Inverse Problems **16** (2000) 59–74. Printed in the UK PII: S0266-5611(00)05919-0

Spectral decomposition for the Dirac system associated to the DSII equation

Dmitry E Pelinovsky and Catherine Sulem

Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 3G3

Received 9 July 1999

Abstract. A new (scalar) spectral decomposition is found for the Dirac system in two dimensions associated to the focusing Davey–Stewartson II (DSII) equation. The discrete spectrum in the spectral problem corresponds to eigenvalues embedded into a two-dimensional essential spectrum. We show that these embedded eigenvalues are structurally unstable under small variations of the initial data. This instability leads to the decay of localized initial data into continuous wavepackets prescribed by the nonlinear dynamics of the DSII equation.

1. Introduction

Gravity-capillary surface wavepackets are described by the Davey–Stewartson (DS) system [1] which is integrable by inverse scattering transform in the limit of shallow water [2]. In this paper, we study the focusing DSII equation which can be written in a complex form

$$\begin{aligned} &\mathrm{i}u_t + u_{zz} + u_{\bar{z}\bar{z}} + 4(g + \bar{g})u = 0, \\ &2g_{\bar{z}} - (|u|^2)_z = 0, \end{aligned} \tag{1.1}$$

where $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$, $u(z, \bar{z}, t)$ and $g(z, \bar{z}, t)$ are complex functions. This equation appears as the compatibility condition for the two-dimensional Dirac system

$$\varphi_{1\bar{z}} = -u\varphi_2, \qquad \varphi_{2z} = \bar{u}\varphi_1, \tag{1.2}$$

coupled to the equations for the time evolution of the eigenfunctions,

$$\begin{aligned} &\mathrm{i}\varphi_{1t} + \varphi_{1zz} + u\varphi_{2\bar{z}} - u_{\bar{z}}\varphi_2 + 4g\varphi_1 = 0, \\ &-\mathrm{i}\varphi_{2t} + \varphi_{2\bar{z}\bar{z}} + \bar{u}_z\varphi_1 - \bar{u}\varphi_{1z} + 4\bar{g}\varphi_2 = 0. \end{aligned} \tag{1.3}$$

The DSII equation was solved formally through the $\bar{\partial}$ problem of complex analysis by Fokas and Ablowitz [3] and Beals and Coifman [4]. Rigorous results on existence and uniqueness of solutions of the initial-value problem were established under a small-norm assumption [5]. The small-norm assumption was used to eliminate homogeneous solutions of equations of the inverse scattering which correspond to bound states and radially symmetric localized waves (lumps) of the DSII equation. When the potential in the linear system becomes weakly localized (in L^2 but not in L^1), homogeneous solutions may exist and the analysis developed in [5] is not applicable.

The lump solutions were included formally in [3], where their weak decay rate was found, $u \sim \mathrm{O}(R^{-1})$ as $R = \sqrt{x^2 + y^2} \to \infty$. This result is only valid for complexified solutions of the DSII equation (when u and $\bar{u}$ are not considered to be complex conjugated). The reality conditions were incorporated in the work of Arkadiev *et al* [6] where lumps were shown to decay like $u \sim \mathrm{O}(R^{-2})$. Multi-lump solutions were expressed as a ratio of two determinants [6], or, in a special case, as a ratio of two polynomials [7] but their dynamical role was left out of consideration.

Recently, structural instability of a single lump of the DSII equation was reported by Gadyl'shin and Kiselev [8, 9]. The authors used methods of perturbation theory based on completeness of *squared* eigenfunctions of the Dirac system [10, 11]. A similar conclusion was announced by Yurov who studied Darboux transformation of the Dirac system [12, 13].

In this paper, we present an alternative solution of the problem of stability of multi-lump solutions of the DSII equation. The approach generalizes our recent work on spectral decomposition of a linear time-dependent Schrödinger equation with weakly localized (not in L^1) potentials [14]. We find a new spectral decomposition in terms of *single* eigenfunctions of the Dirac system. Surprisingly enough, the two-component Dirac system in two dimensions has a scalar spectral decomposition. In contrast, we recall that the Dirac system in one dimension (the so-called AKNS system) has a well known 2×2 matrix spectral decomposition [15].

Using the scalar spectral decomposition, we associate the multi-lump potentials with eigenvalues embedded into a two-dimensional essential spectrum of the Dirac system. Eigenvalues embedded into a one-dimensional essential spectrum occur, for instance, for the time-dependent Schrödinger problem [14, 16]. They were found to be structurally unstable under a small variation of the potential. Depending on the sign of the variation, they either disappear or become resonant poles in the complex spectral plane which correspond to lump solutions of the KPI equation [14].

For the Dirac system in two dimensions, the multi-lump potentials and embedded eigenvalues are more exotic. The discrete spectrum of the Dirac system is separated from the continuous spectrum contribution in the sense that the spectral data satisfy certain constraints near the embedded eigenvalues. These constraints are met for special solutions of the DSII equation such as lumps, but may not be satisfied for a generic combination of lumps and radiative waves. As a result, embedded eigenvalues of the Dirac system generally disappear under a local disturbance of the initial data. Physically, this implies that a localized initial data of the DSII equation decays into radiation except for the cases where the data reduce to special solutions such as lumps.

The paper is organized as follows. Elements of inverse scattering for the Dirac system are reviewed in section 2, where we find that the discrete spectrum of the Dirac system is prescribed by certain constraints on the spectral data. Spectral decomposition is described in section 3 with the proof of orthogonality and completeness relations through a proper adjoint problem. The perturbation theory for lumps is developed in section 4 where some of previous results [8, 9] are recovered. Section 5 contains concluding remarks. The appendix provides a summary of formulae of the complex $\bar{\partial}$-analysis used in proofs of section 3.

2. Spectral data and inverse scattering

Here we review some results on the Dirac system (1.2) and discard henceforth the time dependence of u, g and φ. The potential $u(z, \bar{z})$ is assumed to be non-integrable ($u \notin L^1$) with the boundary conditions, $u \sim \mathrm{O}(|z|^{-2})$ as $|z| \to \infty$.

2.1. Essential spectrum of the Dirac system

We define the fundamental matrix solution of equation (1.2) in the form [6],

$$\varphi = [\boldsymbol{\mu}(z, \bar{z}, k, \bar{k})\mathrm{e}^{\mathrm{i}kz},\ \boldsymbol{\chi}(z, \bar{z}, k, \bar{k})\mathrm{e}^{-\mathrm{i}\bar{k}\bar{z}}], \tag{2.1}$$

where k is a spectral parameter, $\boldsymbol{\mu}(z, \bar{z}, k, \bar{k})$ and $\boldsymbol{\chi}(z, \bar{z}, k, \bar{k})$ satisfy the system

$$\mu_{1\bar{z}} = -u\mu_2, \qquad \mu_{2z} = -\mathrm{i}k\mu_2 + \bar{u}\mu_1, \tag{2.2}$$

$$\chi_{1\bar{z}} = \mathrm{i}\bar{k}\chi_1 - u\chi_2, \qquad \chi_{2z} = \bar{u}\chi_1. \tag{2.3}$$

It follows from equations (2.2) and (2.3) that $\boldsymbol{\mu}$ and $\boldsymbol{\chi}$ are related by the symmetry constraint

$$\boldsymbol{\chi}(z, \bar{z}, k, \bar{k}) = \boldsymbol{\sigma}\bar{\boldsymbol{\mu}}(z, \bar{z}, k, \bar{k}), \qquad \boldsymbol{\sigma} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

We impose the boundary conditions for $\boldsymbol{\mu}(z, \bar{z}, k, \bar{k})$,

$$\lim_{|k|\to\infty} \boldsymbol{\mu}(z, \bar{z}, k, \bar{k}) = \boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.5}$$

Solutions of equation (2.2) with boundary conditions (2.5) can be expressed through Green functions as Fredholm inhomogeneous integral equations [3, 4],

$$\mu_1(z, \bar{z}, k, \bar{k}) = 1 - \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'}{z' - z}(u\mu_2)(z', \bar{z}'), \tag{2.6}$$

$$\mu_2(z, \bar{z}, k, \bar{k}) = \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'}{\bar{z}' - \bar{z}}(\bar{u}\mu_1)(z', \bar{z}')\mathrm{e}^{-\mathrm{i}k(z-z')-\mathrm{i}\bar{k}(\bar{z}-\bar{z}')}. \tag{2.7}$$

Values of k for which the homogeneous system associated to equations (2.6) and (2.7) has bounded solutions are called *eigenvalues* of the discrete spectrum of the Dirac system. Let us suppose that the homogeneous solutions (eigenvalues) are not supported by the potential $u(z, \bar{z})$. We evaluate the departure from analyticity of $\boldsymbol{\mu}$ in the k plane by calculating the derivative $\partial\boldsymbol{\mu}/\partial\bar{k}$ directly from the system (2.6), (2.7) [3] as

$$\frac{\partial\boldsymbol{\mu}}{\partial\bar{k}} = b(k, \bar{k})\boldsymbol{N}_\mu(z, \bar{z}, k, \bar{k}). \tag{2.8}$$

Here $b(k, \bar{k})$ are the spectral data,

$$b(k, \bar{k}) = \frac{1}{2\pi} \iint \mathrm{d}z \wedge \mathrm{d}\bar{z}(\bar{u}\mu_1)(z, \bar{z})\mathrm{e}^{\mathrm{i}(kz+\bar{k}\bar{z})}, \tag{2.9}$$

and $\boldsymbol{N}_\mu(z, \bar{z}, k, \bar{k})$ is a solution of equation (2.2) which is linearly independent of $\boldsymbol{\mu}(z, \bar{z}, k, \bar{k})$ and satisfies the boundary condition

$$\lim_{|k|\to\infty} \boldsymbol{N}_\mu(z, \bar{z}, k, \bar{k})\mathrm{e}^{\mathrm{i}(kz+\bar{k}\bar{z})} = \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.10}$$

This solution can be expressed through the Fredholm inhomogeneous equations,

$$N_{1\mu}(z, \bar{z}, k, \bar{k}) = \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'}{z' - z}(uN_{2\mu})(z', \bar{z}'), \tag{2.11}$$

$$N_{2\mu}(z, \bar{z}, k, \bar{k}) = \mathrm{e}^{-\mathrm{i}(kz+\bar{k}\bar{z})} + \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'}{\bar{z}' - \bar{z}}(\bar{u}N_{1\mu})(z', \bar{z}')\mathrm{e}^{-\mathrm{i}k(z-z')-\mathrm{i}\bar{k}(\bar{z}-\bar{z}')}. \tag{2.12}$$

The following reduction formula [3] connects $\boldsymbol{N}_\mu(z, \bar{z}, k, \bar{k})$ and $\boldsymbol{\mu}(z, \bar{z}, k, \bar{k})$:

$$\boldsymbol{N}_\mu(z, \bar{z}, k, \bar{k}) = \boldsymbol{\chi}(z, \bar{z}, k, \bar{k})\mathrm{e}^{-\mathrm{i}(kz+\bar{k}\bar{z})} = \boldsymbol{\sigma}\bar{\boldsymbol{\mu}}(z, \bar{z}, k, \bar{k})\mathrm{e}^{-\mathrm{i}(kz+\bar{k}\bar{z})}. \tag{2.13}$$

If the potential $u(z, \bar{z})$ has the boundary values $u \sim \mathrm{O}(|z|^{-2})$ as $|z| \to \infty$ ($u \notin L^1$), then the integral kernel in equation (2.9) is not absolutely integrable, while equations (2.6) and (2.7) are still well defined. We specify complex integration in the z-plane of a non-absolutely integrable function $f(z, \bar{z})$ according to the formula

$$\iint \mathrm{d}z \wedge \mathrm{d}\bar{z} f(z, \bar{z}) = \lim_{R\to\infty} \iint_{|z|\leqslant R} \mathrm{d}z \wedge \mathrm{d}\bar{z} f(z, \bar{z}). \tag{2.14}$$

The same formula is valid for integrating eigenfunctions of the Dirac system in the k plane as well. In section 3, we use (2.14) when computing the inner products and completeness relations for the Dirac system and its adjoint.

2.2. The discrete spectrum

Suppose here that integral equations (2.6) and (2.7) have homogeneous solutions at an eigenvalue $k = k_j$. The discrete spectrum associated to multi-lump potentials was introduced in [3, 6]. Here we review their approach and give a new result (proposition 2.1) which clarifies the role of the discrete spectrum in the spectral problem (2.2).

For the discrete spectrum associated to the multi-lump potentials, an isolated eigenvalue $k = k_j$ has double multiplicity with the corresponding two bound states $\mathbf{\Phi}_j(z,\bar z)$ and $\mathbf{\Phi}'_j(z,\bar z)$ [6]. The bound state $\mathbf{\Phi}_j(z,\bar z)$ is a solution of the homogeneous equations,

$$\Phi_{1j}(z,\bar z) = -\frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}z'\wedge \mathrm{d}\bar z'}{z'-z}(u\Phi_{2j})(z',\bar z'), \tag{2.15}$$

$$\Phi_{2j}(z,\bar z) = \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}z'\wedge \mathrm{d}\bar z'}{\bar z'-\bar z}(\bar u\Phi_{1j})(z',\bar z')\mathrm{e}^{-\mathrm{i}k_j(z-z')-\mathrm{i}\bar k_j(\bar z-\bar z')}, \tag{2.16}$$

with the boundary conditions as $|z|\to\infty$,

$$\mathbf{\Phi}_j(z,\bar z) \to \frac{\mathbf{e}_1}{z}. \tag{2.17}$$

Equivalently, this boundary condition can be written as renormalization conditions for equations (2.15) and (2.16),

$$\frac{1}{2\pi\mathrm{i}}\iint \mathrm{d}z\wedge \mathrm{d}\bar z(u\Phi_{2j})(z,\bar z) = 1, \tag{2.18}$$

$$\frac{1}{2\pi\mathrm{i}}\iint \mathrm{d}z\wedge \mathrm{d}\bar z(\bar u\Phi_{1j})(z,\bar z)\mathrm{e}^{\mathrm{i}(k_j z+\bar k_j\bar z)} = 0. \tag{2.19}$$

The other (degenerate) bound state $\mathbf{\Phi}'_j(z,\bar z)$ can be expressed in terms of $\mathbf{\Phi}_j(z,\bar z)$ using equation (2.13),

$$\mathbf{\Phi}'_j(z,\bar z) = \boldsymbol{\sigma}\bar{\mathbf{\Phi}}_j(z,\bar z)\mathrm{e}^{-\mathrm{i}(k_j z+\bar k_j\bar z)}. \tag{2.20}$$

The behaviour of the eigenfunction $\boldsymbol{\mu}(z,\bar z,k,\bar k)$ near the eigenvalue $k = k_j$ becomes complicated due to the fact that the double eigenvalue is embedded into the two-dimensional essential spectrum of the Dirac system (2.2). We prove the following result.

Proposition 2.1. *For smooth data $b(k,\bar k)\in C^1$ at $k\neq k_j$, the eigenfunction $\boldsymbol{\mu}(z,\bar z,k,\bar k)$ has a pole singularity at $k\to k_j$ only if*

$$b_0 = \frac{1}{2\pi}\iint \mathrm{d}z\wedge \mathrm{d}\bar z(\bar z\bar u\Phi_{1j})(z,\bar z)\mathrm{e}^{\mathrm{i}(k_j z+\bar k_j\bar z)} = 0. \tag{2.21}$$

Proof. Suppose $\boldsymbol{\mu}(z,\bar z,k,\bar k)$ has a pole singularity at $k = k_j$. Then, it can be shown from equation (2.2) that the meromorphic continuation of $\boldsymbol{\mu}(z,\bar z,k,\bar k)$ is given by the limiting relation

$$\lim_{k\to k_j}\left[\boldsymbol{\mu}(z,\bar z,k,\bar k) - \frac{\mathrm{i}\mathbf{\Phi}_j(z,\bar z)}{k-k_j}\right] = (z+z_j)\mathbf{\Phi}_j(z,\bar z) + c_j\mathbf{\Phi}'_j(z,\bar z), \tag{2.22}$$

where z_j, c_j are some constants. Using equations (2.8), (2.9) and (2.13), we find the differential relation for $b(k,\bar k)$:

$$\frac{\partial b}{\partial\bar k} = \frac{b(k,\bar k)}{2\pi}\iint \mathrm{d}z\wedge \mathrm{d}\bar z(\bar u\bar\mu_2)(z,\bar z) - \frac{1}{2\pi\mathrm{i}}\iint \mathrm{d}z\wedge \mathrm{d}\bar z(\bar z\bar u\mu_1)(z,\bar z)\mathrm{e}^{\mathrm{i}(kz+\bar k\bar z)}.$$

In the limit $k \to k_j$, this equation reduces with the help of equations (2.18) and (2.22) to the form

$$\frac{\partial b}{\partial \bar{k}} = \frac{b(k, \bar{k})}{\bar{k} - \bar{k}_j} - \frac{b_0}{k - k_j},$$

where b_0 is given in equation (2.21). The reduced equation exhibits the limiting behaviour of $b(k, \bar{k})$ as $k \to k_j$,

$$b(k, \bar{k}) \to -b_0 \frac{\bar{k} - \bar{k}_j}{k - k_j} \ln |\bar{k} - \bar{k}_j|. \tag{2.23}$$

On the other hand, it follows from equations (2.13), (2.20) and (2.22) that $\boldsymbol{N}_\mu(z, \bar{z}, k, \bar{k})$ has the limiting behaviour

$$\boldsymbol{N}_\mu(z, \bar{z}, k, \bar{k}) \to \frac{-\mathrm{i}\boldsymbol{\Phi}'_j(z, \bar{z})}{\bar{k} - \bar{k}_j}. \tag{2.24}$$

According to equations (2.23) and (2.24), the right-hand side of equation (2.8) is of order $\mathrm{O}(b_0|k - k_j|^{-1} \ln |k - k_j|)$ as $k \to k_j$. On the other hand, the left-hand side of equation (2.8) must be of order O(1) in the limit $k \to k_j$ according to equation (2.22). Therefore, the eigenfunction $\boldsymbol{\mu}(z, \bar{z}, k, \bar{k})$ has a pole at $k = k_j$ only if the constraint $b_0 = 0$ holds. □

The limiting relation (2.22) was introduced by Arkadiev *et al* [6]. However, the authors did not notice that the discrete spectrum is supported only by potentials which satisfy the additional constraint (2.21). In particular, such potentials include the multi-lump solutions for which $b(k, \bar{k}) = 0$ everywhere in the k-plane.

2.3. Expansion formulae for inverse scattering

Combining equation (2.8) for the essential spectrum and equation (2.22) for the discrete spectrum, we reconstruct the eigenfunction $\boldsymbol{\mu}(z, \bar{z}, k, \bar{k})$ [3, 6],

$$\boldsymbol{\mu}(z, \bar{z}, k, \bar{k}) = \boldsymbol{e}_1 + \sum_{j=1}^{n} \frac{\mathrm{i}\boldsymbol{\Phi}_j(z, \bar{z})}{k - k_j} + \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}k' \wedge \mathrm{d}\bar{k}'}{k' - k} b(k', \bar{k}') \boldsymbol{N}_\mu(z, \bar{z}, k', \bar{k}'), \tag{2.25}$$

where n is the number of distinct eigenvalues k_j of double multiplicity. At $k \to k_j$, this system is coupled with the algebraic system for the bound states,

$$(z + z_j)\boldsymbol{\Phi}_j(z, \bar{z}) + c_j \boldsymbol{\Phi}'_j(z, \bar{z}) = \boldsymbol{e}_1 + \sum_{l \neq j} \frac{\mathrm{i}\boldsymbol{\Phi}_l(z, \bar{z})}{k_j - k_l} + \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}k \wedge \mathrm{d}\bar{k}}{k - k_j} b(k, \bar{k}) \boldsymbol{N}_\mu(z, \bar{z}, k, \bar{k}). \tag{2.26}$$

Expansion (2.25) can be related to the inverse scattering transform for the potential $u(z, \bar{z})$ [3, 6]. It follows from equation (2.2) that the eigenfunction $\boldsymbol{\mu}(z, \bar{z}, k, \bar{k})$ has the asymptotic expansion as $|k| \to \infty$,

$$\boldsymbol{\mu}(z, \bar{z}, k, \bar{k}) = \boldsymbol{e}_1 + \frac{1}{\mathrm{i}k} \boldsymbol{\mu}_\infty(z, \bar{z}) + \mathrm{O}(|k|^{-2}), \tag{2.27}$$

where $\mu_{2\infty}(z, \bar{z}) = \bar{u}(z, \bar{z})$ and

$$\mu_{1\infty}(z, \bar{z}) = -\frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'}{z' - z} (|u|^2)(z', \bar{z}').$$

We deduce from equations (2.25) and (2.27) that the potential $\bar{u}(z,\bar{z})$ is expressed through the eigenfunctions of the Dirac system in the form [3],

$$\bar{u}(z,\bar{z}) = -\sum_{j=1}^{n} \Phi_{2j}(z,\bar{z}) - \frac{1}{2\pi}\iint \mathrm{d}k \wedge \mathrm{d}\bar{k}\, b(k,\bar{k}) N_{2\mu}(z,\bar{z},k,\bar{k}). \tag{2.28}$$

Formulae (2.6)–(2.28) constitute a standard framework for the inverse scattering transform of the DSII equation with a new relation (2.21). The existence and uniqueness of solutions of the Fredholm integral equations (2.6) and (2.7) and the $\bar{\partial}$ problem (2.8) and (2.25) were proved in [4] and [5] under the small-norm assumption for the potential $u(x,y)$,

$$\left(\sup_{(x,y)\in \boldsymbol{R}^2} |u|(x,y)\right)\left(\iint |u(x,y)|\,\mathrm{d}x\,\mathrm{d}y\right) < \frac{\pi}{8}.$$

In this case $n = 0$ and $b(k,\bar{k}) \neq 0$. The nonlinear two-dimensional Fourier transform associated to this scheme was discussed in examples 8–10 of chapter 7.7 of [17]. Indeed, the connection formula (2.28) implies that there is a scalar spectral decomposition of $\bar{u}(z,\bar{z})$ through $N_{2\mu}(z,\bar{z},k,\bar{k})$ for $n = 0$. In order to close the decomposition, one could use equations (2.9) and (2.13) to construct a 'completeness relation' for the expansion of $\delta(z'-z)$ in the form,

$$\delta(z'-z) = -\frac{1}{2\pi^2 \mathrm{i}}\iint \mathrm{d}k \wedge \mathrm{d}\bar{k}\bar{N}_{2\mu}(z',\bar{z}',k,\bar{k}) N_{2\mu}(z,\bar{z},k,\bar{k}).$$

However, we show in proposition 3.4 below that a completeness theorem for equation (2.2) is different and is based on the set of eigenfunctions of the adjoint Dirac system.

3. Basis for a scalar spectral decomposition

In this section, we specify the adjoint problem for the Dirac system (2.2) and establish orthogonality and completeness relations.

3.1. The adjoint system

The adjoint system for equation (2.2) is

$$\mu^a_{1z} = \mathrm{i}k\mu^a_1 - u\mu^a_2, \qquad \mu^a_{2\bar{z}} = \bar{u}\mu^a_1, \tag{3.1}$$

which provides the balance equation,

$$\mathrm{i}(k'-k)\mu^a_1(k')\mu_2(k) = \frac{\partial}{\partial z}[\mu^a_1(k')\mu_2(k)] - \frac{\partial}{\partial \bar{z}}[\mu^a_2(k')\mu_1(k)]. \tag{3.2}$$

The system (3.1) admits plane solutions $\boldsymbol{\mu}^a(z,\bar{z},k,\bar{k})$ and oscillatory-type solutions $\boldsymbol{N}^a_\mu(z,\bar{z},k,\bar{k})$ with the boundary conditions,

$$\lim_{|k|\to\infty} \boldsymbol{\mu}^a(z,\bar{z},k,\bar{k}) = \boldsymbol{e}_2, \tag{3.3}$$

$$\lim_{|k|\to\infty} \boldsymbol{N}^a_\mu(z,\bar{z},k,\bar{k})\mathrm{e}^{-\mathrm{i}(kz+\bar{k}\bar{z})} = \boldsymbol{e}_1. \tag{3.4}$$

The adjoint eigenfunctions $\boldsymbol{N}^a_\mu(z,\bar{z},k,\bar{k})$ can be expressed through the Green functions,

$$N^a_{1\mu}(z,\bar{z}) = \mathrm{e}^{\mathrm{i}(kz+\bar{k}\bar{z})} - \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}z'\wedge\mathrm{d}\bar{z}'}{\bar{z}'-\bar{z}}(uN^a_{2\mu})(z',\bar{z}')\mathrm{e}^{\mathrm{i}k(z-z')+\mathrm{i}\bar{k}(\bar{z}-\bar{z}')}, \tag{3.5}$$

$$N^a_{2\mu}(z,\bar{z}) = \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}z'\wedge\mathrm{d}\bar{z}'}{z'-z}(\bar{u}N^a_{1\mu})(z',\bar{z}'). \tag{3.6}$$

They are related to the adjoint eigenfunctions $\boldsymbol{\mu}^a(z,\bar{z},k,\bar{k})$ by the formula

$$\boldsymbol{N}^a_\mu(z,\bar{z},k,\bar{k}) = -\boldsymbol{\sigma}\bar{\boldsymbol{\mu}}^a(z,\bar{z},k,\bar{k})\mathrm{e}^{\mathrm{i}(kz+\bar{k}\bar{z})}. \tag{3.7}$$

Using this representation, we prove the following result.

Lemma 3.1. *The spectral data $b(k,\bar{k})$ are expressed in terms of the adjoint eigenfunctions as*

$$b(k,\bar{k}) = \frac{1}{2\pi}\iint \mathrm{d}z\wedge\mathrm{d}\bar{z}(\bar{u}N^a_{1\mu})(z,\bar{z}). \tag{3.8}$$

Proof. Multiplying equation (3.5) by $\bar{u}\mu_1(k)$, integrating over $\mathrm{d}z\wedge\mathrm{d}\bar{z}$ and using equation (2.7), we express $b(k,\bar{k})$ defined in equation (2.9) in the form

$$b(k,\bar{k}) = \frac{1}{2\pi}\iint \mathrm{d}z\wedge\mathrm{d}\bar{z}[\bar{u}\mu_1(k)N^a_{1\mu}(k) - u\mu_2(k)N^a_{2\mu}(k)]. \tag{3.9}$$

On the other hand, multiplying equation (2.6) by $uN^a_{1\mu}(k)$, integrating over $\mathrm{d}z\wedge\mathrm{d}\bar{z}$, and using equations (3.6) and (3.9), we get equation (3.8). $\square$

Suppose now that $k=k_j$ is an isolated double eigenvalue of equation (2.2) with the bound states $\boldsymbol{\Phi}_j(z,\bar{z})$ and $\boldsymbol{\Phi}'_j(z,\bar{z})$ given by equations (2.15)–(2.20). Suppose also that $k=k^a_j$ is an eigenvalue of the adjoint system (3.1) with the adjoint bound states $\boldsymbol{\Phi}^a_j(z,\bar{z})$ and $\boldsymbol{\Phi}^{a\prime}_j(z,\bar{z})$.

Lemma 3.2. *If k_j is a double eigenvalue of the Dirac system (2.2), then k_j is also a double eigenvalue of the adjoint system (3.1).*

Proof. We use equation (3.2) with $\boldsymbol{\mu}=\boldsymbol{\Phi}_j(z,\bar{z})$ and $\boldsymbol{\mu}^a=\boldsymbol{\Phi}^a_j(z,\bar{z})$ at $k=k_j$ and $k'=k^a_j$ and integrate over $\mathrm{d}z\wedge\mathrm{d}\bar{z}$ with the help of equation (A.3) of the appendix. The contour contribution of the integral vanishes due to the boundary conditions (2.17) and (3.12) and the resulting expression is

$$(k^a_j-k_j)\iint \mathrm{d}z\wedge\mathrm{d}\bar{z}(\Phi^a_{1j}\Phi_{2j})(z,\bar{z}) = 0.$$

The relation $k^a_j=k_j$ follows from this formula if the integral is non-zero at $k^a_j=k_j$ (which is proved below in equation (3.22)). The other possibility is when $k^a_j\neq k_j$ but Φ^a_{1j} is orthogonal to Φ_{2j}. We do not consider such a non-generic situation. The other bound state $\boldsymbol{\Phi}^{a\prime}_j$ at $k^a_j=k_j$ can be defined using the symmetry relation (see equation (3.15) below). $\square$

The adjoint bound state $\boldsymbol{\Phi}^a_j(z,\bar{z})$ solves the homogeneous equations,

$$\Phi^a_{1j}(z,\bar{z}) = -\frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}z'\wedge\mathrm{d}\bar{z}'}{\bar{z}'-\bar{z}}(u\Phi^a_{2j})(z',\bar{z}')\mathrm{e}^{\mathrm{i}k_j(z-z')+\mathrm{i}\bar{k}_j(\bar{z}-\bar{z}')}, \tag{3.10}$$

$$\Phi^a_{2j}(z,\bar{z}) = \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}z'\wedge\mathrm{d}\bar{z}'}{z'-z}(\bar{u}\Phi^a_{1j})(z',\bar{z}') \tag{3.11}$$

with the boundary condition as $|z|\to\infty$,

$$\boldsymbol{\Phi}^a_j(z,\bar{z}) \to \frac{\boldsymbol{e}_2}{z}, \tag{3.12}$$

and the normalization conditions,

$$-\frac{1}{2\pi\mathrm{i}}\iint \mathrm{d}z\wedge\mathrm{d}\bar{z}(\bar{u}\Phi^a_{1j})(z,\bar{z}) = 1, \tag{3.13}$$

$$\frac{1}{2\pi\mathrm{i}}\iint \mathrm{d}z\wedge\mathrm{d}\bar{z}(u\Phi^a_{2j})(z,\bar{z})\mathrm{e}^{-\mathrm{i}(k_jz+\bar{k}_j\bar{z})} = 0. \tag{3.14}$$

In addition, the bound state $\boldsymbol{\Phi}_j^{a\prime}(z,\bar{z})$ is related to $\boldsymbol{\Phi}(z,\bar{z})$ according to the symmetry formula

$$\boldsymbol{\Phi}_j^{a\prime}(z,\bar{z}) = -\boldsymbol{\sigma}\bar{\boldsymbol{\Phi}}_j^{a}(z,\bar{z})\mathrm{e}^{\mathrm{i}(k_j z+\bar{k}_j\bar{z})}. \tag{3.15}$$

Using equations (3.1)–(3.15), we see that the adjoint eigenfunction $\boldsymbol{\mu}^a(z,\bar{z},k,\bar{k})$ satisfies relations similar to those for $\boldsymbol{\mu}(z,\bar{z},k,\bar{k})$,

$$\frac{\partial\boldsymbol{\mu}^a}{\partial\bar{k}} = -\bar{b}(k,\bar{k})\boldsymbol{N}_\mu^a(z,\bar{z},k,\bar{k}) \tag{3.16}$$

and

$$\lim_{k\to k_j}\left[\boldsymbol{\mu}^a(z,\bar{z},k,\bar{k}) + \frac{\mathrm{i}\boldsymbol{\Phi}_j^a(z,\bar{z})}{k-k_j}\right] = (z+z_j)\boldsymbol{\Phi}_j^a(z,\bar{z}) - \bar{c}_j\boldsymbol{\Phi}_j^{a\prime}(z,\bar{z}). \tag{3.17}$$

The expansions for inverse scattering transform of the adjoint eigenfunctions can be found in the form

$$\boldsymbol{\mu}^a(z,\bar{z},k,\bar{k}) = \boldsymbol{e}_2 - \sum_{j=1}^{n}\frac{\mathrm{i}\boldsymbol{\Phi}_j^a(z,\bar{z})}{k-k_j} - \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}k'\wedge\mathrm{d}\bar{k}'}{k'-k}\bar{b}(k',\bar{k}')\boldsymbol{N}_\mu^a(z,\bar{z},k',\bar{k}') \tag{3.18}$$

and

$$\begin{aligned}(z'+z_j)\boldsymbol{\Phi}_j^a(z',\bar{z}') - \bar{c}_j\boldsymbol{\Phi}_j^{a\prime}(z',\bar{z}')\\ = \boldsymbol{e}_2 - \sum_{l\neq j}\frac{\mathrm{i}\boldsymbol{\Phi}_l^a(z',\bar{z}')}{k_j-k_l} - \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}k\wedge\mathrm{d}\bar{k}}{k-k_j}\bar{b}(k,\bar{k})\boldsymbol{N}_\mu^a(z,\bar{z},k,\bar{k}).\end{aligned} \tag{3.19}$$

3.2. Orthogonality and completeness relations

Using the Dirac system (2.2) and its adjoint system (3.1), we prove the orthogonality and completeness relations for the set of eigenfunctions $S = [N_{2\mu}(k,\bar{k}), \{\Phi_{2j}\}_{j=1}^n]$ and its adjoint set $S^a = [N_{1\mu}^a(k,\bar{k}), \{\Phi_{1j}^a\}_{j=1}^n]$.

Proposition 3.3. *The eigenfunctions $N_{2\mu}(z,\bar{z},k,\bar{k})$ and $\Phi_{2j}(z,\bar{z})$ are orthogonal to the eigenfunctions $N_{1\mu}^a(z,\bar{z},k,\bar{k})$ and $\Phi_{1j}^a(z,\bar{z})$ as follows:*

$$\langle N_{1\mu}^a(k')|N_{2\mu}(k)\rangle_z = -2\pi^2\mathrm{i}\delta(k'-k), \tag{3.20}$$

$$\langle N_{1\mu}^a(k)|\Phi_{2j}\rangle_z = \langle\Phi_{1j}^a|N_{2\mu}(k)\rangle_z = 0, \tag{3.21}$$

$$\langle\Phi_{1l}^a|\Phi_{2j}\rangle_z = 2\pi\mathrm{i}\delta_{jl}, \tag{3.22}$$

where the inner product is defined as

$$\langle g(k')|f(k)\rangle_z = \iint\mathrm{d}z\wedge\mathrm{d}\bar{z}\,g(z,\bar{z},k',\bar{k}')f(z,\bar{z},k,\bar{k}).$$

Proof. Using equations (2.12) and (3.5), we expand the inner product in equation (3.20) as

$$\begin{aligned}\langle N_{1\mu}^a(k')|N_{2\mu}(k)\rangle_z &= I_0 + \iint\mathrm{d}z\wedge\mathrm{d}\bar{z}(uN_{2\mu}^a)(z,k')\mathrm{e}^{-\mathrm{i}(k'z+\bar{k}'\bar{z})}I_1(z)\\ &\quad - \iint\mathrm{d}z\wedge\mathrm{d}\bar{z}(\bar{u}N_{1\mu})(z,k)\mathrm{e}^{\mathrm{i}(kz+\bar{k}\bar{z})}I_1(z)\\ &\quad + \frac{1}{2\pi\mathrm{i}}\iint\mathrm{d}z\wedge\mathrm{d}\bar{z}(uN_{2\mu}^a)(z,k')\mathrm{e}^{-\mathrm{i}(k'z+\bar{k}'\bar{z})}\\ &\quad\times\iint\frac{\mathrm{d}z'\wedge\mathrm{d}\bar{z}'}{\bar{z}'-\bar{z}}(\bar{u}N_{1\mu})(z',k)\mathrm{e}^{\mathrm{i}(kz'+\bar{k}\bar{z}')}[I_1(z)-I_1(z')],\end{aligned}$$

where

$$I_0 = \iint \mathrm{d}z \wedge \mathrm{d}\bar{z}\, \mathrm{e}^{\mathrm{i}(k'-k)z+\mathrm{i}(\bar{k}'-\bar{k})\bar{z}} = -2\pi^2 \mathrm{i}\delta(k'-k) \tag{3.23}$$

and

$$I_1(z) = \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'}{\bar{z}'-\bar{z}} \mathrm{e}^{\mathrm{i}(k'-k)z'+\mathrm{i}(\bar{k}'-\bar{k})\bar{z}'} = \frac{1}{\mathrm{i}(k'-k)} \mathrm{e}^{\mathrm{i}(k'-k)z+\mathrm{i}(\bar{k}'-\bar{k})\bar{z}}. \tag{3.24}$$

The integrals I_0 and $I_1(z)$ are computed in the appendix. Using these formulae, we find the inner product in equation (3.20) in the form

$$\langle N^a_{1\mu}(k')|N_{2\mu}(k)\rangle_z = -2\pi^2 \mathrm{i}\delta(k'-k) + \frac{1}{\mathrm{i}(k'-k)} \mathcal{R}(k,k'),$$

where the residual term $\mathcal{R}(k,k')$ is expressed in the form

$$\mathcal{R}(k,k') = \iint \mathrm{d}z \wedge \mathrm{d}\bar{z} [u N^a_{2\mu}(k') N_{2\mu}(k) - \bar{u} N^a_{1\mu}(k') N_{1\mu}(k)],$$

with the help of equations (2.12) and (3.5). We show that $\mathcal{R}(k,k') = 0$ by multiplying equation (3.6) by $\bar{u}N_{1\mu}(k)$, integrating over $\mathrm{d}z \wedge \mathrm{d}\bar{z}$ and using equation (2.11).

The zero inner products in equations (3.21) and (3.22) for $j \neq l$ are obtained in a similar way with the help of the Fredholm equations for eigenfunctions $\boldsymbol{\Phi}_j$, $\boldsymbol{\Phi}^a_j$, $\boldsymbol{N}_\mu$, and $\boldsymbol{N}^a_\mu$. In order to find the non-zero inner product (3.22) for $j = l$ we evaluate the following integral by using the same integral equations:

$$\begin{aligned}\iint \mathrm{d}z \wedge \mathrm{d}\bar{z}\, \Phi^a_{1j}\mu_2(k) &= \frac{1}{\mathrm{i}(k_j-k)} \iint \mathrm{d}z \wedge \mathrm{d}\bar{z} [u\Phi^a_{2j}\mu_2(k) - \bar{u}\Phi^a_{1j}\mu_1(k)] \\ &= \frac{1}{\mathrm{i}(k-k_j)} \iint \mathrm{d}z \wedge \mathrm{d}\bar{z}\, \bar{u}\Phi^a_{1j}.\end{aligned}$$

Using equation (3.13), the right-hand side identifies to $\frac{1}{\mathrm{i}(k-k_j)}$. Substituting equation (2.25) in the left-hand side and using the zero inner products (3.21) and (3.22), we find equation (3.22) for $j = l$. □

Proposition 3.4. *The eigenfunctions $N_{2\mu}(z,\bar{z},k,\bar{k})$ and $\Phi_{2j}(z,\bar{z})$ are complete with respect to the adjoint eigenfunctions $N^a_{1\mu}(z,\bar{z},k,\bar{k})$ and $\Phi^a_{1j}(z,\bar{z})$ according to the identity*

$$\begin{aligned}\delta(z'-z) = &-\frac{1}{2\pi^2\mathrm{i}} \iint \mathrm{d}k \wedge \mathrm{d}\bar{k} N^a_{1\mu}(z',\bar{z}',k,\bar{k}) N_{2\mu}(z,\bar{z},k,\bar{k}) \\ &-\frac{1}{\pi} \sum_{j=1}^{n} \Phi^a_{1j}(z',\bar{z}')\Phi_{2j}(z,\bar{z}).\end{aligned} \tag{3.25}$$

Proof. Using the symmetry relations (2.13) and (3.7), we express the integral in equation (3.25) as

$$\langle N^a_{1\mu}(z')|N_{2\mu}(z)\rangle_k = \iint \mathrm{d}k \wedge \mathrm{d}\bar{k}\, \bar{\mu}^a_2(z',\bar{z}',k,\bar{k})\bar{\mu}_1(z,\bar{z},k,\bar{k})\mathrm{e}^{\mathrm{i}k(z'-z)+\mathrm{i}\bar{k}(\bar{z}'-\bar{z})}. \tag{3.26}$$

We use equations (2.25), (2.26), (3.18), and (3.19) and find the pole decomposition for the integrand in equation (3.26),

$$\bar{\mu}^a_2(z')\bar{\mu}_1(z) = 1 + \sum_{j=1}^{n} \frac{\mathrm{i}}{\bar{k}-\bar{k}_j} [\bar{c}_j \bar{\Phi}^a_{2j}(z',\bar{z}')\bar{\Phi}'_{1j}(z,\bar{z}) + c_j \bar{\Phi}^{a\prime}_{2j}(z',\bar{z}')\bar{\Phi}_{1j}(z,\bar{z})]$$

$$+\sum_{j=1}^{n}\frac{\mathrm{i}}{\bar{k}-\bar{k}_j}(\bar{z}-\bar{z}')\bar{\Phi}^a_{2j}(z',\bar{z}')\bar{\Phi}_{1j}(z,\bar{z})+\sum_{j=1}^{n}\frac{\bar{\Phi}^a_{2j}(z',\bar{z}')\bar{\Phi}_{1j}(z,\bar{z})}{(\bar{k}-\bar{k}_j)^2}$$
$$+\frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}k'\wedge\mathrm{d}\bar{k}'}{\bar{k}'-\bar{k}}[\bar{\mu}^a_2(z')\bar{b}\bar{N}_{1\mu}(z)-\bar{\mu}_1(z)b\bar{N}^a_{2\mu}(z')](k',\bar{k}'). \tag{3.27}$$

We substitute (3.27) into equation (3.26) and reduce the integral to the form

$$\langle N^a_{1\mu}(z')|N_{2\mu}(z)\rangle_k = I_0-2\pi\sum_{j=1}^{n}[\bar{c}_j\bar{\Phi}^a_{2j}(z',\bar{z}')\bar{\Phi}'_{1j}(z,\bar{z})+c_j\bar{\Phi}^{a\prime}_{2j}(z',\bar{z}')\bar{\Phi}_{1j}(z,\bar{z})]I_1(k_j)$$
$$+\sum_{j=1}^{n}\bar{\Phi}^a_{2j}(z',\bar{z}')\bar{\Phi}_{1j}(z,\bar{z})[2\pi(\bar{z}'-\bar{z})I_1(k_j)+I_2(k_j)]$$
$$-\iint\mathrm{d}k\wedge\mathrm{d}\bar{k}I_1(k)[\bar{\mu}^a_2(z')\bar{b}\bar{N}_{1\mu}(z)-\bar{\mu}_1(z)b\bar{N}^a_{2\mu}(z')](k,\bar{k})\mathrm{e}^{-\mathrm{i}k(z'-z)-\mathrm{i}\bar{k}(\bar{z}'-\bar{z})}, \tag{3.28}$$

where the integrals I_0 and $I_1(k)$ are given in equations (3.23) and (3.24) respectively, with z and k interchanged, while the integral $I_2(k_j)$ is defined by

$$I_2(k_j)=\lim_{\epsilon\to0}\iint_{|k-k_j|\geqslant\epsilon}\frac{\mathrm{d}k\wedge\mathrm{d}\bar{k}}{(\bar{k}-\bar{k}_j)^2}\mathrm{e}^{\mathrm{i}k(z'-z)+\mathrm{i}\bar{k}(\bar{z}'-\bar{z})}. \tag{3.29}$$

The integral $I_2(k)$ is found in the appendix in the form, $I_2(k_j)=-2\pi(\bar{z}'-\bar{z})I_1(k_j)$, such that the third term in equation (3.28) vanishes. In order to express the second term in equation (3.28) we use equations (2.26), (3.19), (2.20), and (3.15) and derive the relation

$$-\sum_{j=1}^{n}[\bar{c}_j\bar{\Phi}^a_{2j}(z',\bar{z}')\bar{\Phi}'_{1j}(z,\bar{z})+c_j\bar{\Phi}^{a\prime}_{2j}(z'\bar{z}')\bar{\Phi}_{1j}(z,\bar{z})]\mathrm{e}^{\mathrm{i}k_j(z'-z)+\mathrm{i}\bar{k}_j(\bar{z}'-\bar{z})}$$
$$=\sum_{j=1}^{n}[\bar{c}_j\Phi^{a\prime}_{1j}(z',\bar{z}')\Phi_{2j}(z,\bar{z})+c_j\Phi^a_{1j}(z',\bar{z}')\Phi'_{2j}(z,\bar{z})]$$
$$=(z'-z)\sum_{j=1}^{n}\Phi^a_{1j}(z',\bar{z}')\Phi_{2j}(z,\bar{z})$$
$$+\frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{n}\iint\frac{\mathrm{d}k\wedge\mathrm{d}\bar{k}}{k-k_j}[\Phi_{2j}(z)\bar{b}N^a_{1\mu}(z')+\Phi^a_{1j}(z')bN_{2\mu}(z)](k,\bar{k}). \tag{3.30}$$

Using this expression and equations (3.23), (3.24), and (3.29), we rewrite equation (3.28) in the form,

$$\langle N^a_{1\mu}(z')|N_{2\mu}(z)\rangle_k=-2\pi^2\mathrm{i}\delta(z'-z)-2\pi\mathrm{i}\sum_{j=1}^{n}\Phi^a_{1j}(z',\bar{z}')\Phi_{2j}(z,\bar{z})$$
$$+\frac{\mathrm{i}}{z'-z}\iint\mathrm{d}k\wedge\mathrm{d}\bar{k}\frac{\partial}{\partial\bar{k}}\left[\left(\mu^a_1(z')+\sum_{j=1}^{n}\frac{\mathrm{i}\Phi^a_{1j}(z')}{k-k_j}\right)\left(\mu_2(z)-\sum_{j=1}^{n}\frac{\mathrm{i}\Phi_{2j}(z)}{k-k_j}\right)\right].$$

The last integral vanishes according to equation (A.3) of the appendix and the boundary conditions (2.5) and (3.3). □

Our main result for the spectral decomposition associated to the Dirac system (2.2) follows from the above orthogonality and completeness relations.

Proposition 3.5. *An arbitrary scalar function $f(z,\bar{z})$ satisfying the condition $f(z,\bar{z})\sim \mathrm{O}(|z|^{-2})$ as $|z|\to\infty$ can be decomposed through the set $S=[N_{2\mu}(z,\bar{z},k,\bar{k}),\{\Phi_{2j}(z,\bar{z})\}_{j=1}^{n}]$.*

Proof. The spectral decomposition is defined through the orthogonality relations (3.20)–(3.22) as

$$f(z,\bar z) = \iint \mathrm{d}k \wedge \mathrm{d}\bar k\, \alpha(k,\bar k) N_{2\mu}(z,\bar z,k,\bar k) + \sum_{j=1}^{n} \alpha_j \Phi_{2j}(z,\bar z), \tag{3.31}$$

where

$$\alpha(k,\bar k) = -\frac{1}{4\pi^2}\langle N^a_{1\mu}(k)|f\rangle_z, \qquad \alpha_j = \frac{1}{2\pi\mathrm{i}}\langle \Phi^a_{1j}|f\rangle_z. \tag{3.32}$$

Provided the condition on $f(z,\bar z)$ is satisfied, we interchange integration with respect to $\mathrm{d}z\wedge\mathrm{d}\bar z$ and $\mathrm{d}k \wedge \mathrm{d}\bar k$ and use the completeness formula (3.25). □

The spectral decomposition presented here is different from that of Kiselev [10, 11]. In the latter approach, the function $f(z,\bar z)$ is spanned by *squared* eigenfunctions of the original problem (1.2) defined according to oscillatory-type behaviour at infinity. In our approach, we transformed the system (1.2) to the form (2.2) and defined the oscillatory-type eigenfunctions according to the *single* eigenfunctions $\boldsymbol{N}_\mu(z,\bar z,k,\bar k)$. We also notice that the (degenerate) bound states $\boldsymbol{\Phi}'_j(z,\bar z)$ are not relevant for the spectral decomposition, although they appear implicitly through the meromorphic contributions of the eigenfunctions $\boldsymbol{N}_\mu(z,\bar z,k,\bar k)$ at $k = k_j$ (see section 4).

4. Perturbation theory for a single lump

We use the scalar spectral decomposition based on equation (3.31) and develop a perturbation theory for multi-lump solutions of the DSII equation. We present formulae in the case of a single lump ($n = 1$), the case of multi-lump potentials can be obtained by summing along the indices j, l occurring in the expressions below.

The single-lump potential $u(z,\bar z)$ has the form [6]

$$u(z,\bar z) = \frac{c_j}{|z+z_j|^2 + |c_j|^2}\mathrm{e}^{\mathrm{i}(k_j z + \bar k_j \bar z)}, \tag{4.1}$$

where c_j, z_j are complex parameters. The associated bound states follow from equations (2.26) and (3.19) as

$$\boldsymbol{\Phi}_j(z,\bar z) = \frac{1}{|z+z_j|^2 + |c_j|^2}\begin{bmatrix} \bar z + \bar z_j \\ -\bar c_j \mathrm{e}^{-\mathrm{i}(k_j z + \bar k_j \bar z)} \end{bmatrix}, \tag{4.2}$$

$$\boldsymbol{\Phi}^a_j(z,\bar z) = \frac{1}{|z+z_j|^2 + |c_j|^2}\begin{bmatrix} c_j \mathrm{e}^{\mathrm{i}(k_j z + \bar k_j \bar z)} \\ \bar z + \bar z_j \end{bmatrix}. \tag{4.3}$$

We first consider a general perturbation to the single lump subject to the localization condition, $\Delta u \sim \mathrm{O}(|z|^{-2})$ as $|z| \to \infty$. We then derive explicit formulae for a special form of the perturbation term $\Delta u(z,\bar z)$.

4.1. General perturbation of a single lump

Suppose the potential is specified as $u^\epsilon = u(z,\bar z) + \epsilon\Delta u(z,\bar z)$, where $u(z,\bar z)$ is given by equation (4.1) and $\Delta u(z,\bar z)$ is a perturbation term. Two bound states $\boldsymbol{\Phi}_j(z,\bar z)$ and $\boldsymbol{\Phi}'_j(z,\bar z)$ are supported by a single-lump potential $u(z,\bar z)$ at a single point $k = k_j$. The spectral decomposition given by equation (3.31) provides a basis for expansion of $\mu_2^\epsilon(z,\bar z,\kappa,\bar\kappa)$ at $k = \kappa$,

$$\mu_2^\epsilon(z,\bar z,\kappa,\bar\kappa) = \iint \mathrm{d}k \wedge \mathrm{d}\bar k\, \alpha(k,\bar k) N_{2\mu}(z,\bar z,k,\bar k) + \alpha_j \Phi_{2j}(z,\bar z) \tag{4.4}$$

where $\alpha(k,\bar{k})$ and α_j are defined by equation (3.32) and depend on the parameter κ. The other component $\mu_1^\epsilon(z,\bar{z},\kappa,\bar{\kappa})$ can be expressed from equation (2.2) as

$$\mu_1^\epsilon(z,\bar{z},\kappa,\bar{\kappa}) = \iint \mathrm{d}k\wedge\mathrm{d}\bar{k}\,\alpha(k,\bar{k})N_{1\mu}(z,\bar{z},k,\bar{k}) + \alpha_j\Phi_{1j}(z,\bar{z}) + \epsilon\Delta\mu_1(z,\bar{z}), \tag{4.5}$$

where the remainder term $\Delta\mu_1(z,\bar{z})$ solves the equation

$$(\Delta\mu_1)_{\bar{z}} = -\Delta u\mu_2^\epsilon.$$

We write the solution of this equation in the form

$$\Delta\mu_1(z,\bar{z}) = A - \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}z'\wedge\mathrm{d}\bar{z}'}{z'-z}(\Delta u\mu_2^\epsilon)(z',\bar{z}'), \tag{4.6}$$

subject to the boundary condition as $|z|\to\infty$,

$$\Delta\mu_1(z,\bar{z}) \to A + \mathrm{O}(z^{-1}),$$

where A is an arbitrary constant. Using the explicit representation (4.6), we transform equation (2.2) into the system of integral equations for $\alpha(k,\bar{k})$ and α_j,

$$\alpha(k,\bar{k}) = \frac{\epsilon}{4\pi^2\mathrm{i}(k-\kappa)}\left[\iint\mathrm{d}k'\wedge\mathrm{d}\bar{k}'\,K(k,\bar{k},k',\bar{k}')\alpha(k',\bar{k}') + K_j(k,\bar{k})\alpha_j + R(k,\bar{k})A\right] + \mathrm{O}(\epsilon^2), \tag{4.7}$$

$$\alpha_j = \frac{\epsilon}{2\pi(k_j-\kappa)}\left[\iint\mathrm{d}k\wedge\mathrm{d}\bar{k}\,P_j(k,\bar{k})\alpha(k,\bar{k}) + K_{jl}\alpha_l + R_jA\right] + \mathrm{O}(\epsilon^2), \tag{4.8}$$

where

$$K(k,\bar{k},k',\bar{k}') = \langle\boldsymbol{N}_\mu^a(k)|\boldsymbol{N}_\mu(k')\rangle_{\Delta u}, \qquad K_j(k,\bar{k}) = \langle\boldsymbol{N}_\mu^a(k)|\boldsymbol{\Phi}_j\rangle_{\Delta u},$$
$$P_j(k,\bar{k}) = \langle\boldsymbol{\Phi}_j^a|\boldsymbol{N}_\mu(k)\rangle_{\Delta u}, \qquad K_{jl} = \langle\boldsymbol{\Phi}_j^a|\boldsymbol{\Phi}_l\rangle_{\Delta u},$$

and the scalar product for the squared eigenfunction is defined as [10, 11]

$$\langle\boldsymbol{f}(k)|\boldsymbol{g}(k')\rangle_h = \iint\mathrm{d}z\wedge\mathrm{d}\bar{z}[\bar{h}(z,\bar{z})f_1(z,\bar{z},k,\bar{k})g_1(z,\bar{z},k',\bar{k}') + h(z,\bar{z})f_2(z,\bar{z},k,\bar{k})g_2(z,\bar{z},k',\bar{k}')].$$

The non-homogeneous terms $R(k,\bar{k})$ and R_j can be computed exactly as

$$R(k,\bar{k}) = \iint\mathrm{d}z\wedge\mathrm{d}\bar{z}(\bar{u}N_{1\mu}^a)(z,\bar{z}) = 2\pi b(k,\bar{k}),$$
$$R_j = \iint\mathrm{d}z\wedge\mathrm{d}\bar{z}(\bar{u}\Phi_{1j}^a)(z,\bar{z}) = -2\pi\mathrm{i},$$

where $b(k,\bar{k}) = 0$ if $n\neq 0$. We solve the system of equations (4.7) and (4.8) asymptotically for $\kappa = k_j + \epsilon\Delta\kappa$ and $\Delta\kappa \sim \mathrm{O}(1)$. The leading order behaviour of the integral kernels follows from the asymptotic representation (2.24) as $k\to k_j$,

$$K(k,\bar{k},k',\bar{k}') \to \frac{\bar{K}_{jj}}{(\bar{k}-\bar{k}_j)(\bar{k}'-\bar{k}_j)}, \quad K_j(k,\bar{k}) \to \frac{\mathrm{i}P_{jj}}{\bar{k}-\bar{k}_j}, \quad P_j(k,\bar{k}) \to \frac{\mathrm{i}\bar{P}_{jj}}{\bar{k}-\bar{k}_j}, \tag{4.9}$$

where

$$\bar{K}_{jj} = \langle\boldsymbol{\Phi}_j^{a\prime}|\boldsymbol{\Phi}_j'\rangle_{\Delta u}, \qquad P_{jj} = \langle\boldsymbol{\Phi}_j^{a\prime}|\boldsymbol{\Phi}_j\rangle_{\Delta u}, \qquad \bar{P}_{jj} = -\langle\boldsymbol{\Phi}_j^a|\boldsymbol{\Phi}_j'\rangle_{\Delta u}. \tag{4.10}$$

Here we have used the symmetry constraints (2.20) and (3.15). The leading order of $\alpha(k,\bar{k})$ as $k\to k_j$ follows from equation (4.7) as

$$\alpha(k,\bar{k}) \to -\frac{\epsilon\Delta\bar{\kappa}\beta_j}{2\pi(k-\kappa)(\bar{k}-\bar{k}_j)},$$

where β_j is not yet defined. We use equation (A.4) of the appendix to compute the integral term,

$$\iint \frac{\mathrm{d}k \wedge \mathrm{d}\bar{k}}{(k-\kappa)(\bar{k}-\bar{k}_j)^2} = \frac{2\pi\mathrm{i}}{\bar{k}_j - \bar{\kappa}},$$

and reduce the system of integral equations (4.7) and (4.8) to an algebraic system as $k \to k_j$,

$$-2\pi\,\Delta\kappa\alpha_j = K_{jj}\alpha_j - \bar{P}_{jj}\beta_j - 2\pi\mathrm{i}A, \tag{4.11}$$

$$2\pi\,\Delta\bar{\kappa}\beta_j = -P_{jj}\alpha_j - \bar{K}_{jj}\beta_j. \tag{4.12}$$

If $P_{jj} \neq 0$, the determinant of the above system is strictly positive. Therefore, homogeneous solutions at $A = 0$ (bound states) are absent for $\epsilon \neq 0$. This result indicates that the double eigenvalue at $k = k_j$ disappears under a generic perturbation of the potential $u(z, \bar{z})$ with $P_{jj} \neq 0$ (see also [8]).

For $A \neq 0$, we find inhomogeneous solutions of equations (4.11) and (4.12),

$$\alpha_j = \frac{2\pi\mathrm{i}A(\bar{K}_{jj} + 2\pi\,\Delta\bar{\kappa})}{|K_{jj} + 2\pi\,\Delta\kappa|^2 + |P_{jj}|^2}, \qquad \beta_j = \frac{-2\pi\mathrm{i}AP_{jj}}{|K_{jj} + 2\pi\,\Delta\kappa|^2 + |P_{jj}|^2}. \tag{4.13}$$

The eigenfunction $\boldsymbol{\mu}^\epsilon(z, \bar{z}, \kappa, \bar{\kappa})$ given by equations (4.4) and (4.5) satisfies the boundary condition (2.5) if $A = \epsilon^{-1}$ and has the following asymptotic representation,

$$\boldsymbol{\mu}^\epsilon(z, \bar{z}, \kappa, \bar{\kappa}) = \boldsymbol{e}_1 + \frac{2\pi\mathrm{i}[2\pi(\bar{\kappa} - \bar{k}_j) + \epsilon\bar{K}_{jj}]\boldsymbol{\Phi}_j(z, \bar{z})}{|2\pi(\kappa - k_j) + \epsilon K_{jj}|^2 + |\epsilon P_{jj}|^2} - \frac{2\pi\mathrm{i}\epsilon P_{jj}\boldsymbol{\Phi}'_j(z, \bar{z})}{|2\pi(\kappa - k_j) + \epsilon K_{jj}|^2 + |\epsilon P_{jj}|^2}$$
$$+\Delta\boldsymbol{\mu}^\epsilon(z, \bar{z}), \tag{4.14}$$

where the term $\Delta\boldsymbol{\mu}^\epsilon(z, \bar{z})$ is not singular in the limit $\epsilon \to 0$ and $\kappa \to k_j$.

In the limit $\epsilon \to 0$, $\kappa \neq k_j$, we find a meromorphic expansion for $\boldsymbol{\mu}^\epsilon(z, \bar{z}, \kappa, \bar{\kappa})$ as

$$\boldsymbol{\mu}^\epsilon(z, \bar{z}, \kappa, \bar{\kappa}) = \boldsymbol{e}_1 + \frac{\mathrm{i}\boldsymbol{\Phi}_j(z, \bar{z})}{\kappa - k_j} + \epsilon\left[\frac{K_{jj}\boldsymbol{\Phi}_j(z, \bar{z})}{2\pi\mathrm{i}(\kappa - k_j)^2} + \frac{P_{jj}\boldsymbol{\Phi}'_j(z, \bar{z})}{2\pi\mathrm{i}|\kappa - k_j|^2}\right] + \mathrm{O}(\epsilon^2). \tag{4.15}$$

It is clear that the double pole can be incorporated by shifting the eigenvalue k_j to

$$k_j^\epsilon = k_j - \frac{\epsilon K_{jj}}{2\pi}.$$

The other double-pole term in the expansion (4.15) has a non-analytic behaviour in the k-plane and leads to the appearance of the spectral data $b^\epsilon(\kappa, \bar{\kappa}) = \epsilon\Delta b(\kappa, \bar{\kappa})$ which measures the departure of $\boldsymbol{\mu}^\epsilon(z, \bar{z}, \kappa, \bar{\kappa})$ from analyticity according to equation (2.8). We find from equations (2.9) and (4.15) that the spectral data $\Delta b(\kappa, \bar{\kappa})$ has the following singular behaviour as $\kappa \to k_j$:

$$\Delta b(\kappa, \bar{\kappa}) \to \frac{-P_{jj}}{2\pi|\kappa - k_j|^2}. \tag{4.16}$$

Thus, if $P_{jj} \neq 0$ the analyticity of $\boldsymbol{\mu}^\epsilon(z, \bar{z}, \kappa, \bar{\kappa})$ is destroyed and the lump disappears. This conclusion as well as the analytical solution (4.13) agree with the results of Gadyl'shin and Kiselev [8, 9] where the transformation of a single lump into a decaying wavepacket was also studied.

In the other limit $\epsilon \neq 0$ and $\kappa \to k_j^\epsilon$ we find another expansion from equation (4.14),

$$\boldsymbol{\mu}^\epsilon(z, \bar{z}, \kappa, \bar{\kappa}) = \boldsymbol{e}_1 - \frac{2\pi\mathrm{i}}{\epsilon\bar{P}_{jj}}\boldsymbol{\Phi}'_j(z, \bar{z}) + \mathrm{O}(\kappa - k_j^\epsilon). \tag{4.17}$$

We conclude that the eigenfunction $\boldsymbol{\mu}^\epsilon(z, \bar{z}, \kappa, \bar{\kappa})$ is now free of pole singularities [8, 9]. We summarize the main result in the form of a proposition.

Proposition 4.1. *Suppose $u(z, \bar{z})$ is given by equation (4.1) and $\Delta u(z, \bar{z})$ satisfies the constraint*

$$P_{jj} = \langle \mathbf{\Phi}_j^{a'} | \mathbf{\Phi}_j \rangle_{\Delta u} \neq 0.$$

Then, the potential $u^\epsilon = u(z, \bar{z}) + \epsilon \Delta u(z, \bar{z})$ does not support embedded eigenvalues of the Dirac system (2.2) for $\epsilon \neq 0$.

4.2. Explicit solution for a particular perturbation

Here we specify $c_j = c\mathrm{e}^{\mathrm{i}\theta}$, where c and θ are real, and consider a particular perturbation $\Delta u(z, \bar{z})$ to the lump $u(z, \bar{z})$ (4.1) in the form,

$$\Delta u(z, \bar{z}) = Q(z, \bar{z}) \mathrm{e}^{\mathrm{i}(k_j z + \bar{k}_j \bar{z} + \theta)},$$

where $Q(z, \bar{z})$ is a real function. Using equations (4.10), (4.2) and (4.3), we find explicitly the matrix elements K_{jj} and P_{jj},

$$K_{jj} = \iint \mathrm{d}z \wedge \mathrm{d}\bar{z} \frac{c(\bar{z} + \bar{z}_j)[Q(z, \bar{z}) - \bar{Q}(z, \bar{z})]}{[|z + z_j|^2 + c^2]^2} = 0,$$

$$P_{jj} = \iint \mathrm{d}z \wedge \mathrm{d}\bar{z} \frac{|z + z_j|^2 \bar{Q}(z, \bar{z}) + c^2 Q(z, \bar{z})}{[|z + z_j|^2 + c^2]^2} = \frac{1}{2c} \iint \mathrm{d}z \wedge \mathrm{d}\bar{z} (u \Delta \bar{u} + \bar{u} \Delta u).$$

The element P_{jj} can be seen as a correction to the field energy,

$$N = \frac{\mathrm{i}}{2} \iint \mathrm{d}z \wedge \mathrm{d}\bar{z} |u^\epsilon|^2 (z, \bar{z}) = N_0 + \mathrm{i}\epsilon c P_{jj} + \mathrm{O}(\epsilon^2),$$

where $N_0 = \pi$ is the energy of the single lump solution (independent of the lump parameters k_j and c_j). Thus, a perturbation which leads to the destruction of a single lump, that is with $P_{jj} \neq 0$, changes necessarily the value for the lump energy N_0.

5. Concluding remarks

The main result of our paper is the prediction of structural instability of multi-lump potentials in the Dirac system associated to the DSII equation. The multi-lump potentials correspond to eigenvalues embedded into a two-dimensional continuous spectrum with the spectral data $b(k, \bar{k})$ satisfying the additional constraint (2.21). In this case, there is no interaction between lumps and continuous radiation. However, a generic initial perturbation induces coupling between the lumps and radiation and, as a result of their interaction, the embedded eigenvalues disappear. This result indicates that the localized multi-lump solutions decay into continuous wavepackets in the nonlinear dynamics of the DSII equation (see also [8, 9]).

This scenario is different from the two types of bifurcations of embedded eigenvalues discussed in our previous paper [14]. The type-I bifurcation arises from the edge of the essential spectrum when the limiting bounded (non-localized) eigenfunction is transformed into a localized bound state. The type-II bifurcation occurs when an embedded eigenvalue splits off the essential spectrum. Both situations persist in the spectral plane when the essential spectrum is one-dimensional and covers either a half-axis or the whole axis. However, in the case of the DSII equation, the essential spectrum is the whole spectral plane and embedded eigenvalues cannot split off the essential spectrum. As a result, they disappear due to their structural instability.

Acknowledgments

We benefited from stimulating discussions with M Ablowitz, A Fokas, D Kaup, O Kiselev and A Yurov. DP acknowledges support from a NATO fellowship provided by NSERC and CS acknowledges support from NSERC operating grant OGP0046179.

Appendix. Formulae of the ∂-analysis

Here we reproduce some formulae of the complex $\bar{\partial}$-analysis [17] to compute the integrals I_0, $I_1(z)$ and $I_2(z)$ defined in equations (3.23), (3.24), and (3.29). We define the complex integration in the z-plane by

$$\iint \mathrm{d}z \wedge \mathrm{d}\bar{z}\, f(z,\bar{z}) = -\iint \mathrm{d}\bar{z} \wedge \mathrm{d}z\, f(z,\bar{z}),$$

where $\mathrm{d}z \wedge \mathrm{d}\bar{z} = -2\mathrm{i}\,\mathrm{d}x\,\mathrm{d}y$. The complex $\delta(z)$ distribution is defined by

$$\iint \mathrm{d}z \wedge \mathrm{d}\bar{z}\, f(z)\delta(z-z_0) = -2\mathrm{i} f(z_0), \tag{A.1}$$

where $\delta(z) = \delta(x)\delta(y)$. In particular, the δ-distribution appears in the $\bar{\partial}$-analysis according to the relation [17],

$$\frac{\partial}{\partial \bar{z}}\left[\frac{1}{z-z_0}\right] = \pi\delta(z-z_0). \tag{A.2}$$

Computing the integral I_0, we get the formula,

$$I_0 = \iint \mathrm{d}z \wedge \mathrm{d}\bar{z}\mathrm{e}^{\mathrm{i}kz+\mathrm{i}\bar{k}\bar{z}} = -2\mathrm{i}\iint \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{2\mathrm{i}\,\mathrm{Re}(k)x-2\mathrm{i}\,\mathrm{Im}(k)y} = -2\pi^2\mathrm{i}\delta(k),$$

which proves the identity (3.23).

Using the Green theorem [17], one has the integration identity,

$$\iint_D \mathrm{d}z \wedge \mathrm{d}\bar{z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_2}{\partial \bar{z}}\right) = \int_C (f_1\,\mathrm{d}\bar{z} + f_2\,\mathrm{d}z), \tag{A.3}$$

where D is a domain of the complex plane and C its boundary. The generalized Cauchy's formula has the form [17]

$$f(z,\bar{z}) = \frac{1}{2\pi\mathrm{i}}\int_C \frac{f(z',\bar{z}')\,\mathrm{d}z'}{z'-z} + \frac{1}{2\pi\mathrm{i}}\iint_D \frac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'}{z'-z}\frac{\partial f}{\partial \bar{z}'}, \tag{A.4}$$

or, equivalently,

$$f(z,\bar{z}) = -\frac{1}{2\pi\mathrm{i}}\int_C \frac{f(z',\bar{z}')\,\mathrm{d}\bar{z}'}{\bar{z}'-\bar{z}} + \frac{1}{2\pi\mathrm{i}}\iint_D \frac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'}{\bar{z}'-\bar{z}}\frac{\partial f}{\partial z'}. \tag{A.5}$$

In order to find the integral $I_1(z)$ we use equation (A.5) with

$$f(z,\bar{z}) = \frac{1}{\mathrm{i}k}\mathrm{e}^{\mathrm{i}(kz+\bar{k}\bar{z})}, \qquad k \neq 0$$

and choose the domain D to be a large ball of radius R (see equation (2.14)). The boundary value integral vanishes since

$$\lim_{R\to\infty}\int_{|z|=R}\frac{\mathrm{d}\bar{z}}{\bar{z}}\mathrm{e}^{\mathrm{i}(kz+\bar{k}\bar{z})} = -2\pi\mathrm{i}\lim_{R\to\infty} J_0(2|k|R) = 0, \tag{A.6}$$

where $J_0(z)$ is the Bessel function. Equation (A.5) for the function $f(z,\bar{z})$ then reduces to equation (3.24).

In order to compute the integral $I_2(z_0)$, we apply equation (A.3) with $f_1 = 0$ and

$$f_2(z, \bar{z}) = \frac{1}{\bar{z} - \bar{z}_0} \mathrm{e}^{\mathrm{i}(kz+\bar{k}\bar{z})}.$$

The domain D is chosen as above. The boundary value integral vanishes again,

$$\lim_{R\to\infty} \int_{|z|=R} \frac{\mathrm{d}z}{\bar{z}} \mathrm{e}^{\mathrm{i}(kz+\bar{k}\bar{z})} = -2\pi\mathrm{i} \lim_{R\to\infty} J_{-2}(2|k|R) = 0, \tag{A.7}$$

where $J_{-2}(z)$ is the Bessel function. Equation (A.3) for the function $f_2(z, \bar{z})$ reduces to equation (3.29).

References

[1] Davey A and Stewartson K 1974 On three-dimensional packets of surface waves *Proc. R. Soc.* A **388** 101–10
[2] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
[3] Fokas A S and Ablowitz M J 1984 On the inverse scattering transform of multidimensional nonlinear equations related to first-order systems in the plane *J. Math. Phys.* **25** 2494–505
[4] Beals R and Coifman R R 1986 The D-bar approach to inverse scattering and nonlinear evolutions *Physica* D **18** 242–9
[5] Fokas A S and Sung L Y 1992 On the solvability of the N-wave, Davey–Stewartson and Kadomtsev–Petviashvili equations *Inverse Problems* **8** 673–708
[6] Arkadiev V A, Pogrebkov A K and Polivanov M C 1989 Inverse scattering transform method and soliton solutions for Davey–Stewartson II equation *Physica* D **36** 189–97
[7] Arkadiev V A, Pogrebkov A K and Polivanov M C 1989 Closed string-like solutions of the Davey–Stewartson equation *Inverse Problems* **5** L1–6
[8] Gadyl'shin R R and Kiselev O M 1996 On solitonless structure of the scattering data for a perturbed two-dimensional soliton of the Davey–Stewartson equation *Teor. Mat. Phys.* **161** 200–8
[9] Gadyl'shin R R and Kiselev O M 1999 Structural instability of soliton for the Davey–Stewartson II equation *Teor. Mat. Fiz.* **118** 354–61
[10] Kiselev O M 1998 Basic functions associated with a two-dimensional Dirac system *Funct. Anal. Appl.* **32** 56–9
[11] Kiselev O M 1998 Perturbation theory for the Dirac equation in two-dimensional space *J. Math. Phys.* **39** 2333–45
[12] Yurov A 1997 Darboux transformation for Dirac equations with (1 + 1) potentials *Phys. Lett.* A **225** 51–9
[13] Yurov A 1999 Private communication
[14] Pelinovsky D and Sulem C 2000 Eigenvalues and eigenfunctions for a scalar Riemann–Hilbert problem associated to inverse scattering *Commun. Math. Phys.* to appear
[15] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 The inverse scattering transform—Fourier analysis for nonlinear problems *Stud. Appl. Math.* **53** 249–315
[16] Soffer A and Weinstein M I 1998 Time dependent resonance theory *Geom. Funct. Anal.* **8** 1086–128
[17] Ablowitz M J and Fokas A S 1997 *Complex Variables: Introduction and Applications* (Cambridge: Cambridge University Press)