INSTITUTE OF PHYSICS PUBLISHING NONLINEARITY

Nonlinearity **19** (2006) 2277–2312 doi:10.1088/0951-7715/19/10/003

Exact conditions for existence of homoclinic orbits in the fifth-order KdV model

Alexander Tovbis[1] **and Dmitry Pelinovsky**[2]

[1] Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA
[2] Department of Mathematics, McMaster University, Hamilton, Ontario, L8S 4K1, Canada

E-mail: atovbis@pegasus.cc.ucf.edu and dmpeli@math.mcmaster.ca

Received 4 January 2006, in final form 21 June 2006
Published 25 August 2006
Online at stacks.iop.org/Non/19/2277

Recommended by J Lega

Abstract
We consider homoclinic orbits in the fourth-order equation $v^{(\mathrm{iv})} + (1-\varepsilon^2)v'' - \varepsilon^2 v = v^2 + \gamma(2vv'' + v'^2)$, where $(\gamma, \varepsilon) \in \mathbb{R}^2$. Numerical computations [CG97, C01] show that homoclinic orbits exist on certain curves $\gamma(\varepsilon)$ in the parameter plane (γ, ε). We study the dependence $\gamma(\varepsilon)$ in the limit $\varepsilon \to 0$ and prove that a curve $\gamma(\varepsilon)$ passes through the point $(\gamma_0, 0)$ only if $s(\gamma_0) = 0$, where $s(\gamma)$ denotes the Stokes constant for the truncated equation (with $\varepsilon = 0$). The additional condition $s'(\gamma_0) \neq 0$ guarantees the existence of a unique curve $\gamma(\varepsilon)$ passing through the point $(\gamma_0, 0)$. Every homoclinic orbit is proved to be single-humped for sufficiently small ε.

Mathematics Subject Classification: 34M40, 34M30, 35Q53, 37C29

(Figures in this article are in colour only in the electronic version)

1. Introduction

We address existence of homoclinic solutions to the fourth-order equation

$$v^{(\mathrm{iv})}(z) + (1-\varepsilon^2)v''(z) - \varepsilon^2 v(z) = v^2(z) + \gamma(2v(z)v''(z) + v'^2(z)) \tag{1.1}$$

in the limit $\varepsilon \to 0$, where (γ, ε) are real parameters. This problem is equivalent to the problem of persistence of the homoclinic solution

$$y_1(x) = -\frac{3}{2\cosh^2(x/2)} \tag{1.2}$$

to the differential equation

$$y'' - y = y^2 \tag{1.3}$$

that undergoes a singular perturbation

$$\varepsilon^2 y^{(\mathrm{iv})} + (1-\varepsilon^2)y'' - y = y^2 + \varepsilon^2\gamma(2yy'' + (y')^2). \tag{1.4}$$

Indeed, equations (1.1) and (1.4) are related through the change of variables (rescaling)

$$v(z) = \varepsilon^2 y(x), \qquad x = \varepsilon z - c, \tag{1.5}$$

where the arbitrary constant $c \in \mathbb{C}$ is at our disposal.

Persistence of homoclinic solutions for various singularly perturbed problems, including the problems related to dendrite crystal growth, flows in Hele-Shaw cells, solitary waves in the presence of small surface tension, bending losses in optical fibres and shock waves in combustion problems (see [STL92] and references there), has been discussed in the literature during the last two decades. The main difficulty in all such problems is the fact that the difference between perturbed stable and unstable solutions becomes exponentially small in the perturbation parameter. Significant progress in overcoming this difficulty is associated with the paper [KS91] that outlined the approach known nowadays as asymptotics beyond all orders (the preprint of the paper appeared in 1985). This approach was formally applied in [PRG88, GJ95] to show non-existence of homoclinic solutions to (1.4) with $\gamma = 0$ for small $\varepsilon > 0$. The first rigorous proof of non-existence was obtained in [AM91] by a completely different method (see also an earlier paper [HM89]). In the framework of the asymptotic beyond all orders approach, the proof of non-existence was obtained in [T00].

The asymptotic beyond all orders approach to a singularly perturbed problem, for example, to (1.4), consists of three major steps: (a) reduction of (1.4), called *the outer equation*, to (1.1), called *the inner equation*, (b) analysis of solutions $v_{\pm}(z)$ of *the truncated inner equation*

$$v^{(\mathrm{iv})}(z) + v''(z) = v^2(z) + \gamma(2v(z)v''(z) + v'^2(z)) \tag{1.6}$$

that tend to zero as $|z| \to \infty$ in the right or left half-planes of the complex z-plane, respectively, (c) transition from equation (1.1) to (1.6), i.e. the proof that in the limit $\varepsilon \to 0$ stable and unstable solutions of (1.1) approach solutions $v_{\pm}(z)$ of (1.6), respectively. Then the coincidence of solutions $v_+(z) \equiv v_-(z)$, i.e. the fact that (1.6) has a nontrivial solution $v(z)$ analytic at $z = \infty$ and satisfying $v(\infty) = 0$, becomes a necessary condition for persistence of homoclinic solutions to (1.4). Existence of solutions analytic at infinity and their relation with Stokes constants for analytic nonlinear ordinary differential equations was discussed in [T94b]. It was proved in [T00] that $v_+(z) \neq v_-(z)$ for equation (1.6) with $\gamma = 0$. Thus, in the case $\gamma = 0$ equation (1.1) has no homoclinic solutions for small ε. However, there are homoclinic solutions for other values of γ. For example,

$$v(z,\varepsilon) = -\frac{3\varepsilon^2(4+\varepsilon^2)}{8}\operatorname{sech}^2\left(\frac{\varepsilon z}{2}\right), \qquad \gamma(\varepsilon) = \frac{5}{4+\varepsilon^2} \tag{1.7}$$

is a homoclinic solution to (1.1) with $\gamma = \gamma(\varepsilon)$. It corresponds to the smallest real zero $s(\frac{5}{4}) = 0$ of the Stokes constant $s(\gamma)$.

The main results of the present work are listed as follows.

1. Equation (1.1) has homoclinic solutions in a neighbourhood of the point $(\gamma_0, 0)$, $\gamma_0 \in \mathbb{R}$, only if the Stokes constant $s(\gamma)$ of equation (1.6) vanishes at $\gamma = \gamma_0$.
2. If $s(\gamma_0) = 0$ but $s'(\gamma_0) \neq 0$, then there exists a unique smooth curve $\gamma(\varepsilon)$ passing through the point $(\gamma_0, 0)$ in the parameter space (γ, ε), so that equation (1.1) with $\gamma = \gamma(\varepsilon)$ has homoclinic solutions for all sufficiently small ε. Such curves in the parameter space will be referred to as *homoclinic solution curves*.

3. Each homoclinic solution curve corresponds to a single-humped homoclinic orbit in a neighbourhood of the point $(\gamma_0, 0)$ in the parameter space (γ, ε).
4. First zeros of $s(\gamma)$ are approximated numerically on a rigorous basis of a numerical algorithm.

Our technique is based on the results of [T94b, T94c, T00] for the asymptotic beyond all orders approach, applied to equation (1.1). The related problem of distribution of zeros of $s(\gamma)$ for large γ is beyond the scope of this paper.

Our paper is structured as follows. Motivations of our studies of equation (1.1) are described in section 2. The truncated inner equation (1.6) and the Stokes constant $s(\gamma)$ are analysed in section 3. Analysis of the outer equation (1.4) is developed in section 4. Analysis of the inner equation (1.1) is reported in section 5. Numerical computations of homoclinic orbits are described in section 6. Technical proofs can be found in appendix A and appendix B.

2. Motivations

Our work is motivated by the connection of the fourth-order ODE (1.1) to the equation describing travelling waves of the fifth-order Korteweg–de Vries (KdV) equation. Indeed, the transformation

$$z = \omega t, \qquad v(z) = -\frac{\alpha}{\omega^4} u(t), \tag{2.1}$$

reduces (1.1) to the general form

$$u^{(\mathrm{iv})} + (\omega^2 - \kappa^2)u'' - \kappa^2\omega^2 u + \alpha u^2 + \beta(2uu'' + u'^2) = 0, \tag{2.2}$$

where α, β, ω and κ are real parameters. The ODE (2.2) is a travelling wave reduction of the fifth-order KdV equation, which models gravity water waves on a surface of finite depth [CG94]. Note that this equation is not a reduction of the integrable KdV hierarchy and it is not integrable unless it is a linear equation with $\alpha = \beta = 0$. The fifth-order KdV equation leading to the ODE (2.2) generalizes the Kawahara equation ($\beta = 0$), for which there exists a proof that no homoclinic connection to the zero equilibrium state exists [AM91, HM89]. Unfortunately, these proofs were very specific to the ODE (2.2) with $\beta = 0$ and did not allow a direct generalization to a wider class of problems, such as, for example, equation (2.2) with $\beta \neq 0$.

When $\kappa = 0$, the linearization of the zero equilibrium state in the ODE (2.2) admits two purely imaginary eigenvalues $\lambda = \pm i\omega$ and the double zero eigenvalue $\lambda = 0$, which is referred to as the $0^2 i\omega$ resonance. This dynamical system has the vector normal form, given in [IA98] (exercise I.22 and problem 7). Using the formal polynomial transformation, the scalar ODE (2.2) is transformed to the vector normal form [C01]. Non-persistence of homoclinic orbits in the general vector normal form was considered in connection with the original water wave problem by Iooss and Kirchgassner [IK92]. Analysis of exponentially small Melnikov integrals was developed by Lombardi [L99], who derived a criterion for persistence of homoclinic orbits in the vector normal form. Lombardi [L00] also proved that a general vector normal form for the $0^2 i\omega$ resonance admits homoclinic connections to exponentially small periodic orbits and generally no homoclinic connections to the zero equilibrium states.

Altogether, these results do not exclude that there may exist values of parameters of the ODE (2.2) when the true homoclinic orbits exist. Finding such curves numerically happens to be very difficult because one has to distinguish between homoclinic connections to exponentially small periodic orbits and true homoclinic connections. These numerical studies of homoclinic orbits in the scalar ODE (2.2) were undertaken by Champneys [C01] (see also [CG97]). The numerical results reveal families of multi-pulse homoclinic orbits along bifurcation curves of co-dimension one in a two-parameter plane (see figure 12 in [CG97]). Formal applications of the Lombardi's results for the vector normal forms [L99,L00] undertaken in [C01] has indicated existence of infinitely many multi-pulse homoclinic orbits of the ODE (2.2) for $\text{sign}(\alpha\beta) = 1$ and non-existence of homoclinic orbits for $\text{sign}(\alpha\beta) = -1$. However, this formal application cannot be directly applied to the scalar ODE (2.2), because the order of the normal form truncation is not controlled (see remark 7.1.17 in [L00]).

It was further suggested in [C01] that the families of homoclinic orbits intersect the line $\kappa = 0$ at a set of points $\omega = \{\omega_n\}_{n=1}^{\infty}$, according to zeros of a Bessel function, where

$$\lim_{n\to\infty} (\omega_{n+1} - \omega_n) = \frac{\pi}{4}\sqrt{\frac{\alpha}{3\beta}}. \tag{2.3}$$

The numerical approximations are inaccurate in the limit $\kappa \to 0$ and the numerical values of ω_n are not computed to any precision except for the case $n = 1$ (see figure 1 in [C01]). Nevertheless, there is strong numerical evidence that a sequence of values of ω_n does indeed exist, that is consistent with the distribution (2.3) [C01]. In the case $n = 1$, the explicit analytic solution follows from (1.7) and it gives the exact value of ω_1. All other ($n \neq 1$) homoclinic solutions to (1.1) seem to have no explicit expressions. At least, as it is proved in our paper, they cannot be represented by a rational function of $\text{sech}^2(\varepsilon z/2)$. We also show that each homoclinic solution curve in the parameter space represents a single-humped homoclinic orbit near the point $\kappa = 0$ and $\omega = \omega_n$. Therefore, splitting of pulses occurs far from the limiting point (all homoclinic orbits represent multi-pulse solutions in the opposite limit $\omega \to 0$ [C01]). These results are in agreement with the concept from [KCBS02] that homoclinic orbits of the ODE (2.2) are special and could not be treated as bound states of individual pulses.

We will show that the necessary condition for a bifurcation of the homoclinic orbit from the point $\kappa = 0$ is the zero of the corresponding Stokes constant which may occur for discrete values of ω. Moreover, the sufficient condition for existence of a single curve in the parameter space (κ, ω) under fixed values of (α, β) is that the zero of the Stokes constant is simple. The first result (based on [T00]) is of the same type as the one obtained by Lombardi for the $0^2 i\omega$ resonance. The proofs are very different although both proofs are based on a careful description of the analytic continuation of solutions in the complex plane. The second result was expected, but, to our best knowledge, has never been proved before. Moreover, theorem 3.10 in our paper justifies an algorithmic way to compute the Stokes constant numerically. We shall compare our numerical results with previous numerical computations in [CG97], which were carried by different numerical algorithms.

Analysis of this paper can be generalized to the ODE (2.2) with different nonlinear terms or to the general vector form for the $0^2\mathrm{i}\omega$ resonance considered in [L00]. Most proofs admit straightforward generalizations. However, in order to locate zeros of the Stokes constant numerically, one needs to develop individual numerical computations for each given nonlinear function. An example of such numerical computations for a class of differential advance-delay equations was reported recently in [OPB06].

3. The truncated inner equation and the Stokes constant

In this section we study the existence of a solution to the truncated inner equation (1.6) that is analytic at $z = \infty$. Equation (1.6) has the formal series solution

$$\hat{v}(z) = \sum_{k=1}^{\infty} \frac{\alpha_k}{z^{2k}}, \tag{3.1}$$

where the coefficients $\{\alpha_k\}_{k=2}^{\infty}$ satisfy the recurrence equation

$$(2k)(2k+1)(2k+2)(2k+3)\alpha_k + (2k+2)(2k+3)\alpha_{k+1} = \sum_{m=1}^{k+1}\alpha_m\alpha_{k-m+2} + 4\gamma\sum_{m=1}^{k} m(m+k+2)\alpha_m\alpha_{k-m+1}, \qquad k \in \mathbb{N}^+, \tag{3.2}$$

starting with $\alpha_1 = 6$. The first coefficients α_2, α_3, and α_4 can be found from the recurrence equation (3.2) as

$$\alpha_2 = 18(4\gamma - 5),$$
$$\alpha_3 = 18(4\gamma - 5)(16\gamma - 31),$$
$$\alpha_4 = \tfrac{54}{5}(4\gamma - 5)(496\gamma^2 - 2308\gamma + 2759). \tag{3.3}$$

Resolving (3.2) for α_{k+1}, we obtain

$$(2k-1)(2k+6)\alpha_{k+1} = -(2k)(2k+1)(2k+2)(2k+3)\alpha_k + \sum_{m=2}^{k}\alpha_m\alpha_{k-m+2} + 4\gamma\sum_{m=1}^{k} m(m+k+2)\alpha_m\alpha_{k-m+1}, \qquad k \in \mathbb{N}^+, \tag{3.4}$$

where $\alpha_1 = 6$. The main question to be addressed here is whether the formal power series (3.1) converges for finite large values of z. The analysis of convergence is based on the Borel–Laplace transform method (see, for example, [T94b]). The inverse Laplace transform $\mathcal{L}^{-1}$ converts the truncated inner equation (1.6) into the convolution equation:

$$(s^4+s^2)V(s) = V(s)*V(s)+\gamma[2V(s)*(s^2V(s))+(sV(s))*(sV(s))], \tag{3.5}$$

where s is a dual (Borel) variable, $V(s) = [\mathcal{L}^{-1}v](s)$ and $F(s)*G(s) = \int_0^s F(s-\tau)G(\tau)\,\mathrm{d}\tau$. We shall prove the equivalence between convergence of the formal series (3.1), continuations of two analytic solutions $v_+(z)$ and $v_-(z)$ of the truncated inner equation (1.6) and zero values of the Stokes constant $s(\gamma)$ (see corollary 3.5).

Lemma 3.1. *The formal series solution (3.1) with $\alpha_1 = 6$ and $\{\alpha_k\}_{k=2}^{\infty}$ satisfying (3.4) is the only non-trivial formal power series solution to (1.6) in powers of z^{-2}. Moreover, if $\gamma = \gamma_1 = \frac{5}{4}$, then all $\alpha_k = 0$ for $k = 2, 3, \ldots$, so that $\hat{v}(z) = 6/z^2$. If $\gamma \neq \gamma_1$, then, there is no $k_0 \geqslant 2$, such that $\alpha_{k_0} \neq 0$ but $\alpha_k = 0$ for all $k > k_0$.*

Proof. It is straightforward to check that the leading term of any formal power series solution to (1.6) in powers of z^{-2} is $6/z^2$. Then for any $k \in \mathbb{N}^+$ there exists a unique value of α_{k+1}, computed from values $\{\alpha_m\}_{m=1}^{k}$ in the recurrence equation (3.4). If $\gamma = \gamma_1 = \frac{5}{4}$, it is clear from (3.3) that $\alpha_2 = \alpha_3 = \alpha_4 = 0$. By induction, assuming that $\alpha_k = 0$ for $2 \leqslant k \leqslant k_0$, we prove that $\alpha_{k_0+1} = 0$. If $\gamma \neq \gamma_1$, we assume that there exists $k_0 \geqslant 2$, such that all $\alpha_k = 0$ for $k > k_0$. The recurrence equation (3.4) with $k = 2k_0 - 1$ results in

$$\alpha_{2k_0} = \gamma\frac{k_0(3k_0+1)}{(k_0+1)(4k_0-3)}\alpha_{k_0}^2 \neq 0. \tag{3.6}$$

The contradiction proves that there is no $k_0 \geqslant 2$. □

Remark 3.2. Lemma 3.1 shows that $v = 6/z^2$ and $\gamma = \gamma_1 = \frac{5}{4}$ is the only possible (up to a translation) nontrivial solution to (1.6) that is polynomial in inverse powers of z. Combined with results of section 5, it shows that the only possible nontrivial solution to the inner equation (1.1) that is polynomial in $\varepsilon^2\mathrm{sech}^2(\epsilon z/2)$ is the explicit solution (1.7), which matches the solution $v = 6/z^2$ when $\gamma = \gamma_1 = \frac{5}{4}$ and $\varepsilon = 0$. Moreover, the following lemma implies that the same solution (1.7) is the only solution to the inner equation (1.1) that is rational in $\varepsilon^2\mathrm{sech}^2(\epsilon z/2)$.

Lemma 3.3. *The truncated inner equation (1.6) has a nontrivial rational solution only if $\gamma = \gamma_1 = \frac{5}{4}$; this solution is unique (up to a translation) and given by $v(z) = 6/z^2$.*

Proof. Any nontrivial rational function $v(z)$ should have at least one finite pole at $z = z_0$. Direct calculations show that the leading term of the Laurent expansion of $v(z)$ at $z = z_0$ is

$$\frac{15}{2\gamma}(z - z_0)^{-2} \quad \text{if } \gamma \neq 0 \qquad \text{and} \qquad \frac{8!}{4!}(z - z_0)^{-4} \quad \text{if } \gamma = 0. \tag{3.7}$$

We show that the Laurent expansion of $v(z)$ at $z = z_0$ contains only even powers of $z - z_0$. Indeed, in the case $\gamma \neq 0$, let $v(z) = (15/2\gamma)(z - z_0)^{-2} + w(z) + v_k(z - z_0)^k + O((z - z_0)^{k+1})$, where $w(z)$ contains only even powers of $z - z_0$ and $k > -2$ is odd. Substituting this expression into (1.6) and equating the coefficients of the leading odd power term $(z - z_0)^{k-4}$, we obtain

$$k(k-1)(k-2)(k-3)v_k = 15(k^2 - 3k + 6)v_k. \tag{3.8}$$

It is easy to see that $k(k-1)(k-2)(k-3) > 15(k^2 - 3k + 6)$ if $k \geqslant 7$ and that $k(k-1)(k-2)(k-3) \neq 15(k^2 - 3k + 6)$ if $k = -1, 1, 3, 5$. Thus, $v_k = 0$ for all odd $k > -2$. In the case $\gamma = 0$ equation (3.8) becomes $k(k-1)(k-2)(k-3)v_k = 8 \cdot 7 \cdot 6 \cdot 5 v_k$. Since k is odd, it also implies $v_k = 0$ for all odd $k > -2$. Thus, the Laurent expansion of $v(z)$ at $z = z_0$ contains only even powers of $z - z_0$ and $v(z)$ is symmetric with respect to any finite pole $z = z_0$. So, the assumption that $v(z)$ has two different finite poles would imply that $v(z)$ has infinitely many poles. Thus, the only possible rational solution (up to a translation) is $v(z) = 6/z^2$ with $\gamma = \gamma_1 = \frac{5}{4}$. □

Theorem 3.4. *For any $\gamma \in \mathbb{C}$, the convolution equation (3.5) admits a unique nontrivial power series solution in odd powers of s. This solution defines a function $V(s)$ that is analytic at the whole s-plane except, possibly, two vertical cuts: from $s = i$ upwards and from $s = -i$ downwards. The function $V(s)$ may grow at most exponentially (has exponential order 1) along any nonvertical ray in the cut s-plane.*

This statement is a particular case of the main theorem of [T94b], which is valid for systems of convolution equations that are inverse Laplace transforms of so-called 'level one' systems of ODEs.

Corollary 3.5. *Let $v_\pm(z)$ be defined by*

$$v_\pm(z) = \int_0^{\pm\infty} \mathrm{e}^{-zs} V(s)\, \mathrm{d}s. \tag{3.9}$$

These functions are the only analytic solutions of the truncated inner equation (1.6) that satisfy

$$v_\pm(z) \sim \hat{v}(z) \qquad \text{as } z \to \infty, \quad z \in S_\pm, \tag{3.10}$$

where $S_\pm$ are sectors

$$S_+ = \{z : |\arg z| < \pi\}, \qquad S_- = \{z : \ |\arg z - \pi| < \pi\}. \tag{3.11}$$

Moreover,

$$v_+(z) - v_-(z) = -2\pi \mathrm{i} s(\gamma) \mathrm{e}^{\mathrm{i}z}(1 + o(1)), \qquad \text{as } z \to \infty, \quad 0 < \arg z < \pi, \tag{3.12}$$

where the constant $s(\gamma)$ is determined through

$$s(\gamma) = \lim_{s \to -i} (s + \mathrm{i}) V(s). \tag{3.13}$$

Proof. The Taylor expansion of $V(s)$ at $s = 0$ can be obtained by applying the inverse Laplace transform (Borel transform) to the formal series $\hat{v}(z)$. The function $V(s)$ is analytic on a Riemann surface that has possible branch points of the logarithmic type at $s = ik$, where $k \in \mathbb{N}$. The statement on behaviour of $V(s)$ at the singularity $s = i$ follows from theorem 2.2 in [T94c]. The uniqueness of solutions $v_\pm(z)$ satisfying (3.10) follows from theorem 3.4 and properties of the Laplace transform. □

Definition 3.6. *The constant $s(\gamma)$ in (3.13) is called the Stokes constant for the truncated inner equation (1.6).*

Proposition 3.7. *For a given $\gamma \in \mathbb{C}$ the following three conditions are equivalent: (i) the formal power series solution (3.1) has a positive radius of convergence; (ii) $s(\gamma) = 0$ and (iii) $v_+(z) \equiv v_-(z)$.*

Proof. The fact that (i) implies (iii) is obvious. The inverse statement follows from the fact that $v(z) = v_+(z) \equiv v_-(z)$ implies that the function $v(z)$ is single-valued near infinity and has asymptotic expansion $\hat{v}(z)$ in the full neighbourhood of infinity (see [W76]). It is also clear that (i) implies (ii), since in this case $\mathcal{L}^{-1}\hat{v}(s)$ is an entire function. Suppose now $s(\gamma) = 0$. Since $V(s)$ is real-analytic on $\mathbb{R}$, we obtain that $V(s) = o(1/(s-i))$ as $s \to i$. That means, according to corollary 2.2 in [T94c], that $v_+(z)$ coincides with $v_-(z)$ in both lower and upper z half-planes. Thus (ii) implies (iii). □

Remark 3.8. The constant $s(\gamma)$ is related to the limiting behaviour of the coefficients in the formal power series (3.1). In order to show this relation, we define a new sequence $\{\beta_k\}_{k=1}^\infty$ by

$$\alpha_k = (-1)^{k-1}(2k-1)!\beta_k, \qquad k \in \mathbb{N}^+. \tag{3.14}$$

Then the recurrence equation (3.4) becomes

$$\beta_{k+1} - \beta_k = \frac{6\beta_k}{(2k-1)(k+3)} + \frac{1}{2(2k-1)(k+3)} \sum_{m=2}^{k} \frac{(2m-1)!(2k-2m+3)!}{(2k+1)!} \beta_m \beta_{k-m+2}$$
$$- \frac{\gamma}{(2k-1)(k+3)} \sum_{m=1}^{k} \frac{(m+k+2)(2m)!(2k-2m+1)!}{(2k+3)!} \beta_m \beta_{k-m+1}, \tag{3.15}$$

where $k \in \mathbb{N}^+$, and $\beta_1 = 6$.

Proposition 3.9. *The series $\hat{v}(z)$ is divergent and, consequently, $s(\gamma) \neq 0$ for $\gamma \leqslant 0$.*

Proof. It follows from the recurrence equation (3.15) that the sequence $\{\beta_k\}_{k=1}^\infty$ is positive and, consequently, the sequence $\{\alpha_k\}_{k=1}^\infty$ is sign-alternating if $\gamma \leqslant 0$. To prove the lemma by contradiction, we suppose that the series $\hat{v}(z)$ is convergent. Then the solution

$$V(s) = \sum_{k=1}^\infty \frac{\alpha_k}{(2k-1)!} s^{2k-1} = \sum_{k=1}^\infty (-1)^{k-1} \beta_k s^{2k-1} \tag{3.16}$$

of the convolution equation (3.5) is an entire function. Therefore, the left-hand side of (3.5) is zero at $s = -i$. On the other hand, $iV(s)$ is positive along the negative imaginary axis, so that $V(s) * V(s) < 0$ and the expression in the square brackets in (3.5) is positive there. Since $\gamma \leqslant 0$, that means that the right-hand side of (3.5) is strictly negative at $s = -i$, thus leading to contradiction. So, the series $\hat{v}(z)$ is divergent and $s(\gamma) \neq 0$ for $\gamma \leqslant 0$. □

In the particular case $\gamma = 0$ this proposition was proven in [T94c].

Theorem 3.10. *For any $\gamma \in \mathbb{C}$ we have*

$$s(\gamma) = \frac{1}{2}\lim_{k\to\infty}\beta_k = \lim_{k\to\infty}\frac{(-1)^{k-1}\alpha_k}{2(2k-1)!}, \tag{3.17}$$

where the coefficients $\{\alpha_k\}_{k=1}^{\infty}$ and $\{\beta_k\}_{k=1}^{\infty}$ are defined by (3.2) and (3.15), respectively.

Proof. Let us consider first the case when γ is real and $\gamma \leqslant 0$. Then all $\beta_k(\gamma)$ are positive and the sequence $\{\beta_k(\gamma)\}_{k=1}^{\infty}$ is increasing. The arguments above together with (3.13) show that

$$V(s) \sim \frac{s(\gamma)}{s+i} \qquad \text{as } s \to -i, \tag{3.18}$$

where $s(\gamma) \neq 0$. Changing the variable $s = -i\sqrt{\xi}$, we obtain

$$V(s) = -\mathrm{i}\xi^{-1/2}\sum_{k=1}^{\infty}\beta_k(\gamma)\xi^k = -\mathrm{i}\xi^{-1/2}h(\xi)\,. \tag{3.19}$$

Then, according to (3.18),

$$h(\xi) \sim \frac{2s(\gamma)}{1-\xi} \tag{3.20}$$

as $\xi \to 1$. Now we can use the Hardy–Littlewood theorem ([Ti39], section 7.51) that states that under the assumptions that all $\beta_k(\gamma) \geqslant 0$ and $s(\gamma) \neq 0$ we have

$$\sum_{k=1}^{n}\beta_k(\gamma) \sim 2s(\gamma)n \tag{3.21}$$

as $n \to \infty$. Then the statement of the theorem in the case $\gamma \leqslant 0$ follows from (3.21). Let us show that $\beta(\gamma) = \lim_{k\to\infty}\beta_k(\gamma)$ exists for any $\gamma \in \mathbb{C}$. Using (3.15), we obtain

$$\begin{aligned} |\beta_{k+1}(\gamma) - \beta_k(\gamma)| \leqslant{}& \frac{3b_k^2(\gamma)}{(2k+1)2k(2k-1)(k+3)}\sum_{m=2}^{k}\frac{(2m-1)(2m-2)\ldots 4}{(2k-1)(2k-2)\ldots(2k-2m+4)} \\ &+\frac{6|\beta_k(\gamma)|\left(1+\dfrac{2|\gamma|(3k+2)}{2k+1}\right)}{(k+3)(2k+1)} + \frac{3|\gamma|b_k^2(\gamma)}{k(2k-1)^2(k+3)} \\ &\times\sum_{m=2}^{k-1}\frac{(m+k+2)(2m)(2m-1)\ldots 4}{(2k+1)(2k-2)(2k-3)\ldots(2k-2m+2)}, \end{aligned} \tag{3.22}$$

where $k \in N^+$ and $b_k(\gamma) = \max_{1\leqslant j\leqslant k}|\beta_j(\gamma)|$. Note that $|\beta_k(\gamma)| \leqslant \beta_k(-|\gamma|)\ \forall k \in \mathbb{N}^+$ follows immediately from (3.15). Thus, $|\beta_k(\gamma)|$ and $b_k(\gamma)$ do not exceed $M = 2s(|\gamma|)\ \forall k \in \mathbb{N}^+$. Taking into account that the terms in both sums of (3.22) do not exceed one, we can rewrite (3.22) as

$$|\beta_{k+1}(\gamma) - \beta_k(\gamma)| \leqslant \frac{3M(1+4|\gamma|)}{k^2} + \frac{3M^2(1+2|\gamma|)}{8k^3}. \tag{3.23}$$

Then it is clear that $\{\beta_k(\gamma)\}_{k=1}^{\infty}$ is a Cauchy sequence so that $\beta(\gamma) = \lim_{k\to\infty}\beta_k(\gamma)$ exists. It remains to show that $\beta(\gamma) = 0$ implies $s(\gamma) = 0$. Assuming the opposite in the case $\gamma \in \mathbb{R}$ (when $b(\gamma) \in \mathbb{R}$), we see that the geometric series $s(\gamma)\sum_{k=0}^{\infty}\xi^k = s(\gamma)/(1-\xi)$ majorizes the series for $h(\xi)$ given by (3.19) for all sufficiently large $k \in N^+$, i.e. $|s(\gamma)| > |\beta_k(\gamma)|$. Thus, $h(\xi)$ cannot satisfy (3.20) with $s(\gamma) \neq 0$. So, $s(\gamma) = 0$. In the case $\gamma \in \mathbb{C}$ the proof is similar. $\square$

4. Asymptotic analysis of the outer equation

We consider here the outer equation (1.4) and study *stable* and *unstable* solutions $y_s(x, \varepsilon)$ and $y_u(x, \varepsilon)$, i.e. solutions that satisfy

$$\lim_{x\to+\infty} y_s(x, \varepsilon) = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} y_u(x, \varepsilon) = 0, \tag{4.1}$$

respectively. The unperturbed outer equation (1.3) has only one stationary point $(y, y') = (0, 0)$ and it is hyperbolic with two stable and two unstable directions. There exist two different separatrix solutions of the unperturbed equation (1.3)

$$y_1(x) = -\frac{3}{2\cosh^2(x/2)}, \qquad y_2(x) = \frac{3}{2\sinh^2(x/2)}, \tag{4.2}$$

where $y_1(x)$ is bounded on $x \in \mathbb{R}$ and $y_2(x)$ is unbounded on $x \in \mathbb{R}$. However, if x is considered as a complex variable, then the unperturbed equation (1.3) has only one (up to the translation invariance) 'complex' separatrix solution since $y_2(x \pm \mathrm{i}\pi) = y_1(x)$.

In the case $\varepsilon \neq 0$ the outer equation (1.4) has a stationary point $(y, y', y'', y''') = (0, 0, 0, 0)$ with four eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = \mathrm{i}/\varepsilon$, and $\lambda_4 = -\mathrm{i}/\varepsilon$. Let us look for the stable and unstable solutions $y_s(x, \varepsilon)$ and $y_u(x, \varepsilon)$ with the exponential series

$$y_s(x, \varepsilon) = \sum_{k=1}^{\infty} y_k(\varepsilon)\mathrm{e}^{-kx}, \qquad y_u(x, \varepsilon) = \sum_{k=1}^{\infty} y_k(\varepsilon)\mathrm{e}^{kx}, \tag{4.3}$$

where the choice of $y_1(\varepsilon)$ is arbitrary, while the coefficients $\{y_k(\varepsilon)\}_{k=2}^{\infty}$ are uniquely defined from the recurrence equation

$$y_k(\varepsilon) = \frac{1}{(\varepsilon^2k^2 + 1)(k^2 - 1)} \sum_{j=1}^{k-1}(1 + \varepsilon^2\gamma j(j + k))y_j(\varepsilon)y_{k-j}(\varepsilon), \qquad k = 2, 3, \ldots. \tag{4.4}$$

As in the case of complex separatrix for (1.3), the series (4.3) for $y_s(x, \varepsilon)$ with $y_1(\varepsilon) < 0$ and $y_1(\varepsilon) > 0$ represent both bounded and unbounded branches of the same complex stable solution to (1.4). The same is true for unstable solutions $y_u(x, \varepsilon)$. Note that the transformation $y_1(\varepsilon) \mapsto cy_1(\varepsilon)$, where c is an arbitrary constant, results in the transformation: $y_k(\varepsilon) \mapsto c^k y_k(\varepsilon)$, $k \in \mathbb{N}$. Therefore, the series (4.3) with an arbitrary $y_1(\varepsilon)$ can be rewritten as

$$y_s(x, \varepsilon) = \sum_{k=1}^{\infty} c_+^k(\varepsilon)y_k(\varepsilon)\mathrm{e}^{-kx}, \qquad y_u(x, \varepsilon) = \sum_{k=1}^{\infty} c_-^k(\varepsilon)y_k(\varepsilon)\mathrm{e}^{kx}, \qquad y_1(\varepsilon) = 6. \tag{4.5}$$

Here and henceforth we will always assume that $y_1(\varepsilon) = 6$. The normalization $y_1(\varepsilon) = 6$ is chosen for convenience since the separatrix solution $y = \frac{3}{2}\sinh^{-2}(x/2)$ for the unperturbed equation (1.3) has $y_1(0) = 6$ in the representation (4.3). Introducing $x_0(\varepsilon)$ through $c_\pm(\varepsilon) = \mathrm{e}^{\pm x_0(\varepsilon)}$, we represent the series (4.5) in the equivalent form:

$$y_s(x, \varepsilon) = \sum_{k=1}^{\infty} y_k(\varepsilon)\mathrm{e}^{-k(x-x_0(\varepsilon))}, \qquad y_u(x, \varepsilon) = \sum_{k=1}^{\infty} y_k(\varepsilon)\mathrm{e}^{k(x-x_0(\varepsilon))}, \qquad y_1(\varepsilon) = 6. \tag{4.6}$$

Therefore, the arbitrary parameter $c_\pm(\varepsilon)$ in (4.5) corresponds to the translation of the stable and unstable solutions (4.3) by $x_0(\varepsilon)$. We will prove that any stable solution $y_s(x, \varepsilon)$ belongs to a one-parameter family given by the exponential series (4.6). Moreover, if $x_0(\varepsilon) \in \mathbb{R}$, it has a unique minimum on $x \in (-\mu(\varepsilon), \infty)$, where $(-\mu(\varepsilon), \infty)$ is the domain of $y_s(x, \varepsilon)$. Here $\mu(\varepsilon) \leqslant \infty$. Similar results hold for the unstable solution $y_u(x, \varepsilon)$.

Lemma 4.1. *The series (4.3) for $y_s(x, \varepsilon)$ (with $y_1(\varepsilon) = 6$) converges absolutely for $\Re x \geqslant \ln(1 + 2|\gamma|)$.*

Proof. Observe that $\varepsilon^2 j(j+k)/(\varepsilon^2 k^2+1) < 2$ for all $k \in \mathbb{N}$ and all $j = 1, 2, \ldots, k-1$. Therefore, according to (4.4), the series (4.3) is majorized by the series

$$\check{y}(x) = \sum_{k=1}^{\infty} \check{y}_k \mathrm{e}^{-kx}, \tag{4.7}$$

where $\check{y}_1 = y_1 = 6$ and the coefficients $\{\check{y}_k\}_{k=2}^{\infty}$ satisfy the recurrence equation

$$\check{y}_k = \frac{(1+2|\gamma|)}{(k^2-1)} \sum_{j=1}^{k-1} \check{y}_j \check{y}_{k-j}, \qquad k = 2, 3, \ldots. \tag{4.8}$$

It is easy to check that the series $\check{y}(x)$ satisfies the differential equation

$$\check{y}'' - \check{y} = (1+2|\gamma|)\check{y}^2, \tag{4.9}$$

which has the exponential series solution

$$\check{y}^0(x) = \frac{3}{2(1+2|\gamma|)\sinh^2(x/2)} = \sum_{k=1}^{\infty} \frac{6k}{1+2|\gamma|} \mathrm{e}^{-kx}. \tag{4.10}$$

Then the series $\check{y}(x) = \sum_{k=1}^{\infty} \check{y}_k \mathrm{e}^{-kx}$ with $\check{y}_1 = 6$ and $\{\check{y}_k\}_{k=2}^{\infty}$ satisfying (4.8) is given by

$$\check{y}(x) = \check{y}^0(x - \ln[1+2|\gamma|]) = \sum_{k=1}^{\infty} 6k(1+2|\gamma|)^{(k-1)} \mathrm{e}^{-kx}. \tag{4.11}$$

That leads to the estimate

$$|y_k(\varepsilon)| \leqslant 6k(1+2|\gamma|)^{(k-1)}, \qquad k \in \mathbb{N}, \tag{4.12}$$

which proves the statement. □

Lemma 4.2. *If a stable solution $y_s(x, \varepsilon)$ of the outer equation (1.4) has an extremum $x = x_0$, then $y_s(x_0, \varepsilon) \leqslant -\frac{3}{2}$.*

Proof. There exists a conserved quantity for the outer equation (1.4):

$$\varepsilon^2(2y'''y' - y''^2) + (1-\varepsilon^2)y'^2 = y^2 + \tfrac{2}{3}y^3 + 2\varepsilon^2\gamma y y'^2, \tag{4.13}$$

where the integration constant is taken zero because of the zero boundary condition (4.1). Substitution of $y'(x_0) = 0$ into the conserved quantity (4.13) yields

$$-\varepsilon^2 y''^2(x_0) = \tfrac{1}{3}y^2(x_0)(2y(x_0)+3). \tag{4.14}$$

If $2y(x_0)+3 > 0$, then $y(x_0) = y''(x_0) = 0$ and $y'''(x_0) \neq 0$, since otherwise $y(x) \equiv 0$. In this case, the function $y(x)$ is monotonic in a neighbourhood of $x = x_0$. Since x_0 is the point of extremum, we have $2y(x_0)+3 \leqslant 0$. □

Lemma 4.3. *Any stable solution $y_s(x, \varepsilon)$ of the outer equation (1.4) has the form (4.6).*

Proof. Let $y(x)$ denote the stable solution $y_s(x, \varepsilon)$ to the outer equation (1.4). Since $y(x)$ is small for all $x \in [\tilde{x}, \infty)$, where $\tilde{x}$ is a large positive constant, $y'(x)$ is sign-definite by lemma 4.2. Thus $y(x)$ is monotonic and $y(x)y'(x) < 0$ for $x \in [\tilde{x}, \infty)$, such that $y'(x) \in L_1[\tilde{x}, \infty)$. To show that all first four derivatives of y belong to $L_1[\tilde{x}, \infty)$, it is sufficient to show that y', y'', y''' approach zero as $x \to \infty$. We will follow the arguments of [AM91] to prove the latter statement. Indeed, integrating (1.4) once and twice, we see that there exists some $C > 0$ such that

$$|y'(x)|, |y''(x)|, |y'''(x)|, |y''''(x)| \leqslant C(1+x^2) \tag{4.15}$$

for all $x \in [\tilde{x}, \infty)$ and $\gamma \in E$, where E is any finite segment of $\mathbb{R}$. Multiplying (1.4) by e^{-ax} with arbitrary $a > 0$ and taking integral over $[x, \infty)$, where $x \in [\tilde{x}, \infty)$, we obtain after some algebra

$$\begin{aligned}\varepsilon^2 y'''(x) + a\varepsilon^2 y''(x) + [1-\varepsilon^2(1-a^2-2\gamma y(x))]y'(x) = \varepsilon^2\gamma a y^2(x)\\ +e^{ax}\int_x^\infty e^{-at}\{a[1-\varepsilon^2(1-a^2)]y'(t)+\varepsilon^2\gamma y'^2(t) - [y(t)+(1+a^2)y^2(t)]\}\,dt.\end{aligned} \tag{4.16}$$

Existence of the integral in (4.16) follows from (4.15). Since the limit of the right-hand side of (4.16) as $x \to \infty$ is zero, we obtain

$$\varepsilon^2 y'''(x) + a\varepsilon^2 y''(x) + [1-\varepsilon^2(1-a^2-2\gamma y(x))]y'(x) \to 0 \tag{4.17}$$

as $x \to \infty$. Because $a > 0$ is an arbitrary parameter, equation (4.17) for different values of a implies that $y'(x)$, $y''(x)$, and $y'''(x)$ all approach zero as $x \to \infty$. It is now clear that all derivatives of $y(x)$ are absolutely integrable on $[\tilde{x}, \infty)$. Then

$$\begin{aligned}\int_{\tilde{x}}^x [y(t)+y^2(t)]\,dt = \varepsilon^2[y'''(x)-y'''(\tilde{x})] + (1-\varepsilon^2)[y'(x)-y'(\tilde{x})]\\ -\varepsilon^2\gamma\int_{\tilde{x}}^x [2y(t)y''(t)+y'^2(t)]\,dt\end{aligned} \tag{4.18}$$

shows that $y(x) \in L_1[\tilde{x}, \infty)$. We now rewrite the outer equation (1.4) in the vector form

$$Y' = \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \dfrac{1}{\varepsilon^2} & 0 & \dfrac{\varepsilon^2-1}{\varepsilon^2} & 0\end{pmatrix} Y + \begin{pmatrix}0\\0\\0\\ \dfrac{1}{\varepsilon^2}\end{pmatrix}[Y_1^2 + \varepsilon^2\gamma(2Y_1Y_3 + Y_2^2)], \tag{4.19}$$

where $Y = \mathrm{Col}(Y_1, Y_2, Y_3, Y_4)$ and $Y_j = y^{(j-1)}$, $j = 1, 2, 3, 4$. The Jordan form of the coefficient matrix in the linear part of the vector equation (4.19) is $Q = \mathrm{diag}(-1, 1, -i/\varepsilon, i/\varepsilon)$. Using a linear change of variables, we can rewrite the vector equation (4.19) in an equivalent integral form

$$Y(x) = e^{Qx}C + e^{Qx}\int_{\beta(x)} e^{-Qt} f(t, Y)\,dt, \tag{4.20}$$

where $f(t, Y)$ is the transformed nonlinear part of the vector equation (4.19) and $\beta(x) = \{\beta_j(x)\}_{j=1}^4$ is a set of four contours of integration (a specific contour for each entry of the vector integrand). We consider the contour where $\beta_1(x) = [\tilde{x}, x]$ and $\beta_2(x) = \beta_3(x) = \beta_4(x) = (\infty, x]$. It is easy to show that if all components of a vector $Y(x) \in \mathbb{C}^4$ belong to $L_1[\tilde{x}, \infty)$, then

$$\check{Y}(x) = e^{Qx}\int_{\beta(x)} e^{-Qt} Y(t)\,dt \to 0 \qquad \text{as } x \to \infty. \tag{4.21}$$

For all but the first component this limit follows from definitions of $(\beta_2(x), \beta_3(x), \beta_4(x))$. In order to prove the limit (4.21) for the first component, we split the integral

$$\check{Y}_1(x) = e^{-x}\left\{\int_{\tilde{x}}^{(x-\tilde{x})/2} + \int_{(x-\tilde{x})/2}^{x}\right\} e^t Y_1(t)\,dt. \tag{4.22}$$

Then the limit (4.21) follows from the fact that the first integral in (4.22) does not exceed $e^{-(x+\tilde{x})/2}\|Y\|_{L_1[\tilde{x},\infty)}$ while the second integral does not exceed $\|Y\|_{L_1[(x-\tilde{x})/2,\infty)}$. Returning to the integral equation (4.20), we notice that the components of the integrand $f(t, Y) \in L_1[\tilde{x}, \infty)$. Taking the limit $x \to \infty$ in the integral equation (4.20), we observe that only the first component of C can be different from zero if $y(x)$ is a stable solution. Thus, the set of all stable solutions of the outer equation (1.4) coincides with the one-parameter set of solutions of the integral equation (4.20), where $C = ce_1$, $e_1 = \mathrm{Col}(1, 0, 0, 0)$, and $c \in \mathbb{C}$ is a free parameter. Then, the statement follows from theorem 3.1 in [T94a]. □

Lemma 4.4. *There exists a formal power series solution to the outer equation (1.4)*

$$\hat{y}(x, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^{2k} P_{k+1}(p), \tag{4.23}$$

where $P_{k+1}(p)$ are polynomials with real coefficients

$$P_{k+1}(p) = a_{k+1,0}p^{k+1} + a_{k+1,1}p^k + \cdots + a_{k+1,k}p \tag{4.24}$$

and $p = p(x) = \sinh^{-2}(x/2)$. All polynomials $P_k(p)$ are uniquely defined.

Proof. The polynomial $P_1(p)$ is uniquely defined by (4.2) as $P_1(p) = \frac{3}{2}p$. Substitution of (4.23)–(4.24) into the outer equation (1.4) yields the recurrence equation

$$D_x^2 P_{k+1}(p) - (1 + 3p)P_{k+1}(p) = -D_x^4 P_k(p) + D_x^2 P_k(p) + \sum_{j=2}^{k} P_j(p)P_{k+2-j}(p)$$
$$+\gamma \sum_{j=1}^{k} [2P_j(p)D_x^2 P_{k-j}(p) + (D_x P_j(p))(D_x P_{k-j}(p))], \tag{4.25}$$

where $k = 1, 2, \ldots$ and $D_x = \mathrm{d}/\mathrm{d}x$. It is easy to check that the identities

$$D_x^2 p^k = \left(k\left(k + \frac{1}{2}\right)p + k^2\right)p^k, \qquad D_x p^k = -k\sqrt{1 + p}\,p^k \tag{4.26}$$

hold for any $k \in \mathbb{Z}$. Using these identities one can prove by induction that the right-hand side of (4.25) is a polynomial in p of order $k + 1$ with no constant and linear terms. The statement follows from the observation that $j(j + \frac{1}{2}) - 3 \neq 0$ for any $j \in \mathbb{N}$. □

Remark 4.5. The statement of lemma 4.4 is still valid if the outer equation (1.4) is considered with $\gamma = \gamma(\varepsilon)$, where $\gamma(\varepsilon)$ is a smooth function on $[0, \varepsilon_0]$ for some $\varepsilon_0 > 0$ that admits asymptotic expansion

$$\gamma(\varepsilon) \sim \sum_{k=0}^{\infty} \gamma_k \varepsilon^{2k} \tag{4.27}$$

as $\varepsilon \to 0$, where the coefficients γ_k are real numbers.

Remark 4.6. The coefficients of the formal series (4.23) are even functions in x. Therefore, the formal power series (4.23) cannot capture the difference between stable and unstable solutions (i.e. it cannot capture the breakdown of the separatrix solution) in any order of the small parameter ε. This phenomenon is called 'asymptotics beyond all orders' (see review in [STL92]).

Remark 4.7. Introducing $r(x) = \operatorname{sech}^2(x/2)$, we observe that $p(x \pm \mathrm{i}p) = -r(x)$. Therefore, a formal series solution to (1.4) in terms of $r(x)$ is given by

$$\hat{y}(x, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^{2k} P_{k+1}(-r(x)). \tag{4.28}$$

In fact, a stable solution $y_s(x, \varepsilon)$ admits the asymptotic expansion (4.28) in the limit $\varepsilon \to 0$. To be more precise, we need the following definition.

Definition 4.8. *Let A be any positive constant and $\mathcal{B}_j$, $j \in \mathbb{Z}$ be the Banach spaces of continuous functions $y(x)$ on $x \in [-A, \infty)$ with the norm*

$$\|y\|_j = \inf \left\{ M > 0 : \ |y(x)| < \frac{M}{\cosh^{2j}(x/2)} \quad \text{for all } x \in [-A, \infty) \right\}. \tag{4.29}$$

It is clear that $\mathcal{B}_k \subset \mathcal{B}_j$ and $\|y\|_j \leqslant \|y\|_k$ if $k > j$.

Theorem 4.9. *Let $\hat{y}_N(x, \varepsilon)$ denote the first $(N + 1)$ terms in the formal series (4.28), where $N \in \mathbb{N}$. Then there exists some $A > 0$ such that for every $N \in \mathbb{N}$ we can construct a stable solution $y_N(x, \varepsilon)$ satisfying*

$$\|y_N - \hat{y}_N, y_N' - \hat{y}_N', y_N'' - \hat{y}_N'', y_N''' - \hat{y}_N'''\|_{\mathcal{B}_2} = \mathrm{O}(\varepsilon^{2(N+1)}) \tag{4.30}$$

as $\varepsilon \to 0$ uniformly on bounded sets of γ.

The proof and further details can be found in appendix A. According to lemma 4.3, every solution $y_N(x, \varepsilon)$ can be represented in the form (4.6).

Lemma 4.10. *Let $\varepsilon > 0$ be sufficiently small. A stable solution $y_s(x, \varepsilon)$ to the outer equation (1.4) has a unique global minimum on $(-\mu(\varepsilon), \infty)$, where $(-\mu(\varepsilon), \infty)$ is the domain of $y_s(x, \varepsilon)$ and $\mu(\varepsilon) \leqslant \infty$.*

Proof. By lemma 4.3, any stable solution can be represented by the series (4.6) with the proper shift $x_0(\varepsilon)$. Therefore, it is sufficient to prove the statement for a particular stable solution, say $y_0(x, \varepsilon)$, where $\hat{y}_0 = -\frac{3}{2}\cosh^{-2}(x/2)$. By theorem 4.9,

$$\|y_0 - \hat{y}_0, y_0' - \hat{y}_0', y_0'' - \hat{y}_0'', y_0''' - \hat{y}_0'''\|_{\mathcal{B}_2} = \mathrm{O}(\varepsilon^2). \tag{4.31}$$

Since $\hat{y}_0'(x)$ changes sign at $x = 0$, hence $y_0'(x^*(\varepsilon), \varepsilon) = 0$ at some point $x = x^*(\varepsilon)$, such that $|x^*(\varepsilon)| = \mathrm{O}(\varepsilon^2)$. There is only one such point $x^*(\varepsilon)$ in some finite and independent of ε neighbourhood U of $x = 0$. Indeed, the opposite assumption would contradict (4.31) and the fact that $\hat{y}_0''(x, 0) = \frac{3}{4}$. Clearly, $x^*(\varepsilon)$ is a point of local minimum of $y_0(x, \varepsilon)$. To complete the proof, it remains to show that $y_0(x, \varepsilon)$ has no extremal points on $x \in (-\mu(\varepsilon), \infty)$ outside U. The assumption that $x_1 \notin U$ is an extremal point of $y_0(x, \varepsilon)$ leads to a contradiction. Indeed, by lemma 4.2,

$$y_0(x_1, \varepsilon) \leqslant -\tfrac{3}{2}. \tag{4.32}$$

On the other hand, if $x_1 > -A$ then theorem 4.9 implies that $|y_0(x_1, \varepsilon) - \hat{y}_0(x_1)| = \mathrm{O}(\varepsilon^2)$, which contradicts (4.32). Suppose now that $x_1 \in (\mu(\varepsilon), -A)$ and that x_1 is the largest extremal point on this interval. Then $y_0(x, \varepsilon)$ is decreasing on $(x_1, x^*(\varepsilon))$, so that the requirement (4.32) leads to contradiction again. □

Lemma 4.10 provides us with the opportunity to 'normalize' a stable solution by requiring that it has a minimum at $x = 0$. Here and henceforth we will denote by $y_s(x, \varepsilon)$ and will simply call stable solution (unless otherwise specified) this 'normalized' stable solution. The following theorem states that $y_s(x, \varepsilon)$ has asymptotic expansion $\hat{y}(x, \varepsilon)$ as $\varepsilon \to 0$.

Theorem 4.11. *There exists $A > 0$ such that the stable solution $y_s(x, \varepsilon)$ of the outer equation (1.4), which satisfies*

$$y_s'(0, \varepsilon) = 0, \tag{4.33}$$

possesses the asymptotic expansion as $\varepsilon \to 0$ and $x \in [-A, \infty)$:

$$y_s(x, \varepsilon) \sim \hat{y}(x, \varepsilon) \tag{4.34}$$

in $\mathcal{B}_1$ that is uniform in $\gamma \in E$, where E is a bounded subset of $\mathbb{R}$. The exact meaning of the latter statement is that $\forall N \in \mathbb{N}$ and $\forall \gamma \in E$

$$y_s(x, \varepsilon) - \hat{y}_N(x, \varepsilon) = \varepsilon^{2(N+1)} r_n(x, \varepsilon), \tag{4.35}$$

where $r_n \in \mathcal{B}_1$, r_n depends on ε continuously and does not depend on γ, and $\hat{y}_N(x, \varepsilon)$ is the Nth partial sum of the formal power series $\hat{y}(x, \varepsilon)$.

Proof. According to theorem 4.9, for any $N \in \mathbb{N}$ there exists $x_N(\varepsilon) \in \mathbb{R}$, such that

$$y_s(x, \varepsilon) \equiv y_N(x - x_N(\varepsilon), \varepsilon) = \hat{y}_N(x - x_N(\varepsilon), \varepsilon) + O(\varepsilon^{2(N+1)}) \tag{4.36}$$

uniformly in $x \in [-A, \infty)$, $\gamma \in E$. The corresponding expressions also hold for derivatives of $y_s(x, \varepsilon)$ in x. Thus, $x_N(\varepsilon) = O(\varepsilon^{2(N+1)})$. This fact together with (4.36) yield the statement. □

Corollary 4.12. *The coefficient $c_+(\varepsilon)$ in the exponential series (4.5) for the stable solution $y_s(x, \varepsilon)$ has asymptotic expansion*

$$c_+(\varepsilon) \sim 4 \sum_{k=0}^{\infty} \varepsilon^{2k} a_{k+1,k} \qquad \text{as } \varepsilon \to 0. \tag{4.37}$$

where coefficients $a_{k+1,k}$ are defined in (4.24).

Proof. The statement follows from lemma 4.3 and theorem 4.11. □

Remark 4.13. When we move from analysis of the outer equation to analysis of the inner equation, it is convenient to place the singularity of the solution of the unperturbed equation (1.3) at the origin. Therefore, we consider a 'deformation' of the separatrix solution $y_2(x) = \frac{3}{2}\sinh^{-2}(x/2)$. Considering $x \in \mathbb{C}$, the line $\Im x = \pi$ corresponds to the bounded separatrix solution $y_1(x) = -\frac{3}{2}\cosh^{-2}(x/2)$. Then the inner equation (1.1) is related with the outer equation (1.4) through the change of variables (1.5) with the shift $c = 0$. Exponential series solutions to the outer equation (1.4) are given by the series (4.6). Then exponential series solutions to the inner equation (1.1) are given by

$$v_s(z, \varepsilon) = \varepsilon^2 \sum_{k=1}^{\infty} c_+^k(\varepsilon) y_k(\varepsilon) \mathrm{e}^{-\varepsilon k z}, \qquad v_u(z, \varepsilon) = \varepsilon^2 \sum_{k=1}^{\infty} c_-^k(\varepsilon) y_k(\varepsilon) \mathrm{e}^{\varepsilon k z}, \tag{4.38}$$

where $y_k(\varepsilon)$, $k \in \mathbb{N}$, are determined by the recurrence equation (4.4) with $y_1(\varepsilon) = 6$. By lemma 4.1, the series (4.38) for $v_s(x, \varepsilon)$ converge for $\Re z > \ln(1 + 2|\gamma|)/\varepsilon$ and define $2\pi i/\varepsilon$-periodic functions of z. Taking into account the symmetry with respect to the real axis, we restrict our attention to the strip

$$R = \left\{ z : 0 \leqslant \Im z \leqslant \frac{\pi}{\varepsilon} \right\}.$$

By lemma 4.3, any stable solution to the inner equation (1.1) is a translation of the first series in (4.38). Using theorem 4.11, we denote by $v_s(z,\varepsilon)$ the stable solution to the inner equation (1.1) that corresponds to the stable solution $y_s(x,\varepsilon)$ to the outer equation (1.4) shifted by $\mathrm{i}\pi$, i.e.

$$v_s(z,\varepsilon)=\varepsilon^2 y_s(\varepsilon z-\mathrm{i}\pi,\varepsilon),\qquad D_z v_s\left(\frac{\mathrm{i}\pi}{\varepsilon},\varepsilon\right)=0, \tag{4.39}$$

where $D_z=\mathrm{d}/\mathrm{d}z$.

Remark 4.14. Formal solution of the inner equation (1.1) can be written in terms of the new variable

$$q(z,\varepsilon)=\frac{2}{\varepsilon}\sinh\frac{\varepsilon z}{2}, \tag{4.40}$$

such that $\lim_{\varepsilon\to 0} q(z,\varepsilon)=z$. Since $p=\sinh^{-2}(\varepsilon z/2)=(2/\varepsilon q)^2$, the formal solution (4.23)–(4.24) can be transformed as

$$\hat{v}(q,\varepsilon)=\varepsilon^2\sum_{k=0}^{\infty}\varepsilon^{2k}P_{k+1}\left(\frac{2}{\varepsilon q}\right)^2=\sum_{k=0}^{\infty}\varepsilon^{2k}\hat{v}_k(q), \tag{4.41}$$

where polynomials $P_{k+1}(p)=\sum_{j=0}^{k}a_{k+1,j}\,p^{k+1-j}$ and their coefficients $\{a_{j,k}\}$ are defined in lemma 4.4, and

$$\hat{v}_k(q)=\sum_{j=k+1}^{\infty}\frac{2^{2(j-k)}a_{j,k}}{q^{2(j-k)}}. \tag{4.42}$$

As $\varepsilon\to 0$ and $q\to z$, the solution (4.42) matches the power series (3.1) with the correspondence $\alpha_k=2^{2k}a_{k,0}$, $k\in\mathbb{N}$.

5. Analysis of the inner equation

The main result of this section is a proof that a simple zero γ_0 of the Stokes constant $s(\gamma)$ implies a unique curve $\gamma(\varepsilon)$ of homoclinic solutions to the inner equation (1.1) in the parameter space (γ,ε) passing through the point $(\gamma_0,0)$. The central part of this proof is the study of behaviour of the stable and unstable solutions to the inner equation (1.1) in the limit $\varepsilon\to 0$. In particular, we show that

$$\lim_{\varepsilon\to 0} v_s(z,\varepsilon)=v_+(z),\qquad \lim_{\varepsilon\to 0} v_u(z,\varepsilon)=v_-(z),$$

where $v_\pm(z)$ are solutions to the truncated inner equation (1.6) (see corollary 3.5) and find the asymptotic meaning of the formal expansion (4.41).

The exponential series solutions to the inner equation (1.1) are given by (4.38). Taking into account the symmetry with respect to the real axis, it is sufficient to consider the strip $0\leqslant\Im z\leqslant\pi/\varepsilon$ of the complex z-plane. Let R_{z_0}, where $z_0\in(0,\pi/\varepsilon)$, denote the semi-strip $0\leqslant\Im z\leqslant\pi/\varepsilon$, $0\leqslant\Re z$ with the cut square $\{\Re z<z_0\}\cap\{\Im z<z_0\}$. The purpose of this cut is to separate R_{z_0} from the singularity $z=0$ of $q^{-1}(z,\varepsilon)$ (so that $q^{-1}(z,\varepsilon)$ can be estimated from above in R_{z_0}). The domain R_{z_0} is shown in figure 1.

Definition 5.1. *Let $\mathcal{X}$ be the complex Banach space of vector-valued functions $W(z,\varepsilon)\in\mathbb{C}^4$ that are analytic in $z\in R_{z_0}$ and continuous in $\varepsilon\in[0,\varepsilon_0]$ with the norm $\|W\|_{\mathcal{X}}=M_0=\inf_{\mathbb{R}^+}M$, where*

$$\left\|\begin{pmatrix}W_1\\W_2\end{pmatrix}\right\|\leqslant M|q^{-3}(z,\varepsilon)||\mathrm{e}^{-(\varepsilon/2)z}|,\qquad \left\|\begin{pmatrix}W_3\\W_4\end{pmatrix}\right\|\leqslant M|q^{-4}(z,\varepsilon)|,$$
$$z\in R_{z_0},\quad \varepsilon\in[0,\varepsilon_0]. \tag{5.1}$$

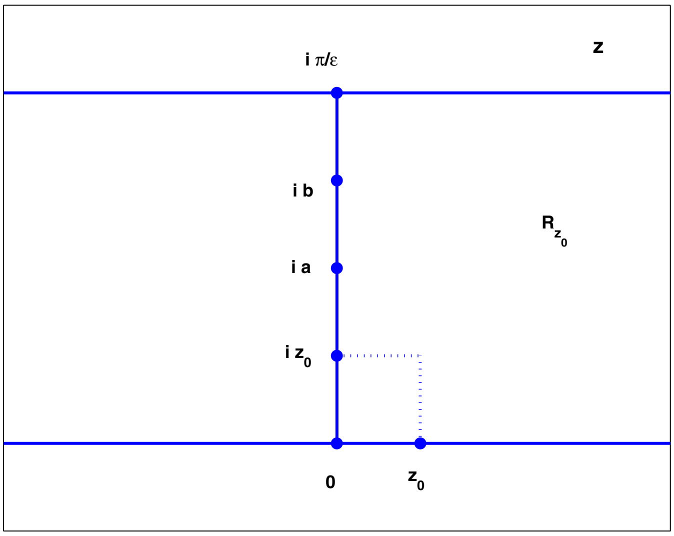

Figure 1. The domain R_{z_0} on the complex plane of z.

The substitutions

$$v(z,\varepsilon) = u(z,\varepsilon) + 6c(\varepsilon)q^{-2}(z,\varepsilon) \tag{5.2}$$

and $U = TW$, where $U = \mathrm{Col}(u, u', u'', u''')$ and

$$T = \begin{pmatrix} 1 & -1 \\ \lambda_1 & -\lambda_2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.3}$$

reduce the inner equation (1.1) to the vector equation

$$W' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \lambda_2 & 0 \end{pmatrix} W - \frac{\theta'}{2\theta} \begin{pmatrix} I & I \\ I & I \end{pmatrix} W + \frac{1}{\theta} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} [(W_1 - W_3)^2 - f(q, W)], \tag{5.4}$$

where

$$\lambda_{1,2} = \frac{\varepsilon^2 - 1 \pm \sqrt{(1+\varepsilon^2)^2 + 48q^{-2}}}{2}, \qquad \theta = \sqrt{(1+\varepsilon^2)^2 + 48q^{-2}}. \tag{5.5}$$

Theorem 5.2. *Let $c(\varepsilon)$ be a continuous function such that there exists some $B > 0$ satisfying*

$$|c(\varepsilon) - 1| \leqslant B\varepsilon \tag{5.6}$$

for all $\varepsilon \in [0, \varepsilon_0]$. There exists a unique solution $W \in \mathcal{X}$ of the vector equation (5.4) such that the function $v(z,\varepsilon)$ in (5.2) solves the inner equation (1.1). This W is a continuous function in $\varepsilon \in [0, \varepsilon_0]$ and an analytic function in $\gamma \in E$.

The statement is proved in appendix B. The proof of the particular case $\gamma = 0$ can be found in [T00].

Remark 5.3. By corollary 4.12 and theorem 5.2, in the case $c_+(\varepsilon) = 6c(\varepsilon)$ the solution (5.2) to the inner equation (1.1) is the stable solution (i.e. $D_z v(\mathrm{i}\pi/\varepsilon, \varepsilon) = 0$), represented by the exponential series (4.38).

Remark 5.4. In the case

$$|f(q,0)| = O(\varepsilon^2 q^{-4}) + O(q^{-6}) \tag{5.7}$$

in $R_{z_0} \times [0, \varepsilon_0]$ the Banach space $\mathcal{X}$ in theorem 5.2 can be replaced by the Banach space $\mathcal{Y}$, where $\|G(z,\varepsilon)\|_{\mathcal{Y}} = \inf_{\mathbb{R}^+} M$, such that

$$\|G(z,\varepsilon)\| \leqslant M|q^{-4}| \tag{5.8}$$

in $R_{z_0} \times [0, \varepsilon_0]$. This observation is also valid in a general case, described in remark B.5. In particular, (5.7) is true when condition (5.6) in theorem 5.2 is replaced by

$$|c(\varepsilon) - 1| \leqslant B\varepsilon^2 \tag{5.9}$$

for some $B > 0$.

It is easy to show that the smoothness of $c(\varepsilon)$ determines the smoothness of $W(z,\varepsilon)$ in ε. To this end one can use the relation between inner and outer solutions $v(z,\varepsilon)$ and $y(x,\varepsilon)$ together with theorem A.1 and corollary 4.12.

Lemma 5.5. *Let $v(z)$ be defined by*

$$v(z) = v(z,0) = \frac{6}{z^2} + W_1(z,0) - W_3(z,0), \tag{5.10}$$

where $W(z,\varepsilon)$ is defined in theorem 5.2. The function $v(z)$ is a solution to the truncated inner equation (1.6) and $v(z) \equiv v_+(z)$.

Proof. When $\varepsilon = 0$ we have $c(0) = 1$ and $\lim_{\varepsilon\to 0} q(z,\varepsilon) = z$ uniformly on a bounded subset of $\mathbb{C}$. Then, by construction (see appendix B), the function $v(z)$ in (5.10) solves the truncated inner equation (1.6). Moreover, according to remark 5.4,

$$W_1(z,0) - W_3(z,0) = O(z^{-4}), \qquad \text{as } z \to \infty, \quad 0 \leqslant \arg z < \frac{\pi}{2}, \tag{5.11}$$

and the vector-function $W(z,\varepsilon)$ is real valued when $z \in \mathbb{R}^+$ and $z > z_0$. Suppose that $W(z,0)$ has an asymptotic expansion in z^{-1} as $z \to \infty$ in $0 \leqslant \arg z < \pi/2$. Then, using the symmetry consideration, we obtain that $v(z)$ has an asymptotic expansion in the right half-plane and $v(z) - (6/z^2) = O(z^{-4})$. The only formal power series solution to (1.6) consistent with the latter condition is given by (3.1). Now the statement of the lemma follows from the fact that $v_+(z)$ is the only solution to (1.6) that has the asymptotic expansion $\hat{v}_0(z)$ in the right half-plane. To prove the existence of a power series for $W(z)$ in z^{-1} in $0 \leqslant \arg z < \pi/2$, we notice that $\theta^{-1} f(z,0)$ in the integral equation (B.9) has an asymptotic expansion and the integral operator $\mathcal{I}$, determined by (B.13) preserves this property. Thus, all $W^{(k)}(z,0)$ in the series (B.15) have asymptotic expansions in powers of z^{-1} when $0 \leqslant \arg z < \pi/2$, such that $W^{(k)}(z,0) = O(z^{-k-3})$. □

Corollary 5.6. *Let $c(\varepsilon)$ in theorem 5.2 be*

$$c(\varepsilon) = 1 + b\varepsilon + o(\varepsilon), \tag{5.12}$$

where $b \in \mathbb{C}$, and let the solution $v(z,\varepsilon)$ be defined by (5.2). Then

$$\lim_{\varepsilon\to 0} v(z,\varepsilon) = v_+(z+b) \tag{5.13}$$

for any $z \in R_{z_0}$.

Proof. In the case $c(\varepsilon) \equiv 1$ the statement follows from theorem 5.2 and lemma 5.5 immediately. In the general case (5.12), the solution $v(z, \varepsilon)$ has form (4.38) by lemma 4.3. Then the statement of the corollary follows from the fact that $\lim_{\varepsilon\to 0}(\ln c(\varepsilon)/\varepsilon) = b$. □

Remark 5.7. Since equation (1.1) is time-reversible, all the obtained results about stable solutions can be extended to unstable solutions by making change of variable $z \mapsto (\mathrm{i}\pi/\varepsilon) - z$. A solution to the inner equation (1.1) is a *homoclinic solution* if it is simultaneously a stable and an unstable solution. It is clear that any homoclinic solution can be turned into an even solution by a proper translation.

Theorem 5.8. *If* $s(\gamma) \neq 0$ *then there exists a neighbourhood* $\mathcal{D}$ *of the point* $(\gamma, 0)$ *so that the inner equation (1.1) has no homoclinic solutions when* $(\gamma, \varepsilon) \in \mathcal{D}$.

Proof. Suppose the opposite. Then there exist a sequence $\{\gamma_n, \varepsilon_n\} \subset \mathcal{D}$, approaching $(\gamma, 0)$ as $n \to \infty$, and the corresponding homoclinic solutions $v_{sn}(z, \varepsilon_n)$ to the inner equation (1.1) with $\gamma = \gamma_n$, satisfying $D_z v_{sn}(\mathrm{i}\pi/\varepsilon_n, \varepsilon_n) = 0$. By corollary 4.12 and remark 4.13, $v_{sn}(z, \varepsilon_n)$ can be represented by (4.38), where $c(\varepsilon_n) = 1 + O(\varepsilon_n^2)$ as $n \to \infty$ uniformly in some neighbourhood E of γ. By theorem 5.2, $v_{sn}(z, \varepsilon_n) \to v_+(z)$ for any $z \in R_{z_0}$. Similar statement is true for $v_-(z)$ in the second quadrant of $\mathbb{C}$. Thus, for any $z \in (\mathrm{i}z_0, \mathrm{i}\pi/\varepsilon)$, we have $v_+(z) \equiv v_-(z)$. By lemma 3.7, the equivalence of $v_+(z)$ and $v_-(z)$ contradicts the assumptions of the theorem. □

Remark 5.9. Let $q(z, \varepsilon)$ be defined by (4.40). The inner equation (1.1) is then rewritten in the equivalent form,

$$\left[\left(1 + \frac{\varepsilon^2 q^2}{4}\right) D_q^2 + \frac{\varepsilon^2 q}{4} D_q + 1\right]\left[\left(1 + \frac{\varepsilon^2 q^2}{4}\right) D_q^2 + \frac{\varepsilon^2 q}{4} D_q - \varepsilon^2\right] v$$
$$= v^2 + \gamma\left[2v\left(D_q^2 + \frac{\varepsilon^2}{4}(qD_q)^2\right) v + \left(1 + \frac{\varepsilon^2 q^2}{4}\right)(D_q v)^2\right], \tag{5.14}$$

where $D_d = \mathrm{d}/\mathrm{d}q$. Equation (5.14) has three singular points $q = \pm 2\mathrm{i}/\varepsilon$ and $q = \infty$ in the complex q plane. Each of these singularities is a regular singular point, i.e. the characteristic exponentials in each singular point are trivial (see [T94a]– [T94c] for details). Thus, every formal power series centred at $q = \pm 2\mathrm{i}/\varepsilon$ or at $q = \infty$ and satisfying (5.14) defines an analytic solution of (5.14). Considering, for example, $q = \infty$, it is easy to see that the coefficient $\phi_1(\varepsilon)$ in the power series

$$v(q, \varepsilon) = \sum_{j=1}^{\infty} \phi_j(\varepsilon) q^{-2j} \tag{5.15}$$

is at our disposal, whereas the remaining coefficients $\phi_j(\varepsilon)$, $j = 2, 3, \ldots$, can be determined by substitution of (5.15) into (5.14). By lemma 4.3, any stable solution can be represented as the series (5.15), where $\phi_1(\varepsilon) = 6c(\varepsilon)$. With a particular choice of $\phi_1(\varepsilon) = c_+(\varepsilon)$ in (5.15), where $c_+(\varepsilon)$ is defined in corollary 4.12, the series (5.15) determines the solution $v_s(q, \varepsilon)$ of the equation (5.14) that corresponds to the solution $v_s(z, \varepsilon)$ of the inner equation (1.1). By theorem 5.3, the series (5.15) converges in the region $|q| \geqslant 2/\varepsilon$, and singularities at $q = \pm 2\mathrm{i}/\varepsilon$ are branch-points. Note that the line $\Im z = \pi/\varepsilon$ is mapped by $q = q(z)$ onto the contour l in the q-plane, that goes from $q = \mathrm{i}\infty$ down to $q = 2\mathrm{i}/\varepsilon$ along the imaginary axis, circles around $q = 2\mathrm{i}/\varepsilon$ and goes back to $q = \mathrm{i}\infty$. Thus, existence of a homoclinic solution to the inner equation (1.1) is equivalent to the requirement that the stable solution $v_s(q, \varepsilon)$ is analytic in a vicinity of $q = 2\mathrm{i}/\varepsilon$ (see [T00]). Figure 2 shows a mapping of the z-plane to the complex q-plane and the deformation of the contour described above.

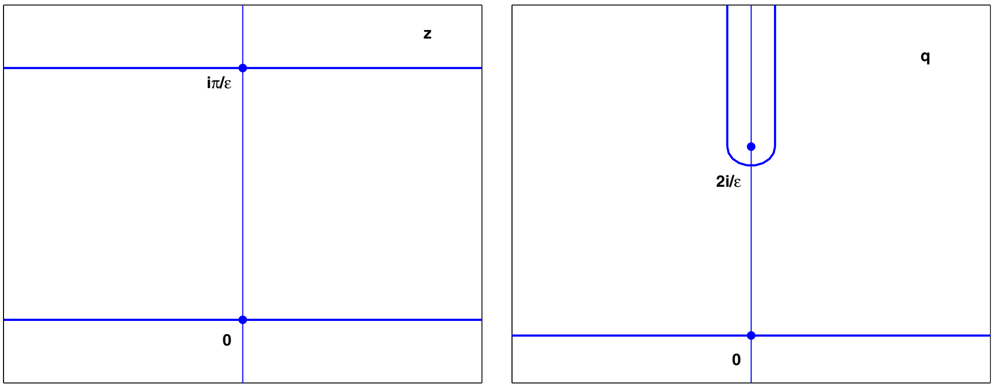

Figure 2. The mapping of the horizontal line $\Im z = \pi/\varepsilon$ by $q = q(z)$.

Remark 5.10. Behaviour of stable and unstable solutions to the inner equation (1.1) in the limit $\varepsilon \to 0$ can be described through the study of collision of regular singular points $q = 2\mathrm{i}/\varepsilon$ and $q = \infty$ of equation (5.14) that form in the limit $\varepsilon \to 0$ an irregular singularity at $q = \infty$ of the truncated inner equation (1.6). Certain results on collision of regular singular points can be found in [G99, G04], but they do not cover the case of equation (5.14).

Remark 5.11. On the formal level, the series (5.15) can be obtained by changing the order of summation in the formal solution (4.41)

$$\hat{v}(q,\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^{2k+2} P_{k+1}\left(\frac{2}{\varepsilon q}\right) = \sum_{j=1}^{\infty} \hat{\phi}_j(\varepsilon) q^{-2j}, \tag{5.16}$$

where

$$\hat{\phi}_j(\varepsilon) = 2^{2j} \sum_{k=j}^{\infty} a_{k,k-j} \varepsilon^{2k}. \tag{5.17}$$

It is easy to show that if $\phi_1(\varepsilon) \sim \hat{\phi}_1(\varepsilon)$ as $\varepsilon \to 0$ for $j = 1$, then $\phi_j(\varepsilon) \sim \hat{\phi}_j(\varepsilon)$ as $\varepsilon \to 0$ for all $j \in \mathbb{N}^+$. Similarly to the formal power series solution (4.23) of the outer equation (1.4), we shall consider the formal power series solution

$$\check{v}(q,\varepsilon) = \sum_{m=0}^{\infty} \varepsilon^{2m} v_m(q) \tag{5.18}$$

of the inner equation (1.1), where $v_0(q)$ satisfies the truncated inner equation (1.6) (note that $q = z$ in the case $\varepsilon = 0$) and the higher-order coefficients $v_m(q)$ satisfy nonhomogeneous linear equations

$$[(\partial_q^2 + 1)\partial_q^2 - 2v_0(q)]v_m(q) = H_m(q), \qquad m \geqslant 1. \tag{5.19}$$

Here H_m depends on $v_0, v_1, \ldots, v_{m-1}$ but not on v_m. The difference between the formal solutions $\check{v}(q,\varepsilon)$ and $\hat{v}(q,\varepsilon)$ in (4.41) is that the coefficients $v_m(q)$ in (5.18) are analytic functions of q (see lemma 5.12 below) whereas the coefficients $\hat{v}_m(q)$ in (4.41) are formal series in q^{-1} given by (4.42).

Lemma 5.12. *Let $S_\pm$ be defined by (3.11). If $v_0(q) = v_\pm(q)$ then there exist uniquely defined solutions $v_m(q) = (v_m)_\pm(q)$, $m = 1, 2, \ldots$, to the nonhomogeneous problem (5.19) that have the same domain of analyticity as $v_0(q)$ and such that*

$$(v_m)_\pm(q) \sim \hat{v}_m(q), \qquad \text{as } q \to \infty, \quad q \in S_\pm. \tag{5.20}$$

Proof. The statement is proved in two steps: (i) verification that the formal series $\hat{v}_m(q)$ satisfies the corresponding equation (5.19) for $m \geqslant 1$ and (ii) application of the main theorem in [T94b] to equation (5.19). Both steps are checked directly. □

Remark 5.13. Similarly to condition (3.12), the solutions $(v_m)_\pm$ satisfy

$$(v_m)_+(z) - (v_m)_-(z) = -2\pi \mathrm{i} s_m(\gamma)\mathrm{e}^{\mathrm{i}z}(1 + o(1)), \qquad \text{as } z \to \infty, \quad 0 < \arg z < \pi, \tag{5.21}$$

where $s_m(\gamma)$ is the Stokes constant for the solution $v_m(z)$, which is determined through

$$s_m(\gamma) = \lim_{s\to -i}(s + \mathrm{i})\mathcal{L}^{-1}[v_m](s). \tag{5.22}$$

It is clear that in the new notations $s(\gamma) = s_0(\gamma)$.

Theorem 5.14. *Let $\mathcal{Y}$ be defined in remark 5.4. Then*

$$v_s(z, \varepsilon) \sim \sum_{m=0}^{\infty} \varepsilon^{2m}(v_m)_+(q(z)), \qquad \text{as } \varepsilon \to 0 \tag{5.23}$$

in $\mathcal{Y}$, i.e. for any $N \in \mathbb{N}$ there exists $M_N > 0$ such that

$$\|v_s(z, \varepsilon) - V_N(q(z), \varepsilon)\|_{\mathcal{Y}} \leqslant \varepsilon^{2N+2}M_N, \tag{5.24}$$

where $V_N(q, \varepsilon) = \sum_{m=0}^{N} \varepsilon^{2m}(v_m)_+(q)$.

The proof of the theorem can be found in appendix B. This theorem proves asymptotic expansion for the inner equation (1.1).

Remark 5.15. Here we state two simple conditions, each of them equivalent to the existence of a homoclinic solution to (1.1). Let $v_\pm(z, \varepsilon)$ be stable and unstable solutions (not necessarily normalized) to the inner equation (1.1) that are analytic (in z) in the regions $R_+ = R_{z_0}$ and $R_- = \mathrm{i}\pi/\varepsilon - R_{z_0}$, respectively (see remark 5.7). By theorem 5.2, we have $\lim_{\varepsilon\to 0} v_\pm(z, \varepsilon) = v_\pm(z)$, where $v_\pm(z)$ is defined by (3.9). Solutions $v_\pm(z, \varepsilon)$ coincide and form a homoclinic solution if and only if for any $a, b, z_0 < a < b < \pi/\varepsilon$, $v_+(z, \varepsilon) \equiv v_-(z, \varepsilon)$ on $[ia, ib]$ (see figure 1). The latter condition can be rewritten as

$$F_0(\gamma, \varepsilon) = \int_{ia}^{ib} |v_+(z, \varepsilon) - v_-(z, \varepsilon)|\, \mathrm{d}z = 0. \tag{5.25}$$

An equivalent condition is given by the system

$$v_s'\left(\frac{\mathrm{i}\pi}{\varepsilon}, \varepsilon\right) = 0, \qquad v_s'''\left(\frac{\mathrm{i}\pi}{\varepsilon}, \varepsilon\right) = 0, \tag{5.26}$$

where $v'(z, \varepsilon) = D_z v(z, \varepsilon)$. Indeed, if both $v'(z_1) = v'''(z_1) = 0$ at some point z_1, then all odd derivatives of $v(z)$ are zero at $z = z_1$, so that $v(z)$ is symmetric with respect to $z = z_1$. The first condition (5.25) is used in theorem 5.16. The second condition (5.26) is used in numerical computations in section 6.

Theorem 5.16. *If $s(\gamma_0) = 0$ for some $\gamma_0 \in \mathbb{R}$ but $s'(\gamma_0) \neq 0$ then there exists a unique smooth curve of homoclinic solutions $\gamma(\varepsilon)$ in the parameter space (γ, ε) with $\gamma(0) = \gamma_0$, such that the inner equation (1.1) has a homoclinic solution in a neighbourhood of the point $(\gamma_0, 0)$ if and only if the parameters of (1.1) are on the curve $\gamma(\varepsilon)$.*

Proof. Without any loss of generality we can assume $v_+ = v_s$, $v_- = v_u$ in (5.25), where v_u is the unstable solution satisfying $D_z v_u(\mathrm{i}\pi/\varepsilon, \varepsilon) = 0$. Since $v_u(\mathrm{i}\pi/2 - \xi) = v_s(\mathrm{i}\pi/2 + \xi)$,

$\xi \in \mathbb{C}$, hence the difference $v_+ - v_-$ is purely imaginary on $[\mathrm{i}z_0, \mathrm{i}\pi/2]$. By theorem 5.14 and the representation (3.12) we know that

$$v_+(z,\varepsilon) - v_-(z,\varepsilon) \neq 0 \qquad \text{when } z \in [ia, ib] \tag{5.27}$$

if $s(\gamma) \neq 0$ and if ε is sufficiently small. In the case $s(\gamma_0) = 0$ condition (5.27) still holds if we additionally require $s_1(\gamma_0) \neq 0$. Then condition $F_0(\gamma, \varepsilon) = 0$ is equivalent to

$$F(\gamma,\varepsilon) = \int_{ia}^{ib} [v_+(z,\varepsilon) - v_-(z,\varepsilon)]\,\mathrm{d}z = 0. \tag{5.28}$$

The proof of the theorem is based on the implicit function theorem applied to (5.28), with the special case $s_1(\gamma_0) = 0$ considered later. According to remark B.5 and theorem 5.14, the solutions $v_\pm$ have continuous derivatives in γ and in ε^2 for $\gamma \in E$, $\varepsilon \in [0, \varepsilon_0]$. Then $F(\gamma, \varepsilon)$ also has continuous derivatives in γ and in ε^2 in the same region. To apply the implicit function theorem, it is sufficient to show that

$$F(\gamma_0, 0) = k_1 s(\gamma_0) \qquad \text{and} \qquad \frac{\partial}{\partial\gamma} F(\gamma,\varepsilon)|_{(\gamma_0,0)} = k_2 s'(\gamma_0), \tag{5.29}$$

where the real coefficients $k_{1,2} = k_{1,2}(a,b) \neq 0$. Indeed, using (3.12) for $z \in [ia, ib]$, we obtain

$$F(\gamma_0, 0) = 2\pi(\mathrm{e}^{-a} - \mathrm{e}^{-b}) s(\gamma_0)(1 + o(1)), \qquad \text{as } a \to \infty. \tag{5.30}$$

It is clear that by choosing some sufficiently large a, b we can guarantee that $k_1 = 2\pi(\mathrm{e}^{-a} - \mathrm{e}^{-b})(1 + o(1)) \neq 0$.

Since $v(z, \varepsilon)$ is continuously differentiable with respect to γ, we obtain

$$\frac{\partial}{\partial\gamma} F(\gamma,\varepsilon) = \int_{ia}^{ib} \left[\frac{\partial v_+}{\partial\gamma}(z,\varepsilon) - \frac{\partial v_-}{\partial\gamma}(z,\varepsilon)\right] \mathrm{d}z. \tag{5.31}$$

Repeating the previous arguments for $\partial v/\partial\gamma$ instead of v, we obtain $\partial/\partial\gamma F(\gamma_0, 0) = k_2 s_{(\gamma)}(\gamma_0)$, where $k_2 = 2\pi(\mathrm{e}^{-a} - \mathrm{e}^{-b})(1 + o(1)) > 0$ and $s_{(\gamma)}(\gamma)$ is the Stokes constant for the solution $v_\gamma(z)$. Thus, it remains to show that $s'(\gamma) = s_{(\gamma)}(\gamma)$ to complete the proof. To this end we convert (1.6) into the integral equation described in theorem 1.3, [T94c]. The desired result follows from differentiating both sides of this equation with respect to γ.

To consider the remaining case $s_1(\gamma_0) = 0$, we start with the linear equation

$$L\frac{\partial v}{\partial\gamma} = (D_z^2 + 1)(D_z^2 - \varepsilon^2)\frac{\partial v}{\partial\gamma} - 2v\frac{\partial v}{\partial\gamma} - 2\gamma\left[\frac{\partial v}{\partial\gamma} D_z^2 v + v D_z^2 \frac{\partial v}{\partial\gamma} + D_z v D_z \frac{\partial v}{\partial\gamma}\right] = \Phi(v) \tag{5.32}$$

for $\partial v/\partial\gamma$, where the free term $\Phi(v)$ does not depend on $\partial v/\partial\gamma$. Note that $\partial v/\partial\varepsilon^2$ satisfies the same nonhomogeneous problem (5.32) but with a different free term, i.e.

$$L\frac{\partial v}{\partial\varepsilon^2} = \Psi(v), \tag{5.33}$$

where $\Psi(v)$, as well as $\Phi(v)$, can be calculated explicitly. The Stokes constant of $\partial v/\partial\gamma(z, 0)$ is $s'(\gamma_0)$ whereas the Stokes constant of $\partial v/\partial\varepsilon^2(z, 0) = v_1(z)$ is $s_1(\gamma_0)$. Let us change the parameter $\gamma = \mu + \varepsilon^2$ in the inner equation (1.1), where μ is a new independent parameter. This change will not affect the truncated equation (1.6), so the Stokes constant of $v_0(z)$ remains the same. However, the higher order equations in (5.19) will be altered. In particular, equation for $\partial v/\partial\varepsilon^2(z, 0) = v_1(z)$ will become

$$L\frac{\partial v}{\partial \varepsilon^2} = \Psi(v) + \Phi(v)\ , \tag{5.34}$$

where $\varepsilon = 0$ and $\mu = \gamma_0$. Then the Stokes constant of the solution $\partial v/\partial\varepsilon^2(z, 0)$ will be $s'(\gamma_0) + s_1(\gamma_0) \neq 0$. Thus, even if $s_1(\gamma_0) = 0$ but $s'(\gamma_0) \neq 0$, the condition $F(\mu, \varepsilon) = 0$ is still equivalent to (5.25). Since $\partial v/\partial\mu(z, 0) = \partial v/\partial\gamma(z, 0)$, we can continue as above to prove the existence of a homoclinic solution curve $\mu(\varepsilon)$, such that $\mu(0) = \gamma_0$. Then $\gamma(\varepsilon) = \mu(\varepsilon) + \varepsilon^2$. □

6. Numerical computations of homoclinic orbits

We report here numerical results on computations of the Stokes constant and homoclinic solutions of the inner equation (1.1). Our numerical results illustrate the analytical theory developed in the paper. We focus on computations of the first five one-parameter families of homoclinic orbits on the parameter plane (γ, ϵ).

The Stokes constant $s(\gamma)$ is computed from the limit of the sequence $\{\beta_k\}_{k=1}^{\infty}$ which converges for any $\gamma \in \mathbb{C}$ according to theorem 3.10. We restricted our attention to the case $\gamma \in \mathbb{R}$. Figure 3(*a*) shows the coefficients β_{60}, β_{70} and β_{80} in the sequence versus parameter γ. Absolute and relative errors between two subsequent elements in the sequence $|\beta_{k+1}(\gamma) - \beta_k(\gamma)|$ are shown in figures 4(*a*) and (*b*) for $k = 60, 70, 80$. It follows from figure 4(*a*) that the truncation error of the numerical approximation is of the order of 10^{-4}. Since the slopes $s'(\gamma)$ are order of $O(1)$ near zeros of $s(\gamma)$, the error of approximating zeros of $s(\gamma)$ by zeros of $\beta_{80}(\gamma)$ is also of the order of 10^{-4}. Numerically, the first five zeros of $\beta_{80}(\gamma)$ have been computed as follows:

$$\gamma_1 = 1.250, \quad \gamma_2 \approx 2.401, \quad \gamma_3 \approx 3.931, \quad \gamma_4 \approx 5.852, \quad \gamma_5 \approx 8.172, \tag{6.1}$$

where the first zero is exact according to lemma 3.1.

By using the numerical approximations for a zero γ_n of the Stokes constant $s(\gamma)$, we can approximate the homoclinic orbit $v(z)$ in the inner equation (1.1). Let us consider the Nth partial sum $\hat{y}_N(x)$ of the formal power series $\hat{y}(x)$, defined in lemma 4.4. The partial sum $\hat{y}_1(x)$ can be computed explicitly as follows:

$$\hat{y}_1(x) = -\frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right) + \varepsilon^2\left[\frac{3(7-6\gamma)}{4}\mathrm{sech}^2\left(\frac{x}{2}\right) + \frac{9(4\gamma-5)}{8}\mathrm{sech}^4\left(\frac{x}{2}\right)\right]. \tag{6.2}$$

Within the partial sum truncation $\hat{y}_1(x)$, we can replace the value of γ by its limiting value $\gamma_n = \lim_{\varepsilon\to 0}\gamma(\varepsilon)$. We also rewrite the approximation $\hat{y}_1(x)$ in the variables (1.5) with $c = 0$ as $\hat{v}_1(z)$. The approximation $\hat{v}_1(z)$ is shown in figure 3(*b*) for the first five zeros (6.1) and $\epsilon = 0.2$. It is clear from figure 3(*b*) that the first five homoclinic orbits $v(z)$ remain single-humped in $z \in \mathbb{R}$. The width and amplitude of the homoclinic orbit increase with larger values of γ.

Homoclinic orbits $v(z)$ have been approximated in a numerical solution of the differential equation (1.1). We solve the differential equation (1.1) by the Runge–Kutta method, starting with the initial value:

$$(v, v', v'', v''')(L) = c_\infty(1, \epsilon, \epsilon^2, \epsilon^3)\mathrm{e}^{\epsilon L}, \tag{6.3}$$

where L is a large negative number and c_∞ is a small negative number, e.g. $L = -50$ and $c_\infty = -0.001$. The solution of the initial-value problem (6.3) would correspond to the unstable

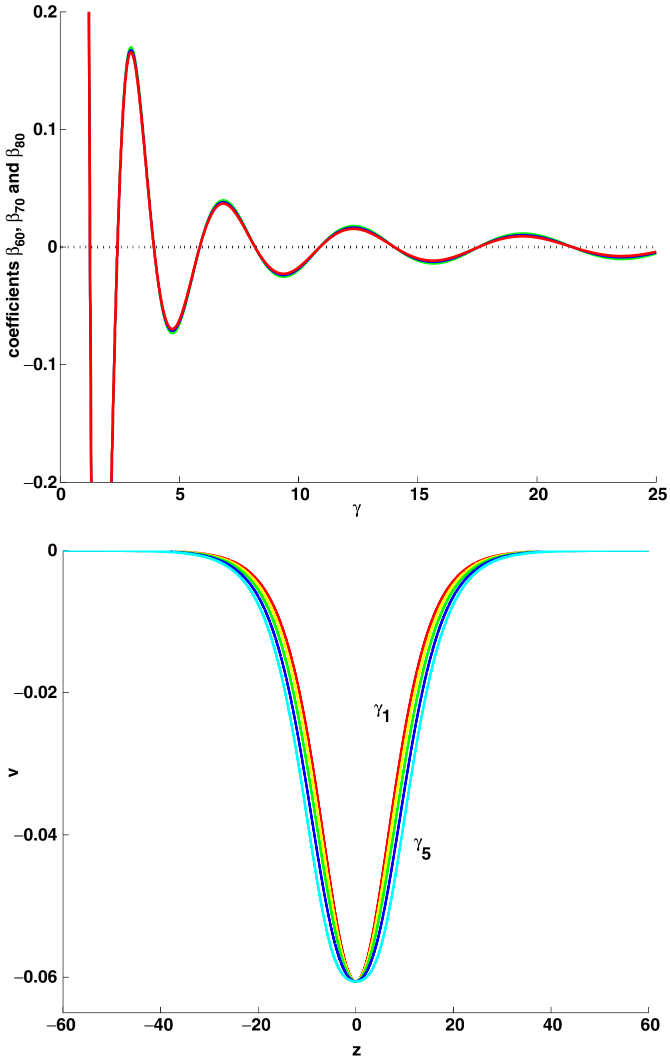

Figure 3. (*a*) The numerical coefficients β_{60}, β_{70}, and β_{80} versus γ from the recurrence equation (3.15). (*b*) Homoclinic orbits obtained from the truncated solution (6.2) for $\epsilon = 0.2$.

solution $v_u(z)$. We continue the numerical solution for $z > L$ until the first minimum point occurs at $z = z_0$ such that $v'(z_0) = 0$. The value of $v'''(z_0)$ is plotted in figure 5(*a*) versus γ for $\epsilon = 0.2$. Zeros of $v'''(z_0)$ in γ give approximations of the values $\gamma_n(\epsilon)$ for location of the homoclinic orbits. The homoclinic orbit solutions are shown in figure 5(*b*) after the shift $v(z - z_0)$, which map all minima points to the origin. The first five homoclinic orbits are single-humped in $z \in \mathbb{R}$ for $\epsilon = 0.2$. Furthermore, the width and amplitude of the homoclinic orbits increase with larger values of γ. Figure 5(*c*) shows the distance between the numerical approximation for the first zero $\gamma_1(\epsilon)$ and the exact solution (1.7) for $\epsilon = 0.2$. The absolute error of the numerical approximation is of the order of 10^{-6}.

Figure 6 shows the five one-parameter curves $\gamma(\epsilon)$ which correspond to the first homoclinic orbits $v(z)$. These curves are obtained from numerical zeros of $v'''(z_0)$ in γ for different values

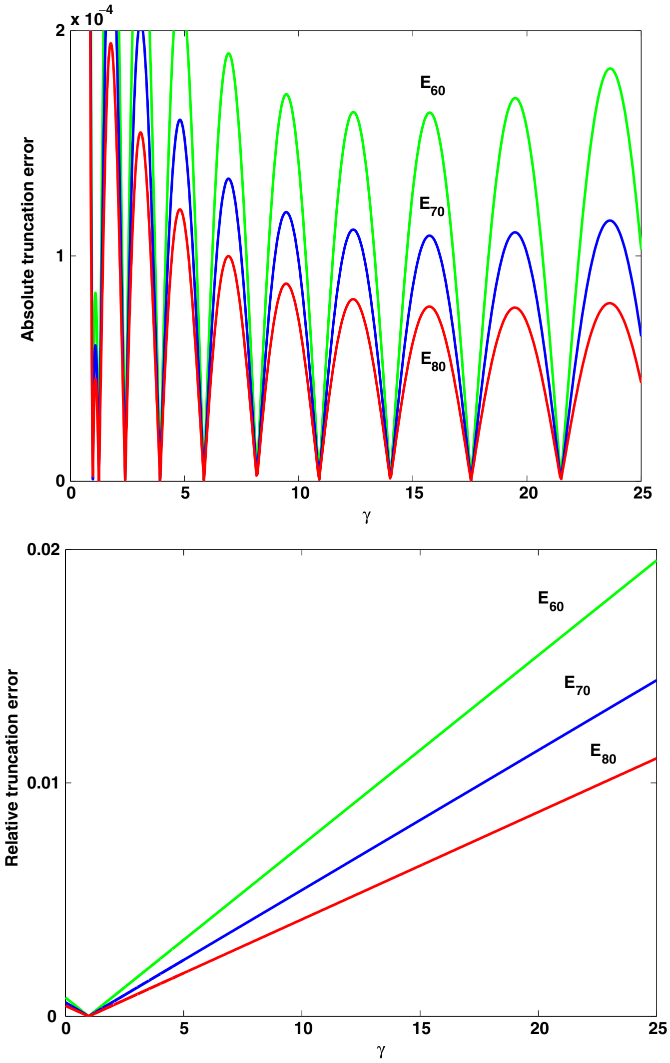

Figure 4. (*a*) Absolute numerical error $E_k(\gamma) = |\beta_{k+1}(\gamma) - \beta_k(\gamma)|$ for $k = 60, 70, 80$. (*b*) Relative numerical error $E_k(\gamma) = |\beta_{k+1}(\gamma) - \beta_k(\gamma)|/|\beta_k(\gamma)|$ for $k = 60, 70, 80$.

of ε (see figure 5(*a*)). The black dots in figure 6 show locations of zeros of the Stokes constant $s(\gamma)$ in (6.1) (see figure 3(*a*))). Dotted curves show results of the first-order approximation of the curve $\gamma(\varepsilon)$:

$$\gamma(\varepsilon) = \gamma_n + \varepsilon^2 \gamma_n^{(1)} + \mathrm{O}(\varepsilon^4), \qquad \gamma_n^{(1)} = -\frac{s_1(\gamma_n)}{s'(\gamma_n)}, \tag{6.4}$$

where $s_1(\gamma)$ is the Stokes constant for the formal power series of the first-order correction:

$$\hat{v}_1(z) = -\frac{1}{2} + \sum_{k=1}^{\infty} \frac{\alpha_k^{(1)}}{z^{2k}} = -\frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-1)!\beta_k^{(1)}}{z^{2k}}, \tag{6.5}$$

such that $s_1(\gamma) = \frac{1}{2}\lim_{k\to\infty} \beta_k^{(1)}(\gamma)$. The Stokes constant $s_1(\gamma)$, defined by (5.21), exists

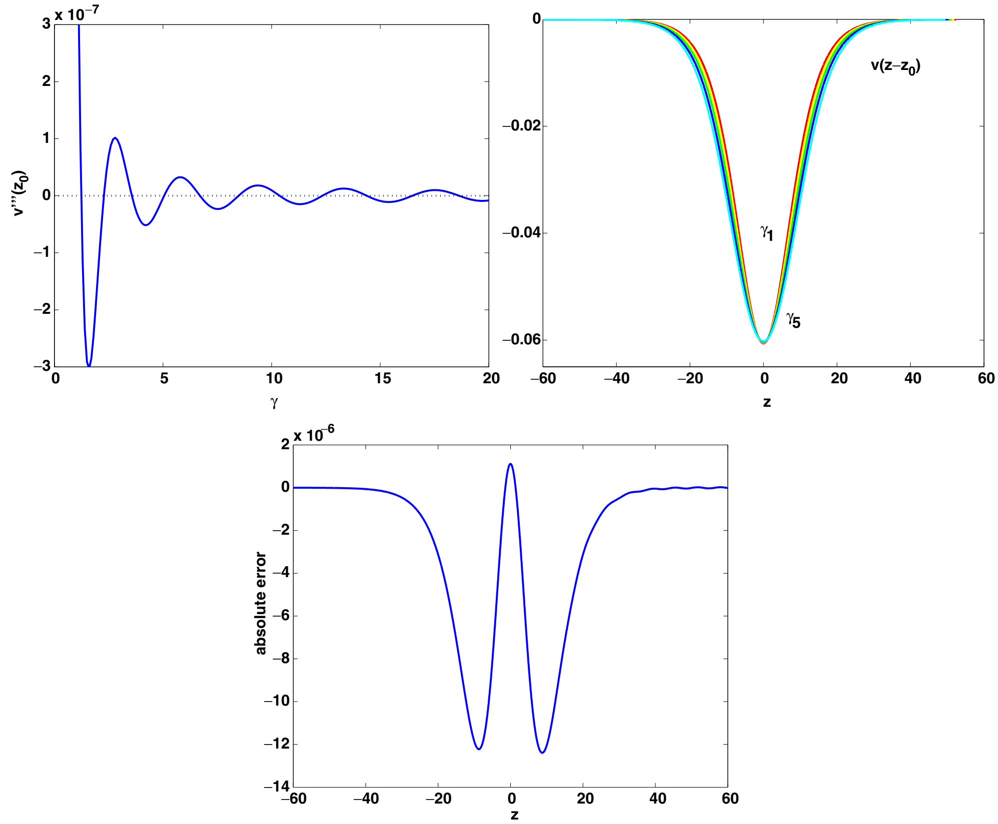

Figure 5. (*a*) Dependence of $v'''(z_0)$ versus γ for $\epsilon = 0.2$ from a numerical solution of the inner equation (1.1) with initial values (6.3), where z_0 is the first minimum of $v(z)$. (*b*) Homoclinic orbits obtained from the numerical solution $v(z)$ after the shift $v(z - z_0)$ for the first five zeros of $v'''(z_0)$ in γ. (*c*) Error between the numerical solution for the first zero of $v'''(z_0)$ and the solution (1.7).

in accordance with lemma 5.12 and remark 5.13. We note that the formal series (6.5) for $\hat{v}_1(z)$ is obtained from the formal series (4.41)–(4.42), where $q(z)$ is expanded in powers of z in (4.40). The graph for elements of the first-order corrections $\beta_k^{(1)}(\gamma)$ looks similar to figure 3(*a*), while the numerical values of $\gamma_n^{(1)}$ for the first five zeros have been computed as follows:

$$\gamma_1^{(1)} \approx -0.316, \quad \gamma_2^{(1)} \approx -3.374, \quad \gamma_3^{(1)} \approx -11.542, \quad \gamma_4^{(1)} \approx -28.846, \quad \gamma_5^{(1)} \approx -58.626.$$

The exact value for $\gamma_1^{(1)}$ can be computed from the analytical solution (1.7) as $\gamma_1^{(1)} = -\frac{5}{16} = -0.3125$. It is clearly seen from figure 6 that the numerical curves $\gamma(\varepsilon)$ approach the asymptotic approximation (6.4) as $\varepsilon \to 0$. Nevertheless, numerical continuation of the homoclinic orbits were stopped at $\varepsilon \approx 0.15$ as the numerical computations of the ODE solve become unreliable for small ε. This numerical problem is similar to the one observed in [C01], where numerical curves have been terminated before they reach the limit $\varepsilon = 0$.

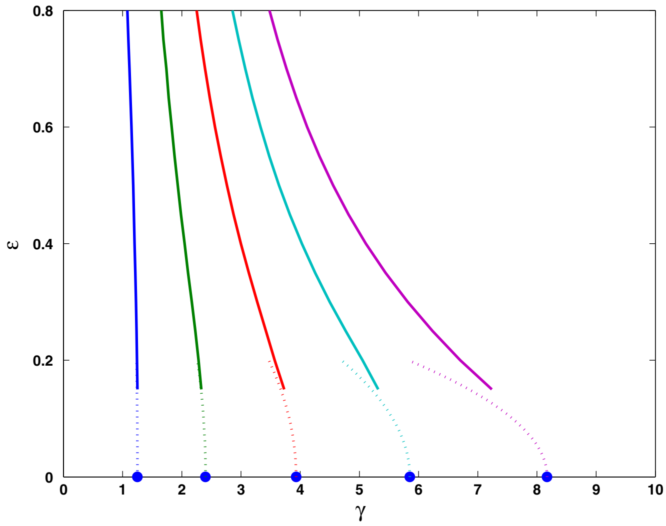

Figure 6. First five families of homoclinic orbits on the plane (γ, ϵ). Black dots show location of zeros of the Stokes constant $s(\gamma)$ computed from data of figure 3(a). Dotted curves show asymptotic approximations (6.4).

Acknowledgments

The authors thank A Champneys and the anonymous referees for a number of useful remarks. The work of AT is supported by the NSF grants DMS 0207201 and DMS 0508779. The work of DP is supported by the NSERC and PREA grants.

Appendix A. Proof of theorem 4.9

Let $\hat{y}_N(x, \varepsilon)$ denote the first $(N+1)$ terms in the formal series (4.28), where $N \in \mathbb{N}$. Linearizing (1.4) around $\hat{y}_N(x, \varepsilon)$ defined by lemma 4.4,

$$y(x, \varepsilon) = \hat{y}_N(x, \varepsilon) + \tilde{y}(x, \varepsilon) \equiv y_N(x, \varepsilon), \tag{A.1}$$

we find the equation for $\tilde{y}(x)$ in the form

$$(\varepsilon^2 D_x^2 + 1)(D_x^2 - 1)\tilde{y} = \tilde{y}^2 + \varepsilon^2\gamma(2\tilde{y}D_x^2\tilde{y} + (D_x\tilde{y})^2) + 2\hat{y}_N\tilde{y} \\ +2\varepsilon^2\gamma(\hat{y}_N D_x^2\tilde{y} + D_x\hat{y}_N D_x\tilde{y} + \tilde{y}D_x^2\hat{y}_N) - \varepsilon^{2(N+1)}R_N, \tag{A.2}$$

where $D_x = \mathrm{d}/\mathrm{d}x$ and the remainder term

$$\varepsilon^{2(N+1)}R_N = (\varepsilon^2 D_x^2 + 1)(D_x^2 - 1)\hat{y}_N - \hat{y}_N^2 - \varepsilon^2\gamma(2\hat{y}_N D_x^2\hat{y}_N + (D_x\hat{y}_N)^2) \tag{A.3}$$

is a polynomial in r. Since the leading term of $\hat{y}_N$ is $-\frac{3}{2}r$, we considered the nonlinear equation (A.2) as a perturbed linear equation

$$(\varepsilon^2 D_x^2 + 1)(D_x^2 - 1)\tilde{y} + 3r\tilde{y} = \tilde{y}^2 + f(\hat{y}_N, \tilde{y}), \tag{A.4}$$

where

$$f(\hat{y}_N, \tilde{y}) = \varepsilon^2\gamma(2\tilde{y}D_x^2\tilde{y} + (D_x\tilde{y})^2) + 2\left(\hat{y}_N + \frac{3}{2}r\right)\tilde{y} \\ +\ 2\varepsilon^2\gamma(\hat{y}_N D_x^2\tilde{y} + D_x\hat{y}_N D_x\tilde{y} + \tilde{y}D_x^2\hat{y}_N) - \varepsilon^{2(N+1)}R_N. \tag{A.5}$$

Note that $f(\hat{y}_N, 0) = -\varepsilon^{2(N+1)} R_N$. Introducing $\tilde{Y} = \mathrm{Col}(\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, \tilde{Y}_4)$, where $\tilde{Y}_j = D_x^{j-1}\tilde{y}$, $j = 1, 2, 3, 4$, we rewrite the perturbed equation (A.4) in the vector form as

$$\tilde{Y}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{1-3r}{\varepsilon^2} & 0 & \dfrac{\varepsilon^2 - 1}{\varepsilon^2} & 0 \end{pmatrix} \tilde{Y} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{\varepsilon^2} \end{pmatrix} [\tilde{Y}_1^2 + f(\hat{y}_N, \tilde{Y})]. \tag{A.6}$$

The notation $f(\hat{y}_N, \tilde{Y})$ is used to emphasize that f depends on the first three components of the vector $\tilde{Y}$. In order to block-diagonalize the matrix in the linear part of the vector equation (A.6), we represent this 4-by-4 matrix by direct products of 2-by-2 matrices:

$$I \otimes H_+ + S \otimes H_-, \tag{A.7}$$

where

$$H_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ \dfrac{(1-3r)}{\varepsilon^2} & \dfrac{\varepsilon^2 - 1}{\varepsilon^2} \end{pmatrix}. \tag{A.8}$$

The matrix S has two eigenvalues

$$\lambda_{1,2} = \frac{\varepsilon^2 - 1 \pm \sqrt{(1+\varepsilon^2)^2 - 12\varepsilon^2 r}}{2\varepsilon^2} \tag{A.9}$$

with the corresponding eigenvectors $\mathrm{Col}(1, \lambda_1)$ and $\mathrm{Col}(-1, -\lambda_2)$. Using a similarity transformation $\tilde{Y} = TW$, where

$$T = \begin{pmatrix} 1 & -1 \\ \lambda_1 & -\lambda_2 \end{pmatrix} \otimes I, \tag{A.10}$$

we transform the vector equation (A.6) to the block-diagonal form:

$$W' = \frac{1}{\varepsilon} \begin{pmatrix} 0 & 1 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \lambda_2 & 0 \end{pmatrix} W - \frac{\theta'}{2\theta} \begin{pmatrix} I & I \\ I & I \end{pmatrix} W + \frac{1}{\theta} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} [(W_1 - W_3)^2 + f(\hat{y}_N, W)], \tag{A.11}$$

where

$$\theta = \sqrt{(1+\varepsilon^2)^2 - 12\varepsilon^2 r}. \tag{A.12}$$

Equation (A.11) will be considered as a perturbation of the trivial solution to the homogeneous linear equation

$$\tilde{W}' = \frac{1}{\varepsilon} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1-3r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\varepsilon^{-2} & 0 \end{pmatrix} \tilde{W}, \tag{A.13}$$

which has the fundamental solution $\tilde{W} = J^{-1}V$, where $J = \mathrm{diag}(1, 1, 1, \varepsilon)$,

$$V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & 0 & 0 \\ v_1' & v_2' & 0 & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}x/\varepsilon} & \mathrm{e}^{-\mathrm{i}x/\varepsilon} \\ 0 & 0 & \mathrm{i}\mathrm{e}^{\mathrm{i}x/\varepsilon} & -\mathrm{i}\mathrm{e}^{-\mathrm{i}x/\varepsilon} \end{pmatrix}, \tag{A.14}$$

and $v_{1,2}$ are linearly independent solutions of the scalar equation $v'' = (1 - 3r)v$ and $v_2(x) = v_1(x)\int_0^x (\mathrm{d}t/v_1^2(t))$. To study stable solutions, we convert the vector equation (A.11) into its integral form

$$W(x,\varepsilon) = \tilde{W}(x)\int_\infty^x \tilde{W}^{-1}(t)\left[\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} - \frac{\theta'}{2\theta}\begin{pmatrix} I & I \\ I & I \end{pmatrix}\right] W(t,\varepsilon)\,\mathrm{d}t$$

$$+\frac{1}{\theta}\tilde{W}(x)\int_\infty^x \tilde{W}^{-1}(t)\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}[(W_1(t,\varepsilon) - W_3(t,\varepsilon))^2 + f(\hat{y}_N, W)]\,\mathrm{d}t, \tag{A.15}$$

where

$$A_j = \begin{pmatrix} 0 & 0 \\ \rho_j & 0 \end{pmatrix}, \qquad j = 1,2 \tag{A.16}$$

and

$$\rho_1 = \lambda_1 - (1 - 3r) = O(\varepsilon^2), \qquad \rho_2 = \lambda_2 + \frac{1}{\varepsilon^2} = O(1). \tag{A.17}$$

The latter estimates hold uniformly with respect to $x \in \mathbb{R}$. The vector equation (A.15) can be written in the operator form as

$$W(x,\varepsilon) = \mathcal{I}\mathcal{R}[W](x,\varepsilon), \tag{A.18}$$

where $\mathcal{I}$ is the solution operator for a nonhomogeneous equation with the homogeneous part (A.13) and $\mathcal{R}[W]$ is the operator in the integrand of (A.15). We consider the solution to the integral equation (A.18) by iterations:

$$W^{(k)}(x,\varepsilon) = \mathcal{I}\mathcal{B}[W^{(k-1)}](x,\varepsilon), \qquad k = 1,2,\ldots, \tag{A.19}$$

where $W^{(0)}(x,\varepsilon) \equiv 0$, $\Delta W^{(k)} = W^{(k)} - W^{(k-1)}$, and

$$W(x,\varepsilon) = \sum_{k=1}^\infty \Delta W^{(k)}(x,\varepsilon), \tag{A.20}$$

provided that the series converges.

Theorem A.1. *For any $N \in \mathbb{N}$, any $A > 0$ and any bounded segment $E \subset \mathbb{R}$, there exists some ε_0 such that the series (A.20) converges in $\mathcal{B}_2$ uniformly in $\varepsilon \in [0,\varepsilon_0]$ and in $\gamma \in E$. The function $W(x,\varepsilon)$ is a smooth function of (ε,γ) in $\varepsilon \in [0,\varepsilon_0]$ and $\gamma \in E$. Moreover, $\|W(x,\varepsilon)\|_{\mathcal{B}_2} = O(\varepsilon^{2(N+1)})$.*

Proof. The proof consists of three main steps: (i) estimates for linear operator $\mathcal{I}$; (ii) estimates for operator $\mathcal{R}$, and; (iii) convergence of iterations.

(i) According to the representation (A.14), we block-diagonalize $\mathcal{I} = \mathrm{diag}(\mathcal{I}_1, \mathcal{I}_2)$. Since the Wronskian of $v_1(x)$ and $v_2(x)$ is identically one, it is easy to see that for a given vector-function $G(x) = \mathrm{Col}(g_1(x), g_2(x))$ we have

$$\mathcal{I}_1[G] = \begin{pmatrix} [-v_1\int v_2 g_2\,\mathrm{d}t + v_2\int v_1 g_2\,\mathrm{d}t] + [v_1\int v_2' g_1\,\mathrm{d}t - v_2\int v_1' g_1\,\mathrm{d}t] \\ [-v_1'\int v_2 g_2\,\mathrm{d}t + v_2'\int v_1 g_2\,\mathrm{d}t] + [v_1'\int v_2' g_1\,\mathrm{d}t - v_2'\int v_1' g_1\,\mathrm{d}t], \end{pmatrix} \tag{A.21}$$

where all integrals are taken over $[x, \infty)$. Using explicit expressions for $v_1(x)$ and $v_2(x)$, we can show that there exist some $L > 0$ (that depends on A) such that

$$\|v_1(x)\|_{\mathcal{B}_1}, \|v_1'(x)\|_{\mathcal{B}_1} < L, \qquad \|v_2(x)\|_{\mathcal{B}_{-1}}, \|v_2'(x)\|_{\mathcal{B}_{-1}} < L. \tag{A.22}$$

Using (A.21) and elementary calculations, it is easy to see that the $\mathcal{I}_1$ is a bounded operator in $\mathcal{B}_j$ when $j \geqslant 2$. We choose a constant K such that $\|\mathcal{I}_1\| \leqslant K$ in the Banach space $\mathcal{B}_2$. In order to consider the integral operator $\mathcal{I}_2$, we use the transformation $\check{\mathcal{I}}_2 = \mathrm{diag}(\varepsilon^{-1}, 1)\mathcal{I}_2\mathrm{diag}(\varepsilon, 1)$ and find the representation for $\check{\mathcal{I}}_2$:

$$\begin{aligned}
\check{\mathcal{I}}_2[G] &= \frac{1}{2}\int_x^\infty \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \varepsilon^{-\mathrm{i}((t-x)/\varepsilon)} & 0 \\ 0 & \varepsilon^{\mathrm{i}((t-x)/\varepsilon)} \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} G(t)\,\mathrm{d}t \\
&= \int_x^\infty \begin{pmatrix} \cos\dfrac{t-x}{\varepsilon} & -\sin\dfrac{t-x}{\varepsilon} \\ \sin\dfrac{t-x}{\varepsilon} & \cos\dfrac{t-x}{\varepsilon} \end{pmatrix} \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \mathrm{d}t \\
&= \begin{pmatrix} \Re \\ \Im \end{pmatrix} \int_x^\infty \varepsilon^{\mathrm{i}((t-x/\varepsilon)}[g_1(t) + \mathrm{i}g_2(t)]\,\mathrm{d}t,
\end{aligned} \tag{A.23}$$

where $G(x) = \mathrm{Col}(g_1(x), g_2(x)) \in \mathcal{B}_2$. It is clear that $\check{\mathcal{I}}_2$ is a bounded operator in $\mathcal{B}_j$ for $j \geqslant 1$. We assume that the constant K, chosen above, is such that it is greater than the norm of $\check{\mathcal{I}}_2$ in $\mathcal{B}_2$. For any four-dimensional vector W let $\check{W} = \mathrm{diag}(1, 1, \varepsilon^{-1}, 1)W$. Let us choose $L > 0$ in (A.22) such that

$$\left\| \frac{R_N}{\theta} \right\|_{\mathcal{B}_2} \leqslant \tfrac{1}{2}L, \tag{A.24}$$

where θ is given by (A.12). Then, according to (A.19), (A.20)

$$\|\check{W}^{(1)}\|_{\mathcal{B}_2} \leqslant \tfrac{1}{2}LK\varepsilon^{2(N+1)}. \tag{A.25}$$

(ii) We use (A.9), (A.10) to calculate

$$\tilde{y} = W_1 - W_3, \qquad \tilde{y}' = W_2 - W_4, \qquad \tilde{y}'' = \lambda_1 W_1 - \lambda_2 W_3. \tag{A.26}$$

Then, according to (A.15), (A.2), the operator $\mathcal{R} = \mathcal{R}_D + \mathcal{R}_I + \mathcal{R}_N + \mathcal{R}_R$, where

$$\mathcal{R}_D W = \begin{pmatrix} A_1 & 0 \\ 0 & \varepsilon A_2 \end{pmatrix} \mathrm{diag}(1, 1, \varepsilon^{-1}, 1)W,$$

$$\mathcal{R}_I W = -\frac{\theta'}{2\theta}\begin{pmatrix} I & I \\ I & I \end{pmatrix} W,$$

$$\mathcal{R}_N W = \boldsymbol{e}_0[(W_1 - W_3)^2 + \varepsilon^2\gamma(2(W_1 - W_3)(\lambda_1 W_1 - \lambda_2 W_3) + (W_2 - W_4)^2)],$$

$$\mathcal{R}_R W = \boldsymbol{e}_0[(2\hat{y}_N + 3r + 2\varepsilon^2\gamma\hat{y}_N'')(W_1 - W_3) + 2\varepsilon^2\gamma(\hat{y}_N'(W_2 - W_4) + \hat{y}_N(\lambda_1 W_1 - \lambda_2 W_3))],$$

where $\boldsymbol{e}_0 = (0, 1, 0, 1)^{\mathrm{T}}$. We rewrite the integral equation (A.18) as

$$\mathrm{diag}(1, 1, \varepsilon^{-1}, 1)W = \check{\mathcal{I}}\mathcal{R}_D W + \check{\mathcal{I}}\mathrm{diag}(1, 1, \varepsilon^{-1}, 1)\mathcal{R}_I W + \check{\mathcal{I}}\mathcal{R}_R W + \check{\mathcal{I}}\mathcal{R}_N W, \tag{A.27}$$

where $\check{\mathcal{I}} = \mathrm{diag}(\mathcal{I}_1, \check{\mathcal{I}}_2)$. To prove the convergence of (A.20), we need to estimate the norm of

$$\begin{aligned}
\Delta\check{W}^{(k)} &= \check{\mathcal{I}}\mathcal{R}_D\Delta W^{(k-1)} + \check{\mathcal{I}}\mathrm{diag}(1, 1, \varepsilon^{-1}, 1)\mathcal{R}_I\Delta W^{(k-1)} + \check{\mathcal{I}}\mathcal{R}_R\Delta W^{(k-1)} \\
&\quad + (\check{\mathcal{I}}\mathcal{R}_N W^{(k-1)} - \check{\mathcal{I}}\mathcal{R}_N W^{(k-2)}).
\end{aligned} \tag{A.28}$$

Lemma A.2. *There exists some $B > 0$ such that*

$$\{\|\mathcal{R}_D W\|_2, \|\mathrm{diag}(1, 1, \varepsilon^{-1}, 1)\mathcal{R}_I W\|_2, \|\mathcal{R}_R W\|_2\} \leqslant \varepsilon B\|\check{W}\|_2, \tag{A.29}$$

where $\check{W} \in \mathcal{B}_2$.

Proof. For operators $\mathcal{R}_D, \mathcal{R}_I$ the statement follows from (A.9), (A.16). The factors in front of $W_1 - W_3$ and $W_2 - W_4$ in the expression for $\mathcal{R}_R$ are of the order $O(\varepsilon^2)$. Replacing W_3 with $\varepsilon\check{W}_3$ we see that the last term in $\mathcal{R}_R$ is of the order $\varepsilon\|\check{W}\|_2$. The proof is completed. □

(iii) Our final goal is to prove that

$$\|\Delta\check{W}^{(n)}\|_{\mathcal{B}_2} \leqslant \frac{LK\varepsilon^{2(N+1)}}{2^n}, \qquad n = 1, 2, \ldots, \tag{A.30}$$

by induction. Suppose that this is true for all $n < k$ for some $k \in \mathbb{N}$. To prove (A.30) for $n = k$, we show that the norm of each term in (A.28) does not exceed $(LK\varepsilon^{2(N+1)})/4 \cdot 2^k$. According to lemma A.2, the condition

$$\varepsilon \leqslant \frac{1}{8BK} \tag{A.31}$$

guarantees the required estimate for the first three terms of (A.28). To estimate the remaining difference we observe that the square brackets expression in $\mathcal{R}_N$ can be rewritten as

$$[1 - \gamma(1 - \varepsilon^2)](W_1 - W_3)^2 + \theta(W_1^2 - W_3^2) + \varepsilon^2\gamma(W_2 - W_4)^2. \tag{A.32}$$

The difference

$$\begin{aligned}
&[W_1^{(k-1)} - W_3^{(k-1)}]^2 - [W_1^{(k-2)} - W_3^{(k-2)}]^2 \\
&\quad = [W_1^{(k-1)} + W_1^{(k-2)} - W_3^{(k-1)} - W_3^{(k-2)}](\Delta W_1^{(k-1)} - \Delta W_3^{(k-1)}).
\end{aligned} \tag{A.33}$$

According to our assumptions, the norm of this difference does not exceed $8((KL\varepsilon^{2N+2})^2/2^{k-1})$. Making similar estimates for other terms in (A.32), we establish that

$$\|\mathcal{R}_N W^{(k-1)} - \mathcal{R}_N W^{(k-2)}\|_{\mathcal{B}_2} \leqslant 24M\frac{(KL\varepsilon^{2N+2})^2}{2^{k-1}}, \tag{A.34}$$

where $M = \max\{1 - \gamma(1 - \varepsilon^2), \theta, \varepsilon^2\gamma\}$. Thus, the condition

$$\varepsilon \leqslant \min\left\{\frac{1}{8BK}, \frac{1}{(48MK^2L)^{1/(2N+2)}}\right\} \tag{A.35}$$

guarantees (A.30) for $n = k$. So, the series $\check{W} = \sum_{k=1}^{\infty}\check{W}^{(k)}$ is strongly convergent in $\mathcal{B}_2$ and $\|\check{W}\|_{\mathcal{B}_2} \leqslant LK\varepsilon^{2N+2}$. □

Appendix B. Existence of inner solutions

We start by constructing an iterative stable solution to the inner equation (1.1) that is a real translation of $v_s(z, \varepsilon)$ defined by (4.40). Our construction will have some common elements with the corresponding construction for the outer equation, described in appendix A. The results of [T00], where the case $\gamma = 0$ was studied, will be used.

Let us linearize the inner equation (1.1) by the substitution

$$v(z, \varepsilon) = u(z, \varepsilon) + 6c(\varepsilon)q^{-2}(z, \varepsilon), \tag{B.1}$$

where $c(\varepsilon)$ is a continuous function and $q(z,\varepsilon)$ is defined by (4.40). Then the inner equation (1.1) is reduced to the form

$$(D_z^2+1)(D_z^2-\varepsilon^2)u-12q^{-2}u=u^2-f(q,U),\tag{B.2}$$

where $U=\mathrm{Col}(u,u',u'',u''')$ and

$$\begin{aligned}f(q,U)&=720cq^{-6}-36c(1-c+4\varepsilon^2)q^{-4}-12(c-1)q^{-2}u-\gamma\\&\quad\times\Big(2uu''+u'^2+12c[(6q^{-4}+\varepsilon^2q^{-2})u-2\cosh(\varepsilon z/2)q^{-3}u'+q^{-2}u'']\\&\quad+36c^2[16q^{-6}+3\varepsilon^2q^{-4}]\Big).\end{aligned}$$

The vector form of equation (B.2) is

$$U'=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\\varepsilon^2+12q^{-2}&0&\varepsilon^2-1&0\end{pmatrix}U+\begin{pmatrix}0\\0\\0\\1\end{pmatrix}[u^2-f(q,U)].\tag{B.3}$$

The change of variables $U=TW$, where

$$T=\begin{pmatrix}1&-1\\\lambda_1&-\lambda_2\end{pmatrix}\otimes I,\tag{B.4}$$

reduces (B.3) to

$$W'=\begin{pmatrix}0&1&0&0\\\lambda_1&0&0&0\\0&0&0&1\\0&0&\lambda_2&0\end{pmatrix}W-\frac{\theta'}{2\theta}\begin{pmatrix}I&I\\I&I\end{pmatrix}W+\frac{1}{\theta}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}[(W_1-W_3)^2-f(q,TW)],\tag{B.5}$$

where

$$\lambda_{1,2}=\frac{\varepsilon^2-1\pm\sqrt{(1+\varepsilon^2)^2+48q^{-2}}}{2},\qquad\theta=\sqrt{(1+\varepsilon^2)^2+48q^{-2}}.\tag{B.6}$$

Vector equation (B.5) is a perturbation of the linear equation

$$V'=\begin{pmatrix}0&1&0&0\\\varepsilon^2+12q^{-2}&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}V,\tag{B.7}$$

which has a fundamental solution

$$V=\begin{pmatrix}V_1&0\\0&V_2\end{pmatrix}=\begin{pmatrix}v_1&v_2&0&0\\v_1'&v_2'&0&0\\0&0&\mathrm{e}^{\mathrm{i}z}&\mathrm{e}^{-\mathrm{i}z}\\0&0&\mathrm{i}\mathrm{e}^{\mathrm{i}z}&-\mathrm{i}\mathrm{e}^{-\mathrm{i}z}\end{pmatrix}.\tag{B.8}$$

Here $v_{1,2}$ are linearly independent solutions of $v''=(\varepsilon^2+12q^{-2})v$, such that $v_1(z)=6(q^{-2})'=-12q^3\cosh(\varepsilon z/2)$ and $v_2(z)=v_1(z)\int_0^z(\mathrm{d}t/v_1^2(t))$. The differential equation (B.5) can be converted into the integral equation

$$W(z,\varepsilon) = V(z)\int_{\infty}^{z} V^{-1}(t)\left[\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} - \frac{\theta'}{2\theta}\begin{pmatrix} I & I \\ I & I \end{pmatrix}\right] W(t,\varepsilon)\,\mathrm{d}t$$

$$+\frac{1}{\theta}V(z)\int_{\infty}^{z} V^{-1}(t)\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}[(W_1(t,\varepsilon) - W_3(t,\varepsilon))^2 - f(q,TW)]\,\mathrm{d}t, \qquad \text{(B.9)}$$

where the contour of integration is a horizontal ray $\Im t = \Im z$, $\Re t \geqslant \Re z$,

$$A_j = \begin{pmatrix} 0 & 0 \\ \rho_j & 0 \end{pmatrix}, \qquad j = 1, 2, \qquad \text{(B.10)}$$

and

$$\rho_1 = \frac{\theta - (1-\varepsilon^2)}{2} - (\varepsilon^2 + 12q^{-2}), \qquad \rho_2 = -\frac{\theta - (1+\varepsilon^2)}{2}. \qquad \text{(B.11)}$$

The vector integral equation (B.9) can be written in the operator form as

$$W(z,\varepsilon) = \mathcal{I}\mathcal{R}[W](z,\varepsilon), \qquad \text{(B.12)}$$

where

$$\mathcal{I}[F] = V(z)\int_{\gamma(z)} V^{-1}(t)F(t)\,\mathrm{d}t \qquad \text{(B.13)}$$

acts on four-dimensional vector-valued functions that are analytic in R_{z_0} and $\mathcal{R}[W]$ is the operator in the integrand of (B.9). The block-diagonal structure of $V(z)$ induces the corresponding block-diagonal structure of $\mathcal{I}[F] = \mathrm{diag}(\mathcal{I}_1[F], \mathcal{I}_2[F])$, where $\mathcal{I}_1[F]$ and $\mathcal{I}_2[F]$ act on the first (last) two components of F respectively.

Let

$$W^{(0)}(z,\varepsilon) \equiv 0, \qquad W^{(k)}(z,\varepsilon) = \mathcal{I}\mathcal{B}[W^{(k-1)}](z,\varepsilon), \qquad \text{(B.14)}$$

where $k = 1, 2, \ldots$, and $\Delta W^{(k)} = W^{(k)} - W^{(k-1)}$. Then

$$W(z,\varepsilon) = \sum_{k=1}^{\infty} \Delta W^{(k)}(z,\varepsilon) \qquad \text{(B.15)}$$

is a solution to (B.12) provided that the series converges. We need the following statements to prove convergence of iterations for the integral equation (B.9).

Lemma B.1. [T00] *The inequality*

$$\frac{\varepsilon}{2}|\mathrm{e}^{-\varepsilon z/2}| \leqslant |q^{-1}(z)| \leqslant \frac{2}{z_0}|\mathrm{e}^{-\varepsilon z/2}| \qquad \text{(B.16)}$$

holds for all $z \in R_{z_0}$ and all $\varepsilon \geqslant 0$ provided $\varepsilon z_0 \leqslant \ln 2$.

Remark B.2. Note that $q(z) = -(2/\varepsilon)\cosh(\varepsilon \Re z/2)$ along the upper boundary $\Im z = \pi/\varepsilon$ of the region R_{z_0}. Therefore, on this boundary the inequality (B.16) turns into

$$\frac{\varepsilon}{2}|\mathrm{e}^{-\varepsilon z/2}| \leqslant |q^{-1}(z)| \leqslant \varepsilon|\mathrm{e}^{-\varepsilon z/2}|. \qquad \text{(B.17)}$$

Based on lemma B.1, the constant ε_0 is chosen as $\varepsilon_0 = \ln 2/z_0$. This implies that $\mathrm{i}z_0 \in R_{z_0}$ for any $\varepsilon \in [0, \varepsilon_0]$.

Lemma B.2. *The inequality*

$$|\rho_1| \leqslant 36|q^{-2}||\varepsilon^2 + 12q^{-2}|, \qquad |\rho_2| \leqslant 24|q^{-2}| \tag{B.18}$$

holds for all $z \in R_{z_0}$ and $\varepsilon \in [0, \varepsilon_0]$ provided $z_0 \geqslant 14$.

Proof. According to (B.11),

$$\rho_1 = -\frac{48q^{-2}(\varepsilon^2 + 12q^{-2})}{2(\theta + 1 + \varepsilon^2 + 24q^{-2})}, \qquad \rho_2 = -\frac{48q^{-2}}{2(\theta + 1 + \varepsilon^2)}. \tag{B.19}$$

Taking into account (B.16), we obtain $48|q^{-2}| \leqslant 192/z_0^2$ in R_{z_0}. Thus the choice $z_0 \geqslant 14$ guarantees $\arg\theta < \pi/4$, so that the denominator for ρ_2 is greater than 2 by absolute value. This implies the second inequality in (B.18). Similar arguments can be applied for the first inequality. □

Lemma B.4. [T00] *If $G(z)$ is a two-dimensional vector such that*

$$\|G(z)\| \leqslant |q^{-4}| \qquad \text{or} \qquad \|G(z)\| \leqslant |q^{-6}| \tag{B.20}$$

for $z \in R_{z_0}$ then, respectively,

$$\|\mathcal{I}_1 G(z)\| \leqslant \frac{4}{\varepsilon}|q^{-3}(z)||\mathrm{e}^{-(\varepsilon/2)z}|, \qquad \|\mathcal{I}_2 G(z)\| \leqslant 8|q^{-3}(z)||\mathrm{e}^{-(\varepsilon/2)z}|$$

or

$$\|\mathcal{I}_1 G(z)\| \leqslant 24|q^{-4}(z)||\mathrm{e}^{-(\varepsilon/2)z}|, \qquad \|\mathcal{I}_2 G(z)\| \leqslant 8|q^{-5}(z)||\mathrm{e}^{-(\varepsilon/2)z}|.$$

Proof of theorem 5.2. By lemma B.1 and lemma B.4, there exists some $M > 0$, such that

$$\|\Delta W^{(1)}\|_{\mathcal{X}} \leqslant \frac{M}{2}. \tag{B.21}$$

Similarly to the proof of theorem A.1, we represent the operator $\mathcal{R} = \mathcal{R}_D + \mathcal{R}_I + \mathcal{R}_N + \mathcal{R}_R$, where

$$\mathcal{R}_D W = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} W,$$

$$\mathcal{R}_I W = -\frac{\theta'}{2\theta} \begin{pmatrix} I & I \\ I & I \end{pmatrix} W,$$

$$\mathcal{R}_N W = \boldsymbol{e}_0[(W_1 - W_3)^2 + \gamma(2(W_1 - W_3)(\lambda_1 W_1 - \lambda_2 W_3) + (W_2 - W_4)^2)],$$

$$\begin{aligned}\mathcal{R}_R W = {} & 12\boldsymbol{e}_0 q^{-2}[1 - c + c\gamma(6q^{-2} + \varepsilon^2)](W_1 - W_3) \\ & - 12\boldsymbol{e}_0 c\gamma[2\cosh(\varepsilon z/2)q^{-3}(W_2 - W_4) - q^{-2}(\lambda_1 W_1 - \lambda_2 W_3)],\end{aligned}$$

and $\boldsymbol{e}_0 = (0, 1, 0, 1)^{\mathrm{T}}$. To prove the convergence of (B.15) we need to estimate the norm of

$$\Delta W^{(k)} = \mathcal{I}\mathcal{R}_D \Delta W^{(k-1)} + \mathcal{I}\mathcal{R}_I \Delta W^{(k-1)} + \mathcal{I}\mathcal{R}_R \Delta W^{(k-1)} + (\mathcal{I}\mathcal{R}_N W^{(k-1)} - \mathcal{I}\mathcal{R}_N W^{(k-2)}). \tag{B.22}$$

According to lemma 5.1 from [T00], the norms of operators $\mathcal{I}\mathcal{R}_D, \mathcal{I}\mathcal{R}_I : \mathcal{X} \to \mathcal{X}$ are bounded by K/z_0, where $K > 0$ is some constant. To prove the same estimate for $\mathcal{I}\mathcal{R}_R$ (K can be increased, if necessary) we use (B.6), (B.22) and lemma B.1 to observe that the expression in the brackets in $\mathcal{R}_R W$ can be estimated by $O(|q^{-6}|)$ uniformly in ε on $[0, \varepsilon_0]$. Now the desired estimate follows from lemma B.4. Our goal is to prove that

$$\|\Delta W^{(n)}\|_{\mathcal{X}} \leqslant \frac{M}{2^n}, \qquad n = 1, 2, \ldots, \tag{B.23}$$

by induction. Suppose that this is true for all $n < k$ for some $k \in \mathbb{N}$. To prove (B.23) for $n = k$, we want to show that the norm of each term in (B.22) does not exceed $M/4 \cdot 2^k$. For the first three terms that can be achieved by choosing $z_0 > 1/8K$. For the nonlinear term $\mathcal{IR}_N W^{(k-1)} - \mathcal{IR}_N W^{(k-2)}$ of (B.22) we repeat the corresponding arguments from lemma A.2 to obtain that $z_0 > 64MLK$, where the constant $L > 0$ depends on γ, yields the necessary estimate on this term. Then convergence of the series (B.15) follows by induction. ■

Remark B.5. It is clear that the statements of theorem 5.2 remain true if the expression $(W_1 - W_3)^2 - f(q, TW)$ in (B.9) is replaced by any function $h(q, W)$ that is analytic in W in a vicinity of $0 \in \mathbb{C}^4$ and satisfies the following properties: (1) $\mathcal{IR}[0] \in \mathcal{X}$; (2) the norm of the corresponding operator $\mathcal{IR}_R : \mathcal{X} \mapsto \mathcal{X}$ in (B.22), where $\mathcal{R}_R W = (\partial h/\partial W)|_{W=0} \cdot W$, can be made as small as necessary by choosing sufficiently small z_0^{-1} and ε; (3) if $\|X\|_{\mathcal{X}}, \|Y\|_{\mathcal{X}} \leqslant \frac{1}{2}\|\mathcal{IR}[0]\|_{\mathcal{X}}$, then $\|\mathcal{IR}_N X - \mathcal{IR}_N Y\|_{\mathcal{X}} < \delta\|X - Y\|_{\mathcal{X}}$, where $\mathcal{R}_N X$ denotes the nonlinear part of $h(q, X)$ and the value of $\delta > 0$ can be made as small as necessary by choosing sufficiently small z_0^{-1} and ε.

A particular example of the equation satisfying the requirements of remark B.5 can be obtained by substituting $v_\gamma(z, e) = u(z, \varepsilon) - 4(a^1_{2,1}(\gamma))c(\varepsilon)q^{-2}(z, \varepsilon)$, where $c(\varepsilon)$ is described in theorem 5.2, into the equation

$$v_\gamma'''' + (1 - \varepsilon^2)v_\gamma'' - \varepsilon^2 v_\gamma = 2vv_\gamma + (2vv'' + v'^2) + 2\gamma(v_\gamma v'' + v'v_\gamma' + vv_\gamma'') \tag{B.24}$$

for the derivative $v_\gamma = \partial v/\partial\gamma$.

Proof of theorem 5.14. Let us fix some $N \in \mathbb{N}$. Substitution of

$$v_s(z, \varepsilon) = V_N(q(z), \varepsilon) + \varepsilon^{2N+2}v(z, \varepsilon). \tag{B.25}$$

into inner equation (1.1) yields

$$\begin{aligned}(D_z^2 + 1)(D_z^2 - \varepsilon^2)v &= 2(V_N v + \gamma[V_N'' v + V_N' v' + V_N v'']) \\ &\quad + \varepsilon^{2N+2}(v^2 + \gamma[2vv'' + v'^2]) + \rho_N,\end{aligned} \tag{B.26}$$

where

$$\rho_N = \varepsilon^{-2N-2}(V_N^2 + \gamma[2V_N V_N'' + V_N'^2] - (D_z^2 + 1)(D_z^2 - \varepsilon^2)V_N). \tag{B.27}$$

According to (4.40) and (4.26),

$$(D_z^2 + 1)(D_z^2 - \varepsilon^2) = \left[D_q^2 + 1 + \frac{\varepsilon^2}{4}(qD_q)^2\right]\left[D_q^2 + \frac{\varepsilon^2}{4}\left((qD_q)^2 - 4\right)\right]. \tag{B.28}$$

Substituting (B.28) into (B.27) and taking into account lemma 5.12, we obtain after some algebra

$$\begin{aligned}\rho_N(q, \varepsilon) &= \tfrac{1}{4}(D_q^2 + 1)[((qD_q)^2 - 4) + (qD_q)^2 D_q^2]v_N + \tfrac{1}{16}(qD_q)^2[(qD_q)^2 - 4](v_{N-1} + \varepsilon^2 v_N) \\ &\quad - \sum_{i+j=N+1}[v_i v_j + \gamma(2v_i D_q^2 v_j + (D_q v_i)(D_q v_j))] \\ &\quad + \frac{\gamma}{4}\sum_{i+j=N}[2v_i(qD_q)^2 v_j + q^2(D_q v_i)(D_q v_j)] + O(\varepsilon^2 q^{-4}),\end{aligned} \tag{B.29}$$

where v_j denotes $(v_j)_+$, $j = 0, 1, \ldots, N$. It is easy to see now that $\rho_N(q, \varepsilon) = O(q^{-4})$ uniformly on R_{z_0}.

Substituting ρ_N together with

$$v = \beta(\varepsilon)q^{-2} + u \tag{B.30}$$

into (B.26), we obtain

$$[(D_z^2+1)(D_z^2-\varepsilon^2)-12q^{-2}]u = 2\{(V_N-6q^{-2})u+\gamma[V_N''u+V_N'u'+V_Nu'']\}$$
$$+\varepsilon^{2N+2}\{(\beta(\varepsilon)q^{-2}+u)^2+\gamma[2(\beta(\varepsilon)q^{-2}+u)(\beta(\varepsilon)q^{-2}+u)''+(\beta(\varepsilon)q^{-2}+u)'^2]\}$$
$$+2\beta(\varepsilon)\{q^{-2}V_N+\gamma[q^{-2}V_N''+V_N'(q^{-2})'+V_N(q^{-2})'']\}+\rho_N. \qquad \text{(B.31)}$$

Similarly to (B.2), we rewrite (B.31) as

$$[(D_z^2+1)(D_z^2-\varepsilon^2)-12q^{-2}]u = \varepsilon^{2N+2}u^2 - f(q,U), \qquad \text{(B.32)}$$

where $U = \mathrm{Col}(u, u', u'', u''')$. Let us define

$$\beta(\varepsilon) = \left(c_+(\varepsilon) - 4\sum_{k=0}^{N}\varepsilon^{2k}a_{k+1,k}\right)\varepsilon^{-2N-2} \qquad \text{(B.33)}$$

where $c_+(\varepsilon)$ is the coefficient in the exponential series expansion (4.38) of v_s. Then, according to corollary 4.12 and remark 5.3, $v_s = V_N + \varepsilon^{2N+2}(\beta(\varepsilon)q^{-2}+u)$, where u is the solution to (B.32) constructed by iterations as in theorem 5.2, provided that $u = O(q^{-4})$ uniformly on R_{z_0}. To demonstrate the latter fact, we first note that $f(q,0) = O(q^{-4})$ uniformly on R_{z_0}. Since $\beta(\varepsilon) = 4a_{N+2,N+1} + O(\varepsilon^2)$ and $\hat{v}_{N+1}(q) = 4a_{N+2,N+1}q^{-2} + O(q^{-4})$, hence, according to lemma 5.12, $f(q,0) = O(\varepsilon^2 q^{-4}) + O(q^{-6})$ uniformly on R_{z_0}. It follows now from remark B.5, remark 5.4 that the solution u to (B.32) can be constructed by iterations as in theorem 5.2 in the region R_{z_N} with some $z_N \geqslant z_0$, and that $u = O(q^{-4})$ uniformly on R_{z_N}. However, we can replace the region R_{z_N} by R_{z_0}, since functions $v_s(z,\varepsilon)$ and $v_j(z)$, $j = 0, 1, \ldots, N$ are analytic in R_{z_0}. ■

References

[AM91] Amick C J and McLeod J B 1991 A singular perturbation problem in water waves *Stab. Appl. Anal. Cont. Media* **1** 127–48

[CG97] Champneys A R and Groves M D 1997 A global investigation of solitary-wave solutions to a two-parameter model for water waves *J. Fluid Mech.* **342** 199–229

[C01] Champneys A R 2001 Codimension-one persistence beyond all orders of homoclinic orbits to singular saddle centres in reversible systems *Nonlinearity* **14** 87–112

[CG94] Craig W and Groves M D 1994 Hamiltonian long-wave approximations to the water-wave problem *Wave motion* **19** 367–89

[IA98] Iooss G and Adelmeyer M 1998 *Topics in Bifurcation Theory and Applications* (Singapore: World Scientific)

[IK92] Iooss G and Kirchgassner K 1992 Water waves for small surface tension: an approach via normal forms *Proc. R. Soc. Edin.* A **112** 267–300

[HM89] Hammersley J M and Mazzarino G 1989 Computational aspects of some autonomous differential equations *Proc. R. Soc. Lond.* A **424** 19–37

[G99] Glutsuk A A 1999 Stokes operators via limit monodromy of generic perturbation *J. Dyn. Control Syst.* **5** 101–35

[G04] Glutsyuk A A 2004 Resonant confluence of singular points and Stokes phenomena *J. Dyn. Control Syst.* **10** 253–302

[GJ95] Grimshaw R and Joshi N 1995 Weakly nonlocal solitary waves in a singularly perturbed Korteweg–de Vries equation *Stud. J. Appl. Math.* **55** 124–35

[KCBS02] Kolossovski K, Champneys A R, Buryak A V and Sammut R A 2002 Multi-pulse embedded solitons as bound states of quasi-solitons *Physica* D **171** 153–77

[KS91] Kruskal M D and Segur H 1991 Asymptotics beyond all orders in a model of crystal growth *Stud. Appl. Math.* **85** 129–81

[L99] Lombardi E 1999 Non-persistence of homoclinic connections for perturbed integrable reversible systems *J. Dyn. Diff. Eqns* **11** 124–208

[L00] Lombardi E 2000 Oscillatory integrals and phenomena beyond all algebraic orders (*Lecture Notes in Mathematics* vol 1741) (Berlin: Springer)

[OPB06] Oxtoby O F, Pelinovsky D E and Barashenkov I V 2006 Travelling kinks in discrete ϕ^4 models *Nonlinearity* **19** 217–35

[PRG88] Pomeau Y, Ramani A and Grammaticos B 1988 Structural stability of the Korteweg–de Vries solitons under a singular perturbation *Physica* D **31** 127–34

[STL92] Segur H, Tanveer S and Levine H (ed) 1992 *Asymptotics Beyond all Orders* (*NATO ASI Series B Physics* vol 284) (New York: Plenum)

[Ti39] Titchmarsh E C 1939 *The Theory of Functions* (New York: Oxford University Press)

[T94a] Tovbis A 1994 Nonlinear ordinary differential equations resolvable with respect to an irregular singular point *J. Diff. Eqns* **109** 201–21

[T94b] Tovbis A 1994 Asymptotics beyond all orders and analytic properties of inverse Laplace transforms of solutions *Commun. Math. Phys.* **163** 245–55

[T94c] Tovbis A 1994 On exponentially small terms of solutions to nonlinear ordinary differential equations *Methods Appl. Anal.* **1** 57–74

[T00] Tovbis A 2000 Breaking homoclinic connections for a singularly perturbed differential equation and the Stokes phenomenon *Stud. Appl. Math.* **104** 353–86

[W76] Wasow W 1976 *Asymptotic Expansions for Ordinary Differential Equations* (Huntington, New York: Krieger)