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On a wave field transformation described by the two-dimensional Kadomtsev–Petviashvili equation

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Abstract. The problem of interaction of a smooth nonlinear two-dimensional wave field and a large-amplitude quasi-plane solitary wave is considered in the framework of the Kadomtsev–Petviashvili equation by means of the asymptotic multiscale technique. It is shown that their interaction results in an essential transformation of a nonlinear wave field including the birth, death and translation of soliton components of the field spectrum.

1. Introduction

In two-dimensional isotropic media with weak positive dispersion, the dynamics of quasi-one-dimensional waves is described by the Kadomtsev–Petviashvili equation (KP) [1]

$$(4u_t + 12uu_x + u_{xxx})_x = 3u_{yy} \tag{1}$$

which belongs to the class of integrable equations of mathematical physics [2]. Recent investigations of this equation [3–8] reveal that the interaction of plane and two-dimensional solitary waves possesses a number of properties which are different from those of the usual soliton interactions in one-dimensional integrable systems [9, 10]. Firstly, it was found [3, 6, 7] that one-dimensional solitons can transform into periodic chains of two-dimensional solitons due to instability with respect to long-wave transversal perturbations which has been known for positive-dispersion media since the pioneering work of Kadomtsev and Petviashvili [1]. However, the chains of two-dimensional solitons are also unstable with respect to modulation of their fronts and decay into new chains with longer periods [6]. On the other hand, the inverse processes of merging solitary waves are also possible as a result of their resonant interaction [4, 5]. Such cascades of soliton decaying instabilities and their resonant mergings should lead to the formation of a complex multiperiodic structure of a nonlinear wave field in positive-dispersion media. Analysis of the appearance of such a structure has not been carried out until now.

In the present paper we consider the problem of a quasi-plane solitary wave transformation on a smooth extended wave field. Using a modification of an asymptotic multiscale technique (see [11, 12]) we analyse the general features of soliton–wave field interaction which becomes more complicated due to development of unstable transversal perturbations on the solitary wave front. A new approach to a description of the wave

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processes in two spatial dimensions is based on a separation of the nonlinear wave field into two components. The first ('fast') component is concentrated at a planar large-amplitude soliton with small transversal perturbations and is described by the inhomogeneous Eckhaus equation [12–14]. The second ('slow') component is distributed within a small smooth wave field outside the large-amplitude soliton and is described by the original KP equation.

All equations derived in the framework of our asymptotic multiscale approach are integrable, which fits well into a general ideology developed in the papers [11, 12]. Therefore, the soliton–wave field interaction can be described in an explicit form. Indeed, the homogeneous Eckhaus equation was found to be integrable by Kundu [13] and its explicit soliton solutions were investigated in detail by Calogero and de Lillo [14]. However, in our case we deal with the equation where spatial and temporal variables are mutually replaced [8]. This difference has an important consequence. We will show that the inhomogeneous Eckhaus equation with replaced spatial and temporal variables possesses active and dissipative properties caused by the energy flow which enters a fast soliton from the wave field initially given and re-emits from the soliton as soliton radiation.

Our analysis reveals the essential transformation of a two-dimensional nonlinear wave field as a result of interaction with an individual large-amplitude soliton. Such a transformation includes the birth, death and translation of soliton components of the extended wave field. From a mathematical point of view, the fast soliton generates the Backlund–Darboux transformation [15] which transforms the unperturbed field in front of the soliton into a perturbed field behind it. On the other hand, the corresponding transformation of the quasi-plane soliton also takes place and is represented by the change of transversal modulation of its front. As a result, the fast and slow components of a common wave field represent a close self-consistent system with full momentum and energy conserved.

2. Asymptotic description of soliton propagation

Let us consider the following multiscale expansion of solutions to (1)

$$u = u_0(\xi) + \sum_{n=1}^{\infty} \epsilon^{n/2} u_{n/2}(\xi, y, X, Y, T, \tau) . \qquad (2)$$

Here the basic term $u_0 = 1/\cosh^2(\xi)$ describes the quasi-plane soliton with unit amplitude, the variable $\xi = x - t - x_0(Y, T, \tau)$ corresponds to its moving coordinate, the slow variables $X = \epsilon x$, $Y = \epsilon^2 y$, $T = \epsilon t$, $\tau = \epsilon^3 t$ determine the spatial and temporal variations of the soliton and small-amplitude wave field, and $\epsilon \ll 1$ is a small parameter.

By substituting the expansion (2) into (1) we obtain the system of linear equations for consecutive calculation of corrections $u_{n/2}$:

$$L u_{n/2} = H_{n/2}(u_0, u_{1/2}, \ldots, u_{(n-1)/2}) \qquad n \geqslant 1 . \qquad (3)$$

Here the linearized operator L on the soliton u_0 has the form

$$L = -4\frac{\partial^2}{\partial \xi^2} + 12\frac{\partial^2}{\partial \xi^2}(u_0) + \frac{\partial^4}{\partial \xi^4} - 3\frac{\partial^2}{\partial y^2}$$

and the right-hand side operators $H_{n/2}$ are expressed by the lower-order corrections, except the case $n = 1$ for which $H_{1/2} = 0$. Let us choose the solution of the homogeneous equation (3) at $n = 1$ in the following form:

$$u_{1/2} = (a(Y, T, \tau)\exp(\mathrm{i}y) + \mathrm{CC})\frac{\partial^2}{\partial \xi^2}\left(\frac{1}{\cosh(\xi)}\right) . \qquad (4)$$

The function $u_{1/2}$ describes a perturbation to the quasi-plane soliton u_0, which is oscillating intensively along the transversal coordinate and is localized along the longitudinal coordinate. Slow variables Y, T and τ determine the modulation of such a quasi-harmonic transversal perturbation and the non-stationary effects caused by development of the solitary wave instability.

Using the explicit form of functions u_0, $u_{1/2}$ and solving the inhomogeneous equations (3) for $n \geqslant 2$ by means of the technique of separating variables, we can consecutively find higher-order terms of the series (2). The corresponding expressions are presented in our previous paper [8]. It is important to point out that, in a general case, solutions of (3) contain terms which grow algebraically and exponentially as $\xi \to \pm\infty$. The algebraically growing terms appear in the order of $O(\epsilon^2)$ and describe a wave field extending outside the soliton and varying along the X-axis. The exponentially growing terms must be removed because they lead to divergences of the asymptotic series.

The conditions of removing the exponentially growing terms are represented by the equations imposed on the parameters of the soliton and its perturbation. For our problem, the non-trivial equations appear when we include terms up to the order of $O(\epsilon^3)$. Note that the amplitude of soliton perturbation has an order of $O(\epsilon^{1/2})$ that corresponds to the situation in the nonlinear Schrödinger equation where the coefficient before a cubic nonlinear term is identically equal to zero (see [12]).

On applying the technique outlined above, we found that the varying part of soliton velocity (x_{0T}) is expressed by an explicit formula

$$x_{0T} = |a|^2 + \epsilon\big(3|a|^4 + \tfrac{1}{2}(u^+ + u^-)|_{X=X_s}\big) + O(\epsilon^2) \tag{5}$$

and the amplitude of transversal perturbations on the soliton front (a) obeys the equation

$$-\,\mathrm{i}a_Y + a_{TT} + a|a|^4 + a\big(u^+ + u^-\big)|_{X=X_s} + O(\epsilon) = 0\,. \tag{6}$$

Here $u^{\pm} = \epsilon^{-2}u|_{\xi\to\pm\infty}$ are the components of extended wave field in front of the quasi-plane soliton and behind it, respectively, and $X_s(T)$ is the soliton coordinate in a fixed reference frame so that $X_s = T + O(\epsilon)$. The scheme of the problem under consideration is depicted in figure 1.

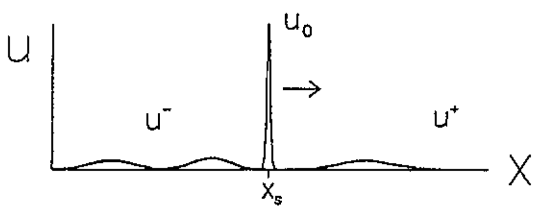

Figure 1. Scheme of asymptotic approach for description of the fast soliton-smooth wave field interaction.

It can readily be shown that, outside the soliton, components $u^{\pm} = u^{\pm}(X, Y, \tau)$ satisfy (1) which is rewritten in slow variables as follows

$$\big(4u^{\pm}_{\tau} + 12u^{\pm}u^{\pm}_{X} + u^{\pm}_{XXX}\big)_X = 3u^{\pm}_{YY}\,. \tag{7}$$

On the other hand, at the soliton position, these components are related by boundary condition

$$\big(u^- - u^+\big)|_{X=X_s} = 2\big(|a|^2\big)_T + \epsilon\big[\tfrac{9}{2}\big(|a|^4\big)_T + \big(u^+_X + u^-_X\big)|_{X=X_s}\big] + O(\epsilon^2)\,. \tag{8}$$

Since components $u^{\pm}$ do not depend upon T, the radiation escaping from the fast soliton remains behind it. As a result, the component $u^{+} = U(X, Y, \tau)$ is completely determined by an initial distribution of the wave field. Therefore, the set of equations (5)–(8) is closed and, in the leading order of our asymptotic expansions, it reduces to the following Eckhaus equation with a given potential $U(X, Y, \tau)$:

$$-\mathrm{i}a_Y + a_{TT} + a|a|^4 + 2a\left(|a|^2\right)_T + 2aU(X_s(T), Y, \tau) = 0. \tag{9}$$

Note that the dependence of function $a(Y, T, \tau)$ on τ is implicit because dynamics of the perturbations localized on the fast soliton occurs mainly on the scale of the time variable T.

Let us consider the integral properties of the set of equations (7)–(9). If the function a is supposed to be localized along the transversal coordinate Y, we can introduce a functional which has the sense of the varying part of the projection of soliton momentum on the x-axis:

$$P_s = \int_{-\infty}^{+\infty} \left[\tfrac{1}{2}\mathrm{i}\left(aa_Y^* - a_Y a^*\right) + |a_T|^2 + \tfrac{1}{3}|a|^6\right] \mathrm{d}Y .$$

Then, we can readily find from (9) that

$$\frac{\mathrm{d}P_s}{\mathrm{d}T} = -2\int_{-\infty}^{+\infty} \left(|a|^2\right)_T\left[\left(|a|^2\right)_T + U(X_s(T), Y, \tau)\right] \mathrm{d}Y .$$

This formula implies that the radiation escaping from the fast soliton always leads to loss of soliton momentum, while the incoming field U can lead both to loss and to storing of momentum on the fast soliton. On the other hand, a change of momentum in a localized component of the wave field should be compensated by the corresponding change of momentum in an extended component.

Indeed, the x-projection of momentum of the smooth wave field described by (7) has the form

$$P_r = \tfrac{1}{2}\iint_{-\infty}^{+\infty} (u_r)^2 \,\mathrm{d}X\,\mathrm{d}Y$$

where

$$u_r = u^{+}(X, Y, \tau)\theta(X - X_s) + u^{-}(X, Y, \tau)\left(1 - \theta(X - X_s)\right)$$

and $\theta(x)$ is the unit (Heaviside) function. Then, direct calculations give

$$\frac{\mathrm{d}P_r}{\mathrm{d}T} = \int_{-\infty}^{+\infty} \tfrac{1}{2}\left(u^- - u^+\right)|_{X=X_s}\left(u^- + u^+\right)|_{X=X_s} \,\mathrm{d}Y = -\frac{\mathrm{d}P_s}{\mathrm{d}T} .$$

Thus, the fast soliton and the extended wave field represent a close system so that the full momentum (and energy) of this system do not change as a result of nonlinear wave transformations.

3. Reduction to the Backlund–Darboux scheme

We now consider solutions to the set of equations (5), (7)–(9) and assume that function U is smooth throughout the X-axis. As the soliton coordinate X_s is proportional to the time variable T in a leading order of asymptotic expansions and functions $u^{\pm}$ do not depend upon T, variable T can be replaced in (8) and (9) by variable X. It corresponds to the choice of a spatial scale in the fixed reference frame for the description of dynamics of the perturbations localized on the moving soliton. Let us introduce the transformation of dependent variables [14]

$$a = \frac{g}{\sqrt{2f}} \qquad |a|^2 = \frac{f_x}{2f} \qquad x_0 = \tfrac{1}{2}\log(f)\,. \tag{10}$$

Since the function $|a|^2$ is real, f is real, too. On replacing the dependent variables, equation (9) transforms to the linear inhomogeneous Schrödinger equation for function g:

$$\mathrm{i}g_Y = g_{XX} + 2Ug \tag{11}$$

while function f is related to g by the formula

$$f = C + \int_{-\infty}^{X} |g|^2\,\mathrm{d}X\,. \tag{12}$$

Here C is an arbitrary constant of integration.

There exists the direct relationship between a general solution to (9) and exact solutions to the KP equation. First of all, equation (7) for function $U(X, Y, \tau)$ is a solvability condition of a pair of linear partial differential equations (Lax pair) one of which is presented by (11) [2]. It means that the implicit dependence of functions a, f and g on τ can be determined by the other linear differential equation

$$g_\tau + g_{XXX} + 3Ug_X + \tfrac{3}{2}\left(U_X + \int U_Y\,\mathrm{d}X\right)g = 0\,. \tag{11$'$}$$

Furthermore, equation (8) can be rewritten in new variables as

$$u^- = U + (\log(f))_{XX} \tag{13}$$

where we neglect terms of the order of $\mathrm{O}(\epsilon)$. Equation (13) determines the instant profile of wave field u^- which is formed behind the moving quasi-plane soliton as a result of its radiation. We remind the reader that the evolution of this distribution along the slow variable τ obeys the KP equation (7).

Together with the formulae (11), (11′) and (12), the expression (13) is nothing other than the Backlund–Darboux transformation [15] from one solution of the KP equation $u^+ = U$ to the other solution u^-. Thus, the interaction of a large-amplitude soliton and an extended wave field can be regarded as the transformation of smooth small-amplitude nonlinear waves occurring in accordance with the Backlund–Darboux transformation. On the other hand, this interaction also results in transformation of transversal perturbations localized on the large-amplitude soliton and described by amplitude a.

4. General solutions of approximate equations and related transformation of the nonlinear wave field

In a one-dimensional case, interaction of a fast soliton and a smooth wave field reduces to a small phase shift of nonlinear waves along the x-axis. Indeed, equation (8) at $a = 0$ implies that the wave field behind the soliton u^- differs from the wave field in front of the soliton u^+ by a small value of the order of $O(\epsilon)$. Moreover, this small correction corresponds exactly to the coordinate translation of the smooth wave field appearing under the action of the fast soliton

$$u^- = u^+\left(X + \epsilon \int_{-\infty}^{+\infty} u_0 \, d\xi, \tau\right).$$

On the other hand, it follows from (5) that the fast soliton also acquires a phase shift when it moves along the smooth distribution of u^+. This phase shift is found to be

$$\Delta x = \epsilon \int_{-\infty}^{X_s} u^+ \, dX .$$

After interaction, the fast soliton is described by the function

$$u = u_0\left(x - t - \epsilon \int_{-\infty}^{+\infty} u^+ \, dX\right).$$

Note that such a trivial result of the interaction of one-dimensional nonlinear waves is well known for many integrable equations of mathematical physics [9, 10].

The dynamics of nonlinear waves in a two-dimensional case is cardinally different. In what follows we analyse the characteristic types of the soliton–wave field interaction in the presence of transversal perturbations at the soliton front ($a \neq 0$). For this purpose we express potential u^+ by a superposition of two terms (the so-called solvable and unsolvable components [16]):

$$u^+ = U_0 + (\log(F_N))_{XX} . \tag{14}$$

Here the term U_0 describes wave disturbances of the continuous spectrum while the function F_N corresponds to soliton solutions and is expressed in a general form by the determinant of the Nth order

$$F_N = \det\left(C_n \delta_{nm} + \int_{-\infty}^{X} g_n g_m^* \, dX\right)_{1 \leqslant n,m \leqslant N} . \tag{15}$$

Each partial function g_n in determinant F_N satisfies the shortened equations (11) and (11′) where the function U is replaced by U_0. Furthermore, the dressing method as well as the Backlund–Darboux transformation method [15, 16] allow us to express a general solution to the complete equations (11), (11′) and (12) by the same set of partial functions g_n for $1 \leqslant n \leqslant N$ and an additional function g_{N+1} which represents a general solution to the shortened equations (11) and (11′). Omitting details of applying this technique, we write the corresponding expressions for functions g and f:

$$g = \frac{G_N}{F_N} \qquad f = \frac{F_{N+1}}{F_N} \tag{16}$$

where

$$G_N = \det \begin{cases} C_n\delta_{nm} + \int_{-\infty}^{X} g_n g_m^* \, \mathrm{d}X & 1 \leqslant n \leqslant N \quad 1 \leqslant m \leqslant N \\ \int_{-\infty}^{X} g_{N+1} g_m^* \, \mathrm{d}X & n = N+1 \quad 1 \leqslant m \leqslant N \\ g_n & 1 \leqslant n \leqslant N+1 \quad m = N+1 \end{cases} \tag{17}$$

and F_{N+1} is given by (15) with the substitution $N \to N+1$. Using eqution (16) we easily find that the wave field behind the soliton u^- is expressed by the determinant F_{N+1} of the $(N+1)$th order:

$$u^- = U_0 + (\log(F_{N+1}))_{XX} \, . \tag{18}$$

Thus, in a general case, the interaction of a fast soliton with a small transversal perturbation and a smooth wave field gives rise to a new component of the nonlinear wave spectrum which is described by the function g_{N+1}. Note that each function g_n being a solution of linear partial differential equations generates a non-stationary multisoliton distribution of a nonlinear wave field [6]. Therefore, the wave field u^- behind the fast soliton can have a more complicated form than the wave field u^+ and may contain new plane solitons and two-dimensional solitary waves with transversely modulated fronts which are generated by a new function g_{N+1}. Such an enlargement of the nonlinear wave spectrum is caused by the development of unstable perturbations on the background of a large-amplitude soliton and its transformation to a quasi-plane modulated wave. During this transformation part of the soliton momentum goes into the extended field in the form of small-amplitude radiation [8].

Besides such a general type of soliton-wave field interaction, there exist other types when the spectrum of nonlinear waves either transforms in a trivial manner like that in the one-dimensional case or transforms with the death of a soliton component. These types are also described by (16) and (18) but the function g_{N+1} becomes dependent on g_n for some $n \leqslant N$.

Indeed, such a dependence decreases the order of determinants G_N and F_{N+1}. So, for $g_{N+1} = g_N$ we can readily show that the following relations are met:

$$G_N = C_N G_{N-1} \qquad F_{N+1} = (C_N + C_{N+1})F_N - C_N^2 F_{N-1} \, .$$

As a result, at $C_{N+1} \neq -C_N$ the field u^- is given by the same expression (14) as u^+ but the constant C_N in determinant F_N is replaced by $C_N C_{N+1}/(C_N + C_{N+1})$. In this case, the wave field structure does not change excepting the trivial shift of phase constants in soliton components. However, if $C_{N+1} = -C_N$ we reveal a more essential transformation of the smooth wave field because now the distribution of u^- is determined by the determinant F_{N-1} which is different from F_N by 'loss' of the function g_N. The corresponding soliton component of the original wave field u^+ is absorbed by the large-amplitude soliton when it moves along this field.

Thus, in positive-dispersion media the interaction of quasi-plane solitons and extended wave fields may lead to the transformation of a number of soliton components as well as of their qualitative structure due to processes of decaying instability and resonant merging of solitary waves. The asymptotic approach described in this paper enables us to regard these processes as a self-consistent transformation of a nonlinear wave field occurring under the action of an individual solitary wave moving along the extended field.

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