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ON THE ASYMPTOTIC STABILITY OF LOCALIZED MODES IN THE DISCRETE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. Asymptotic stability of localized modes in the discrete nonlinear Schrödinger equation was earlier established for septic and higher-order nonlinear terms by using Strichartz estimate. We use here pointwise dispersive decay estimates to push down the lower bound for the exponent of the nonlinear terms.

1. **Introduction.** We consider the discrete nonlinear Schrödinger (DNLS) equation with a power nonlinearity and a bounded potential,

$$i\dot{u}_n = (-\Delta + V_n + |u_n|^{2p})u_n, \quad n \in \mathbb{Z}, \tag{1.1}$$

where $p > 0$, $\{V_n\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, and

$$\Delta u_n := u_{n+1} - 2u_n + u_{n-1}, \quad n \in \mathbb{Z}.$$

We shall assume that $\{V_n\}_{n\in\mathbb{Z}}$ decays to zero as $|n| \to \infty$ exponentially fast and that the discrete Schrödinger operator,

$$H := -\Delta + V : \quad l^2(\mathbb{Z}) \to l^2(\mathbb{Z}),$$

supports only one eigenvalue $\omega_0 < 0$ outside the continuous spectrum $\sigma_c(H) = [0, 4]$ and no resonance at the end points $\{0, 4\}$. We note that the operator H has no eigenvalues in $(0, 4)$ (see [16, Theorem 1]).

A localized mode of the DNLS equation (1.1) is a real-valued solution of the stationary DNLS equation,

$$(-\Delta + V_n + \phi_n^{2p})\phi_n = \omega\phi_n, \quad n \in \mathbb{Z}. \tag{1.2}$$

It is known from the variational methods (see Pankov [14] for details) that under the above assumptions on V and H, there is a localized mode of the stationary DNLS equation (1.2) in $l^2(\mathbb{Z})$ for any $\omega \in (\omega_0, 0)$. Asymptotic stability of the small localized mode for a small value of $|\omega-\omega_0|$ was considered recently by Cuccagna & Tirulli [4] and Kevrekidis, Pelinovsky & Stefanov [5]. Both papers employed the analysis based on the Strichartz estimate, which are deduced from the l^2-conservation law of the DNLS equation (1.1) and the pointwise dispersive decay estimate,

$$\|e^{-itH}P_{\text{a.c.}}(H)\mathbf{f}\|_{l^\infty} \le C(1+|t|)^{-\frac{1}{3}}\|\mathbf{f}\|_{l^1}, \quad t \in \mathbb{R}, \tag{1.3}$$

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where $P_{\mathrm{a.c.}}(H)$ is the orthogonal projection to the continuous spectrum $\sigma_c(H) = [0,4]$.

Using the stationary phase method for $H = -\Delta + V$, the decay estimate (1.3) was obtained by Stefanov & Kevrekidis for $V = 0$ [19], by Komech, Kopylova & Kunze for a compactly-supported potential V [8], and by Pelinovsky & Stefanov for a potential V in a weighted l^1 space [16]. Using the Strichartz estimate, asymptotic stability of localized modes with small $|\omega - \omega_0|$ was established in the DNLS equation (1.1) with $p \geq 3$, that is, for septic and higher-order nonlinear terms [4, 5]. The cases of cubic and quintic nonlinearity, which have important applications in the context of the Gross–Pitaevskii equation with periodic coefficients (see Belmonte–Beitia & Pelinovsky [1] for details), were excluded from the previous analysis. It is worth mentioning in this context the work of Cuccagna [3], where the author showed that if H supports two eigenvalues $\omega_0 < 0$ and $\omega_1 > 4$, then localized modes undertake long-time oscillations without dispersive decay.

Improved pointwise dispersive decay estimates for the linear DNLS equation were recently derived by Mielke & Patz [11] using a modification of the stationary phase method and an approximation of discrete sums by continuous integrals. The main result of [11] is the pointwise dispersive decay estimate,

$$\|e^{it\Delta}\mathbf{f}\|_{l^s} \leq C(1+|t|)^{-\alpha_s}\|\mathbf{f}\|_{l^1}, \quad \alpha_s = \begin{cases} \frac{s-2}{2s}, & 2 \leq s < 4, \\ \frac{s-1}{3s}, & 4 < s \leq \infty. \end{cases} \tag{1.4}$$

We note that the Riesz–Thorin interpolation between the l^2-conservation law and the decay estimate (1.3) gives the bound,

$$\|e^{-itH}P_{\mathrm{a.c.}}(H)\mathbf{f}\|_{l^s} \leq C(1+|t|)^{-\beta_s}\|\mathbf{f}\|_{l^{s'}}, \quad \beta_s = \frac{s-2}{3s}, \quad s' = \frac{s}{s-1}. \tag{1.5}$$

Compared to (1.5), bound (1.4) gives a faster decay $\alpha_s > \beta_s$ for any $2 < s < \infty$ if $\mathbf{f} \in l^1(\mathbb{Z})$ and $H = -\Delta$ ($V = 0$).

Using the pointwise dispersive decay estimate (1.4) and interpolation of the nonlinear terms in l^s spaces, Mielke & Patz [11] proved scattering to zero of a small initial data in l^1 for the DNLS equation (1.1) with $2p+1 > 4$ ($p > \frac{3}{2}$) and $V = 0$. These results improved the earlier computations of Stefanov & Kevrekidis [19], who proved nonlinear scattering of solutions with small initial data in l^2 for $2p+1 \geq 7$ ($p \geq 3$) using the Strichartz estimate. These authors also proved nonlinear scattering of solutions with small initial data in $l^{s'}$ by using the decay estimates (1.5) with

$$1 + \frac{1}{2p+1} \leq s' < \frac{2p+1}{p+2},$$

provided that $2p+1 > 2+\sqrt{7} \approx 4.65$ ($p > \frac{1+\sqrt{7}}{2} \approx 1.875$). (Actually, $s' = \frac{5}{4}$ and $p > 2$ were claimed in [19, Theorem 7] but a sharper condition $p > \frac{1+\sqrt{7}}{2}$ can be obtained using the same proof.)

We shall employ the improved dispersive decay estimate (1.4) to push down the exponent p of the nonlinear terms that guarantees asymptotic stability of localized modes of the stationary DNLS equation (1.2) with small $|\omega - \omega_0|$. In this way, we obtained that p must exceed 2.75. The exponent is still too high to include the cubic and quintic DNLS equations because we need integrability in time of a small linear term in our argument. We expect that dispersive estimates for time-dependent potentials such as Kirr & Mizrak [6] and Kirr & Zarnescu [7] could be useful to overcome this difficulty.

We anticipate that the results of this paper can be extended to the DNLS equation (1.1) with $V \equiv 0$. Weinstein [20] proved orbital stability of localized modes in this equation for any $p > 0$ provided the l^2-norm is large enough. The limit of a large l^2-norm can be rescaled to the anti-continuous limit of the DNLS equation, where the resolvent analysis for localized modes was recently developed by Pelinovsky & Sakovich [17]. It was shown that the linearization spectrum of a single-peaked localized mode near the anti-continuum limit has no isolated eigenvalues outside the continuous spectrum, excluding the double zero eigenvalue that persists due to the gauge symmetry of the DNLS equation (1.1). Because the spectral assumption for the standard asymptotic stability theory is satisfied, it is natural to expect that the details of this work can be extended to the DNLS equation (1.1) with $V \equiv 0$ near the anti-continuum limit, for the price of working with dispersive estimates for non-self-adjoint linearized operators.

The paper is organized as follows. Section 2 presents the main result after a decomposition of the solution of the DNLS equation (1.1) into the localized mode with slowly varying parameters and the remainder term. Section 3 gives the proof of the main result using pointwise dispersive decay estimates (1.4) and continuation arguments in weighted l^s spaces. To develop nonlinear analysis, we use the l^1_α–$l^\infty_{-\alpha}$ dispersive decay estimate for any $\alpha \in [0, 1]$. A continuous analogue of this estimate was developed in a similar context by Krieger & Schlag [9, 18] (see also Buslaev & Perelman [2]). The proof of the improved estimate (1.4) for $H = -\Delta + V$ with an exponentially decaying potential is given in Section 4 and the l^1_α–$l^\infty_{-\alpha}$ estimate is proved in Section 5.

2. **Preliminaries and the main result.**

In what follows, we use bold-faced notations for vectors in discrete space $l^s(\mathbb{Z})$ defined by their norms

$$\|\mathbf{u}\|_{l^s} := \left(\sum_{n\in\mathbb{Z}} |u_n|^s \right)^{1/s}, \quad s \geq 1.$$

Components of $\mathbf{u}$ are denoted by u_n for $n \in \mathbb{Z}$. We denote $< n >= (1 + n^2)^{1/2}$ and use $< n > \mathbf{u}$ to denote $\{< n > u_n\}_{n\in\mathbb{Z}}$.

Let $l^s_\sigma(\mathbb{Z})$ be a Banach space whose norm is defined by $\|\mathbf{u}\|_{l^s_\sigma} = \| < n >^\sigma \mathbf{u}\|_{l^s}$ for $1 \leq s \leq \infty$ and $\sigma \in \mathbb{R}$. We denote by $\langle \cdot, \cdot \rangle$ the inner product of $l^2(\mathbb{Z})$.

We recall the continuous embedding $l^{s_1} \subset l^{s_2}$ for any $1 \leq s_1 < s_2 \leq \infty$ with the bound $\|\mathbf{u}\|_{l^{s_2}} \leq \|\mathbf{u}\|_{l^{s_1}}$. We also recall the interpolation inequality for any $s_0, s_1 \in [1, \infty]$ and $\theta \in [0, 1]$,

$$\|\mathbf{u}\|_{l^s} \leq \|\mathbf{u}\|^{1-\theta}_{l^{s_0}} \|\mathbf{u}\|^{\theta}_{l^{s_1}}, \quad \text{where} \quad \frac{1}{s} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}.$$

Our convention is that if a positive constant depends on a parameter α, we denote it by C_α. We use a generic positive constant C and change it from one line to another line.

To formulate the problem, we use the previous work of Kevrekidis *et al.* [5]. In particular, we consider a localized mode of the stationary DNLS equation (1.2) for small $|\omega - \omega_0|$ and derive the modulation equations for slowly varying parameters of the localized mode.

The following result is standard and proved in many papers by using the local bifurcation analysis.

Lemma 1. *Assume that* $\{V_n\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ *and that* H *has a simple eigenvalue* $\omega_0 < 0$ *with a normalized eigenfunction* $\boldsymbol{\psi}_0 \in l^2$ *such that* $\|\boldsymbol{\psi}_0\|_{l^2} = 1$. *For any* $p > 0$, *there exist positive constants* ϵ_0, κ *and* C *such that for all* $\omega \in [\omega_0, \omega_0 + \epsilon_0)$, *there exists a unique real-valued solution* $\boldsymbol{\phi}(\omega) \in C([\omega_0, \omega_0+\epsilon_0), l^2) \cap C^2((\omega_0, \omega_0+\epsilon_0), l^2)$ *of the stationary DNLS equation (1.2) satisfying*

$$\left\| e^{\kappa|\cdot|}\partial_\omega^i \left(\boldsymbol{\phi}(\omega) - \frac{(\omega-\omega_0)^{\frac{1}{2p}}\boldsymbol{\psi}_0}{\|\boldsymbol{\psi}_0\|_{l^{2p+2}}^{1+\frac{1}{p}}} \right) \right\|_{l^2} \leq C(\omega-\omega_0)^{1-i+\frac{1}{2p}} \quad \textit{for } i = 0,\ 1. \tag{2.1}$$

Recall that the DNLS equation (1.1) is globally well-posed in any polynomially weighted l^2 space (see, e.g., [12, 13, 15]). Using Lemma 1, we decompose a suitable solution to the DNLS equation (1.1) into a family of stationary solutions with time varying parameters and a radiation part using the substitution,

$$\mathbf{u}(t) = e^{-i\theta(t)}\left(\boldsymbol{\phi}(\omega(t)) + \mathbf{z}(t)\right), \tag{2.2}$$

where $(\omega, \theta) \in \mathbb{R}^2$ represents a two-dimensional orbit of stationary solutions (their time evolution will be specified later) and $\mathbf{z}(t) \in C^1(\mathbb{R}, l^2(\mathbb{Z}))$ solves the time-evolution equation,

$$i\dot{\mathbf{z}} = (H-\omega)\mathbf{z} - (\dot\theta - \omega)(\boldsymbol{\phi}(\omega)+\mathbf{z}) - i\dot\omega\partial_\omega\boldsymbol{\phi}(\omega) + \mathbf{N}(\boldsymbol{\phi}(\omega)+\mathbf{z}) - \mathbf{N}(\boldsymbol{\phi}(\omega)), \tag{2.3}$$

where $H = -\Delta + V$ and $[\mathbf{N}(\boldsymbol{\psi})]_n = |\psi_n|^{2p}\psi_n$.

The linearized operator around a stationary solution $\mathbf{u}(t) = e^{-it\omega}\boldsymbol{\phi}(\omega)$ of the DNLS equation (1.1), where ω is fixed, is given by

$$\mathcal{L}(\omega)\mathbf{z} := (H-\omega)\mathbf{z} + \mathcal{W}(\omega)\mathbf{z} + p\mathcal{W}(\omega)(\mathbf{z}+\bar{\mathbf{z}}), \tag{2.4}$$

where $\mathcal{W}(\omega) : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is a diagonal operator with $\mathcal{W}_n(\omega) = \phi_n^{2p}(\omega)$. The generalized kernel of the linearized operator (2.4) is spanned by $i\boldsymbol{\phi}(\omega)$ and $\partial_\omega\boldsymbol{\phi}(\omega)$.

To determine uniquely the time evolution for (ω, θ), we assume that $\mathbf{z}$ satisfies the symplectic orthogonality conditions to the generalized kernel of the linearized operator at each time $t \in \mathbb{R}$:

$$\mathrm{Re}\langle \mathbf{z}(t), \boldsymbol{\psi}_1(\omega(t))\rangle = \mathrm{Im}\langle \mathbf{z}(t), \boldsymbol{\psi}_2(\omega(t))\rangle = 0\,, \tag{2.5}$$

where

$$\boldsymbol{\psi}_1(\omega) = \frac{\boldsymbol{\phi}(\omega)}{\|\boldsymbol{\phi}(\omega))\|_{l^2}}, \quad \boldsymbol{\psi}_2(\omega) = \frac{\partial_\omega\boldsymbol{\phi}(\omega)}{\|\partial_\omega\boldsymbol{\phi}(\omega)\|_{l^2}}.$$

Under this condition, $\mathbf{z}(t)$ belongs to the subspace associated to the continuous spectrum of the linear operator $\mathcal{L}(\omega(t))$ for each $t \in \mathbb{R}$.

Lemma 1 implies $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ are locally close to $\boldsymbol{\psi}_0$, the normalized eigenfunction of H for eigenvalue ω_0, that is, for any $\omega \in (\omega_0, \omega_0+\epsilon_0)$, $\alpha \geq 0$, and $s \geq 1$, there exists $C_{\alpha,s} > 0$ such that

$$\| < n >^\alpha (\boldsymbol{\psi}_1 - \boldsymbol{\psi}_0)\|_{l^s} + \| < n >^\alpha (\boldsymbol{\psi}_2 - \boldsymbol{\psi}_0)\|_{l^s} \leq C_{\alpha,s}(\omega-\omega_0). \tag{2.6}$$

The following result has been proved by Kevrekidis *et al.* [5].

Lemma 2. *Fix* $\omega_* \in (\omega_0, \omega_0+\epsilon_0)$. *There exist* δ_0, $C > 0$ *such that for any* $\delta \in (0, \delta_0)$ *and any* $\mathbf{u} \in l^2$ *satisfying*

$$\|\mathbf{u} - \boldsymbol{\phi}(\omega_*)\|_{l^2} \leq \delta(\omega_* - \omega_0)^{\frac{1}{2p}}, \tag{2.7}$$

there exist unique $(\omega, \theta) \in \mathbb{R}^2$ *and* $\mathbf{z} \in l^2(\mathbb{Z})$ *in the decomposition*

$$\mathbf{u} = e^{-i\theta}\left(\boldsymbol{\phi}(\omega) + \mathbf{z}\right)$$

subject to the symplectic orthogonality conditions

$$\mathrm{Re}\langle \mathbf{z}, \boldsymbol{\psi}_1(\omega)\rangle = \mathrm{Im}\langle \mathbf{z}, \boldsymbol{\psi}_2(\omega)\rangle = 0\,,$$

and the bound

$$|\omega - \omega_*| \le C\delta(\omega_* - \omega_0), \quad |\theta| \le C\delta, \quad \|\mathbf{z}\|_{l^2} \le C\delta(\omega_* - \omega_0)^{\frac{1}{2p}}. \tag{2.8}$$

The mapping $l^2(\mathbb{Z}) \ni \mathbf{u} \mapsto (\omega, \theta, \mathbf{z}) \in \mathbb{R}^2 \times l^2(\mathbb{Z})$ is a C^1 diffeomorphism.

Assuming $(\omega, \theta) \in C^1(\mathbb{R}, \mathbb{R}^2)$ and using the decomposition (2.2), we define the time evolution of (ω, θ) from the projections of the time evolution equation (2.3) with the symplectic orthogonality conditions (2.5). The resulting system is written in the matrix–vector form

$$\mathbf{A}(\omega, \mathbf{z}) \begin{bmatrix} \dot{\omega} \\ \dot{\theta} - \omega \end{bmatrix} = \mathbf{f}(\omega, \mathbf{z}), \tag{2.9}$$

where

$$\mathbf{A}(\omega, \mathbf{z}) = \begin{bmatrix} \langle \partial_\omega \boldsymbol{\phi}(\omega), \boldsymbol{\psi}_1(\omega)\rangle & 0 \\ 0 & \langle \boldsymbol{\phi}(\omega), \boldsymbol{\psi}_2(\omega)\rangle \end{bmatrix} + \begin{bmatrix} -\mathrm{Re}\langle \mathbf{z}, \partial_\omega \boldsymbol{\psi}_1(\omega)\rangle & \mathrm{Im}\langle \mathbf{z}, \boldsymbol{\psi}_1(\omega)\rangle \\ \mathrm{Im}\langle \mathbf{z}, \partial_\omega \boldsymbol{\psi}_2(\omega)\rangle & \mathrm{Re}\langle \mathbf{z}, \boldsymbol{\psi}_2(\omega)\rangle \end{bmatrix}$$

and

$$\mathbf{f}(\omega, \mathbf{z}) = \begin{bmatrix} \mathrm{Im}\langle \mathbf{N}(\boldsymbol{\phi}(\omega) + \mathbf{z}) - \mathbf{N}(\boldsymbol{\phi}(\omega)) - \mathcal{W}(\omega)\mathbf{z}, \boldsymbol{\psi}_1(\omega)\rangle \\ \mathrm{Re}\langle \mathbf{N}(\boldsymbol{\phi}(\omega) + \mathbf{z}) - \mathbf{N}(\boldsymbol{\phi}(\omega)) - (2p+1)\mathcal{W}(\omega)\mathbf{z}, \boldsymbol{\psi}_2(\omega)\rangle \end{bmatrix}.$$

Using an elementary property for power functions, we see that for $p \ge 1$, there exists a $C_p > 0$ such that

$$||a+b|^{2p}(a+b) - |a|^{2p}a| \le C_p(|a|^{2p}|b| + |b|^{2p+1}), \quad \forall a, b \in \mathbb{C}. \tag{2.10}$$

As a result, for any $s \ge 1$, the vector fields of system (2.3) and (2.9) are bounded by

$$\|\mathbf{N}(\boldsymbol{\phi}(\omega) + \mathbf{z}) - \mathbf{N}(\boldsymbol{\phi}(\omega)\|_{l^s} \le C_s \left(\|\boldsymbol{\phi}^{2p}(\omega)\mathbf{z}\|_{l^s} + \|\mathbf{z}\|_{l^s}^{2p+1} \right), \tag{2.11}$$

and

$$\|\mathbf{f}(\omega, \mathbf{z})\| \le C \sum_{j=1}^{2} \left(\|\boldsymbol{\phi}(\omega)^{2p-1}\boldsymbol{\psi}_j(\omega)\mathbf{z}^2\|_{l^1} + \|\boldsymbol{\psi}_j(\omega)\mathbf{z}^{2p+1}\|_{l^1} \right), \tag{2.12}$$

for some $C_s, C > 0$, where the pointwise multiplication of vectors on $\mathbb{Z}$ is understood in the sense of $(\boldsymbol{\phi}\boldsymbol{\psi})_n = \phi_n \psi_n$.

Thanks to Lemma 1, $\mathbf{A}(\omega, \mathbf{z})$ is invertible for small $\mathbf{z} \in l^2$ and any $\omega \in (\omega_0, \omega_0 + \epsilon_0)$. Using bounds (2.1), (2.6), and (2.12), we obtain that for any $\mathbf{z}$ such that $\|\mathbf{z}\|_{l^2} \le C_0(\omega - \omega_0)^{\frac{1}{2p}}$ for a $C_0 > 0$, there exists a $C > 0$ such that

$$|\dot{\omega}| \le C(\omega - \omega_0)^{2 - \frac{1}{p}} \|e^{-\kappa|n|}\mathbf{z}^2\|_{l^1}, \tag{2.13}$$

$$|\dot{\theta} - \omega| \le C(\omega - \omega_0)^{1 - \frac{1}{p}} \|e^{-\kappa|n|}\mathbf{z}^2\|_{l^1}. \tag{2.14}$$

The following theorem describes our main result.

Theorem 1. *Assume that* $\{V_n\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ *decays exponentially fast to zero as* $|n| \to \infty$ *and that* H *has only one eigenvalue* $\omega_0 < 0$ *outside of the continuous spectrum* $\sigma_c(H) = [0,4]$ *and has no resonance at the end points* $\{0,4\}$*. For any* $p > 2.75$*, there exist an* $\epsilon_0 > 0$ *and a* $\delta > 0$ *such that if* $\epsilon := \omega_* - \omega_0 \in (0, \epsilon_0)$ *and*

$$\| < n > (\mathbf{u}(0) - \boldsymbol{\phi}(\omega_*))\|_{l^1} \le \delta \epsilon^{\frac{1}{2p}}, \tag{2.15}$$

then there exist $C > 0$*,* $\theta_\infty \in \mathbb{R}$*,* $\omega_\infty \in (\omega_0, \omega_0 + \epsilon_0)$*,* $(\omega, \theta) \in C^1(\mathbb{R}_+, \mathbb{R}^2)$*, such that*

$$\lim_{t\to\infty} \left(\theta(t) - \int_0^t \omega(s) ds \right) = \theta_\infty , \quad \lim_{t\to\infty} \omega(t) = \omega_\infty , \tag{2.16}$$

and

$$\sup_{t \ge 0} |\omega(t) - \omega_*| \le C\delta\epsilon , \tag{2.17}$$

whereas $\mathbf{u}(t) \in C^1(\mathbb{R}_+, l^2)$ *solves the DNLS equation (1.1) and, for any* $s \in (2,4) \cup (4, \infty]$ *and* $t \ge 0$*, there exists* $C_s > 0$ *such that*

$$\|\mathbf{u}(t) - e^{-i\theta(t)} \boldsymbol{\phi}(\omega(t))\|_{l^s} \le C_s \delta \epsilon^{\frac{1}{2p}} (1+t)^{-\alpha_s} ,$$

where $\alpha_s \in \left(0, \frac{1}{3}\right]$ *is given by (1.4).*

3. **Proof of Theorem 1.** Let $\mathbf{y}(t) = e^{-i\theta(t)}\mathbf{z}(t)$ and write the time-evolution problem for $\mathbf{y}(t)$ in the form,

$$i\dot{\mathbf{y}} = H\mathbf{y} + \mathbf{g}, \quad \mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}_3, \tag{3.1}$$

where

$$\begin{aligned} \mathbf{g}_1 &:= \left(\mathbf{N}(\boldsymbol{\phi}(\omega) + \mathbf{y}e^{i\theta}) - \mathbf{N}(\boldsymbol{\phi}(\omega)) \right) e^{-i\theta}, \\ \mathbf{g}_2 &:= -(\dot{\theta} - \omega)\boldsymbol{\phi}(\omega) e^{-i\theta}, \\ \mathbf{g}_3 &:= -i\dot{\omega}\partial_\omega \boldsymbol{\phi}(\omega) e^{-i\theta}. \end{aligned}$$

Let $P_0 = \langle \cdot, \boldsymbol{\psi}_0 \rangle \boldsymbol{\psi}_0$, $Q_0 = (I - P_0) \equiv P_{a.c.}(H)$, where $\boldsymbol{\psi}_0$ is defined in Lemma 1. We decompose the solution $\mathbf{y}(t)$ into two orthogonal parts,

$$\mathbf{y}(t) = a(t)\boldsymbol{\psi}_0 + \boldsymbol{\eta}(t), \tag{3.2}$$

where $\langle \boldsymbol{\eta}(t), \boldsymbol{\psi}_0 \rangle = 0$ and $a(t) = \langle \mathbf{y}(t), \boldsymbol{\psi}_0 \rangle$. The new coordinates $a(t)$ and $\boldsymbol{\eta}(t)$ satisfy the time evolution problem,

$$\left\{ \begin{array}{rcl} i\dot{a} &=& \omega_0 a + \langle \mathbf{g}, \boldsymbol{\psi}_0 \rangle, \\ i\dot{\boldsymbol{\eta}} &=& H\boldsymbol{\eta} + Q_0 \mathbf{g}. \end{array} \right. \tag{3.3}$$

The time-evolution problem for $\boldsymbol{\eta}(t)$ can be written in the integral form,

$$\boldsymbol{\eta}(t) = e^{-itH} Q_0 \boldsymbol{\eta}(0) - i \int_0^t e^{-i(t-s)H} Q_0 \mathbf{g}(s) ds. \tag{3.4}$$

We shall use the following lemmas to estimate decay of $\boldsymbol{\eta}(t)$.

Lemma 3. *Assume that* $\{V_n\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ *decays exponentially fast to zero as* $|n| \to \infty$ *and that* H *has only one eigenvalue* $\omega_0 < 0$ *outside the continuous spectrum* $\sigma_c(H) = [0,4]$ *and no resonance at the end points* $\{0,4\}$*. For any* $s \ge 2$*, there is* $C_s > 0$ *such that for all* $t \in \mathbb{R}$*,*

$$\|e^{-itH} Q_0 \mathbf{f}\|_{l^s} \le C_s (1+|t|)^{-\alpha_s} \|\mathbf{f}\|_{l^1}, \quad \alpha_s = \left\{ \begin{array}{ll} \frac{s-2}{2s}, & 2 \le s < 4, \\ \frac{s-1}{3s}, & 4 < s \le \infty. \end{array} \right. \tag{3.5}$$

The proof of Lemma 3 will be given in Section 4.

Lemma 4. *Assume that* $\{V_n\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ *decays exponentially fast to zero as* $|n| \to \infty$ *and that* H *has only one eigenvalue* $\omega_0 < 0$ *outside the continuous spectrum* $\sigma_c(H) = [0,4]$ *and no resonance at the end points* $\{0,4\}$*. There is* $C > 0$ *such that*

$$\| < n >^{-1} e^{-itH} Q_0 \mathbf{f}\|_{l^\infty} \leq C(1+|t|)^{-\frac{4}{3}} \| < n > \mathbf{f}\|_{l^1}, \quad t \in \mathbb{R}, \tag{3.6}$$

The proof of Lemma 4 will be given in Section 5. Note that a continuous analogue of the bound (3.6) for the one-dimensional Schrödinger operator was proved by Schlag [18].

Corollary 1. *Under assumptions of Lemma 4, for any* $\alpha \in [0,1]$*, there is* $C_\alpha > 0$ *such that for all* $t \in \mathbb{R}$*,*

$$\| < n >^{-\alpha} e^{-itH} Q_0 \mathbf{f}\|_{l^\infty} \leq C_\alpha (1+|t|)^{-\frac{1}{3}-\alpha} \| < n >^{\alpha} \mathbf{f}\|_{l^1}. \tag{3.7}$$

Proof. The corollary is proved by the interpolation between bounds (1.3) and (3.6). □

Lemma 5. *Let* $\mathbf{u}(t) \in C(\mathbb{R}, l^2(\mathbb{Z}))$ *be a solution of the DNLS equation (1.1) and assume that* $< n > \mathbf{u}(0) \in l^2(\mathbb{Z})$*. For any* $\alpha \in [0,1]$*, there is* $C_\alpha > 0$ *such that for all* $t \in \mathbb{R}$*,*

$$\| < n >^{\alpha} \mathbf{u}(t)\|_{l^2} \leq C_\alpha (1+|t|)^{\alpha} \| < n > \mathbf{u}(0)\|_{l^2}. \tag{3.8}$$

Proof. We first recall the l^2-conservation law for the DNLS equation (1.1),

$$\|\mathbf{u}(t)\|_{l^2} = \|\mathbf{u}(0)\|_{l^2}, \quad t \in \mathbb{R}. \tag{3.9}$$

It follows from the DNLS equation (1.1) that

$$\begin{aligned} \frac{d}{dt} \sum_{n\in\mathbb{Z}} (1+n^2)|u_n(t)|^2 &= i \sum_{n\in\mathbb{Z}} (1+n^2) \left(\bar{u}_n u_{n+1} - u_n \bar{u}_{n+1} + \bar{u}_n u_{n-1} - u_n \bar{u}_{n-1}\right) \\ &= i \sum_{n\in\mathbb{Z}} (1+2n) \left(\bar{u}_{n+1} u_n - u_{n+1} \bar{u}_n\right). \end{aligned}$$

By the Cauchy-Schwarz inequality and the l^2-conservation law (3.9), there exists a $C > 0$ such that

$$\left| \frac{d}{dt} \| < n > \mathbf{u}(t)\|_{l^2} \right| \leq C \|\mathbf{u}(0)\|_{l^2}, \tag{3.10}$$

which yields

$$\| < n > \mathbf{u}(t)\|_{l^2} \leq C(1+|t|) \| < n > \mathbf{u}(0)\|_{l^2}, \tag{3.11}$$

that is, a bound (3.8) for $\alpha = 1$. To get (3.8) for an arbitrary $\alpha \in [0,1]$, we represent

$$\| < n >^{\alpha} \mathbf{u}\|_{l^2}^2 = \sum_{n\in\mathbb{Z}} < n >^{2\alpha} |u_n|^{2\alpha} |u_n|^{2(1-\alpha)}$$

and use the Hölder inequality to obtain

$$\| < n >^{\alpha} \mathbf{u}(t)\|_{l^2} \leq \| < n > \mathbf{u}(t)\|_{l^{2\alpha q}}^{\alpha} \|\mathbf{u}(t)\|_{l^{2(1-\alpha)q'}}^{1-\alpha}, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

The choice $q = \frac{1}{\alpha} \geq 1$ and $q' = \frac{1}{1-\alpha} \geq 1$, the l^2-conservation law (3.9), and the estimate (3.11) imply (3.8) for any $\alpha \in [0,1]$. □

We shall now set up function spaces, where the solution of the time-evolution equation (3.1) is bounded. We define

$$\begin{aligned} M_1(t) &:= \sup_{0\le\tau\le t}(1+\tau)^{\alpha_{2p+1}}\|\mathbf{y}(\tau)\|_{l^{2p+1}} + \sup_{0\le\tau\le t}(1+\tau)^{\alpha_{4p}}\|\mathbf{y}(\tau)\|_{l^{4p}}\,, \\ M_2(t) &:= \sup_{0\le\tau\le t}(1+\tau)^{\nu_\alpha}\| <n>^{-\alpha}\mathbf{y}(\tau)\|_{l^\infty}, \\ M_3(t) &:= \sup_{0\le\tau\le t}|\omega(\tau)-\omega_*|, \end{aligned} \tag{3.12}$$

where $\alpha_s = \frac{s-1}{3s}$ for $s>4$ and α and ν_α are positive constants that will be fixed later (see (3.21) and (3.25) below).

To control (ω,θ) along the solution, we use estimates (2.13) and (2.14) that follow from the modulation equations (2.9). We need to show that $\dot\omega, \dot\theta-\omega\in L^1(\mathbb{R})$ so that there exist limits

$$\omega_\infty := \lim_{t\to\infty}\omega(t), \quad \theta_\infty := \lim_{t\to\infty}\left(\theta(t) - \int_0^t \omega(s)ds\right).$$

It follows from the bound (2.13) that for any $t\ge 0$ and any $\alpha\in[0,1]$,

$$\begin{aligned} |\dot\omega(t)| &\le C(\omega-\omega_0)^{2-\frac{1}{p}}\| <n>^{2\alpha} e^{-\kappa|n|}\|_{l^1}\| <n>^{-\alpha}\mathbf{y}(t)\|^2_{l^\infty} \\ &\le C(\omega-\omega_0)^{2-\frac{1}{p}}(1+t)^{-2\nu_\alpha}M_2^2(t). \end{aligned} \tag{3.13}$$

Therefore, $\dot\omega\in L^1(\mathbb{R})$ if $M_2(t)$ is bounded on $\mathbb{R}_+$ and $\nu_\alpha>\frac{1}{2}$.

Similarly, the bound (2.14) implies that

$$|\dot\theta(t)-\omega(t)| \le C(\omega-\omega_0)^{1-\frac{1}{p}}(1+t)^{-2\nu_\alpha}M_2^2(t), \tag{3.14}$$

hence we have $\dot\theta-\omega\in L^1(\mathbb{R})$ if $M_2(t)$ is bounded on $\mathbb{R}_+$ and $\nu_\alpha>\frac{1}{2}$.

Now we need to estimate $\| <n>^{-\alpha}\mathbf{y}(t)\|_{l^\infty}$ by using the decomposition (3.2) and the integral equation (3.4). First, the projection of $\mathbf{y}(t)$ to $\boldsymbol{\psi}_0$ is controlled by the orthogonality condition (2.5) and an approximation formula (2.6). For any $\alpha\in[0,1]$, there is $C_\alpha>0$ such that

$$\begin{aligned} |a(t)| &= |\langle \mathbf{y},\boldsymbol{\psi}_0\rangle| \le |\mathrm{Re}\langle \mathbf{z},\boldsymbol{\psi}_0-\boldsymbol{\psi}_1\rangle| + |\mathrm{Im}\langle \mathbf{z},\boldsymbol{\psi}_0-\boldsymbol{\psi}_2\rangle| \\ &\le (\| <n>^{\alpha}(\boldsymbol{\psi}_1-\boldsymbol{\psi}_0)\|_{l^1} + \| <n>^{\alpha}(\boldsymbol{\psi}_2-\boldsymbol{\psi}_0)\|_{l^1})\| <n>^{-\alpha}\mathbf{z}(t)\|_{l^\infty} \\ &\le C_\alpha(\omega-\omega_0)\| <n>^{-\alpha}\mathbf{y}(t)\|_{l^\infty}. \end{aligned} \tag{3.15}$$

Using the bound

$$\| <n>^{-\alpha}\mathbf{y}(t)\|_{l^\infty} \le |a(t)|\| <n>^{-\alpha}\boldsymbol{\psi}_0\|_{l^\infty} + \| <n>^{-\alpha}\boldsymbol{\eta}(t)\|_{l^\infty} \tag{3.16}$$

and the estimate (3.15) with small $(\omega-\omega_0)$, we have

$$\| <n>^{-\alpha}\mathbf{y}(t)\|_{l^\infty} \le 2\| <n>^{-\alpha}\boldsymbol{\eta}(t)\|_{l^\infty}. \tag{3.17}$$

Next, we consider the dispersive term $\boldsymbol{\eta}(t)$. Using the bound (2.11), the decay estimate in Corollary 1, and the integral equation (3.4), we obtain for any $\alpha\in[0,1]$ and all $t\in\mathbb{R}$,

$$\begin{aligned} \| <n>^{-\alpha}\boldsymbol{\eta}(t)\|_{l^\infty} \le\ & C_\alpha(1+|t|)^{-\frac{1}{3}-\alpha}\| <n>^{\alpha}\boldsymbol{\eta}(0)\|_{l^1} \\ & + C_\alpha\int_0^t (1+|t-s|)^{-\frac{1}{3}-\alpha}\| <n>^{\alpha}\mathbf{g}(s)\|_{l^1}ds. \end{aligned} \tag{3.18}$$

By (2.1), (2.6), (2.11), (3.13), and (3.14), we have

$$\begin{aligned} \| <n>^{\alpha} \mathbf{g}_1(s)\|_{l^1} &\leq C\| <n>^{2\alpha} \boldsymbol{\phi}^{2p}(\omega)\|_{l^1} \| <n>^{-\alpha} \mathbf{y}(s)\|_{l^\infty} \\ &\qquad + C\| <n>^{\alpha} \mathbf{y}^{2p+1}(s)\|_{l^1} \\ &\leq C(I_1(s) + I_3(s)) \end{aligned}$$

and

$$\begin{aligned} \| <n>^{\alpha} \mathbf{g}_{2,3}(s)\|_{l^1} &\leq C(|\dot{\theta} - \omega| \| <n>^{\alpha} \boldsymbol{\phi}(\omega)\|_{l^1} + |\dot{\omega}| \| <n>^{\alpha} \partial_\omega \boldsymbol{\phi}(\omega)\|_{l^1}) \\ &\leq C I_2(s)\,, \end{aligned}$$

where

$$\begin{aligned} I_1(s) &:= (\omega - \omega_0)(1+s)^{-\nu_\alpha} M_2(s), \\ I_2(s) &:= (\omega - \omega_0)^{1-\frac{1}{2p}}(1+s)^{-2\nu_\alpha} M_2^2(s), \\ I_3(s) &:= \|\mathbf{y}(s)\|_{l^{4p}}^{2p} \| <n>^{\alpha} \mathbf{y}(s)\|_{l^2}. \end{aligned}$$

We recall the standard pointwise estimate. For any $\beta_1, \beta_2 > 0$ such that $\beta_1, \beta_2 \neq 1$, there exists a $C > 0$ such that for all $t > 0$,

$$\int_0^t \frac{1}{(1+t-s)^{\beta_1}} \frac{1}{(1+s)^{\beta_2}} ds \leq \frac{C}{(1+t)^{\gamma}}, \quad \gamma = \min(\beta_1, \beta_2, \beta_1 + \beta_2 - 1). \tag{3.19}$$

This estimate is useful only if either $\beta_1 > 1$ or $\beta_2 > 1$ (or both), in which case $\gamma = \min(\beta_1, \beta_2)$. If both $\beta_1, \beta_2 < 1$, then $\gamma = \beta_1 + \beta_2 - 1 < \min(\beta_1, \beta_2)$ and the decay estimates for the bound (3.18) cannot be closed.

In view of Corollary 1, we expect that $0 < \nu_\alpha \leq \alpha + \frac{1}{3}$. Thus, in order to obtain

$$\begin{aligned} &\textstyle\int_0^t (1+t-s)^{-\frac{1}{3}-\alpha}(I_1(s) + I_2(s))ds \\ \leq \quad & C(1+t)^{-\nu_\alpha}\left[(\omega - \omega_0)M_2(t) + (\omega - \omega_0)^{1-\frac{1}{2p}} M_2^2(t)\right] \end{aligned}$$

from (3.19) with $\beta_1 = \frac{1}{3} + \alpha$ and $\beta_2 = \nu_\alpha$, we need $\alpha > \frac{2}{3}$.

The nonlinear term I_3 can be estimated if $\mathbf{y}(t)$ inherits the dispersive decay estimate (3.5) for $s = 4p$. Indeed by Lemma 5, we have

$$\begin{aligned} I_3(t) &\leq C(1+t)^{-2p\alpha_{4p}} M_1^{2p}(t)((1+t)^{\alpha}\| <n> \mathbf{u}(0)\|_{l^2} + \| <n>^{\alpha} \boldsymbol{\phi}(\omega)\|_{l^2}) \\ &\leq C(1+t)^{-2p\alpha_{4p}+\alpha} M_1^{2p}(t)\left(\epsilon^{\frac{1}{2p}} + (\omega - \omega_0)^{\frac{1}{2p}}\right). \end{aligned} \tag{3.20}$$

Applying (3.19) with $\beta_1 = \frac{1}{3} + \alpha > 1$ and $\beta_2 = 2p\alpha_{4p} - \alpha = \frac{2}{3}p - \frac{1}{6} - \alpha$, we obtain

$$\int_0^t (1+t-s)^{-\alpha-\frac{1}{3}} I_3(s) ds \leq C(1+t)^{-\nu_\alpha} M_1^{2p}(t)\left(\epsilon^{\frac{1}{2p}} + (\omega - \omega_0)^{\frac{1}{2p}}\right)$$

with

$$\nu_\alpha = \min\left(\frac{1}{3} + \alpha, \frac{2}{3}p - \frac{1}{6} - \alpha\right). \tag{3.21}$$

To ensure that $\nu_\alpha > \frac{1}{2}$, we require $\frac{2}{3}p - \frac{1}{6} - \alpha > \frac{1}{2}$, that is, $p > 1 + \frac{3}{2}\alpha > 2$. Hence, it follows from (3.16), (3.18), and (3.20) that

$$\begin{aligned} M_2(t) \quad &\leq \quad C\| <n>^{\alpha} \boldsymbol{\eta}(0)\|_{l^1} + C(\omega - \omega_0)M_2(t) + C(\omega - \omega_0)^{1-\frac{1}{2p}} M_2^2(t) \\ &\qquad + C M_1^{2p}(t)\left(\epsilon^{\frac{1}{2p}} + (\omega - \omega_0)^{\frac{1}{2p}}\right). \end{aligned} \tag{3.22}$$

Now we turn to estimate $M_1(t)$. We have

$$\|\mathbf{y}(t)\|_{l^s} \leq 2\|\boldsymbol{\eta}(t)\|_{l^s} \tag{3.23}$$

in the same way as (3.17). Applying Lemma 3 to the integral equation (3.4), we obtain for any $s \in [2,\infty]\backslash\{4\}$ and $t \geq 0$,

$$\|\boldsymbol{\eta}(t)\|_{l^s} \leq C_s(1+t)^{-\alpha_s}\|\boldsymbol{\eta}(0)\|_{l^1} + C_s\int_0^t (1+t-s)^{-\alpha_s}\|\mathbf{g}(s)\|_{l^1}ds. \tag{3.24}$$

By the bound (2.11), there exists a $C_\alpha > 0$ for any $\alpha \in [0,1]$ such that

$$\begin{aligned}\|\mathbf{g}_1(s)\|_{l^1} &\leq C_\alpha\left(\| <n>^\alpha \boldsymbol{\phi}^{2p}(\omega)\|_{l^1}\| <n>^{-\alpha}\mathbf{y}(s)\|_{l^\infty} + \|\mathbf{y}(s)\|^{2p+1}_{l^{2p+1}}\right)\\ &\leq C_\alpha I_1(s) + C_\alpha(1+s)^{-(2p+1)\alpha_{2p+1}}M_1^{2p+1}(s)\,.\end{aligned}$$

Furthermore,

$$\|\mathbf{g}_2(s)\|_{l^1} + \|\mathbf{g}_3(s)\|_{l^1} \leq C_\alpha I_2(s)\,.$$

Now we apply (3.19) with $(\beta_1,\beta_2) = (\alpha_s,\nu_\alpha)$, $(\alpha_s,(2p+1)\alpha_{2p+1})$, $(\alpha_s, 2\nu_\alpha)$ to obtain

$$\begin{aligned}&\int_0^t (1+t-s)^{-\alpha_s}\|\mathbf{g}(s)\|_{l^1}ds\\ &\leq C(1+t)^{-\alpha_s}\left[(\omega-\omega_0)M_2(t) + (\omega-\omega_0)^{1-\frac{1}{2p}}M_2^2(t) + M_1^{2p+1}(t)\right].\end{aligned}$$

Because $\beta_1 = \alpha_s < 1$, we now need

$$\nu_\alpha > 1 \quad \text{and} \quad (2p+1)\alpha_{2p+1} = \frac{2}{3}p > 1.$$

The condition $p > \frac{3}{2}$ $(2p+1 > 4)$ corresponds to an exponent that appears in Mielke–Patz [11] so that small solutions to the DNLS equation (1.1) with $V \equiv 0$ decay like $e^{it\Delta}\mathbf{u}(0)$. Since $\alpha > \frac{2}{3}$, the condition $\nu_\alpha > 1$ is attained if

$$\frac{2}{3}p - \frac{1}{6} - \alpha > 1 \quad \Rightarrow \quad p > \frac{7}{4} + \frac{3}{2}\alpha > \frac{11}{4} = 2.75,$$

in which case we can fix α for any

$$\frac{2}{3} < \alpha < \min\left(1, \frac{2}{3}p - \frac{7}{6}\right). \tag{3.25}$$

Thus for α and ν_α satisfying (3.25) and (3.21) respectively, we have

$$M_1(t) \leq C\|\boldsymbol{\eta}(0)\|_{l^1} + C(\omega-\omega_0)M_2(t) + C(\omega-\omega_0)^{1-\frac{1}{2p}}M_2^2(t) + CM_1^{2p+1}(t). \tag{3.26}$$

By Lemma 2 and the bound (3.13), we obtain

$$M_3(t) \leq |\omega(0)-\omega_*| + \int_0^t |\dot{\omega}(s)|ds \leq C\delta\epsilon + \sup_{0\leq\tau\leq t}|\omega(\tau)-\omega_0|^{2-\frac{1}{p}}M_2^2(t)\,.$$

By the triangle inequality,

$$|\omega(t)-\omega_0| \leq |\omega(t)-\omega_*| + |\omega_*-\omega_0| \leq M_3(t) + \epsilon,$$

we have $\omega(t) - \omega_0 = \mathcal{O}(\varepsilon)$ as long as $M_2(t)$ remains bounded. Thanks to the smallness of $\omega - \omega_0$, it follows from (3.22) and (3.26) that

$$M_1(t) + M_2(t) \leq C\| <n>^\alpha \boldsymbol{\eta}(0)\|_{l^1} + CM_2^2(t) + CM_1^{2p}(t)\left(\epsilon^{\frac{1}{2p}} + M_1(t)\right).$$

Since $M_1(t)$ and $M_2(t)$ are continuous, we have

$$\sup_{t\geq 0}\left(M_1(t) + M_2(t)\right) \leq 2C\| <n>^\alpha \mathbf{y}(0)\|_{l^1} \leq 2C\delta\epsilon^{\frac{1}{2p}} \tag{3.27}$$

and

$$\sup_{t\geq 0} M_3(t) \leq 2C\delta\epsilon, \tag{3.28}$$

if δ is chosen sufficiently small in (2.15). Once we prove (3.27), decay estimates

$$\|\mathbf{y}(t)\|_{l^s} \leq C_{\alpha,s}(1+t)^{-\alpha_s}\| <n>^{\alpha} \mathbf{y}(0)\|_{l^1}, \quad s \in [2,\infty] \setminus \{4\}, \quad t \geq 0,$$

follows immediately from (3.24). Thus the proof of Theorem 1 is now complete. □

We finish this section with a remark that the bound (3.8) can be extended to $\alpha > 1$. To be precise, for any $\alpha \geq 1$, there exists $C_\alpha > 0$ such that for all $t \in \mathbb{R}$,

$$\| <n>^{\alpha} \mathbf{u}(t)\|_{l^2} \leq C_\alpha \left(\| <n>^{\alpha} \mathbf{u}(0)\|_{l^2} + |t|^{\alpha} \| <n> \mathbf{u}(0)\|_{l^2}\right). \tag{3.29}$$

Indeed, for any $\alpha \in \mathbb{R}$, there is $C_\alpha > 0$ such that

$$\left|(2+2n+n^2)^{\alpha} - (1+n^2)^{\alpha}\right| \leq C_\alpha (1+n^2)^{\alpha-1/2}, \quad n \in \mathbb{Z}.$$

Using the same technique as in the proof of Lemma 5, we obtain from the DNLS equation (1.1),

$$\left|\frac{d}{dt}\sum_{n\in\mathbb{Z}}(1+n^2)^{\alpha}|u_n(t)|^2\right| \leq C_\alpha \sum_{n\in\mathbb{Z}}(1+n^2)^{\alpha-1/2}\left|\bar{u}_{n+1}u_n - u_{n+1}\bar{u}_n\right|,$$

so that

$$\left|\frac{d}{dt}\| <n>^{\alpha} \mathbf{u}(t)\|_{l^2}\right| \leq C_\alpha \| <n>^{\alpha-1} \mathbf{u}(t)\|_{l^2}. \tag{3.30}$$

Bound (3.29) follows from the integration of (3.30) in t using the bound (3.8) and recursion in α.

4. **Proof of Lemma 3.** In this section, we prove Lemma 3 by using the Jost function for the discrete Schrödinger operator. First, let us introduce the free resolvent $R_0(\lambda) = (-\Delta - \lambda)^{-1}$ for $\lambda \in \mathbb{C}\backslash[0,4]$ and the perturbed resolvent $R_V(\lambda) = (H-\lambda)^{-1}$ for $\lambda \in \mathbb{C}\backslash\sigma(H)$. Under the assumption of the fast decay of $\{V_n\}_{n\in\mathbb{Z}}$ to zero at infinity, for any fixed $\omega \in (0,4)$, there exist

$$R_0^{\pm}(\omega) = \lim_{\epsilon\downarrow 0} R_0(\omega \pm i\epsilon), \quad R_V^{\pm}(\omega) = \lim_{\epsilon\downarrow 0} R_V(\omega \pm i\epsilon)$$

as bounded operators from $l^1(\mathbb{Z})$ to $l^\infty(\mathbb{Z})$. In particular, we have

$$(R_0^{\pm}(\omega)\mathbf{f})_n = \pm \frac{1}{2i\sin(\theta)} \sum_{m\in\mathbb{Z}} e^{\mp i\theta|n-m|} f_m, \tag{4.1}$$

where $\theta \in (-\pi, 0)$ is uniquely defined from the roots of $2 - 2\cos(\theta) = \omega$ with $\omega \in (0,4)$. By the Cauchy formula, we write

$$e^{-itH}Q_0\mathbf{f} = \frac{1}{2\pi i}\int_0^4 e^{-it\omega}\left[R_V^{+}(\omega) - R_V^{-}(\omega)\right]\mathbf{f}\,d\omega, \tag{4.2}$$

where $Q_0 = P_{a.c.}(H)$.

To begin with, let us recall fundamental properties of the Jost functions. Let $\boldsymbol{\psi}^+(\theta)$ and $\boldsymbol{\psi}^-(\theta)$ be solutions of the linear discrete Schrödinger equation,

$$H\boldsymbol{\psi} = \omega\boldsymbol{\psi}, \quad \omega = 2 - 2\cos(\theta), \tag{4.3}$$

satisfying boundary conditions

$$\lim_{n\to\infty}|\psi_n^+ - e^{-in\theta}| = 0, \quad \lim_{n\to-\infty}|\psi_n^- - e^{in\theta}| = 0.$$

The Wronskian

$$W[\boldsymbol{\phi}, \boldsymbol{\psi}] := \phi_n\psi_{n+1} - \phi_{n+1}\psi_n$$

is independent of n if both $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are solutions of the same equation (4.3). Moreover, we have

$$W[\boldsymbol{\psi}^+(\theta),\boldsymbol{\psi}^+(-\theta)] = 2i\sin(\theta)\,, \quad W[\boldsymbol{\psi}^-(\theta),\boldsymbol{\psi}^-(-\theta)] = -2i\sin(\theta)\,,$$

and we define

$$W(\theta) := W[\boldsymbol{\psi}^+(\theta),\boldsymbol{\psi}^-(\theta)].$$

Furthermore, the scattering relation holds by the linear dependence of one set of two fundamental solutions from another set of two fundamental solutions of the discrete Schrödinger equation (4.3),

$$\left\{ \begin{array}{l} \boldsymbol{\psi}^-(\theta) = a(\theta)\boldsymbol{\psi}^+(\theta) + b(\theta)\boldsymbol{\psi}^+(-\theta)\,, \\ \boldsymbol{\psi}^+(\theta) = -b(-\theta)\boldsymbol{\psi}^-(\theta) + a(-\theta)\boldsymbol{\psi}^-(-\theta)\,, \end{array} \right. \tag{4.4}$$

where $\overline{a(\theta)} = a(-\theta)$, $\overline{b(\theta)} = b(-\theta)$, $1 + |a(\theta)|^2 = |b(\theta)|^2$ and

$$a(\theta) = \frac{W[\boldsymbol{\psi}^-(\theta),\boldsymbol{\psi}^+(-\theta)]}{2i\sin(\theta)}\,, \quad b(\theta) = \frac{W(\theta)}{2i\sin(\theta)}\,.$$

Note that $\overline{\boldsymbol{\psi}^\pm(\theta)} = \boldsymbol{\psi}^\pm(-\theta)$ for $\theta \in \mathbb{R}$ because the potential V is real-valued.

Because $|b(\theta)| \geq 1$ for all $\theta \in [-\pi,\pi]$, the Wronskian $W(\theta)$ does not vanish unless $\theta = 0, \pm\pi$. Since neither 0 nor 4 is a resonance of H, the Wronskian $W(\theta)$ does not vanish at $\theta = 0$, $\pm\pi$. Therefore, solutions $\boldsymbol{\psi}^+$ and $\boldsymbol{\psi}^-$ of the discrete Schrödinger equation (4.3) are linearly independent for all $\theta \in [-\pi,\pi]$.

Let $f_n^\pm(\theta) = e^{\pm in\theta}\psi_n^\pm(\theta)$ for $n \in \mathbb{Z}$. Then $f_n^\pm$ are solutions of

$$f_n^+(\theta) = 1 - \frac{i}{2\sin\theta}\sum_{m=n}^{\infty}(1 - e^{-2i(m-n)\theta})V_m f_m^+(\theta)\,,$$

$$f_n^-(\theta) = 1 - \frac{i}{2\sin\theta}\sum_{m=-\infty}^{n}(1 - e^{2i(m-n)\theta})V_m f_m^-(\theta)\,.$$

There exists a $C_0 > 0$ such that

$$\sup_{\theta\in[-\pi,\pi]}\left|\frac{1 - e^{-2iN\theta}}{\sin(\theta)}\right| \leq C_0 N \quad \text{for any } N \in \mathbb{Z}. \tag{4.5}$$

By the estimate (4.5) and the exponential decay of V as $n \to \pm\infty$, we see that for any $k \geq 0$,

$$\sup_{n\geq 0}\sum_{m=n}^{\infty}(m-n)^k|V_m| \leq \sum_{m=0}^{\infty} m^k|V_m| < \infty.$$

Thus $f_n^\pm(\theta)$ are smooth and

$$\sup_{\theta\in[-\pi,\pi]}\|\partial_\theta^k f^\pm(\theta)\|_{l^\infty(\mathbb{Z}_\pm)} < \infty \quad \text{for any } k \geq 0, \tag{4.6}$$

where $\mathbb{Z}_+$ is the set of nonnegative integers and $\mathbb{Z}_-$ is the set of nonpositive integers.

Now we are ready to prove Lemma 3.

Proof of Lemma 3. For $t \in [0,1]$, the bound (3.5) follows from the fact that

$$\|e^{-itH}Q_0\mathbf{f}\|_{l^s} \leq \|e^{-itH}Q_0\mathbf{f}\|_{l^2} = \|\mathbf{f}\|_{l^2} \leq \|\mathbf{f}\|_{l^1}$$

for any $\mathbf{f} \in l^1$, $s \geq 2$ and $t \in \mathbb{R}$.

For $t \geq 1$, we will prove the bound (3.5) by using the representation

$$\left(\int_0^4 e^{-it\omega}\left[R_V^+(\omega) - R_V^-(\omega)\right]\mathbf{f}\,d\omega\right)_n =: \sum_{m\in\mathbb{Z}} S^{m,n}(t) f_m\,.$$

By Minkowski's inequality, it suffices to show

$$\sup_{m\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}}|S^{m,n}|^s\right)^{\frac{1}{s}} \le Ct^{-\alpha_s}, \quad t\ge 1\,, \tag{4.7}$$

to prove Lemma 3.

Using the Jost functions, we can represent $S^{m,n}$ as

$$S^{m,n}(t) = \begin{cases} -2\displaystyle\int_{-\pi}^{\pi} e^{it(2-2\cos\theta)}\frac{\psi_m^+(\theta)\psi_n^-(\theta)}{W(\theta)}\sin(\theta)d\theta & \text{if } m>n,\\ -2\displaystyle\int_{-\pi}^{\pi} e^{it(2-2\cos\theta)}\frac{\psi_m^-(\theta)\psi_n^+(\theta)}{W(\theta)}\sin(\theta)d\theta & \text{if } m\le n. \end{cases} \tag{4.8}$$

Since $W(\theta)$ is continuous on $[-\pi,\pi]$ and $W(\theta)\ne 0$ for all $\theta\in[-\pi,\pi]$, there exists a positive constant c such that $|W(\theta)|\ge c$ for all $\theta\in[-\pi,\pi]$.

Let δ be a positive constant and let

$$\Gamma_m(t) = \{n\in\mathbb{Z}\ :\ \min_{\omega\in[0,4]}\min_{\sigma_1\,,\,\sigma_2=\pm1}|\sigma_1 m+\sigma_2 n-\omega t|\le\delta t\}\,.$$

Now we will show a decay estimate of $S^{m,n}(t)$ for $m\notin\Gamma_m(t)$ to prove (4.7). By symmetry, it suffices to consider the case $m\ge n$. As in the proof of [16, Lemma 2], the cases $m\ge 0\ge n$, $m\ge n>0$ and $0>m\ge n$ can be shown differently.

Suppose $n\notin\Gamma_m(t)$ and $m\ge 0\ge n$. Substituting

$$\psi_m^+(\theta)=e^{-im\theta}f_m^+(\theta),\quad \psi_n^-(\theta)=e^{in\theta}f_n^-(\theta) \tag{4.9}$$

into the integral representation (4.8) and integrating the resulting equation $2N$ times by parts, we see from the bound (4.6) that there exists a positive constant $C_{N,\delta}$ depending only on $N\in\mathbb{Z}_+$ and $\delta>0$ such that for $t\ge1$,

$$|S^{m,n}(t)|\le C_{N,\delta}t^{-2N}\le C_{N,\delta}t^{-N}\left(1+||m|-|n||\right)^{-N}\,. \tag{4.10}$$

Suppose $n\notin\Gamma_m(t)$ and $m\ge n>0$. We substitute (4.9) and the first equation of system (4.4) into the integral representation (4.8). Noting that the singularity of $a(\theta)$ and $b(\theta)$ at $\theta=0$ and $\theta=\pm\pi$ cancels out with zeros of $\sin(\theta)$ and that the derivatives of $a(\theta)\sin(\theta)$ and $b(\theta)\sin(\theta)$ are bounded, we can prove (4.10) in the same way as above.

The case $0>m\ge n$ can be shown in the similar manner. Therefore for an arbitrary N, we have

$$\sup_{m\in\mathbb{Z}}\sum_{n\notin\Gamma_m(t)}|S^{m,n}(t)|^s=\mathcal{O}(t^{-sN})\quad\text{as } t\to\infty.$$

For $n\in\Gamma_m(t)$, we rely on the argument of Mielke and Patz [11]. Substituting (4.9) into the integral representation (4.8) and using system (4.4) again, we obtain

$$\sup_{m,n\in\mathbb{Z}}|S^{m,n}(t)|\le Ct^{-\frac{1}{3}}\quad\text{for any } t\ge1, \tag{4.11}$$

and

$$\sup_{m,n\in\mathbb{Z}}|S^{m,n}(t)|\le Ct^{-\frac{1}{2}}\left(1+\sum_{\sigma=\pm1}\left|\frac{|m+\sigma n|}{t}-2\right|^{-\frac{1}{4}}\right) \tag{4.12}$$

for any $t \geq 1$ and m, n satisfying $|m \pm n| \neq 2t$ in exactly the same way as [11, Lemmas 3.5 and 3.6]. Let

$$A_m(t) = \left\{ n \in \mathbb{Z} : \quad \min_{\sigma=\pm 1} ||m+\sigma n| - 2t| \leq t^{\frac{1}{3}} \right\},$$
$$B_m(t) = \left\{ n \in \mathbb{Z} : \quad t^{\frac{1}{3}} \leq \min_{\sigma=\pm 1} ||m+\sigma n| - 2t| \leq (4+\delta)t \right\}.$$

Then $\Gamma_m(t) \subset A_m(t) \cup B_m(t)$ and it follows from (4.11) and (4.12) that

$$\sum_{n \in A_m(t)} |S^{m,n}(t)|^s \leq C t^{-\frac{s-1}{3}},$$
$$\sum_{n \in B_m(t)} |S^{m,n}(t)|^s \leq \begin{cases} Ct^{-\frac{s-1}{3}} & \text{if } s > 4, \\ Ct^{-\frac{s-2}{2}} & \text{if } s \in (2,4), \end{cases}$$

where C is a constant that does not depend on m. The proof of Lemma 3 is complete. $\square$

5. **Proof of Lemma 4.** In this section, we prove Lemma 4 by using the inverse Laplace transformation (4.2).

Fix $\omega_0 \in (0,1)$ and let $\chi \in C_0^\infty(0,4)$ be such that $\chi(\omega) = 0$ for $\omega \in [0,\omega_0] \cup [4-\omega_0, 4]$ and $\chi(\omega) = 1$ for $\omega \in [2\omega_0, 4-2\omega_0]$. Decay estimate (3.6) follows from the following lemma.

Lemma 6. *Under assumptions of Lemma 4, there is $C > 0$ such that for all $t \in \mathbb{R}$,*

$$\left\| < n >^{-1} \int_0^4 \chi(\omega) e^{-it\omega} \left[R_V^+(\omega) - R_V^-(\omega) \right] \mathbf{f} d\omega \right\|_{l^\infty}$$
$$\leq C < t >^{-\frac{4}{3}} \| < n > \mathbf{f} \|_{l^1} \tag{5.1}$$

and

$$\left\| < n >^{-1} \int_0^4 (1-\chi(\omega)) e^{-it\omega} \left[R_V^+(\omega) - R_V^-(\omega) \right] \mathbf{f} d\omega \right\|_{l^\infty}$$
$$\leq C < t >^{-\frac{3}{2}} \| < n > \mathbf{f} \|_{l^1}. \tag{5.2}$$

To prove the bound (5.1), we use the technique of Lemma 3 in [16], where the dispersive decay $|t|^{-1/3}$ is obtained in the l^1-l^∞ norm without weights. We note that the bound (5.1) needs not be proved for $|t| \leq 1$ because $\chi(\omega) R_V^{\pm}(\omega)$ are bounded operators from $l^1(\mathbb{Z})$ to $l^\infty(\mathbb{Z})$. For any $|t| \geq 1$, we integrate by parts and obtain

$$< n >^{-1} \int_0^4 \chi(\omega) e^{-it\omega} \left(\left[R_V^+(\omega) - R_V^-(\omega) \right] \mathbf{f} \right)_n d\omega$$
$$= \frac{1}{it} < n >^{-1} \int_0^4 e^{-it\omega} \frac{\partial}{\partial \omega} \chi(\omega) \left(\left[R_V^+(\omega) - R_V^-(\omega) \right] \mathbf{f} \right)_n d\omega. \tag{5.3}$$

Substituting the finite Born series

$$R_V(\lambda) = R_0(\lambda) - R_0(\lambda) V R_0(\lambda) + R_0(\lambda) V R_V(\lambda) V R_0(\lambda) \tag{5.4}$$

into the integral (5.3), we obtain three terms, which we estimate separately using a particular version of the van der Corput Lemma.

Lemma 7. *Assume that $\psi \in C^1(-\pi,\pi)$ and there is $\theta_0 \in \left(0, \frac{\pi}{4}\right)$ such that*

$$\text{supp}(\psi) \subset [-\pi + \theta_0, -\theta_0] \cup [\theta_0, \pi - \theta_0].$$

Then, for any $|t| \geq 1$, there is $C > 0$ such that

$$\sup_{a\in\mathbb{R}} \left| \int_{-\pi}^{\pi} e^{-2it(1-\cos(\theta))+ia\theta} \psi(\theta) d\theta \right| \leq C|t|^{-1/3} \left(\|\psi\|_{L^\infty} + \|\partial_\theta \psi\|_{L^1} \right).$$

Proof. Conditions of the van der Corput Lemma (see, e.g., Corollary 1.1 in [10]) are satisfied with $k = 3$ because the smooth phase function,

$$\phi(\theta) = 2(1 - \cos(\theta)) - \frac{a\theta}{t}$$

satisfies $\phi'(\theta_\pm) = \phi''(\theta_\pm) = 0$ and $\phi'''(\theta_\pm) = \mp 2$ for $\theta_\pm = \pm\frac{\pi}{2}$ and $a = \pm 2t$. As a result, there is $C > 0$ such that $|\phi''(\theta)| + |\phi'''(\theta)| \geq C$ for all $\theta \in \mathrm{supp}(\psi)$. $\square$

The first term in (5.3)–(5.4) is estimated using the change of variables in the integration and the representation (4.1),

$$\begin{aligned} &\frac{1}{it} < n >^{-1} \int_0^4 e^{-it\omega} \frac{\partial}{\partial\omega} \chi(\omega) \left(\left[R_0^+(\omega) - R_0^-(\omega) \right] \mathbf{f} \right)_n d\omega \\ &= -\frac{1}{2t} \sum_{m\in\mathbb{Z}} < m > f_m \int_{-\pi}^{\pi} e^{-2it(1-\cos(\theta))-i\theta|n-m|} \psi_{n,m}(\theta) d\theta, \end{aligned}$$

where

$$\begin{aligned} \psi_{n,m}(\theta) =& \frac{|n-m|}{< n >< m >} \frac{\chi(2-2\cos(\theta))}{i\sin(\theta)} \\ &+ \frac{1}{< n >< m >} \frac{\partial}{\partial\theta} \frac{\chi(2-2\cos(\theta))}{\sin(\theta)}. \end{aligned}$$

Using Lemma 7, bound

$$\sup_{n,m\in\mathbb{Z}} \frac{|n-m|}{< n >< m >} < \infty,$$

and the fact that a C^∞-function χ is compactly supported in the region where $\sin(\theta)$ is bounded away from zero, we obtain for any $|t| \geq 1$,

$$\left\| < n >^{-1} \int_0^4 \chi(\omega) e^{-it\omega} \left(\left[R_0^+(\omega) - R_0^-(\omega) \right] \mathbf{f} \right)_n d\omega \right\|_{l^\infty} \leq C|t|^{-4/3} \| < n > \mathbf{f} \|_{l^1}.$$

The second term in (5.3)–(5.4) yields

$$-\frac{1}{4it} \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} < m > V_m < k > f_k \int_{-\pi}^{\pi} e^{-2it(1-\cos(\theta))-i\theta|n-m|-i\theta|m-k|} \psi_{n,m,k}(\theta) d\theta,$$

where

$$\begin{aligned} \psi_{n,m,k}(\theta) =& \frac{|n-m| + |m-k|}{< n >< m >< k >} \frac{\chi(2-2\cos(\theta))}{i\sin^2(\theta)} \\ &+ \frac{1}{< n >< m >< k >} \frac{\partial}{\partial\theta} \frac{\chi(2-2\cos(\theta))}{\sin^2(\theta)}. \end{aligned}$$

Using Lemma 7 and bound

$$\sup_{n,m,k\in\mathbb{Z}} \frac{|n-m| + |m-k|}{< n >< m >< k >} < \infty,$$

this term in l^∞-norm is estimated by

$$C|t|^{-4/3} \| < n > V \|_{l^1} \| < n > \mathbf{f} \|_{l^1}.$$

Finally, we will prove a bound for

$$< n >^{-1} \int_0^4 e^{-it\omega} \frac{\partial}{\partial \omega} \left[\chi(\omega) R_0^{\pm}(\omega) V R_V^{\pm}(\omega) V R_0^{\pm}(\omega) \mathbf{f} \right]_n d\omega \,.$$

Let $B =< n >^{\sigma} \; V$ with $\sigma > \frac{5}{2}$ and $G^{\pm}(\omega) =< n >^{-\sigma} \; R_V^{\pm}(\omega) \; < n >^{-\sigma}$ and represent $G^{\pm}(\omega)$ by components,

$$(G^{\pm}(\omega)\mathbf{f})_n = \sum_{k \in \mathbb{Z}} G^{\pm}_{m,n}(\omega) f_m.$$

Using this representation, we can write the third term in (5.3)–(5.4) as a sum of

$$\frac{\pm i}{4t} \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} < m > B_m < k > B_k < l > f_l \\ \times \int_0^{\pm\pi} e^{-2it(1-\cos(\theta)) - i\theta|n-m| - i\theta|k-l|} \psi^{\pm}_{n,m,k,l}(\theta) d\theta,$$

where

$$\begin{aligned} \psi^{\pm}_{n,m,k,l}(\theta) &= \frac{|n-m| + |k-l|}{< n >< m >< k >< l >} \frac{\chi(2 - 2\cos(\theta))}{i \sin^2(\theta)} [G^{\pm}(\omega)]_{k,m} \\ &\quad + \frac{1}{< n >< m >< k >< l >} \frac{\partial}{\partial \theta} \frac{\chi(2 - 2\cos(\theta))}{\sin^2(\theta)} [G^{\pm}(\omega)]_{k,m}. \end{aligned}$$

By the limiting absorption principle under the assumption of the exponential decay of V (Theorem 1 in [16]), it follows that $\partial^k_\omega R^{\pm}_V(\omega) : l^2_s \to l^2_{-s}$ ($k = 0, 1, \cdots$) are bounded for any $s > k + \frac{1}{2}$ and continuous in $\omega \in \mathrm{supp}(\chi)$. Thus there exists a positive constant C such that

$$|\partial^k_\omega G^{\pm}_{m,n}(\omega)| \le \sup_{\|f\|_{l^2} = 1} \|\partial^k_\omega G^{\pm}(\omega)\mathbf{f}\|_{l^2} \le C \quad \text{for } k = 0,\, 1,\, 2 \text{ and } \omega \in \mathrm{supp}(\chi),$$

since $G^{\pm}_{m,n}(\omega) = (G^{\pm}(\omega)\mathbf{e}_m)_n$ for $\mathbf{e}_m$ such that $(\mathbf{e}_m)_k = 1$ if $k = m$ and $(\mathbf{e}_m)_k = 0$ if $m \neq k$. Combining this argument with the bound

$$\sup_{n,m,k,l \in \mathbb{Z}} \frac{|n-m| + |k-l|}{< n >< m >< k >< l >} < \infty,$$

and using Lemma 7 again, we estimate the last term in l^∞-norm by

$$C|t|^{-4/3} \| < n >^{\sigma+1} V\|^2_{l^1} \| < n > \mathbf{f}\|_{l^1}.$$

All together, the bound (5.1) is now proved.

The proof of the bound (5.2) relies on the technique of Lemma 2 in [16], where the dispersive decay $|t|^{-1/2}$ is obtained in the l^1-l^∞ norm without weights. To do this task, we use properties of Jost functions (4.4) and (4.6). Computations near the end points of the continuous spectrum for the discrete Schrödinger operator can be done in the same way as those for the continuous Schrödinger operator, which have been described in full details in the proof of Theorem 3.1 by Schlag [18]. Since the proof is analogous but lengthy, we omit it here. We remark that $a(\theta) + b(\theta)$ is bounded near $\theta = 0$ and $\theta = \pm\pi$ although $a(\theta)$ and $b(\theta)$ are unbounded near these points.

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