omega| + o(|\varepsilon_\omega^2| \log \varepsilon_\omega|), & d = 4, \\ \varepsilon_\omega^2 \|U\|^2_{L^2(\mathbb{R}^5)} + o(\varepsilon_\omega^2), & d \geqslant 5, \end{cases}$$

where $\alpha \in (0, 1)$ *and* $\sigma > 0$ *is a fixed constant which is sufficiently small.*

Proof. Since it follows from (1.7) that

$$U_\varepsilon(x) \sim \varepsilon^{\frac{d-2}{2}} |x|^{2-d} \quad \text{for } |x| \gtrsim 1, \tag{2.11}$$

the classical L^p-theory of elliptic equations and the Sobolev embedding theorem imply that the unique solution of the inhomogeneous equation (1.10) exists and satisfies $PU_\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}^d \backslash \{0\})$. In particular, $PU_\varepsilon \lesssim 1$ for $|x| \gtrsim 1$ and $\varepsilon \lesssim 1$. Since

$$-\Delta|x|^{-(4+d)} + (|x|^2 - \omega_*)|x|^{-(4+d)} \sim |x|^{-(2+d)} \quad \text{for } |x| \gtrsim 1,$$

it follows from (2.11) that $\varepsilon^{\frac{d+2}{2}}|x|^{-(4+d)}$ is a supersolution of equation (1.10) for $|x| \gtrsim 1$. Now, by the fact that $PU_\varepsilon \lesssim 1$ for $|x| \gtrsim 1$ and $\varepsilon \lesssim 1$, the fact that $PU_\varepsilon \to 0$ and $\varepsilon^{\frac{d+2}{2}}|x|^{-(4+d)} \to 0$ as $|x| \to +\infty$ and the maximum principle, we obtain (2.8).

To obtain (2.7), we write

$$\varphi_\varepsilon := U_\varepsilon - PU_\varepsilon, \tag{2.12}$$

then by (1.8) and (1.10), φ_ε is the unique solution of the following equation:

$$-\Delta u + (|x|^2 - \omega_*)u = (|x|^2 - \omega_*)U_\varepsilon, \quad u \in X. \tag{2.13}$$

By (1.3) and the maximum principle, $\varphi_\varepsilon > 0$ in $\mathbb{R}^d$ for $d \geqslant 4$. For $d=3$, since $PU_\varepsilon > 0$ in $\mathbb{R}^3$, there exists a unique $r_0 > 0$ such that φ_ε is strictly increasing with respect to $r = |x|$ in $[0, r_0)$ and is strictly decreasing in $[r_0, +\infty)$. Moreover, it follows from (1.11) by using the maximum principle that

$$|G(x)| \lesssim \mathrm{e}^{-\sigma|x|^2} \quad \text{for some } \sigma > 0, \tag{2.14}$$

so that $H(x) = |x|^{2-d} + \mathcal{O}(\mathrm{e}^{-\sigma|x|^2})$ as $|x| \to \infty$. Thus, by (1.3) and the classical L^p-theory of elliptic equations, we know that $H \in W^{2,s}_{loc}(\mathbb{R}^d)$ for $1 < s < 3$ in the case of $d=3$, $1 < s < +\infty$ in the case of $d=4$ and $1 < s < \frac{d}{d-4}$ in the case of $d \geqslant 5$. It follows from the Sobolev embedding theorem that $H \in L^\infty(\mathbb{R}^d) \cap C^\alpha_{\mathrm{loc}}(\mathbb{R}^d)$ for $3 \leqslant d \leqslant 5$ and $0 < \alpha < 1$ and $H \in L^{\frac{3s}{3-s}}_{\mathrm{loc}}(\mathbb{R}^6)$ for $d=6$ and $1 < s < 3$. Next we define

$$\eta_\varepsilon := \varphi_\varepsilon - \varepsilon^{\frac{d-2}{2}}[d(d-2)]^{\frac{d-2}{4}} H. \tag{2.15}$$

It follows from (1.12) and (2.13) that η_ε is the unique solution of the following equation:

$$\begin{cases} -\Delta u + (|x|^2 - \omega_*)u = \varepsilon^{\frac{d-2}{2}}[d(d-2)]^{\frac{d-2}{4}}(|x|^2 - \omega_*)g_\varepsilon & \text{in } \mathbb{R}^d, \\ u(x) \to 0 & \text{as } |x| \to +\infty, \end{cases}$$

where $g_\varepsilon = (\varepsilon^2 + |x|^2)^{\frac{2-d}{2}} - |x|^{2-d}$ satisfies

$$g_\varepsilon(x) \sim \begin{cases} -|x|^{2-d}, & |x| \leqslant \frac{\varepsilon}{\sqrt{2}}, \\ -\varepsilon^2|x|^{-d}, & |x| \geqslant \frac{\varepsilon}{\sqrt{2}}. \end{cases} \tag{2.16}$$

As in the previous estimates, by the classical L^p-theory of elliptic equations, the Sobolev embedding theorem and the maximum principle, we obtain

$$|\eta_\varepsilon(x)| \lesssim \varepsilon^{\frac{d+2}{2}}|x|^{-(2+d)} \quad \text{for } |x| \gtrsim 1. \tag{2.17}$$

Let $h_\varepsilon = \varepsilon^{\frac{d-2}{2}}[d(d-2)]^{\frac{d-2}{4}}(|x|^2 - \omega_*)g_\varepsilon$. It follows from (2.16) that

$$\|h_\varepsilon\|_{L^s_{\mathrm{loc}}(\mathbb{R}^d)} \lesssim \begin{cases} \varepsilon^{\frac{3}{s}-\frac{1}{2}}, & d=3, \\ \varepsilon^{2+\frac{d}{s}-\frac{d-2}{2}}, & 4 \leqslant d \leqslant 6. \end{cases} \tag{2.18}$$

Thus, by (2.17) and the classical L^p-theory of elliptic equations, we know that $\eta_\varepsilon \in W^{2,s}(\mathbb{R}^d)$ for $1 < s < 3$ in the case of $d=3$, $1 < s < +\infty$ in the case of $d=4$ and $1 < s < \frac{d}{d-4}$ in the case of $d \geqslant 5$. The Sobolev embedding theorem implies that $\eta_\varepsilon \in L^\infty(\mathbb{R}^d) \cap C^\alpha_{\mathrm{loc}}(\mathbb{R}^d)$ for $3 \leqslant d \leqslant 5$ and $0 < \alpha < 1$ and $\eta_\varepsilon \in L^{\frac{3s}{3-s}}(\mathbb{R}^6)$ for $d=6$ and $1 < s < 3$. Representation (2.7) follows from (2.12) and (2.15). Estimates (2.9) and (2.10) follow from (2.17) and (2.18), the classical L^p-theory and the Sobolev embedding theorem by choosing $s=2$ for $d=3$ and $s = \frac{d}{d-2}$ for $d=4,5,6$. By the regularity of H for $d=4,5$, $H(x) = H(0) + \mathcal{O}(|x|^\alpha)$ near $|x|=0$. For $d=3$ and $d=6$, we need to expand $H(x)$ as done in [17]. We define $\psi = H(x) - \frac{1}{2}|x|$ for $d=3$, then by (1.12), ψ satisfies

$$-\Delta\psi + (|x|^2 - 1)\psi = \frac{1}{2}|x|(1 - |x|^2) \quad \text{in } \mathbb{R}^3.$$

Since the data $\frac{|x|-|x|^3}{2}$ belongs to $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^3)$. Thus, by the classical regularity theory, $\psi \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ for some $\alpha \in (0,1)$, which, together with ψ being radial, implies $\nabla\psi(0) = 0$. It follows that

$$H(x) = H(0) + \frac{1}{2}|x| + \mathcal{O}(|x|^2) \quad \text{near } |x| = 0$$

for $d = 3$. For $d = 6$, since $\Delta(\log|x|) = \frac{4}{|x|^2}$ in $\mathbb{R}^6$ in the sense of distributions, it follows from (1.12) that $\widehat{H} := H + \frac{1}{4}\log|x|$ satisfies the following equation:

$$-\Delta\widehat{H} + |x|^2\widehat{H} = |x|^2\log|x| \quad \text{in } \mathbb{R}^6,$$

in the sense of distributions. Since $|x|^2\log|x| \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^6)$, by the classical elliptic regularity, $\widehat{H} \in C^2_{\mathrm{loc}}(\mathbb{R}^6)$. It follows that $H = -\frac{1}{4}\log|x| + \mathcal{O}(1)$ in B_R for any $R > 0$. The computation of $\|PU_{\varepsilon_\omega}\|^2_{L^2(\mathbb{R}^d)}$ is standard (see [16, 17]), so we omit it here. □

By (2.6) and Lions' theorem (see [43, theorem 1.41]), there exists $\{\varepsilon_\omega\} \subset \mathbb{R}_+$ such that $u_\omega \to U_{\varepsilon_\omega}$ strongly in $D^{1,2}(\mathbb{R}^d)$ as $\omega \to \omega_*^+$. Since $u_\omega \to 0$ strongly in $L^2(\mathbb{R}^d)$ as $\omega \to \omega_*^+$, it is easy to see that $\varepsilon_\omega \to 0$ as $\omega \to \omega_*^+$. The following lemma specifies a precise decomposition of u_ω near U_{ε_ω}.

Lemma 2.2. *As $\omega \to \omega_*^+$, there exists $\varepsilon_\omega > 0$ such that*

$$u_\omega = \begin{cases} PU_{\varepsilon_\omega} + \hat{u}_\omega & \text{for } 3 \leqslant d \leqslant 6, \\ U_{\varepsilon_\omega} + \hat{u}_\omega & \text{for } d \geqslant 7, \end{cases} \tag{2.19}$$

where $\varepsilon_\omega \to 0$ and $\hat{u}_\omega \to 0$ in X as $\omega \to \omega_*^+$ and

$$\hat{u}_\omega = \begin{cases} (\alpha_\omega - 1)PU_{\varepsilon_\omega} + \hat{u}_{\omega,*} & \text{for } 3 \leqslant d \leqslant 6, \\ (\alpha_\omega - 1)U_{\varepsilon_\omega} + \hat{u}_{\omega,*} & \text{for } d \geqslant 7, \end{cases}$$

with $\alpha_\omega \to 1$ and $\hat{u}_{\omega,*} \in \mathcal{M}_\omega^\perp$ defined by

$$\mathcal{M}_\omega = \begin{cases} \{PU_{\varepsilon_\omega}, \partial_{\varepsilon_\omega}PU_{\varepsilon_\omega}, \partial_{x_1}PU_{\varepsilon_\omega}, \ldots, \partial_{x_d}PU_{\varepsilon_\omega}\} & \text{for } 3 \leqslant d \leqslant 6, \\ \{U_{\varepsilon_\omega}, \partial_{\varepsilon_\omega}U_{\varepsilon_\omega}, \partial_{x_1}U_{\varepsilon_\omega}, \ldots, \partial_{x_d}U_{\varepsilon_\omega}\} & \text{for } d \geqslant 7, \end{cases}$$

and the orthogonality holds simultaneously in X and $L^2(\mathbb{R}^d)$.

Proof. It follows from the explicit formula (1.7) for $d \geqslant 7$ that

$$\int_{\mathbb{R}^d} |x|^2 U_\varepsilon^2 \mathrm{d}x = \varepsilon^4 \int_{\mathbb{R}^d} |x|^2 U^2 \mathrm{d}x, \qquad \int_{\mathbb{R}^d} U_\varepsilon^2 \mathrm{d}x = \varepsilon^2 \int_{\mathbb{R}^d} U^2 \mathrm{d}x. \tag{2.20}$$

Moreover, for all $d \geqslant 3$,

$$\int_{\mathbb{R}^d} U_\varepsilon^q \mathrm{d}x = \varepsilon^{d - \frac{(d-2)q}{2}} \int_{\mathbb{R}^d} U^q \mathrm{d}x \quad \text{for } q > \frac{d}{d-2} \tag{2.21}$$

and

$$\int_{B_1} U_\varepsilon^{\frac{d}{d-2}} \mathrm{d}x \sim \varepsilon^{\frac{d}{2}} |\log\varepsilon|. \tag{2.22}$$

Thus, by the fact that $u_\omega \to U_{\varepsilon_\omega}$ strongly in $D^{1,2}(\mathbb{R}^d)$ as $\omega \to \omega_*^+$ and (2.6), we have for $d \geqslant 7$

$$\|u_\omega - U_{\varepsilon_\omega}\|_X^2 \to 0 \quad \text{as } \omega \to \omega_*^+. \tag{2.23}$$

On the other hand, since $H, \eta_{\varepsilon_\omega} \in L^\infty(\mathbb{R}^d) \cap C^\alpha_{\mathrm{loc}}(\mathbb{R}^d)$ for $3 \leqslant d \leqslant 5$ and $0 < \alpha < 1$ and $H, \eta_{\varepsilon_\omega} \in L^{\frac{3s}{3-s}}(\mathbb{R}^6)$ for $d = 6$ and $1 < s < 3$ by lemma 2.1, it follows from (2.7)–(2.10) that $PU_{\varepsilon_\omega} \to U_{\varepsilon_\omega}$ strongly in $D^{1,2}(\mathbb{R}^d)$ as $\omega \to \omega_*^+$. Thus, it is also easy to see for $3 \leqslant d \leqslant 6$ that

$$\|u_\omega - PU_{\varepsilon_\omega}\|_X^2 \to 0 \quad \text{as } \omega \to \omega_*^+. \tag{2.24}$$

Now, we define

$$e(\omega) := \begin{cases} \inf_{\varepsilon\in\mathbb{R}_+,\alpha\in\mathbb{R}} \|u_\omega - \alpha P U_\varepsilon\|_X^2 & \text{for } 3 \leqslant d \leqslant 6, \\ \inf_{\varepsilon\in\mathbb{R}_+,\alpha\in\mathbb{R}} \|u_\omega - \alpha U_\varepsilon\|_X^2 & \text{for } d \geqslant 7. \end{cases}$$

By (2.23) and (2.24), it is standard (see [3, 17, 35]) to show that $e(\omega) = o_\omega(1)$ is attained by some ε_ω satisfying $\varepsilon_\omega \to 0$ as $\omega \to \omega_*^+$, which implies that (2.19) hold with $\hat{u}_\omega \to 0$ in X as $\omega \to \omega_*^+$. The orthogonality conditions in X for $\hat{u}_{\omega,*} \in \mathcal{M}_\omega^\perp$ are obtained from

$$\frac{\partial}{\partial\varepsilon}\|u_\omega - \alpha P U_\varepsilon\|_X^2|_{\varepsilon=\varepsilon_\omega,\alpha=\alpha_\omega} = \frac{\partial}{\partial\alpha}\|u_\omega - \alpha P U_\varepsilon\|_X^2|_{\varepsilon=\varepsilon_\omega,\alpha=\alpha_\omega} = 0 \quad \text{for } 3 \leqslant d \leqslant 6,$$

and

$$\frac{\partial}{\partial\varepsilon}\|u_\omega - \alpha U_\varepsilon\|_X^2|_{\varepsilon=\varepsilon_\omega,\alpha=\alpha_\omega} = \frac{\partial}{\partial\alpha}\|u_\omega - \alpha U_\varepsilon\|_X^2|_{\varepsilon=\varepsilon_\omega,\alpha=\alpha_\omega} = 0 \quad \text{for } d \geqslant 7.$$

The orthogonality conditions in $L^2(\mathbb{R}^d)$ follows from the fact that the eigenfunctions of $-\Delta + |x|^2$ form an orthogonal basis of $L^2(\mathbb{R}^d)$. □

2.3. Estimates on $\hat{u}_\omega$

By [35, appendix D],

$$\int_{\mathbb{R}^d} \left(|\nabla v|^2 - (2^* - 1)U_{\varepsilon_\omega}^{2^*-2}|v|^2\right) \mathrm{d}x \geqslant \frac{4}{d+4}\int_{\mathbb{R}^d} |v|^2 \mathrm{d}x \tag{2.25}$$

for all $v \in D^{1,2}(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} \nabla v \nabla U_{\varepsilon_\omega} \mathrm{d}x = \int_{\mathbb{R}^d} \nabla v \nabla \partial_{\varepsilon_\omega} U_{\varepsilon_\omega} \mathrm{d}x = \int_{\mathbb{R}^d} \nabla v \nabla \partial_{x_l} U_{\varepsilon_\omega} \mathrm{d}x = 0$$

where $l = 1, 2, \ldots, d$. By lemma 2.2, we have

$$\int_{\mathbb{R}^d} \nabla \hat{u}_{\omega,*} \nabla U_{\varepsilon_\omega} \mathrm{d}x = \int_{\mathbb{R}^d} \nabla \hat{u}_{\omega,*} \nabla \partial_{\varepsilon_\omega} U_{\varepsilon_\omega} \mathrm{d}x = \int_{\mathbb{R}^d} \nabla \hat{u}_{\omega,*} \nabla \partial_{x_l} U_{\varepsilon_\omega} \mathrm{d}x = o(1)$$

for all $l = 1, 2, \ldots, d$ as $\omega \to \omega_*$. Thus,

$$\int_{\mathbb{R}^d} \left(|\nabla \hat{u}_{\omega,*}|^2 - (2^* - 1)U_{\varepsilon_\omega}^{2^*-2}|\hat{u}_{\omega,*}|^2\right) \mathrm{d}x \geqslant \left(\frac{4}{d+4} + o(1)\right)\int_{\mathbb{R}^d} |\hat{u}_{\omega,*}|^2 \mathrm{d}x \tag{2.26}$$

for $d \geqslant 7$. On the other hand, by (1.3), (2.7)–(2.10), (2.25) and $\hat{u}_{\omega,*} \in \mathcal{M}_\omega^\perp$, it is also standard (see [35, appendix D]) to show that

$$\int_{\mathbb{R}^d} \left(|\nabla \hat{u}_{\omega,*}|^2 + (|x|^2 - \omega_*)|\hat{u}_{\omega,*}|^2 - (2^* - 1)PU_{\varepsilon_\omega}^{2^*-2}|\hat{u}_{\omega,*}|^2\right) \mathrm{d}x \gtrsim \int_{\mathbb{R}^d} |\hat{u}_{\omega,*}|^2 \mathrm{d}x \tag{2.27}$$

for $3 \leqslant d \leqslant 6$. For $d = 3$, we need to use the fact that $\omega_* = 1$ and $\lambda = 3$ is the first eigenvalue of the operator $-\Delta + |x|^2$ in $L^2(\mathbb{R}^3)$.

The following lemma gives the asymptotic estimate on the X norm of $\hat{u}_\omega$. The proofs are simpler for $d \geqslant 7$ but get more technically involved for $3 \leqslant d \leqslant 6$.

Lemma 2.3. *Let $d \geqslant 3$, Then as $\omega \to \omega_*^+$,*

$$\|\hat{u}_\omega\|_X \lesssim \begin{cases} (\omega-1)\varepsilon_\omega^{\frac{1}{2}} + \varepsilon_\omega & \text{for } d=3, \\ \omega\varepsilon_\omega^{\frac{d-2}{2}} |\log \varepsilon_\omega|^{\frac{d-2}{d}} + \varepsilon_\omega^{d-2} & \text{for } 4 \leqslant d \leqslant 5, \\ \omega\varepsilon_\omega^2 |\log \varepsilon_\omega|^{\frac{2}{3}} + \varepsilon_\omega^{4-\sigma} & \text{for } d=6, \\ \omega\varepsilon_\omega^2 + \varepsilon_\omega^3 & \text{for } d \geqslant 7, \end{cases} \tag{2.28}$$

where $\sigma > 0$ is a small fixed constant.

Proof. For $3 \leqslant d \leqslant 6$, we obtain from (1.1), (1.10), and (2.19) that $\hat{u}_\omega$ satisfies

$$\begin{cases} -\Delta\hat{u}_\omega + (|x|^2-\omega)\hat{u}_\omega - \frac{d+2}{d-2}(PU_{\varepsilon_\omega})^{\frac{4}{d-2}}\hat{u}_\omega = E_\omega + N_\omega(\hat{u}_\omega) & \text{in } \mathbb{R}^d, \\ \hat{u}_\omega(x) \to 0 & \text{as } |x| \to 0 \end{cases} \tag{2.29}$$

where the inhomogeneous term is

$$E_\omega := (\omega-\omega_*)PU_{\varepsilon_\omega} + (PU_{\varepsilon_\omega})^{\frac{d+2}{d-2}} - U_{\varepsilon_\omega}^{\frac{d+2}{d-2}}$$

and the nonlinear term satisfies

$$|N_\omega(\hat{u}_\omega)| \lesssim (PU_{\varepsilon_\omega})^{\frac{6-d}{d-2}}|\hat{u}_\omega|^2 + |\hat{u}_\omega|^{\frac{d+2}{d-2}}. \tag{2.30}$$

It follows from (2.8) and (2.11) that

$$|E_\omega| \lesssim \varepsilon_\omega^{\frac{d+2}{2}}((\omega-\omega_*)|x|^{-(4+d)} + |x|^{-(2+d)}) \quad \text{for } |x| \gtrsim 1. \tag{2.31}$$

For $|x| \lesssim 1$, it follows from (2.7) for $4 \leqslant d \leqslant 6$ (for which $\omega_* = 0$) that

$$|E_\omega| \lesssim \omega U_{\varepsilon_\omega} + U_{\varepsilon_\omega}^{\frac{4}{d-2}}\left(\varepsilon_\omega^{\frac{d-2}{2}}|H| + |\eta_{\varepsilon_\omega}|\right) + U_{\varepsilon_\omega}^{\frac{6-d}{d-2}}\left(\varepsilon_\omega^{d-2}|H|^2 + |\eta_{\varepsilon_\omega}|^2\right) + \varepsilon_\omega^{\frac{d+2}{2}}|H|^{\frac{d+2}{d-2}} + |\eta_{\varepsilon_\omega}|^{\frac{d+2}{d-2}},$$

where we have used the fact that $\varphi_{\varepsilon_\omega} > 0$ which is given by (2.12) and (2.15). By lemma 2.1, we obtain from (2.21) and (2.22) for $4 \leqslant d \leqslant 5$ and $R > 0$ sufficiently large,

$$\begin{aligned} \left|\int_{B_R} E_\omega \hat{u}_{\omega,*}\mathrm{d}x\right| &\lesssim \omega\|\hat{u}_{\omega,*}\|_{L^{\frac{2d}{d-2}}(B_R)}\|U_{\varepsilon_\omega}\|_{L^{\frac{d}{d-2}}(B_R)} + \int_{B_R} U_{\varepsilon_\omega}^{\frac{4}{d-2}}\left(\varepsilon_\omega^{\frac{d-2}{2}}|H| + |\eta_{\varepsilon_\omega}|\right)|\hat{u}_{\omega,*}|\mathrm{d}x \\ &\quad + \int_{B_R} U_{\varepsilon_\omega}^{\frac{6-d}{d-2}}\left(\varepsilon_\omega^{d-2}|H|^2 + |\eta_{\varepsilon_\omega}|^2\right)|\hat{u}_{\omega,*}|\mathrm{d}x \\ &\quad + \int_{B_R}\left(\varepsilon_\omega^{\frac{d+2}{2}}|H|^{\frac{d+2}{d-2}} + |\eta_{\varepsilon_\omega}|^{\frac{d+2}{d-2}}\right)|\hat{u}_{\omega,*}|\mathrm{d}x \\ &\lesssim \left(\omega\|U_{\varepsilon_\omega}\|_{L^{\frac{d}{d-2}}(B_R)} + \varepsilon_\omega^{\frac{d+2}{2}}\right)\|\hat{u}_{\omega,*}\|_{L^{\frac{2d}{d-2}}(B_R)} \\ &\quad + \int_{B_R} U_{\varepsilon_\omega}^{\frac{4}{d-2}}\left(\varepsilon_\omega^{\frac{d-2}{2}}|H| + |\eta_{\varepsilon_\omega}|\right)|\hat{u}_{\omega,*}|\mathrm{d}x \\ &\quad + \int_{B_R} U_{\varepsilon_\omega}^{\frac{6-d}{d-2}}\left(\varepsilon_\omega^{d-2}|H|^2 + |\eta_{\varepsilon_\omega}|^2\right)|\hat{u}_{\omega,*}|\mathrm{d}x \\ &\lesssim \left(\omega\varepsilon_\omega^{\frac{d-2}{2}}|\log\varepsilon_\omega|^{\frac{d-2}{d}} + \varepsilon_\omega^{\frac{d+2}{2}}\right)\|\hat{u}_{\omega,*}\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} + |I|, \end{aligned} \tag{2.32}$$

where we have use the fact that $\frac{2d}{d+2} \leqslant \frac{d}{d-2}$ for $d = 4, 5, 6$ and

$$I = \int_{B_R} U_{\varepsilon_\omega}^{\frac{4}{d-2}}\left(\varepsilon_\omega^{\frac{d-2}{2}}|H| + |\eta_{\varepsilon_\omega}|\right)|\hat{u}_{\omega,*}|\mathrm{d}x + \int_{B_R} U_{\varepsilon_\omega}^{\frac{6-d}{d-2}}\left(\varepsilon_\omega^{d-2}|H|^2 + |\eta_{\varepsilon_\omega}|^2\right)|\hat{u}_{\omega,*}|\mathrm{d}x.$$

Note that $H, \eta_{\varepsilon_\omega} \in L^\infty(\mathbb{R}^d)$ for $4 \leqslant d \leqslant 5$ and $H, \eta_{\varepsilon_\omega} \in L^{\frac{3s}{3-s}}(\mathbb{R}^d)$ for $d = 6$ with $1 < s < 3$ by lemma 2.1. Since $\frac{8d}{d^2-4} > \frac{d}{d-2}$ for $4 \leqslant d \leqslant 5$, it follows from (2.9) and (2.21) that

$$
\begin{aligned}
|I| &\lesssim \varepsilon_\omega^{\frac{d-2}{2}} \|U_{\varepsilon_\omega}\|_{L^{\frac{8d}{d^2-4}}(B_R)}^{\frac{4}{d-2}} \|\hat{u}_{\omega,*}\|_{L^{\frac{2d}{d-2}}(B_R)} + \varepsilon_\omega^{d-2} \|U_{\varepsilon_\omega}\|_{L^{\frac{2d(6-d)}{d^2-4}}(B_R)}^{\frac{6-d}{d-2}} \|\hat{u}_{\omega,*}\|_{L^{\frac{2d}{d-2}}(B_R)} \\
&\lesssim \varepsilon_\omega^{d-2} \left(1 + \|U_{\varepsilon_\omega}\|_{L^{\frac{d}{d-2}+\sigma}(B_R)}^{\frac{6-d}{d-2}}\right) \|\hat{u}_{\omega,*}\|_{L^{\frac{2d}{d-2}}(B_R)} \\
&\lesssim \varepsilon_\omega^{d-2} \|\hat{u}_{\omega,*}\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)},
\end{aligned}
$$

and for $d = 6$, it follows by (2.10) that

$$
\begin{aligned}
|I| &\lesssim \varepsilon_\omega^{2} \|U_{\varepsilon_\omega}\|_{L^{\frac{d+2}{d-2}+\sigma}(B_R)}^{\frac{d+2}{d-2}} \|\hat{u}_{\omega,*}\|_{L^3(B_R)} + \varepsilon_\omega^4 \|\hat{u}_{\omega,*}\|_{L^3(B_R)} \\
&\lesssim \varepsilon_\omega^{4-\sigma} \|\hat{u}_{\omega,*}\|_{L^3(\mathbb{R}^d)}.
\end{aligned}
$$

It follows from (2.31) and (2.32) that for $4 \leqslant d \leqslant 6$,

$$
\left|\int_{\mathbb{R}^d} E_\omega \hat{u}_{\omega,*} \mathrm{d}x\right| \lesssim \begin{cases} \left(\omega \varepsilon_\omega^{\frac{d-2}{2}} |\log \varepsilon_\omega|^{\frac{d-2}{d}} + \varepsilon_\omega^{d-2}\right) \|\hat{u}_{\omega,*}\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}, & d = 4, 5, \\ \left(\omega \varepsilon_\omega^2 |\log \varepsilon_\omega|^{\frac{2}{3}} + \varepsilon_\omega^{4-\sigma}\right) \|\hat{u}_{\omega,*}\|_{L^3(\mathbb{R}^6)}, & d = 6. \end{cases} \tag{2.33}
$$

For $d = 3$, the estimates are similar to that of $d = 4, 5$. The difference is that $\omega_* = 1$ and we do not know if $\varphi_{\varepsilon_\omega} > 0$ in $\mathbb{R}^3$. Thus, we write

$$
\begin{aligned}
|E_\omega| \lesssim{}& (\omega - 1) U_{\varepsilon_\omega} + U_{\varepsilon_\omega}^4 (\varepsilon_\omega^{\frac{1}{2}} |H| + |\eta_{\varepsilon_\omega}|) + U_{\varepsilon_\omega}^3 (\varepsilon_\omega |H|^2 + |\eta_{\varepsilon_\omega}|^2) \\
&+ \varepsilon_\omega^{\frac{5}{2}} |H|^5 + |\eta_{\varepsilon_\omega}|^5 + (\omega - 1) 3^{\frac{1}{4}} \varepsilon_\omega^{\frac{1}{2}} |H| + (\omega - 1) |\eta_{\varepsilon_\omega}|,
\end{aligned}
$$

which implies that

$$
\left|\int_{\mathbb{R}^d} E_\omega \hat{u}_{\omega,*} \mathrm{d}x\right| \lesssim \left((\omega - 1) \varepsilon_\omega^{\frac{1}{2}} + \varepsilon_\omega\right) \|\hat{u}_{\omega,*}\|_{L^6(\mathbb{R}^3)} \tag{2.34}
$$

for $d = 3$. By multiplying (2.29) with PU_{ε_ω} and integrating by parts, we can use lemma 2.2 and similar estimates as above to show that

$$
|\alpha_\omega - 1| \lesssim \|\hat{u}_{\omega,*}\|_X^2 + (\omega - \omega_*) \|PU_{\varepsilon_\omega}\|_{L^2(\mathbb{R}^d)}^2 + \begin{cases} \varepsilon_\omega^{d-2}, & d = 3, 4, 5, \\ \varepsilon_\omega^{4-\sigma}, & d = 6, \end{cases}
$$

which together with lemma 2.1 and (2.27), (2.29), (2.30), (2.33) and (2.34), imply (2.28) for $3 \leqslant d \leqslant 6$. For $d \geqslant 7$, we obtain from (1.1), (1.8), and (2.19) that $\hat{u}_\omega$ satisfies

$$
\begin{cases} -\Delta \hat{u}_\omega + (|x|^2 - \omega) \hat{u}_\omega - \frac{d+2}{d-2} U_{\varepsilon_\omega}^{\frac{4}{d-2}} \hat{u}_\omega = E_\omega + N_\omega(\hat{u}_\omega) & \text{in } \mathbb{R}^d, \\ \hat{u}_\omega(x) \to 0 & \text{as } |x| \to 0, \end{cases} \tag{2.35}
$$

where $E_\omega := (\omega - |x|^2) U_{\varepsilon_\omega}$ and

$$
|N_\omega(\hat{u}_\omega)| \lesssim |\hat{u}_\omega|^{\frac{d+2}{d-2}}. \tag{2.36}
$$

It follows from (2.21) and the fact that $\frac{2d}{d+2} > \frac{d}{d-2}$ for $d \geqslant 7$ that

$$
\begin{aligned}
\left|\int_{B_R} \widehat{E}_\omega \hat{u}_{\omega,*} \mathrm{d}x\right| &\lesssim \left(\omega \|U_{\varepsilon_\omega}\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)} + \|x U_{\varepsilon_\omega}\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}\right) \|\hat{u}_{\omega,*}\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \\
&\lesssim (\omega \varepsilon_\omega^2 + \varepsilon_\omega^3) \|\hat{u}_{\omega,*}\|_{L^2(\mathbb{R}^d)}.
\end{aligned} \tag{2.37}
$$

By multiplying (2.29) with U_{ε_ω} and integrating by parts, we can use lemma 2.2 and similar estimates as above to show that

$$|\alpha_\omega - 1| \lesssim \|\hat{u}_{\omega,*}\|_X^{\frac{d+2}{d-2}} + \varepsilon_\omega^4 + \omega\varepsilon_\omega^2,$$

which, together with (2.26) and (2.35)–(2.37), implies (2.28) for $d \geqslant 7$. □

2.4. Asymptotic behaviors of $\mathcal{I}_\omega$ and ε_ω as $\omega \to \omega_*^+$

It follows from (1.1) and (1.6) that if $u_\omega = (\mathcal{I}_\omega)^{\frac{d-2}{4}} v_\omega$, then

$$\mathcal{I}_\omega = \frac{\|u_\omega\|_X^2 - \omega\|u_\omega\|_{L^2(\mathbb{R}^d)}^2}{\|u_\omega\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^2} = \|u_\omega\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^{\frac{4}{d-2}},$$

which yields

$$\mathcal{I}_\omega = \left(\|u_\omega\|_X^2 - \omega\|u_\omega\|_{L^2(\mathbb{R}^d)}^2\right)^{\frac{2}{d}}. \tag{2.38}$$

The following four lemmas give details of estimates for different values of $d \geqslant 3$. The estimates are simpler for $d \geqslant 7$ and become computationally challenging for $3 \leqslant d \leqslant 6$ due to different leading order terms in the expansion of $\mathcal{I}_\omega$ and due to different regularity of the non-singular part H of Green's function. Some similar computations can be found in [5, 6, 16, 17, 35] for $d \geqslant 4$ and in [13, 14, 17, 21] for $d = 3$.

Lemma 2.4. *For $d \geqslant 7$, we have*

$$\mathcal{I}_\omega = \mathcal{S} - \mathcal{S}^{-\frac{d-2}{2}} \frac{\|U\|_{L^2(\mathbb{R}^d)}^4}{4\|xU\|_{L^2(\mathbb{R}^d)}^2}\omega^2 + o(\omega^2) \tag{2.39}$$

and

$$\varepsilon_\omega = \left(\frac{\|U\|_{L^2(\mathbb{R}^d)}^2}{2\|xU\|_{L^2(\mathbb{R}^d)}^2}\omega\right)^{\frac{1}{2}} + o(\omega^{\frac{1}{2}}) \tag{2.40}$$

as $\omega \to 0^+$.

Proof. By (2.19), (2.20) and the estimates of lemma 2.3,

$$\begin{aligned}\|u_\omega\|_X^2 - \omega\|u_\omega\|_{L^2(\mathbb{R}^d)}^2 = \|U_{\varepsilon_\omega}\|_X^2 - \omega\varepsilon_\omega^2\|U\|_{L^2(\mathbb{R}^d)}^2 + o(\omega\varepsilon_\omega^2 + \varepsilon_\omega^4)\\ + 2\int_{\mathbb{R}^d} \nabla U_{\varepsilon_\omega}\nabla\hat{u}_\omega + (|x|^2 - \omega)U_{\varepsilon_\omega}\hat{u}_\omega \mathrm{d}x\end{aligned} \tag{2.41}$$

By (1.8) and the estimates of lemma 2.3,

$$\int_{\mathbb{R}^d} \nabla U_{\varepsilon_\omega}\nabla\hat{u}_\omega + (|x|^2 - \omega)U_{\varepsilon_\omega}\hat{u}_\omega \mathrm{d}x = \int_{\mathbb{R}^d} U_{\varepsilon_\omega}^{\frac{d+2}{d-2}}\hat{u}_\omega \mathrm{d}x + o(\omega\varepsilon_\omega^2 + \varepsilon_\omega^4). \tag{2.42}$$

By (2.35) and the estimates of lemma 2.3,

$$\begin{aligned}\int_{\mathbb{R}^d} \nabla U_{\varepsilon_\omega}\nabla\hat{u}_\omega + (|x|^2 - \omega)U_{\varepsilon_\omega}\hat{u}_\omega \mathrm{d}x = \omega\varepsilon_\omega^2\|U\|_{L^2(\mathbb{R}^d)}^2 - \|xU_{\varepsilon_\omega}\|_{L^2(\mathbb{R}^d)}^2\\ + \frac{d+2}{d-2}\int_{\mathbb{R}^d} U_{\varepsilon_\omega}^{\frac{d+2}{d-2}}\hat{u}_\omega \mathrm{d}x + o(\omega\varepsilon_\omega^2 + \varepsilon_\omega^4).\end{aligned} \tag{2.43}$$

It follows from (2.38) and (2.41)–(2.43) that

$$\begin{aligned}\mathcal{I}_\omega &= \left(\|U_{\varepsilon_\omega}\|_X^2 - \omega\varepsilon_\omega^2\|U\|_{L^2(\mathbb{R}^d)}^2 + o(\omega\varepsilon_\omega^2+\varepsilon_\omega^4)\right)^{\frac{2}{d}}\\ &= \left(\mathcal{S}^{\frac{d}{2}} + \frac{d}{2}\left(\varepsilon_\omega^4\|xU_{\varepsilon_\omega}\|_{L^2(\mathbb{R}^d)}^2 - \omega\varepsilon_\omega^2\|U\|_{L^2(\mathbb{R}^d)}^2\right) + o(\omega\varepsilon_\omega^2+\varepsilon_\omega^4)\right)^{\frac{2}{d}}\\ &= \mathcal{S} + \mathcal{S}^{-\frac{d-2}{2}}\left(\varepsilon_\omega^4\|xU\|_{L^2(\mathbb{R}^d)}^2 - \omega\varepsilon_\omega^2\|U\|_{L^2(\mathbb{R}^d)}^2\right) + o(\omega\varepsilon_\omega^2+\varepsilon_\omega^4). \qquad (2.44)\end{aligned}$$

On the other hand, by using $\{U_\varepsilon\}_{\varepsilon>0}$ as a test function of $\mathcal{I}_\omega$ for $d \geqslant 7$, we obtain

$$\begin{aligned}\mathcal{I}_\omega &\leqslant \frac{\|U_\varepsilon\|_X^2 - \omega\|U_\varepsilon\|_{L^2(\mathbb{R}^d)}^2}{\|U_\varepsilon\|^2_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}}\\ &= \mathcal{S} + \mathcal{S}^{-\frac{d-2}{2}}\left(\varepsilon^4\|xU\|_{L^2(\mathbb{R}^d)}^2 - \omega\varepsilon^2\|U\|_{L^2(\mathbb{R}^d)}^2\right). \qquad (2.45)\end{aligned}$$

Minimising the right hand side of (2.45) in terms of ε implies that

$$\mathcal{I}_\omega \leqslant \mathcal{S} - \mathcal{S}^{-\frac{d-2}{2}}\frac{\|U\|_{L^2(\mathbb{R}^d)}^4}{4\|xU\|_{L^2(\mathbb{R}^d)}^2}\omega^2. \qquad (2.46)$$

Thus, combining (2.44) and (2.46), we have (2.39) and (2.40). □

Lemma 2.5. *For $d=6$, we have*

$$\mathcal{I}_\omega = \mathcal{S} - \mathcal{S}^{-2}\frac{\|U\|_{L^2(\mathbb{R}^d)}^4\omega^2}{8\times 24^2|\mathbb{S}^5||\log\omega|} + o\left(\frac{\omega^2}{\log\omega}\right) \qquad (2.47)$$

and

$$\varepsilon_\omega = \left(\frac{\|U\|_{L^2(\mathbb{R}^d)}^2\omega}{12\times 24^2|\mathbb{S}^5||\log\omega|}\right)^{\frac{1}{2}} + o\left(\left(\frac{\omega}{|\log\omega|}\right)^{\frac{1}{2}}\right) \qquad (2.48)$$

as $\omega \to 0^+$.

Proof. With $d=6$, expression (2.38) becomes

$$\mathcal{I}_\omega = \left(\|u_\omega\|_X^2 - \omega\|u_\omega\|_{L^2(\mathbb{R}^6)}^2\right)^{\frac{1}{3}}.$$

By lemmas 2.2 and 2.3 and similar arguments as that used in the proof of lemma 2.4, we have

$$\begin{aligned}\|u_\omega\|_X^2 - \|u_\omega\|_{L^2(\mathbb{R}^6)}^2 &= 3\int_{\mathbb{R}^6} U_{\varepsilon_\omega}^2 PU_{\varepsilon_\omega}\mathrm{d}x - 2\|PU_{\varepsilon_\omega}\|_{L^3(\mathbb{R}^6)}^3\\ &\quad - 3\omega\|PU_{\varepsilon_\omega}\|_{L^2(\mathbb{R}^6)}^2 + o(\omega\varepsilon_\omega^2 + \varepsilon_\omega^4|\log\varepsilon_\omega|)\\ &= \mathcal{S}^3 + 72\varepsilon_\omega^2\int_{\mathbb{R}^6} U_{\varepsilon_\omega}^2 H\mathrm{d}x - 3\omega\|PU_{\varepsilon_\omega}\|_{L^2(\mathbb{R}^6)}^2\\ &\quad + o(\omega\varepsilon_\omega^2 + \varepsilon_\omega^4|\log\varepsilon_\omega|). \qquad (2.49)\end{aligned}$$

By (2.22) and lemma 2.1,

$$\begin{aligned}\int_{B_R} U_{\varepsilon_\omega}^2 H\mathrm{d}x &= -\frac{1}{4}\int_{B_R} U_{\varepsilon_\omega}^2 \log|x|\mathrm{d}x + \mathcal{O}\left(\int_{B_R} U_{\varepsilon_\omega}^2\mathrm{d}x\right)\\ &= 144|\mathbb{S}^5|\varepsilon_\omega^2|\log\varepsilon_\omega| + \mathcal{O}(\varepsilon_\omega^{2+\sigma}),\end{aligned}$$

where $R>0$ is sufficiently large. This, together with (2.49) and lemma 2.1, implies

$$\mathcal{I}_\omega = \mathcal{S}+\mathcal{S}^{-2}(6\times 24^2|\mathbb{S}^5|\varepsilon_\omega^4|\log\varepsilon_\omega| - \omega\varepsilon_\omega^2\|U\|^2_{L^2(\mathbb{R}^d)} + o(|\varepsilon_\omega^4|\log\varepsilon_\omega|+\omega\varepsilon_\omega^2)). \tag{2.50}$$

On the other hand, by using $W_\varepsilon := (U_\varepsilon - 24\varepsilon^2 H)\phi_R$, where $\phi_R\in[0,1]$ is a smooth cut-off function such that $\phi_R=1$ for $|x|\leqslant R$ and $\phi_R=0$ for $|x|\geqslant R+1$, as a test function of $\mathcal{I}_\omega$ for $d=6$, we have

$$\mathcal{I}_\omega \leqslant \frac{\|W_\varepsilon\|_X^2 - \omega\|W_\varepsilon\|^2_{L^2(\mathbb{R}^d)}}{\|W_\varepsilon\|^2_{L^{2^*}(\mathbb{R}^d)}},$$

which implies

$$\mathcal{I}_\omega \leqslant \mathcal{S}+\mathcal{S}^{-2}(6\times 24^2|\mathbb{S}^5|\varepsilon^4|\log\varepsilon| - \omega\varepsilon^2\|U\|^2_{L^2(\mathbb{R}^6)} + o(|\varepsilon^4|\log\varepsilon|+\omega\varepsilon^2)). \tag{2.51}$$

Minimising the right hand side of (2.51) in terms of ε implies that

$$\mathcal{I}_\omega \leqslant \mathcal{S}-\mathcal{S}^{-2}\frac{\|U\|^4_{L^2(\mathbb{R}^d)}\omega^2}{8\times 24^2|\mathbb{S}^5||\log\omega|} + o\left(\left|\frac{\omega^2}{\log\omega}\right|\right). \tag{2.52}$$

Thus, by combining (2.50) and (2.52), we have (2.47) and (2.48). □

Lemma 2.6. *For $d=4,5$, we have*

$$\mathcal{I}_\omega = \begin{cases} \mathcal{S}-2\sqrt{2}\mathcal{S}^{-2}H(0)\|U\|^3_{L^3(\mathbb{R}^d)}\mathrm{e}^{\frac{3\sqrt{2}H(0)\|U\|^3_{L^3(\mathbb{R}^d)}}{2\omega|\mathbb{S}^3|}} + o(\mathrm{e}^{-\frac{1}{\omega}}), & d=4, \\ \mathcal{S}-\mathcal{S}^{-\frac{5}{2}}\dfrac{54\|U\|^6_{L^2(\mathbb{R}^5)}}{686\times 15^{\frac{3}{2}}(H(0)\|U\|^{\frac{7}{3}}_{L^{\frac{7}{3}}(\mathbb{R}^d)})^2}\omega^3 + o(\omega^3), & d=5, \end{cases} \tag{2.53}$$

and

$$\varepsilon_\omega = \begin{cases} \mathrm{e}^{-\frac{3\sqrt{2}H(0)\|U\|^3_{L^3(\mathbb{R}^d)}}{4\omega|\mathbb{S}^3|}} + o(\mathrm{e}^{-\frac{1}{\omega}}), & d=4, \\ \dfrac{3\|U\|^2_{L^2(\mathbb{R}^5)}}{7\times 15^{\frac{3}{4}}H(0)\|U\|^{\frac{7}{3}}_{L^{\frac{7}{3}}(\mathbb{R}^d)}}\omega + o(\omega), & d=5 \end{cases} \tag{2.54}$$

as $\omega\to 0^+$.

Proof. The proof is very similar to that of lemma 2.5. The only difference is that by lemma 2.1,

$$\begin{aligned}\int_{B_R} U_{\varepsilon_\omega}^{\frac{d+2}{d-2}} H\mathrm{d}x &= H(0)\int_{B_\rho} U_{\varepsilon_\omega}^{\frac{d+2}{d-2}}\mathrm{d}x + \mathcal{O}\left(\int_{B_\rho} U_{\varepsilon_\omega}^{\frac{d+2}{d-2}}|x|^\alpha\mathrm{d}x\right) + \mathcal{O}\left(\int_{B_R\backslash B_\rho} U_{\varepsilon_\omega}^{\frac{d+2}{d-2}}\mathrm{d}x\right) \\ &= \varepsilon_\omega^{\frac{d-2}{2}} H(0)\|U\|^{\frac{d+2}{d-2}}_{L^{\frac{d+2}{d-2}}(\mathbb{R}^d)} + \mathcal{O}\left(\varepsilon_\omega^{\frac{d-2}{2}+\alpha}\right),\end{aligned}$$

where $\alpha\in(0,1)$ and $\rho<R$ are two positive constants. Now, by similar arguments as that used for lemma 2.5, we obtain (2.53) and (2.54). □

Lemma 2.7. *For $d=3$, we have*

$$\mathcal{I}_\omega = \mathcal{S}-\mathcal{S}^{-\frac{3}{2}}\frac{3^{\frac{7}{4}}\|G\|^4_{L^2(\mathbb{R}^3)}}{80\pi}(\omega-1)^2 + o((\omega-1)^2) \tag{2.55}$$

and

$$\varepsilon_\omega = \frac{3^{\frac{5}{4}}\|G\|^2_{L^2(\mathbb{R}^3)}}{20\pi}(\omega-1)+o(\omega-1) \tag{2.56}$$

as $\omega \to 1^+$.

Proof. The main idea of the proof is still similar to that of lemmas 2.4–2.6. However, a significant difference for $d=3$ is that by similar arguments as that used for lemmas 2.4–2.6, we have

$$\mathcal{I}_\omega = \mathcal{S} - CH(0)\varepsilon_\omega^{\frac{1}{2}} + o(\varepsilon_\omega^{\frac{1}{2}}).$$

It follows from (2.1) that $H(0) \geqslant 0$. On the other hand, by using $W_\varepsilon := (U_\varepsilon - \sqrt[4]{3}\varepsilon^2 H)\phi_R$, where $\phi_R \in [0,1]$ is a smooth cut-off function such that $\phi_R = 1$ for $|x| \leqslant R$ and $\phi_R = 0$ for $|x| \geqslant R+1$, as a test function of $\mathcal{I}_\omega$ for $d=3$ and $\omega=1$, we have

$$\mathcal{I}_1 \leqslant \mathcal{S} - C'H(0)\varepsilon_\omega^{\frac{1}{2}} + o(\varepsilon_\omega^{\frac{1}{2}}),$$

which, together with (2.1), implies that $H(0) \leqslant 0$. Hence, we have $H(0)=0$ and we need to expand

$$\begin{aligned}
\|u_\omega\|_X^2 - \|u_\omega\|^2_{L^2(\mathbb{R}^3)} &= \frac{3}{2}\int_{\mathbb{R}^3} U^5_{\varepsilon_\omega} PU_{\varepsilon_\omega}\mathrm{d}x - \frac{1}{2}\|PU_{\varepsilon_\omega}\|^6_{L^6(\mathbb{R}^3)} \\
&\quad - \frac{3}{2}\omega\|PU_{\varepsilon_\omega}\|^2_{L^2(\mathbb{R}^6)} + o\left((\omega-1)\varepsilon_\omega + \varepsilon_\omega^2\right) \\
&= \mathcal{S}^{\frac{3}{2}} + \int_{\mathbb{R}^3}\left(4\sqrt[4]{3}\varepsilon_\omega^{\frac{1}{2}} U^5_{\varepsilon_\omega} H + \frac{15\sqrt{3}}{2}\varepsilon_\omega U^4_{\varepsilon_\omega} H^2\right)\mathrm{d}x - \frac{3}{2}\omega\|PU_{\varepsilon_\omega}\|^2_{L^2(\mathbb{R}^3)} \\
&\quad + o\left((\omega-1)\varepsilon_\omega + \varepsilon_\omega^2\right).
\end{aligned} \tag{2.57}$$

By lemma 2.1,

$$\int_{B_R} U^5_{\varepsilon_\omega} H\mathrm{d}x = \frac{4\pi}{3}H(0)\varepsilon_\omega^{\frac{1}{2}} - \frac{4\pi}{3}\varepsilon_\omega^{\frac{3}{2}} + \mathcal{O}\left(\varepsilon_\omega^{\frac{5}{2}}|\log\varepsilon_\omega|\right) \tag{2.58}$$

and

$$\int_{B_R} U^4_{\varepsilon_\omega} H^2 dx = H(0)^2\pi^2\varepsilon_\omega + \mathcal{O}\left(\varepsilon_\omega^2|\log\varepsilon_\omega|\right). \tag{2.59}$$

Now, by using (2.57)–(2.59) and similar arguments as that used in the proof of lemma 2.5, we obtain (2.55) and (2.56). □

Remark 2.1. We note that $H(0)$ is a global minimum of $H(x)$ in $\mathbb{R}^3$. Indeed, by the maximum principle, it is easy to see that there exists $r_0 \geqslant 1$ such that $H(r)$ is strictly increasing in $[0, r_0]$ and is strictly decreasing in $[r_0, +\infty)$. Thus, by $H(0)=0$ and $H(x) \to 0$ as $|x| \to +\infty$, we have that $H(0)$ is actually a global minimum of $H(x)$.

The proof of theorem 1.1 follows immediately from lemmas 2.4–2.7. □

3. The energy-supercritical case

3.1. Preliminaries

Let u_∞ be the singular solution of the stationary equation (1.13) for some $\omega_\infty \in (d-4, d)$ satisfying (1.14) for $d \geqslant 5$. Let L_∞ be the associated linear operator given by

$$L_\infty := -\Delta + |x|^2 - \omega_\infty - 3u_\infty^2.$$

Since $u_\infty(r) = \mathcal{O}(r^{-1})$ as $r \to 0$, $u_\infty \in C^\infty(0, \infty)$, and $u_\infty(r) \to 0$ exponentially fast as $r \to +\infty$, we consider L_∞ in the form domain $X_{\text{rad}} := \{f \in X : f \text{ is radial}\}$. The singular potential is controlled in the form domain by using the following Hardy inequality for every $d \geqslant 3$:

$$\| |\cdot|^{-1} f \|_{L^2(\mathbb{R}^d)} \leqslant \frac{2}{d-2} \|\nabla f\|_{L^2(\mathbb{R}^d)}, \quad \forall f \in D^{1,2}(\mathbb{R}^d). \tag{3.1}$$

where $D^{1,2}(\mathbb{R}^d)$ is the same as in (1.8).

In order to justify the definition of Morse index $\mathfrak{m}(u_\infty)$ according to definition 1.2, we show that the linear operator L_∞ has a compact resolvent, which implies that the spectrum of L_∞ in X_{rad} is purely discrete and consists of (isolated) simple eigenvalues.

Lemma 3.1. *For every $d \geqslant 5$, the linear operator L_∞ has a compact resolvent in X_{rad}.*

Proof. Consider the following variational problem:

$$\tau_1 = \inf_{\phi \in X_{\text{rad}}} \frac{\int_{\mathbb{R}^d} (|\nabla \phi|^2 + (|x|^2 - 3u_\infty^2)|\phi|^2) \mathrm{d}x}{\int_{\mathbb{R}^d} |\phi|^2 \mathrm{d}x}.$$

Since $F(r) := r u_\infty(r)$ is monotonically decreasing (see [4, 38]), we have $F(r) < F(0) = \sqrt{d-3}$, which implies that $u_\infty(r) < \frac{\sqrt{d-3}}{r}$ for every $r > 0$. By Hardy's inequality (3.1), we obtain

$$\int_{\mathbb{R}^d} 3u_\infty^2 |\phi|^2 \mathrm{d}x \leqslant \int_{\mathbb{R}^d} \frac{3(d-3)}{|x|^2} |\phi|^2 \mathrm{d}x \leqslant \frac{12(d-3)}{(d-2)^2} \|\nabla \phi\|_{L^2(\mathbb{R}^d)}.$$

By classical variational arguments and the fact that X is compactly embedded into $L^2(\mathbb{R}^d)$, we can see that $\tau_1 > -\infty$ is attained. Since the linear operator

$$L_\infty + \omega_\infty - \tau_1 + 1 = -\Delta + |x|^2 - 3u_\infty^2 - \tau_1 + 1$$

is strictly positive in X_{rad}, the linear equation

$$-\Delta \psi + (|x|^2 - 3u_\infty^2 - \tau_1 + 1)\psi = \varphi \quad \text{in } X_{\text{rad}}, \tag{3.2}$$

is unique solvable for every $\varphi \in X_{\text{rad}}$. Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be bounded in X_{rad}, then it follows by the compactness of the embedding from X to $L^2(\mathbb{R}^d)$ that $\varphi_n \to \varphi_*$ as $n \to \infty$ strongly in $L^2(\mathbb{R}^d)$. Since the equation (3.2) is linear, we may assume that $\varphi_* = 0$. By the positivity of $L_\infty + \omega_\infty - \tau_1 + 1$ in X_{rad}, the sequence of the corresponding solutions of (3.2) given by $\{\psi_n\}_{n \in \mathbb{N}}$ is bounded in X_{rad}. Since $\varphi_n \to 0$ strongly in $L^2(\mathbb{R}^d)$ as $n \to \infty$, then $\psi_n \to 0$ as $n \to \infty$ strongly in X_{rad}. Therefore, $L_\infty + \omega_\infty - \tau_1 + 1$ has a compact resolvent in X_{rad}, and so does L_∞. □

Remark 3.1. The mapping $d \mapsto \frac{12(d-3)}{(d-2)^2}$ is monotonically decreasing for $d \geqslant 5$. Since $\frac{12(d-3)}{(d-2)^2} < 1$ for $d \geqslant 13$, we have $\tau_1 > 0$ for $d \geqslant 13$. However, $\tau_1 < 0$ for $5 \leqslant d \leqslant 12$.

Let $\mathfrak{m}(u_\infty)$ be the Morse index of u_∞ in X_{rad} according to definition 1.2. It is well-defined for $d \geqslant 5$ because L_∞ has a purely discrete spectrum of (isolated) simple eigenvalues by lemma 3.1.

3.2. Morse index in the oscillatory case

The following lemma shows that the Morse index of u_∞ is infinite for $5 \leqslant d \leqslant 12$, for which ω_b oscillates near ω_∞ as $b \to \infty$.

Lemma 3.2. *For $5 \leqslant d \leqslant 12$, we have $\mathfrak{m}(u_\infty) = \infty$.*

Proof. We consider the following two cases:

(1) There exists $b_n \to +\infty$ as $n \to \infty$ such that $\omega_{b_n} - \omega_\infty > 0$.
(2) $\omega_b \leqslant \omega_\infty$ for $b > 0$ sufficiently large.

Case (1)**.** By using equations (5.4), (6.30), and (6.47) from [4], we obtain

$$u_\infty(r) = \frac{\sqrt{d-3}}{r} - \frac{\omega_\infty\sqrt{d-3}}{4d-10} r + \mathcal{O}(r^3) \tag{3.3}$$

and

$$u_{b_n}(r) = \frac{\sqrt{d-3}}{r} + C(\omega_{b_n} - \omega) r^{-\beta-1} \sin(\alpha \log r + \delta) + \mathcal{O}(b_n^{-2(1-a)} + \varepsilon^2), \tag{3.4}$$

for $r = \mathcal{O}(b_n^{a-1})$, where $|\omega_{b_n} - \omega_\infty| = \mathcal{O}(\varepsilon b_n^{-\beta(1-a)})$, $C \in \mathbb{R}$, $\delta \in \mathbb{R}$, $\varepsilon > 0$ is sufficiently small, $a \in (0,1)$ and

$$\beta = \frac{d-4}{2}, \qquad \alpha = \frac{\sqrt{-d^2 + 16d - 40}}{2}.$$

Let $\varphi_{b_n} := u_\infty - u_{b_n}$. Since $u_{b_n}(0) = b_n$ and $\omega_{b_n} - \omega_\infty > 0$, it follows from (3.3) and (3.4) that there exists $r_{b_n} \to 0$ such that $\varphi_{b_n}(r) > 0$ for $r \in (0, r_{b_n})$ and $\varphi_{b_n}(r_{b_n}) = 0$. It follows from (1.13) that φ_{b_n} satisfies for $r \in (0, r_{b_n})$:

$$\begin{aligned} -\Delta\varphi_{b_n} + |x|^2\varphi_{b_n} &= (\omega_\infty + u_\infty^2)\varphi_{b_n} - (u_{b_n}^2 - u_\infty^2 + \omega_{b_n} - \omega_\infty) u_{b_n} \\ &= (\omega_\infty + 3u_\infty^2)\varphi_{b_n} - (2u_\infty^2 - u_\infty u_{b_n} - u_{b_n}^2)\varphi_{b_n} - (\omega_{b_n} - \omega_\infty) u_{b_n} \\ &< (\omega_\infty + 3u_\infty^2)\varphi_{b_n}. \end{aligned} \tag{3.5}$$

Let

$$\widetilde{\varphi}_{b_n} = \begin{cases} \varphi_{b_n}, & 0 < r < r_{b_n}, \\ 0, & r \geqslant r_{b_n}. \end{cases}$$

Then by multiplying (3.5) with $\widetilde{\varphi}_{b_n}$ on both sides and integrating by parts, we have

$$\int_{\mathbb{R}^d} \left(|\nabla\widetilde{\varphi}_{b_n}|^2 + |x|^2|\widetilde{\varphi}_{b_n}|^2\right) \mathrm{d}x < \int_{\mathbb{R}^d} (\omega_\infty + 3u_\infty^2)|\widetilde{\varphi}_{b_n}|^2 \mathrm{d}x.$$

Since $r_{b_n} \to 0$ as $n \to \infty$, $\{\widetilde{\varphi}_{b_n}\}$ is linearly independent up to a subsequence. Hence, $\mathfrak{m}(u_\infty) = \infty$.

Case (2)**.** We follow the idea in [20]. Let $W_b = u_b(\mathrm{e}^t)$ and $W_\infty = u_\infty(e^t)$, then $Z_b = \frac{W_b}{W_\infty}$ satisfies

$$Z_b'' + \left(d - 2 + \frac{2W_\infty'}{W_\infty}\right) Z_b' + \mathrm{e}^{2t} Z_b(\omega_b - \omega_\infty + W_\infty^2(Z_b^2 - 1)) = 0. \tag{3.6}$$

It follows from the convergence $u_b \to u_\infty$ in Σ by [38, theorem 1.2] that $Z_{b_n}(t) \to 1$ as $n \to +\infty$ for every fixed t. Moreover, by classical elliptic regularity, we also have $u_b \to u_\infty$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^d \backslash \{0\})$ as $b \to +\infty$. We claim that there exists $b_n \to +\infty$ as $n \to \infty$ such that $1 - Z_{b_n}(t)$

has at least n zeros, say $t_{n,n} < \ldots < t_{2,n} < t_{1,n}$, such that $t_{n,n} \to 0$ as $n \to +\infty$. In other words, we claim that Z_{b_n} is oscillatory around 1 as $n \to \infty$ on $(-\infty, 0)$, in agreement with (3.4).

Suppose the contrary. Then for every sequence $\{b_n\}$ satisfying $b_n \to +\infty$ as $n \to \infty$, there exists $N > 0$, independent of n, such that $1 - Z_{b_n}(t)$ has at most N zeros for all n. Since $Z_b(t) = \mathcal{O}(\mathrm{e}^t)$ as $t \to -\infty$ by [4, (3.9)] for every $b > 0$ there exists $t_0 > 0$, independent of n, such that $0 < Z_{b_n}(t) < 1$ for all $t < t_0$ and n. If $V_{b_n} = 1 - Z_{b_n}$, then $0 < V_{b_n}(t) < 1$ for $t < t_0$. Moreover, by (3.6), V_{b_n} satisfies

$$V''_{b_n} + \left(d - 2 + \frac{2W'_\infty}{W_\infty}\right) V'_{b_n} - \mathrm{e}^{2t} Z_{b_n}(\omega_{b_n} - \omega_\infty - W_\infty^2 (Z_{b_n} + 1) V_{b_n}) = 0.$$

Since $u_{b_n} \to u_\infty$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^d \backslash \{0\})$ as $n \to \infty$, we know that $Z_{b_n}(t) \to 1$ as $n \to \infty$ uniformly in every compact set of the interval $(-\infty, t_0]$. Note that we also have $\mathrm{e}^{2t} W_\infty^2 \to (d-3)$ as $t \to -\infty$, thus, there exists $t'_0 < t_0$ which is independent of n, such that $\mathrm{e}^{2t} W_\infty^2 = (d-3) + o(1)$ for $t < t'_0$ where $o(1) \to 0$ as $t'_0 \to -\infty$. Thus, without loss of generality, we may assume that $\mathrm{e}^{2t} Z_{b_n} W_\infty^2 (Z_{b_n} + 1) = 2(d-3) + o(1)$ uniformly in every compact set of the interval $(-\infty, t'_0]$, where $o(1)$ could be arbitrary small if necessary by taking t'_0 sufficiently close to $-\infty$ and n sufficiently large. Note that by (3.3),

$$\frac{2W'_\infty(t)}{W_\infty(t)} \to -2, \quad \text{as } t \to -\infty.$$

Since we have $\omega_{b_n} \leqslant \omega_\infty$ by assumption, we can write the equation of V_{b_n} as follows:

$$V''_{b_n} + (d - 4 + o(1)) V'_{b_n} + (2(d-3) + o(1)) V_{b_n} \leqslant 0$$

in every compact set of the interval $(-\infty, t'_0]$ by taking t'_0 sufficiently close to $-\infty$ if necessary. Since $5 \leqslant d \leqslant 12$, the fundamental solution of the linear equation,

$$\phi'' + (d-4)\phi' + 2(d-3)\phi = 0,$$

is given by $\phi = C\mathrm{e}^{-\beta t} \sin(\alpha t + \delta)$ for some $C \in \mathbb{R}$ and $\delta \in \mathbb{R}$. By the Sturm–Liouville theorem, V_{b_n} must have zeros in a sufficiently large compact set of the interval $(-\infty, t'_0]$. But this contradicts the assumption that $V_{b_n}(t) > 0$ for all $t < t'_0$. Thus, there exists $b_n \to +\infty$ as $n \to \infty$ such that $1 - Z_{b_n}(t)$ has at least n zeros for $t \ll -1$. We denote the zeros of Z_{b_n} by $0 < a_{1,n} < a_{2,n} < \ldots < a_{k_n,n}$ with $k_n \geqslant n$. For the sake of simplicity, we also denote $a_{0,n} = 0$. Then we can define

$$\widehat{\varphi}_{n,j} = \begin{cases} 0, & 0 < r \leqslant a_{j-1,n}, \\ u_\infty - u_{b_n}, & a_{j-1,n} < r < a_{j,n}, \\ 0, & r \geqslant a_{j,n} \end{cases}$$

and by the convexity of t^3 for $t \geqslant 0$, we have

$$\int_{\mathbb{R}^d} \left(|\nabla \widehat{\varphi}_{n,j}|^2 + |x|^2 |\widehat{\varphi}_{n,j}|^2 \right) \mathrm{d}x < \int_{\mathbb{R}^d} (\omega_\infty + 3u_\infty^2) |\widehat{\varphi}_{n,j}|^2 \mathrm{d}x.$$

It follows from $k_n \to \infty$ as $n \to \infty$ that $\mathfrak{m}(u_\infty) = \infty$.

□

3.3. Morse index in the monotone case

By theorems 1.1 and 1.2 in [31], the Morse index of u_∞ is finite for $d \geqslant 13$ for which ω_b converges to ω_∞ monotonically as $b \to \infty$. Here we will give a more precise estimates on $\mathfrak{m}(u_\infty)$ for $d \geqslant 13$.

Let us consider the confluent hypergeometric function, which is also called Kummer's function, given by

$$M(a;b;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!},$$

where $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$ are Pochhammer symbols. It is well known (see [42]) that $M(a;b;x)$ is a solution of the confluent hypergeometric differential equation, which is also called the Kummer equation:

$$x\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + (b-x)\frac{\mathrm{d}u}{\mathrm{d}x} + au = 0.$$

Let

$$W_{a,l}(r) = r^l \mathrm{e}^{-\frac{r^2}{2}} M\left(a; l+\frac{d}{2}; r^2\right),$$

then it can be directly verified that $W_{a,l}$ satisfies

$$-W''_{a,l} - \frac{d-1}{r}W'_{a,l} + \frac{l(l+d-2)}{r^2}W_{a,l} + r^2 W_{a,l} = (d-4a+2l)W_{a,l}.$$

Let

$$l_\pm = \frac{2-d\pm\sqrt{d^2-16d+40}}{2} \tag{3.7}$$

then $W_{a,l_\pm}$ satisfies

$$-\Delta W_{a,l_\pm} + |x|^2 W_{a,l_\pm} - \frac{3(d-3)}{|x|^2} W_{a,l_\pm} = (d-4a+2l_\pm)W_{a,l_\pm}. \tag{3.8}$$

Remark 3.2. It is easy to see that $b = l_\pm + \frac{d}{2} \neq 0,-1,-2,\ldots$. Otherwise, we have

$$\frac{d^2-16d+40}{4} - p^2 = 0$$

for some $p \in \mathbb{Z}$, which implies

$$d = 2(4 \pm \sqrt{p^2+6}) \in \mathbb{N}.$$

It follows that $(\frac{q}{2})^2 - p^2 = 6$ for some $q \in \mathbb{Z}$. Thus, either $\frac{q}{2} - p = 2k$ or $\frac{q}{2} + p = 2k$ for some $k \in \mathbb{N}$, which implies $4k^2 \pm 4pk = 6$. But this is impossible since $2k^2$ is even but $3 \pm 2kp$ is odd.

If $a \neq 0,-1,-2,\ldots$ then

$$M(a; l_\pm + \frac{d}{2}; r^2) \sim \sum_{n=0}^{\infty} n^{l_\pm+\frac{d}{2}-a}\frac{r^{2n}}{n!} \gtrsim \sum_{n=0}^{\infty} \frac{(\frac{2}{3}r^2)^n}{n!} = \mathrm{e}^{\frac{2}{3}r^2}.$$

If $-a \in \mathbb{N}$, then $M(-n; l_\pm + \frac{d}{2}; r^2) = P_n(r^2)$ is a polynomial of order $2n$. Therefore, $W_{a,l_\pm} \in L^2(\mathbb{R}^d)$ if and only if $-a \in \mathbb{N}$. On the other hand, if $W_{a,l_\pm} \in L^2(\mathbb{R}^d)$ is a eigenfunction of the operator $-\Delta+|x|^2-\frac{3(d-3)}{|x|^2}$ in $L^2(\mathbb{R}^d)$, then $W_{a,l_\pm}\in L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ by the Hardy inequality for $d\geqslant 13$. However, as $r\to 0$,

$$|r^{l_-}\mathrm{e}^{-\frac{r^2}{2}}M\left(-n;l_-+\frac{d}{2};r^2\right)|^{2^*}\sim r^{2^*l_-}\sim r^{-d-\frac{d\sqrt{d^2-16d+40}}{d-2}}>r^{-d}.$$

Thus, by (3.8),

$$W_{-n,l_+}=r^{l_+}\mathrm{e}^{-\frac{r^2}{2}}M\left(-n;l_++\frac{d}{2};r^2\right)$$

is the only eigenfunctions of the operator $-\Delta+|x|^2-\frac{3(d-3)}{|x|^2}$ in X_{rad} with eigenvalues $(d+4n+2l_+)$, for all $n\in\mathbb{N}$. By (3.7), the third eigenvalue σ_3 is given by

$$\sigma_3=10+\sqrt{d^2-16d+40} \tag{3.9}$$

and the fourth eigenvalue σ_4 is given by

$$\sigma_4=14+\sqrt{d^2-16d+40}. \tag{3.10}$$

The following lemma gives the estimate on $\mathfrak{m}(u_\infty)$ for $d\geqslant 13$.

Lemma 3.3. *For $d\geqslant 13$, we have*

$$\mathfrak{m}(u_\infty)=\begin{cases}1 \text{ or } 2, & 13\leqslant d\leqslant 15,\\ 1, & d\geqslant 16.\end{cases}$$

Proof. **Case $d\geqslant 16$.** Since $F(r):=ru_\infty(r)$ is monotonically decreasing (see [4, 38]), we have $F(r)<F(0)=\sqrt{d-3}$, which implies that $u_\infty(r)<\frac{\sqrt{d-3}}{r}$ for every $r>0$. Note that $\omega_\infty\in(d-4,d)$ by [4, theorem 1.2]. Then by $\sigma_3>d$ for $d\geqslant 16$, as is clear from (3.9), we have

$$\omega_\infty+3u_\infty^2<\sigma_3+3\frac{d-3}{r^2}\quad\text{in }\mathbb{R}^d\text{ for }d\geqslant 16. \tag{3.11}$$

Since L_∞ has a compact resolvent in X_{rad} by lemma 3.1, the spectrum of $-\Delta+|x|^2-3u_\infty^2$ in X_{rad} consists of (isolated) simple eigenvalues $\{\tau_j\}_{j\in\mathbb{N}}$ such that $\tau_j\to\infty$ as $j\to\infty$. For each simple eigenvalue τ_j, there exists a unique eigenfunction $\phi_j\in X_{\mathrm{rad}}$ (up to scalar multiplication) which satisfies

$$-\Delta\phi_j+|x|^2\phi_j-3u_\infty^2\phi_j=\tau_j\phi_j\quad\text{in }\mathbb{R}^d.$$

Moreover, ϕ_j has exact $j-1$ zeros. Since

$$\int_{\mathbb{R}^d}\left(|\nabla u_\infty|^2+|x|^2u_\infty^2\right)\mathrm{d}x=\int_{\mathbb{R}^d}\left(\omega_\infty u_\infty^2+u_\infty^4\right)\mathrm{d}x<\int_{\mathbb{R}^d}\left(\omega_\infty u_\infty^2+3u_\infty^4\right)\mathrm{d}x,$$

we have $\mathfrak{m}(u_\infty)\geqslant 1$ so that $\tau_1<\omega_\infty$. Suppose that $\tau_2\leqslant\omega_\infty$, then it follows from (3.11) that

$$\tau_2+3u_\infty^2<\sigma_3+3\frac{d-3}{r^2}\quad\text{in }\mathbb{R}^d\text{ for }d\geqslant 16. \tag{3.12}$$

Recall that ϕ_2 has exact one zero on $(0,\infty)$ so that we can define

$$\phi_{2,f}=\begin{cases}\phi_2, & 0\leqslant r<r_u,\\ 0, & r\geqslant r_u,\end{cases}\quad\text{and}\quad \phi_{2,l}=\begin{cases}0, & 0\leqslant r<r_u,\\ \phi_2, & r\geqslant r_u,\end{cases}$$

where r_u is the unique zero of ϕ_2. Then by (3.12), we have

$$\int_{\mathbb{R}^d}\left(|\nabla\phi_1|^2+|x|^2\phi_1^2\right)\mathrm{d}x=\int_{\mathbb{R}^d}(\tau_1+3u_\infty^2)\phi_1^2\,\mathrm{d}x<\int_{\mathbb{R}^d}\left(\sigma_3+3\frac{d-3}{r^2}\right)\phi_1^2\,\mathrm{d}x,$$
$$\int_{\mathbb{R}^d}\left(|\nabla\phi_{2,f}|^2+|x|^2\phi_{2,f}^2\right)\mathrm{d}x=\int_{\mathbb{R}^d}(\tau_2+3u_\infty^2)\phi_{2,f}^2\,\mathrm{d}x<\int_{\mathbb{R}^d}\left(\sigma_3+3\frac{d-3}{r^2}\right)\phi_{2,f}^2\,\mathrm{d}x$$

and

$$\int_{\mathbb{R}^d}\left(|\nabla\phi_{2,l}|^2+|x|^2\phi_{2,l}^2\right)\mathrm{d}x=\int_{\mathbb{R}^d}(\tau_2+3u_\infty^2)\phi_{2,l}^2\,\mathrm{d}x<\int_{\mathbb{R}^d}\left(\sigma_3+3\frac{d-3}{r^2}\right)\phi_{2,l}^2\,\mathrm{d}x.$$

Since ϕ_1 is sign-constant and $\phi_{2,f}$ and $\phi_{2,l}$ share the unique zero at r_u, the functions ϕ_1, $\phi_{2,f}$ and $\phi_{2,l}$ are linearly independent. Indeed, if there exists c_1, $c_{2,f}$ and $c_{2,l}$ such that

$$c_1\phi_1+c_{2,f}\phi_{2,f}+c_{2,l}\phi_{2,l}\equiv 0\quad\text{in }\mathbb{R}^d,$$

then by $\phi_{2,f}(r_u)=\phi_{2,l}(r_u)=0$, we have $c_1=0$. On the other hand, since $\phi_{2,f}\phi_{2,l}\equiv 0$, then we also have $c_{2,f}=c_{2,l}=0$, which implies ϕ_1, $\phi_{2,f}$ and $\phi_{2,l}$ are linearly independent. However, σ_3 is the third eigenvalue of the operator $-\Delta+|x|^2-3\frac{d-3}{r^2}$ in X_{rad}, thus, $\mathfrak{m}(W_{-2,l_+})=2$, which is a contradiction. Therefore, $\tau_2>\omega_\infty$ for $d\geqslant 16$, which implies $\mathfrak{m}(u_\infty)=1$.

Case $13\leqslant d\leqslant 15$**.** We use the same idea to show that $1\leqslant\mathfrak{m}(u_\infty)\leqslant 2$. Indeed, since $\sigma_4>\omega_\infty$ for $13\leqslant d\leqslant 15$, as follows from (3.10), we have

$$\omega_\infty+3u_\infty^2<\sigma_4+3\frac{d-3}{r^2}\quad\text{in }\mathbb{R}^d\text{ for }13\leqslant d\leqslant 15.\tag{3.13}$$

If $\tau_3\leqslant\omega_\infty$, then by (3.13),

$$\tau_3+3u_\infty^2<\sigma_4+3\frac{d-3}{r^2}\quad\text{in }\mathbb{R}^d\text{ for }13\leqslant d\leqslant 15.\tag{3.14}$$

The third eigenfunction ϕ_3, corresponding to τ_3, has exact two zeros $\widetilde{r}_f<\widetilde{r}_l$. Moreover, by the Sturm–Liouville theorem, it is well known that $\widetilde{r}_f<r_u<\widetilde{r}_l$. Let

$$\phi_{3,f}=\begin{cases}\phi_3, & 0\leqslant r<\widetilde{r}_f,\\ 0, & r\geqslant\widetilde{r}_f,\end{cases}\qquad \phi_{3,l}=\begin{cases}0, & 0\leqslant r<\widetilde{r}_l,\\ \phi_3, & r\geqslant\widetilde{r}_l,\end{cases}$$

and

$$\phi_{3,m}=\begin{cases}0, & 0\leqslant r<\widetilde{r}_f,\\ \phi_3, & r_f\leqslant r<\widetilde{r}_l,\\ 0, & r\geqslant\widetilde{r}_l.\end{cases}$$

Then by similar arguments as used above, we can show from (3.14) that $\mathfrak{m}(W_{-3,l_+})\geqslant 6$, which contradicts the fact that $\mathfrak{m}(W_{-3,l_+})=3$. Thus, we must have $\tau_3>\omega_\infty$ for $13\leqslant d\leqslant 15$, which implies that $\mathfrak{m}(u_\infty)\leqslant 2$.

□

Remark 3.3. As a by-product, the proof of lemma 3.3 shows that $\tau_1<\omega_\infty<\tau_2$ for $d\geqslant 16$. Therefore, the homogeneous equation $L_\infty Z=0$ has only trivial solutions in X_{rad} for $d\geqslant 16$. This implies that u_∞ is nondegenerate in X_{rad} for $d\geqslant 16$ in the following sense. The radial solution Z satisfies $Z=\mathcal{O}(r^{\omega_\infty-d}\mathrm{e}^{-\frac{r^2}{2}})$ as $r\to+\infty$ and there exists $L_-\neq 0$ such that

$$Z=L_-r^{J_-}+O(r^{J_+},r^{J_-+2})\quad\text{as }r\to 0.$$

This argument verifies the non-degeneracy assumption 2.2 in [31] for $d\geqslant 16$. It is not clear if this assumption can be verified for $13\leqslant d\leqslant 15$.

The proof of theorem 1.2 follows immediately from lemmas 3.2 and 3.3. □

Data availability statement

No new data were created or analysed in this study.

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