Self-focusing instability of nonlinear plane waves in shear flows

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The instability of plane periodic waves and solitons in near-surface shear flows of deep water is investigated within the framework of the Shrira model. The discrete of modes spectrum growing in time superposed on an arbitrary nonlinear wave and their growth rate are found explicitly for small wave numbers of the transversal modulation. It is shown that the nonlinear wave instability is involved with a decay dispersion relation for linear perturbations, which allows the resonance condition to be met for three-wave interaction processes.

1. INTRODUCTION

The mechanism of modulational and self-focusing instabilities of weakly nonlinear quasiharmonic waves is well known for different dispersive media (see, for example, Refs. 1–5). However, the instability of essentially nonlinear waves including solitons has not been studied well enough until now.

Plane nonlinear waves and solitons in weakly dispersive isotropic media are stable with respect to small longitudinal and transverse perturbations if the phase velocity of quasiharmonic waves decreases with increasing wave number (i.e., the dispersion is negative).[6,7] In this case the linear dispersion relation does not meet the conditions of three-wave resonance and the spectrum of weakly nonlinear waves is referred to as a nondecaying spectrum. However, the presence of an external field or flow that introduces anisotropic properties in the medium can lead to a decaying spectrum even for negative dispersion and have a destabilizing effect on the stability of quasiplane waves with respect to transverse modulation of their fronts.[8–12]

Our work is also concerned with the problem of nonlinear wave stability in anisotropic media with negative dispersion. We consider the Shrira model[13–15] describing large-scale moderate-amplitude perturbations in shear flows of a nonstratified fluid. The corresponding integro-differential equation is written in the form

$$\partial_t A + c\partial_x A - \alpha A \partial_x A - \beta \partial_x \iint_{-\infty}^{\infty} Q(x-x', y-y') \times A(x',y')dx'dy' = 0. \quad (1.1)$$

In the context of this problem, the variable A is the amplitude of the longitudinal velocity perturbation of the fluid particles; c, α and β are positive constants which can be calculated from the velocity profile of the main flow, and the kernel of the integral operator $Q(x,y)$ is expressed by the Fourier transform

$$Q(x,y) = \frac{1}{4\pi^2} \iint_{-\infty}^{+\infty} \hat{Q}(k_x,k_y) \exp(ik_x x + ik_y y) dk_x dk_y,$$

of the function

$$\hat{Q}(k_x,k_y) = k \coth(kH), \quad (1.2)$$

where $k = \sqrt{k_x^2 + k_y^2}$ and H is a free parameter corresponding the fluid depth in the original hydrodynamic problem.

It is readily verified that the dispersion relation for linear perturbations in the model has the form

$$\omega = k_x [c - \beta k \coth(kH)]. \quad (1.3)$$

The spectrum (1.3) is non-decaying in the one-dimensional (1D) case. However, three quasiharmonic perturbations propagating at different angles can form a resonant triplet. Therefore, it is to be expected that plane nonlinear waves should be unstable with respect to transverse perturbations of their fronts in the Shrira model, resulting in the formation of two-dimensional (2D) solitons. Indeed, 2D soliton solutions were revealed by numerical calculations based on Eq. (1.1) (Refs. 14,15). According to the estimates given in Ref. 14 these solitons can be associated with the coherent structures of the boundary layer observed in experiments in a wind tunnel.[16] It is important to emphasize that the structures are formed spontaneously as a result of the development of the self-focusing instability of plane nonlinear waves and live for a long time as characteristic entities of the boundary layer.

Note that in the limiting case $H \to 0$, Eq. (1.1) transforms into a 2D analog of the Zakharov–Kuznetsov equation describing ion–acoustic waves in a magnetized plasma.[17] In this paper we consider the other limiting case $H \to \infty$. However, our results are qualitatively similar to the ones found in the Refs. 8–12 for the Zakharov–Kuznetsov model. This fact enables us to speak about the general mechanism of plane nonlinear wave instability in shear flows of fluid of arbitrary depth H described by equation (1.1).

2. PLANE NONLINEAR WAVES AND THEIR STABILITY

Below, we shall consider only the limiting case of deep water ($H \to \infty$). In the reference frame propagating with velocity c, equation (1.1) can be simplified by replacing

$$u(\tilde{x},y,\tilde{t}) = -\frac{\alpha}{2\beta} A(\tilde{x},y,\tilde{t}), \quad \tilde{x} = x - ct, \quad \tilde{t} = \beta t.$$

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As a result, equation (1.1) takes on the form (the tilde is omitted below):

$$\partial_t u+2u\partial_x u+\partial_x\iint_{-\infty}^{\infty} Q(x-x',y-y')\times u(x',y')dx'dy'=0, \quad (2.1)$$

where $Q(x,y)=1/2\pi(x^2+y^2)^{3/2}$.

In a 1D case, this equation transforms to the well-known Benjamin–Ono (BO) equation, which has a complete family of steady solutions with the following explicit form[18]

$$u_0=\frac{V\tanh^2\phi}{1-\cos(\mathbf{Kx}-\omega t)\,\mathrm{sech}\,\phi}, \quad (2.2)$$

where V and ϕ are positive real constants, and ω and $\mathbf{K}(K_x,K_y)$ are related by the nonlinear dispersion equation

$$\omega=VK_x,\quad V=\sqrt{K_x^2+K_y^2}\coth\phi. \quad (2.3)$$

The solution (2.2) describes a plane nonlinear periodic wave with frequency ω and wave vector $\mathbf{K}$, where $|\mathbf{K}|\leqslant V$. The parameter ϕ characterizes the nonlinear properties of the wave. In the limit $\phi\to\infty$, the wave has an vanishingly small amplitude and a sinusoidal shape superposed on the constant background V:

$$u_0=V+a\cos(\mathbf{Kx}-\omega t)+O(a^2), \quad (2.2a)$$

$$\omega=K_x\sqrt{K_x^2+K_y^2}+O(a^2),\quad a=V/\,\mathrm{sech}\,\phi\ll V. \quad (2.3a)$$

In the other limiting case $\phi\to 0$, $\mathbf{K}\to 0$, the solution (2.2) yields a plane "algebraic" soliton falling off rationally in the direction of the vector $\mathbf{S}=\mathbf{K}/\phi$:

$$u_0=\frac{2V}{1+(\mathbf{Sx}-\Omega t)^2}+O(\phi^2), \quad (2.2b)$$

$$\Omega\equiv\frac{\omega}{\phi}=S_x\sqrt{S_x^2+S_y^2}+O(\phi^2). \quad (2.3b)$$

As is known (see, for example, Ref. 18), the entire family of stationary waves (2.2) is stable with respect to the longitudinal perturbations whose evolution is described by the 1D BO equation. Below, we consider the stability of plane wave with respect to the transverse modulations of their fronts using Eq. (2.1). We introduce an angle α between the direction of wave propagation and the x–axis so that $\mathbf{K}=K(\cos\alpha,\sin\alpha)$, and rotate the coordinate system propagating with the wave velocity V by this angle:

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}=\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x-Vt\\ y\end{pmatrix}. \quad (2.4)$$

Further, by linearizing Eq. (2.1) about the solution (2.2) and looking for the perturbation in the form $u_1(\xi,\eta,t)=w(\xi)\exp(\lambda t+i\kappa_\eta\eta)$, where κ_η is the wave number of transverse modulation, we obtain a linear integro–differential equation with variable coefficients for the function $w(\xi)$:

$$\lambda w(\xi)=\partial_x\left(Vw-2u_0w-\int_{-\infty}^{\infty}R(\xi-\xi')w(\xi')d\xi'\right), \quad (2.5)$$

where $\partial_x\equiv\cos\alpha\partial_\xi-\sin\alpha i\kappa_\eta$ and $R(\xi)$ is expressed by the first-order modified Bessel function of the second kind

$$R(\xi)=\frac{|\kappa_\eta|}{\pi|\xi|}K_1(|\kappa_\eta\xi|).$$

If the plane wave $u_0(\xi)$ is unstable, a spatially bounded perturbation exists in the problem (2.5) for the eigenvalue λ with a positive real part $\mathrm{Re}\,\lambda>0$.

The eigenvalue problem (2.5) is rather complicated. In the general case we have to use numerical calculations to investigate its solutions. However, the problem may be investigated analytically under certain assumptions. Namely, the approximation of a long transverse modulation scale leads to essential simplification of the problem for the waves propagating along the main flow ($\alpha=0$).

Note that the waves propagating at small angles α to the flow are the most critical ones from the standpoint of their possible instability. On the other hand, the waves propagating transversely to the flow are neutrally stable (i.e., the parameter λ is a purely imaginary value) because the integro-differential operator on the right-hand side of (2.5) is anti-Hermitian for $\alpha=\pi/2$. Moreover, in this case the original wave $u_0(y)$ can have an arbitrary form and may in fact be regarded as a static inhomogeneity in the distribution $U(z,y)$ of the shear flow. Therefore, in what follows we shall consider mainly the most unstable waves with $\alpha=0$.

Using the known form of the modified Bessel function[19] at small values of its argument, we expand the kernel of the integral operator in the series of κ_η:

$$R(\xi)=\frac{1}{\pi\xi^2}\left(1+\frac{\kappa_\eta^2\xi^2}{2}\left[\ln\frac{|\kappa_\eta\xi|}{2}-C-\frac{1}{2}\right]+O(\kappa_\eta^4\xi^4)\right), \quad (2.6)$$

where C, the Euler constant, is unimportant in what follows.

Substituting the expansion (2.6) into Eq. (2.5) at $\alpha=0$ and neglecting the terms that are $O(\kappa_\eta^4\xi^4)$ and higher, we find a simpler eigenvalue problem

$$\lambda w(x)=\partial_x(Vw-2u_0w)-\partial_{xx}^2\mathbf{H}w+\frac{k_y^2}{2}\mathbf{H}w, \quad (2.7)$$

where $\mathbf{H}w=(1/\pi)\int_{-\infty}^{\infty}[w(x')dx'/x'-x]$ is the Hilbert transform of the function $w(x)$.

This simplified equation is valid when the transverse modulation length is much greater than the characteristic length of the original wave. For the problem (2.7) we succeed in constructing a complete set of time-growing modes of the discrete spectrum for an arbitrary nonlinear wave described by the solution (2.2), (2.3) at $K_y=0$.

3. PLANE WAVE INSTABILITY TO LONG TRANSVERSE MODULATIONS

We consider an auxiliary equation possessing some remarkable properties:

$$\partial_t\psi+|\nabla\psi|^2+\nabla^2\mathbf{H}\psi=0. \quad (3.1)$$

This equation helps us investigate the spectrum of eigenvalues and to construct the eigenfunctions of the problem (2.7). First of all, we consider the family of exact solutions describing plane stationary waves. It is convenient to express them by the vector variable $\mathbf{u}=\nabla\psi$ [cf. (2.2)]:

$$\mathbf{u}_0=\frac{\mathbf{V}\tanh^2\phi}{1-\cos(\mathbf{Kx}-\omega t)/\text{sech}\,\phi}, \quad (3.2)$$

where the parameters ϕ, ω, $\mathbf{K}$ and $\mathbf{V}$ are related by the nonlinear dispersion equation

$$\omega=\mathbf{KV}, \quad \mathbf{V}=\mathbf{K}\coth\phi, \quad (3.3)$$

and

$$\text{sign}\,\phi=\text{sign}\,K_x. \quad (3.4)$$

It is not difficult to show that there exists a close analogy between the solutions (2.2) and (3.2), on the one hand, and the corresponding dispersion Eqs. (2.3) and (3.3), (3.4), on the other hand. In the paraxial approximation $(K_y^2/K_x^2\ll 1)$, Eq. (2.3) leads to the expression $\omega\approx(K_x^2+1/2K_y^2)\coth\phi$ that coincides with the dispersion dependence (3.3) to within a factor 1/2 (which can be easily eliminated by scaling transformation of the variables).

By differentiating Eq. (3.1) with respect to x and linearizing it about the plane wave $\mathbf{u}_0=(u_0,0)$ with the wave vector $\mathbf{K}=(K,0)$ we find an equation for the linear perturbation $u_1=w(x-Vt)\exp(\lambda t+ik_y y)$ which coincides with the eigenvalue problem (2.7).

The auxiliary equation (3.1) is remarkable in that it can be reduced to a bilinear form

$$(iD_t+D_x^2+D_y^2)f^+f^-=0 \quad (3.5)$$

by the substitution of the dependent variable

$$\psi(x,y,t)=i\ln\left(\frac{f^-(x,y,t)}{f^+(x,y,t)}\right). \quad (3.6)$$

Here we use the Hirota operator (see, for example, [18]):

$$(D_x^n)fg=(\partial_x-\partial_{x'})^n f(x)g(x')|_{x=x'}.$$

For transformation to equation (3.5) the functions $f^\pm$ are supposed to satisfy the additional relation

$$\mathbf{Hu}=-\nabla\ln(f^+f^-). \quad (3.7)$$

Following Ref. 18, we can use the Hirota direct method to construct the partial solutions of Eq. (3.5) in the form of a polynomial of exponential functions. One of them is necessary for our analysis of plane wave stability and has the form

$$f^\pm=1+\exp(i\eta_1\pm\phi_1)+\exp(i\eta_2\pm\phi_2)+\exp(i\eta_1+i\eta_2 \pm\phi_1\pm\phi_2+A_{12}), \quad (3.8)$$

where $\eta_i=\mathbf{k}_i\,(\mathbf{x}-\mathbf{v}_i t-\mathbf{x}_i)+i\eta_{0i}$ and η_{0i} are arbitrary constants,

$$\mathbf{v}_i=\mathbf{k}_i\coth\phi_1,\quad \text{and}\ \exp A_{12}=\frac{|\mathbf{v}_1-\mathbf{v}_2|^2-|\mathbf{k}_1-\mathbf{k}_2|^2}{|\mathbf{v}_1-\mathbf{v}_2|^2-|\mathbf{k}_1+\mathbf{k}_2|^2}.$$

For $\eta_{02}=+\infty$ and $\eta_{01}=0$, the solution (3.6), (3.8) of Eq. (3.1) is a family of plane stationary waves. We set $\exp(-\eta_{02})=\varepsilon\ll 1$ and write the expression (3.8) in the form $f^\pm=f_0^\pm+\varepsilon f_1^\pm$. Substituting it into (3.6) and neglecting the terms of order $O(\varepsilon^2)$ and higher, we find the solution of Eq. (3.1) linearized about the plane wave (3.2): $\mathbf{u}=\mathbf{u}_0+\varepsilon\mathbf{u}_1+O(\varepsilon^2)$. The condition (3.7) is fulfilled for the two first terms of the expansion if the zeros of the function f_0^+ (f_0^-) of the complex variable x lie in the upper-half (lower-half) of the plane. The relation (3.4) meets this condition. In addition, the functions $f_1^\pm$ must satisfy the limiting relations:

$$\lim_{\text{Im}\,x\to\mp\infty}\left(\frac{f_1^\pm(x,y,t)}{f_0^\pm(x,y,t)}\right)=\text{const.} \quad (3.9)$$

Now we consider the plane wave $\mathbf{u}_0(u_0,0)$ propagating along the x-axis: $\mathbf{k}_1=\mathbf{K}=(K,0)$, $\mathbf{v}_1=\mathbf{V}=(V,0)$, and $\phi_1=\phi$. We remind our reader that only for this wave can the linearized analog of equation (3.1) be reduced to the eigenvalue problem (2.7). We denote the perturbation parameters by $\mathbf{k}_2\equiv\mathbf{k}=(k_x,k_y)$, $\mathbf{v}_2\equiv\mathbf{V}+i\lambda\mathbf{k}/|\mathbf{k}|^2$, and $\phi_2\equiv\theta$.

Analysis shows that nontrivial solutions of the linearized problem may be found only under the condition that the factor $\exp(A_{12})$ in (3.8) is equal to zero or to infinity. Otherwise, the limiting relations (3.9) hold only at $k_x=0$, but then $f_1^\pm=f_0^\pm$ and $\mathbf{u}_1=0$.

Let us set $\exp(A_{12})=0$. The corresponding solution of equation (3.1) can be written explicitly as

$$u_x=u_0+\varepsilon w(x-Vt)\exp(\lambda t+ik_y y)+O(\varepsilon^2),$$
$$w(x,k_x) =i\partial_x\left(\frac{\sinh(\phi-\theta)\exp(ik_x x)+\sinh\theta\exp[i(k_x-K)x]}{\cosh\phi-\cos(Kx)}\right), \quad (3.10)$$

where $\tanh\theta=(k_x^2+k_y^2)/(k_xV+i\lambda)$, and the growth rate $\lambda(k_x,k_y,K,V)$ is found from the equation $\exp A_{12}=0$:

$$\lambda^2=V^2k_y^2-(k_x^2+k_y^2)[(K-k_x)^2+k_y^2]. \quad (3.11)$$

It is not difficult to show that for the case $\exp A_{12}=\infty$ the solution (3.8) transforms to the same discrete-spectrum mode w after renormalization of the parameters $\mathbf{k}$ and θ.

The function w describes a linear perturbation of the nonlinear periodic stationary wave $\mathbf{u}_0=(u_0,0)$ at the values of the parameters k_x, k_y, quasi-periodic in space and growing in time, for which $\lambda^2>0$. The period of the carrier wave is determined by the parameter K varying in the range from 0 to V. The wave number k_x, which together with K determine the two other periods of the perturbation w, can be regarded as the Floquet-parameter in the theory of linear equations with periodic coefficients.[20] This parameter must vary in the range $0\leqslant k_x<K$. Indeed, direct calculation revealed that the relations (3.9) are fulfilled only for this range of k_x variations.

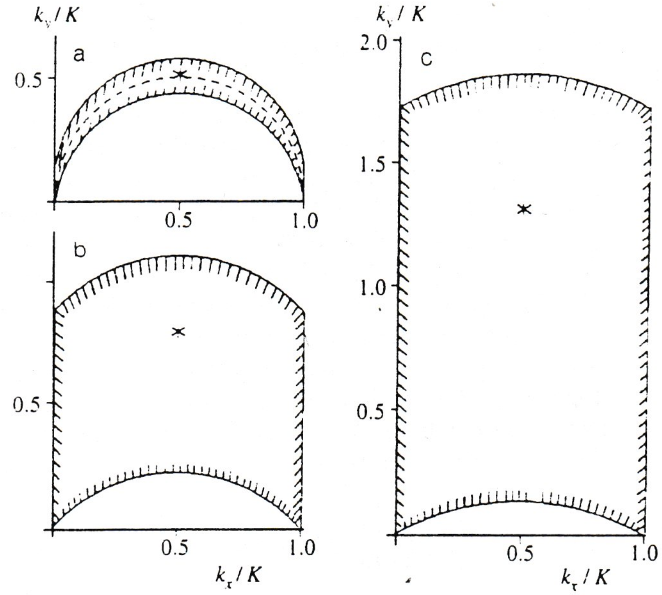

FIG. 1. The instability regions of a plane periodic wave in model (2.7) for $K/V=0.99$ (a); 0.75 (b) and 0.5 (c). The asterisk indicates the position of the global maximum of the growth rate. In Fig. 1a the dashed line corresponds to the resonant curve of quasiharmonic waves.

This time–growing mode has some symmetries. First of all, the symmetry about the axis $k_y=0$ as $w(-k_y)=w(k_y)$ follows from the formulas (3.10), (3.11). Further, using the identity

$$\sinh(\bar{\theta}(k_x)-\phi)\sinh(\theta(K-k_x)-\phi)$$
$$=\sinh\bar{\theta}(k_x)\sinh\theta(K-k_x),$$

which is valid for real λ, we can prove that

$$w(K-k_x)=\frac{\sinh(\theta(K-k_x)-\phi)}{\sinh\bar{\theta}(k_x)}\bar{w}(k_x). \tag{3.12}$$

The equality (3.12) means that the plane wave perturbations growing in time are also symmetrical about $k_x=K/2$. Moreover, as follows from the relation (3.11), the global maximum of the growth rate lies on this curve where $k_y^2=V^2/2-K^2/4$ and is determined by the equality $\max(\lambda^2)=V^2(V^2-K^2)/4$.

The instability regions on the k_x k_y plane with the global maximum of the growth rate labeled by the asterisk are hatched in Fig. 1 for three values of the parameter K/V: 0.99, 0.75, and 0.5. (Bear in mind that the nonlinear properties of the waves increase as the parameter $K/V=\tanh\phi$ decreases.) It is obvious from Fig. 1 that the instability region expands with increasing wave amplitude. The solitary waves (for which $K/V=0$) are the most unstable with respect to transverse modulations of their fronts. The dependence of the growth rate λ on the wave number of the transverse modulations k_y is shown in Fig. 2 for the particular cases $k_x=0$ and $k_x=K/2$.

In the limiting case of weakly nonlinear waves $K/V\to 1$ the instability region degenerates to the curve $k_y^2=k_x(K-k_x)$ shown in Fig. 1a by a dashed line. It is easy to verify from the dispersion relation in the limit (3.3) $\phi\to\infty$ that the resonance conditions for the triplet of waves ω_0, $\mathbf{K}$; ω_1, $\mathbf{k}$; ω_2, $(\mathbf{K}-\mathbf{k})$ are fulfilled on this curve so that

$$\omega_0(\mathbf{K})=\omega_1(\mathbf{k})+\omega_2(\mathbf{K}-\mathbf{k}).$$

Thus, this instability of the whole family of nonlinear stationary waves generalizes the well-known phenomenon of the decay instability of a quasilinear wave with parameters ω_0, $\mathbf{K}$ caused by its resonant interaction with the satellite waves which have the parameters ω_1, $\mathbf{k}$ and ω_2, $\mathbf{K}-\mathbf{k}$. In fact, it is obvious from Eq. (3.10) that the wave vectors $\mathbf{k}$ and $\mathbf{K}-\mathbf{k}$ determine the two-dimensional profile of the perturbation superposed on a nonlinear wave with finite ϕ.

Note that λ has no imaginary part at any point of the instability region. It means that the growing mode does not change the frequency of the carrier wave and propagates with the same velocity. For weakly nonlinear waves this phenomenon is related to the fact that the wave-satellites ω_1, $\mathbf{k}$ and ω_2, $\mathbf{K}-\mathbf{k}$ have the same velocity V along the x-axis as the carrier wave ω_0, $\mathbf{K}$.

For perturbations with the period of the carrier wave ($k_x=0$), we found that the growing mode $w(x)$ can be

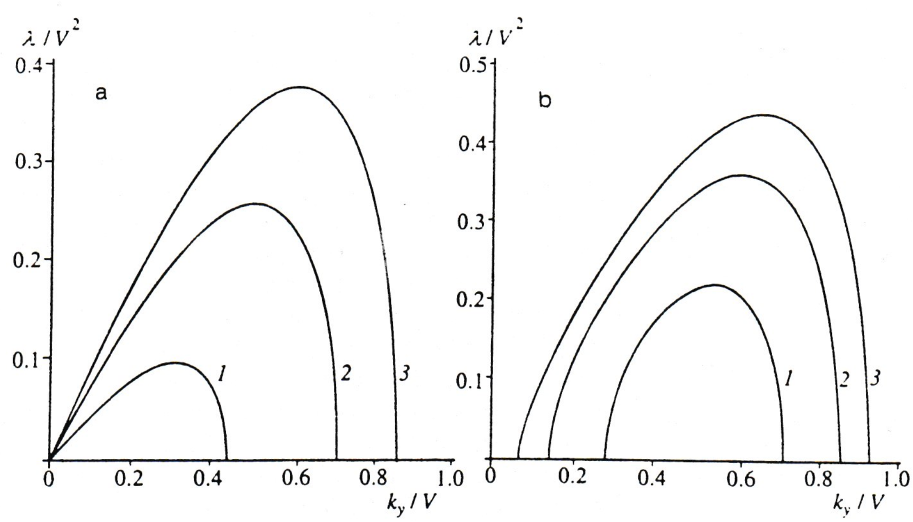

FIG. 2. The growth rate versus wave number of transverse modulation in the model (3.1) for perturbations with the same and double periods compared to the carrier wave [Fig. (a) and (b), respectively] for the cases $K/V=0.9$ (curves *1*); $K/V=0.7$ (curves *2*); $K/V=0.5$ (curves *3*).

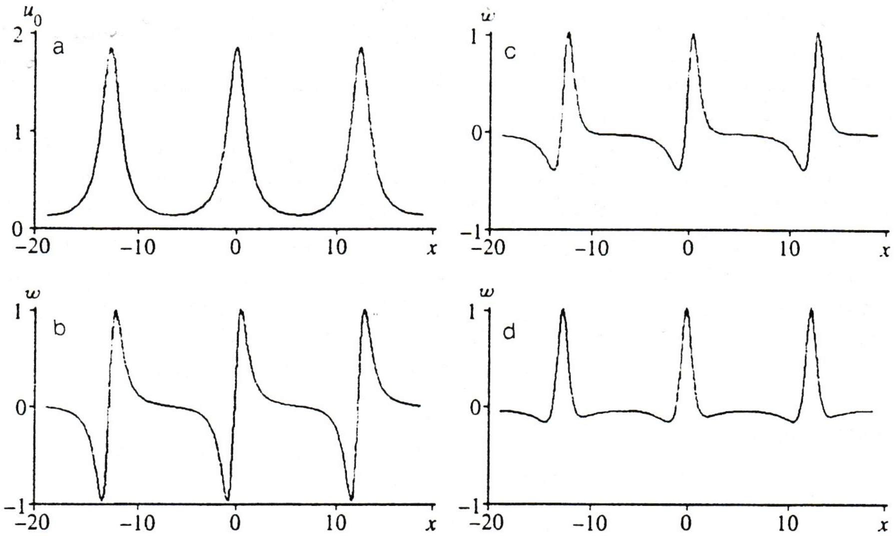

FIG. 3. The profile of the periodic wave u_0 for $K/V=0.5$ (a) and the structure of the growing mode w with the same period at the left (b) and right (d) edges of the instability region and near the region of peak growth rate (c) for the model (2.7).

expressed as a superposition of odd and even components with respect to u_0:

$$w \propto \lambda \partial_x u_0 - \frac{K^2 k_y^2}{\sinh^2 \phi} \partial_V u_0. \tag{3.13}$$

The first term in fact describes phase modulation of the carrier wave and the second term describes modulation of its velocity (amplitude). The mode structure along the x-axis is presented in Fig. 3 for three different values of k_y: near the left (b) and right (d) edges of the instability region and near the region of maximum growth rate (c). The profile of the wave u_0 with the parameter $K/V=0.5$ is shown in Fig. 3a. Evidently the instability of a plane wave can be regarded mainly as phase instability in the region of long-wave transverse perturbations and as amplitude instability in the region of short-wave perturbations. The wave instability mechanism is accounted for the positive feedback between amplitude and phase modulations: wave front bending is accompanied by increasing amplitude in its convex portions and by decreasing amplitude in the concave portions. The convex portions move faster and the concave ones move slower, which leads to further bending of the wave front.

For arbitrary k_x, we found that the structure of the time-growing perturbations at each crest of the carrier wave has the same features in different portions of the instability region. However, the perturbation phases are different at the neighboring crests. In the symmetrical case $k_x=K/2$, the linear mode has a period double that of the carrier wave and the perturbations of the neighboring crests have opposite phases. Using (3.12) we also found for this case an explicit form of the growing mode for an arbitrary nonlinear wave u_0:

$$w \propto \sin\left(\frac{Kx+\delta}{2}\right)\partial_x u_0 + \frac{K}{2}\cos\left(\frac{Kx+\delta}{2}\right)u_0. \tag{3.14}$$

The parameter $\delta=\mathrm{Arg}[\sinh(\theta-\phi)/\sinh\theta]$ is real inside the instability region and varies from π at the left edge to zero at the right one. The profile of the mode (3.14) is drawn in Fig. 4 for three values of k_y.

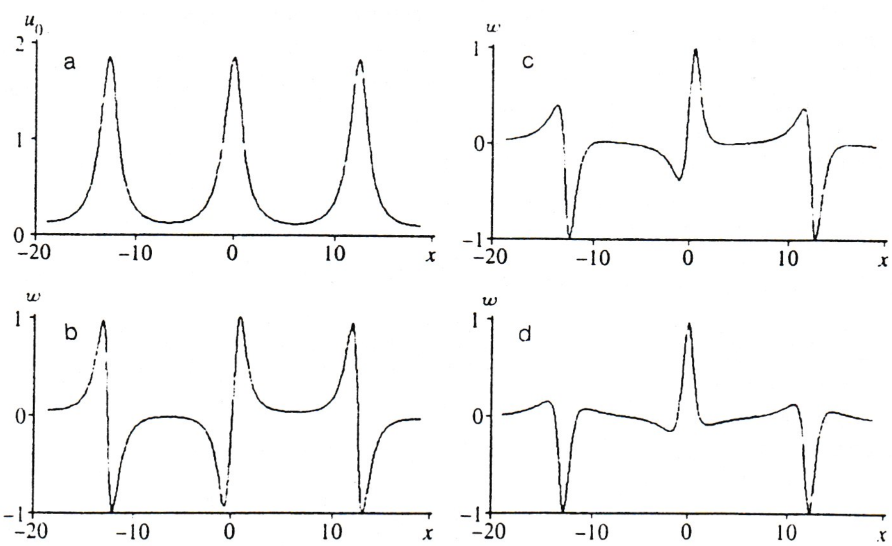

FIG. 4. The same for the mode with period twice that of the carrier wave.

We would like to point out the existence of a critical value of the wave number k_y at which the wave modulation does not grow. At this point one should expect the plane stationary solution to bifurcate and a branch of 2D structures periodic in the transverse direction to appear.[12,21] The features of the linear modes allow us to conclude that the 2D structures with large transverse periods look like phase-modulated nonlinear waves while the ones with small periods look like amplitude-modulated waves. Formation of such 2D structures may be observed in the nonlinear stage of the self-focusing instability of plane waves.

4. PLANE WAVE INSTABILITY TO TRANSVERSE MODULATION WITH AN ARBITRARY PERIOD

The approach used in reducing Eq. (2.5) to (2.7) is valid to within k_y^2/V^2. Therefore, in order to extend the results discussed in the previous section to Eq. (2.5) we need to neglect the higher-order terms, which does not allow us to take into account the cut-off instability at large transverse modulation wave numbers ($k_y/V\sim 1$). Nevertheless, results of analytical and numerical investigations of Eq. (2.5) reveal that there is good qualitative correspondence between the solutions of both equations for the wave $u_0(x)$ propagating along the x-axis.

4.1. The decaying instability of weakly nonlinear waves

In the limiting case $\phi=\infty$ when there is no periodic structure in the carrier wave u_0, the linear integro-differential equation (2.5) transforms to an equation with constant coefficients and admits an explicit solution in the form of neutral perturbations of a certain background V:

$$w_0(\xi,\kappa_\xi)=W(\kappa_\xi)\exp(i\kappa_\xi\xi),$$
$$\lambda_0(\kappa_\xi,\kappa_\eta)=i(\kappa_\xi\cos\alpha-\kappa_\eta\sin\alpha)(v-V), \tag{4.1}$$
$$v=\sqrt{\kappa_\xi^2+\kappa_\eta^2}.$$

Here and in what follows the Greek symbols denote as before, the coordinates and wave numbers of perturbations in the reference frame rotated at the angle α in accordance with the transformation (2.4), and the Latin symbols denote the coordinates in the laboratory frame.

In the solution (4.1), the wave number component κ_ξ is an arbitrary parameter determining the imaginary eigenvalue λ. When the harmonic wave (2.2a) in (2.5) has a small but finite amplitude, the parameter κ_ξ can have only a countable number of values differing by the value NK, where N is an arbitrary integer. This follows from the theory of linear differential equations with periodic coefficients.[20] The instability of the wave $u_0(\xi)$ can arise from merging of an eigenvalue pair $\lambda_0(\kappa_\xi,\kappa_\eta)$ and $\lambda_0(-\kappa'_\xi,-\kappa'_\eta)$, where $\kappa'_\xi=-(\kappa_\xi-NK)$, $\kappa'_\eta=-\kappa_\eta$ and their transition from the imaginary axis to the complex plane.[10,22] By equating the eigenvalues for two modes we find the resonant Nth order condition for the carrier wave ω_0, $\mathbf{K}$ and two satellite waves ω_1, $\mathbf{k}$ and ω_2, $\mathbf{k}'$ in the reference frame ξ, η:

$$\mathbf{k}+\mathbf{k}'=N\mathbf{K},$$
$$\omega_1(\mathbf{k})+\omega_2(\mathbf{k}')=V(\kappa_\xi+\kappa'_\xi)\cos\alpha=N\omega_0(\mathbf{K}), \tag{4.2}$$

where the dependence $\omega(\mathbf{k})$ is expressed by the relation (2.3a).

Analysis of the functional equations (4.2) reveals that the higher order ($N>1$) resonance conditions are not fulfilled for harmonic waves in the Shrira model (2.1). However, at $N=1$ such a solution exists for any α and gives the dependence $\kappa_\eta\equiv\kappa_{\eta0}(\kappa_\xi)$ which has a form of two broken curves. This dependence is shown in Fig. 5 by dashed lines for some values of α in the laboratory frame. At $\alpha=0$, the two curves are connected and form a single curve determined by the expression:

$$k_{y0}^2(k_x)=4k_xk'_x\frac{k_x^2+k_x'^2-(k_x+k'_x)\sqrt{k_xk'_x}}{(k_x-k'_x)^2}. \tag{4.3}$$

Comparison of this curve and the ellipse where the wave resonance conditions in the simplified model (3.1) hold (the ellipse is represented by the dot-dash line in Fig. 5a) shows that the two curves are qualitatively similar but the resonance of weakly nonlinear waves occurs at larger values of $|k_y|$ in Eq. (2.1). The maximum deviation of the curves $|\Delta k_y/k_y|$ is reached on the line $k_x=k'_x=K/2$ is 18.3%.

In the other special case $\alpha=\pi/2$, the dependence $k_{y0}(k_x)$ degenerates into the line $k_{y0}=K/2$.

So, only the simplest interaction of three coupled weakly nonlinear waves leads to decay instability of the plane stationary solution (2.2a). As is known,[3,10] the characteristic features of harmonic wave instability can be found in this case in the first order of wave amplitude a which is supposed to be a small parameter.

By substituting the expansions

$$w(\xi)=w_0(\xi,\kappa_\xi)+w_0(\xi,-\kappa'_\xi)+aw_1(\xi)+O(a^2),$$
$$\lambda=\lambda_0(\kappa_\xi,\kappa_{\eta0})+a\lambda_1(\kappa_\xi,\kappa_{\eta0},\kappa_{\eta1})+O(a^2), \tag{4.4}$$
$$\kappa_\eta=\kappa_{\eta0}(\kappa_\xi)+a\kappa_{\eta1}+O(a^2),$$

into Eq. (2.5) with the function $u_0(\xi)$ in the form (2.2a) we obtain a linear inhomogeneous equation for the first correction $w_1(\xi)$. Elimination of the resonant terms in the right-hand side of this equation, which gives rise to secular growth[3] of $w_1(\xi)$, allows us to find the ratio $W(k_\xi)/W(-k'_\xi)$ and the first correction $\lambda_1(\kappa_\xi,\kappa_{\eta0},\kappa_{\eta1})$ in the form

$$\lambda_1=i\omega_1+q, \tag{4.5}$$

where

$$\omega_1=\kappa_{\eta1}\left[\frac{\kappa_{\eta0}}{2}\left(\frac{\kappa_\xi\cos\alpha-\kappa_{\eta0}\sin\alpha}{v}\right.\right.$$
$$\left.\left.-\frac{\kappa'_\xi\cos\alpha+\kappa_{\eta0}\sin\alpha}{v'}\right)+\sin\alpha\left(V-\frac{v+v'}{2}\right)\right],$$

$$q^2=(\kappa_\xi\cos\alpha-\kappa_{\eta0}\sin\alpha)(\kappa'_\xi\cos\alpha+\kappa_{\eta0}\sin\alpha)$$
$$-\kappa_{\eta1}^2\left[\frac{\kappa_{\eta0}}{2}\left(\frac{\kappa_\xi\cos\alpha-\kappa_{\eta0}\sin\alpha}{v}\right.\right.$$

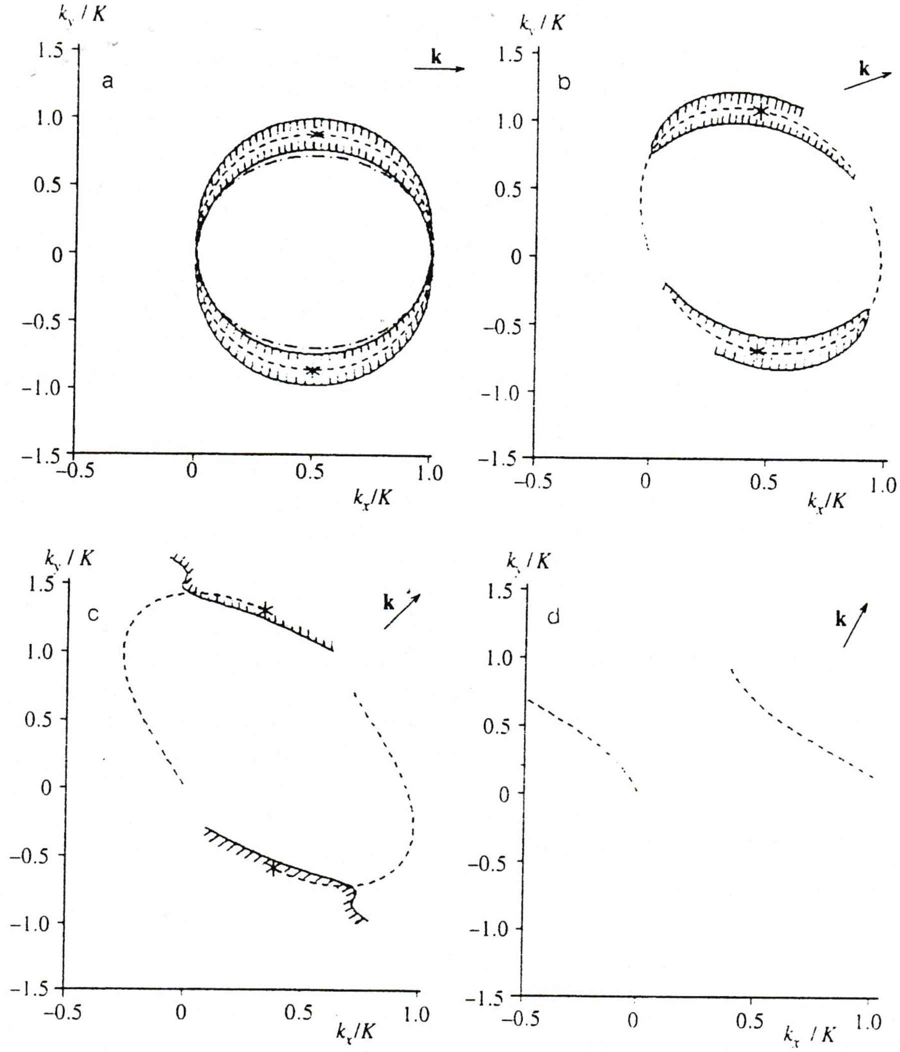

FIG. 5. The region of instability of a plane quasiharmonic wave in the laboratory frame with respect to small transverse perturbations with amplitude $a=0.1$ within the model (2.5) at $\alpha=0$ (a); $\alpha=\pi/8$ (b); $\alpha=\pi/4$ (c); $\alpha=3\pi/8$ (d). The asterisks and dashed lines have the same meaning as in Fig. 1. The dot-dash line corresponds to the resonant curve in the model (2.7).

$$+\frac{\kappa_\xi' \cos\alpha+\kappa_{\eta0}\sin\alpha}{v'}\Bigg)+\sin\alpha\left(\frac{v'-v}{2}\right)\Bigg]^2.$$

The instability regions $q^2\geqslant 0$ are hatched in Fig. 5. We would like to emphasize that for waves propagating in the x direction, the instability region expands from the resonant curve $k_{y0}(k_x)$ just as in Fig. 1a, which is drawn for weakly nonlinear waves perturbed by long transverse modulation.

As α increases, the region $q^2>0$ shrinks and there is no harmonic wave instability in the first-order expansion of its amplitude for the portions of the resonant curves with $k_x<0$ and $k_x'<0$. For $\alpha\geqslant 0.297\pi$, the instability region of plane quasilinear waves vanishes. It is obvious from Eq. (4.5) that the perturbations with large transverse wave numbers k_y are the most unstable ones and the maximum of the growth rate lies on the line $k_x=k_x'=K_x/2$ if it intersects with the resonant curves. Such an intersection is absent at large α and the maximum coincides with the edges of the resonant curves (Fig. 5c).

It is important to note that the imaginary part of the eigenvalues does not vanish in the model (2.1) anywhere in the instability region except on two special lines: $\kappa_\xi=0$ and $\kappa_\xi=K/2$. Only the perturbations with the same and double periods as the carrier wave in the ξ, η frame have velocities v and v' equal to the wave velocity V so that their amplitudes grow monotonically due to the developing instability. At the edges of the instability region, such perturbations represent neutral modes from which new stationary solutions in the form of 2D modulated structures with the same and double periods may appear.

4.2. The transversal instability of plane solitons

The eigenfunctions $w(\xi)$ and eigenvalues λ of Eq. (2.5) for the solitary wave $u_0(\xi)$ specified by the solution (2.2b) are investigated by the numerical procedure proposed in Ref. 23. The idea is based on discretizing the Fourier-space of the integral equation (2.5) and subsequently finding the eigenvalues and eigenvectors of the matrix obtained as a result of the discretization by the projective method. This method gives small numerical errors if the spectrum of the function $w(\xi)$ falls off rapidly with the order of the Fourier harmonics, which was observed in our numerical calculations.

Using this procedure we found the unique discrete mode of a plane soliton that grows in time at $\alpha\neq\pi/2$ over a broad interval of transverse wave numbers κ_η. The de-

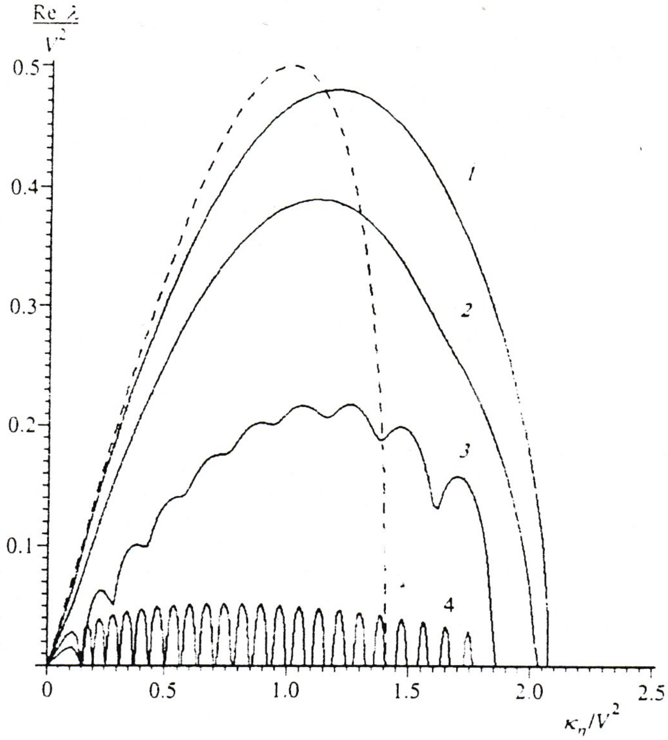

FIG. 6. The growth rate versus wave number of transverse modulation in the model (2.5) for a plane soliton at $\alpha=0$ (curve *1*); $\alpha=\pi/8$ (curve *2*); $\alpha=\pi/4$ (curve *3*); $\alpha=3\pi/8$ (curve *4*). The dashed line corresponds to the same dependence in model (2.7) at $\alpha=0$.

pendence $\mathrm{Re}\,\lambda(\kappa_\eta)$ is presented in Fig. 6 for some values of α. The instability region and the magnitude of the growth rate $\mathrm{Re}\lambda>0$ decrease with increasing α but the transverse modulation with wave numbers $\kappa_\eta\sim V$ is always the most unstable. The dashed line in Fig. 6 indicates the dependence $\lambda(\kappa_\eta)$ which we found in the framework of the simplified equation (2.7) for $\alpha=0$. Obviously, the quantitative correspondence is good in the region of long transverse perturbations but becomes worse with increasing κ_η. The qualitative characteristic features of the dependence $\lambda(\kappa_\eta)$ are presented correctly by the simplified model (2.7) as seen from Fig. 6. Note that the asymptotic dependence $\lambda(k_y)$ for a solitary wave was also found recently in the long-wave limit.[24]

The structure of the mode growing out of a plane soliton is shown in Fig. 7(b–d) for three values of κ_η at $\alpha=0$. It also agrees well with the approximate solution (3.13) and can be expressed by the superposition of the odd and even components describing transverse modulation of the phase and velocity of a plane soliton.

The eigenvalue λ for the found mode is a complex value except the special case $\alpha=0$. So, the appearance at $\alpha\lesssim\pi/2$ of numerous critical points $\kappa_{\eta c}$ at which $\mathrm{Re}\,\lambda$ vanishes does not cause bifurcation of different branches of the stationary solutions.

Thus, comparison of the characteristic features of growing modes in the limiting cases of weakly nonlinear and solitary waves for the simplified (2.7) and original (2.5) models shows that they are in good qualitative agreement with the description of the instability of waves propagating along the main flow. When the original wave propagates at some angle α to the flow, the instability region and the peak value of the growth rate decrease. There is no instability for the waves propagating transversely to the flow.

5. CONCLUSION

The analysis presented in this paper revealed the important role of resonant interactions of plane periodical waves propagating at certain angles to each other in the problem of self-focusing instability of nonlinear perturbations of a shear flow. The structure of a growing linear mode superposed on the original plane wave gives us grounds to expect the formation of 2D nonlinear waves

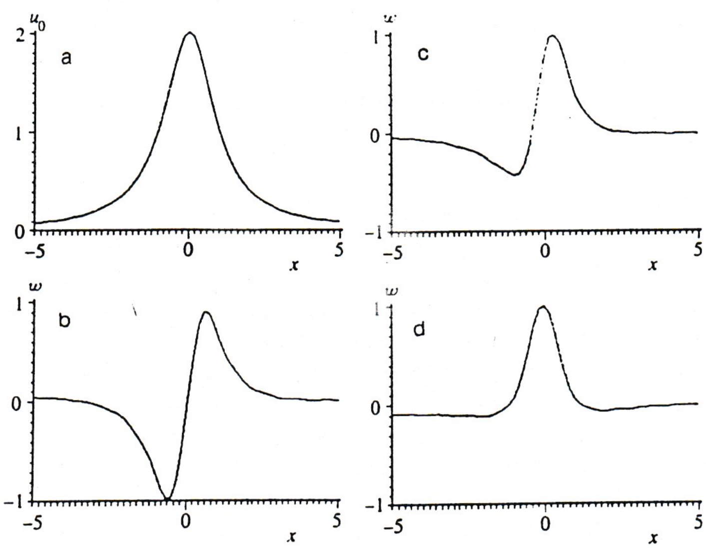

FIG. 7. The profile of the soliton u_0 (a) and the structure of the mode w at the left (b) and right (d) edges of the instability region and near the region of maximum growth rate (c) for the model (2.7) at $\alpha=0$.

that are phase- and amplitude-modulated along the front as a result of developing self-focusing. The study of the growth rate of transverse perturbations shows that the perturbations having a period twice that of the original wave are the most unstable ones. In this case the transition from plane waves to a modulated structure is probably not accompanied by the formation of perturbations with different periods. The description of the nonlinear stage of developing self-focusing instability of periodic waves and solitons needs further investigation.

The transverse instability of plane waves occurs in a medium with weak negative dispersion due to the destabilizing influence of a shear fluid flow. It should be recalled that plane waves are stable against small transverse perturbations in isotropic media with negative dispersion.[6,7] The presence of an external flow introduces anisotropic properties in the medium, and the spectrum of small perturbations becomes decaying with respect to three-wave processes. This circumstance may be a cause of transverse instability of essentially nonlinear waves.

Note that similar wave processes are also observed in other anisotropic media. For instance, analysis of the ion-acoustic wave instability in a magnetized plasma[8–12] has many features in common with the results found in our paper.

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