On the asymptotic integrability of a higher-order evolution equation describing internal waves in a deep fluid

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A higher-order nonlocal evolution equation describing internal waves in a deep fluid is shown to be asymptotically integrable only if the coefficients of the higher-order terms satisfy certain constraints. In this case, the nonlocal equation can be transformed to the integrable Benjamin–Ono equation. The asymptotic integrability of the reductions of the higher-order evolution equation to a complex Burgers equation, to an envelope-wave equation, and to a finite-dimensional dynamical system is also considered. © *1996 American Institute of Physics.*
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I. INTRODUCTION

The evolution of small amplitude waves of certain nonlinear dispersive systems can be studied asymptotically using the so-called multiscale expansion method (see, e.g., Ref. 1). This method yields a basic evolution equation which is formally valid at the leading asymptotic order, as well as a sequence of evolution equations at higher asymptotic orders. It turns out that for many important physical systems the basic evolution equation is an *integrable* equation (see Ref. 2 for a discussion of this remarkable fact). Each integrable evolution equation is a member of a hierarchy of infinitely many integrable equations. It is interesting that the evolution equation valid at the next asymptotic order, differs from the next member of the associated integrable hierarchy, only in the value of the numerical coefficients of the nonlinear terms. For example, idealized unidirectional water waves of small amplitude and large wave length satisfy[3] the equation

$$\eta_t+\eta_{xxx}+6\eta\eta_x+\epsilon(\alpha_1\eta_{xxxxx}+\alpha_2\eta\eta_{xxx}+\alpha_3\eta_x\eta_{xx}+\alpha_4\eta^2\eta_x)+O(\epsilon^2)=0, \tag{1.1}$$

where $\alpha_1,\ldots,\alpha_4$ are certain numbers. As $\epsilon\rightarrow 0$, this equation becomes the Korteweg–deVries (KdV) equation, which is an integrable equation. Furthermore, if $\alpha_2=10\alpha_1$, $\alpha_3=20\alpha_1$, $\alpha_4=30\alpha_1$, then the $O(\epsilon)$ term of Eq. (1.1) becomes the right-hand side of the next member of the hierarchy of integrable equations associated with the KdV equation.

If the basic evolution equation is integrable, we say that the underlying physical system is *asymptotically integrable* to $O(\epsilon)$. It turns out that in certain cases it is possible to formally extend the asymptotic integrability of the system to $O(\epsilon^2)$. For example, in the case of water waves Kodama found[4] an explicit transformation which maps Eq. (1.1) to the integrable equation obtained by Eq. (1.1) when $\alpha_2=10\alpha_1$, $\alpha_3=20\alpha_1$, and $\alpha_4=30\alpha_1$. A generalization of Kodama's transformation which actually maps Eq. (1.1) to KdV equation itself, and an extension of this result to the case of water waves without the unidirectionalization assumption, are given in Ref. 5. It is also shown in Ref. 5 that the concept of the mastersymmetries (see Ref. 6 and references therein) provides an algorithmic approach to finding the transformations which map the physical equations to the integrable ones. Similar results are valid for the case that the basic evolution equation is the nonlinear Schrödinger (NLS) equation.

In this paper we study the asymptotic integrability of the systems whose basic equation is the Benjamin–Ono (BO) equation, i.e., we study the equation

$$u_t = 2uu_x + Hu_{xx} + \epsilon[\alpha u_{xxx} + \beta_1 (uHu_x)_x + \beta_2 uHu_{xx} + \beta_3 H(uu_x)_x + \gamma u^2 u_x] + O(\epsilon^2), \quad (1.2)$$

where ϵ is the small parameter of the multiscale expansion, H is the Hilbert transform,

$$(Hf)(x) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x}\, d\xi, \quad (1.3)$$

and $\fint$ denotes the principal value integral.

Equation (1.2) occurs in the modeling of long internal waves in a deep continuously stratified fluid, and was recently derived in Ref. 7. In this case u denotes the horizontal velocity of the fluid and the coefficients $\alpha,...,\gamma$ can be expressed through the parameters of the fluid stratification. The particular case that the stratification profile can be approximated by a two-layer model with density ρ_1 in the upper (shallow) layer and ρ_2 in the lower (deep) layer, was studied in Ref. 8 and is described by Eq. (1.2) with

$$\alpha = \frac{27}{4}\left(\frac{4\delta^2}{9} - 1\right), \quad \beta_1 = 6, \quad \beta_2 = \frac{3}{2}, \quad \beta_3 = \frac{27}{2}, \quad \gamma = -3, \quad (1.4)$$

where $\delta = \rho_1/\rho_2 < 1$.

The structure of our paper is as follows. In Sec. II we present two main results for Eq. (1.2). (a) We show that if the coefficients satisfy the numerical constraints

$$3\alpha + \beta_1 + \beta_2 + \beta_3 = 0, \quad (1.5)$$

and

$$\beta_1 + \beta_3 - \gamma = 0, \quad (1.6)$$

then Eq. (1.2) is asymptotically integrable to $O(\epsilon^2)$ (see Propositions 2.1 and 2.2).

(b) We study the pole-decomposition solution of Eq. (1.2) and establish that if Eqs. (1.5) and (1.6) are satisfied, then the system describing these solutions is asymptotically integrable to $O(\epsilon^2)$ (see Proposition 2.3). This implies that in this case the algebraic solitary waves interact without a phase shift to $O(\epsilon^2)$.

Unfortunately, in the physically important case that the coefficients of Eq. (1.2) are given by Eq. (1.4), the constraints (1.5) and (1.6) are not satisfied. This is consistent with the fact that in this case the interaction of the algebraic solitary waves exhibits phase shifts to $O(\epsilon^2)$.[9]

Although we have only shown that the validity of Eqs. (1.5) and (1.6) is a sufficient condition for asymptotic integrability, we conjecture that it is also a necessary condition. This conjecture is supported by the following arguments. There exists an exact reduction from Eq. (1.2) to a complex perturbed Burgers equation. In this case, if the coefficients of Eq. (1.2) satisfy a single constraint, denoted here by $\nu=0$, then the perturbed Burgers equation can be mapped to the integrable Burgers equation. Furthermore, it was shown in Ref. 7 that there exists an asymptotic limit from Eq. (1.2) to a certain modulation equation for envelope waves. This equation contains a free parameter, denoted by χ and is an integrable equation if $\chi=0$.[10] It is remarkable that the equations $\nu=0$ and $\chi=0$ are equivalent to Eqs. (1.5) and (1.6). The reductions related to Eq. (1.2) are discussed in Sec. III.

II. MAIN RESULTS

Proposition 2.1: Let $v(x,t)$ satisfy the BO equation,

$$v_t = 2vv_x + Hv_{xx}. \quad (2.1)$$

Let $u(x,t)$ be defined by

$$u=v+\frac{\epsilon}{2}(\lambda_1 v^2+\lambda_2 Hv_x+\lambda_3 x[2vv_x+Hv_{xx}]). \tag{2.2}$$

Then, u solves Eq. (1.2) where the coefficients of Eq. (1.2) satisfy the constraints given by Eqs. (1.5) and (1.6). Actually, these coefficients can be parametrized by λ_1, λ_2, λ_3 through the equations

$$\alpha=\lambda_3, \quad \beta_1=-\lambda_2, \quad \beta_2=\lambda_1-\lambda_3, \quad \beta_3=-\lambda_1+\lambda_2-2\lambda_3, \quad \gamma=-\lambda_1-2\lambda_3. \tag{2.3}$$

Proof: This result can be verified by a direct calculation. However, we choose to derive a more general result which contains the above as a particular case.

Let $\mathscr{H}(v)$ denote the ring consisting of smooth functions of $v(x,t)$, of its x derivatives, of the action of H on these functions, and of the multiplication by x on these functions. Let v solve the equation,

$$v_t=K(v)+\epsilon\tilde{K}(v), \tag{2.4}$$

where $K,\tilde{K}\in\mathscr{H}$. Define u by the transformation,

$$u=v+\epsilon\tilde{\tau}(v), \tag{2.5}$$

where $\tilde{\tau}\in\mathscr{H}$. Then by direct substitution it follows that u solves the equation,

$$u_t=K(u)+\epsilon(\tilde{K}(u)+[\tilde{\tau}(u),K(u)]_L)+O(\epsilon^2). \tag{2.6}$$

Here $[A,B]_L$ denotes the Lie bracket of $A,B\in\mathscr{H}$, defined by

$$[A(u),B(u)]_L=A'[B](u)-B'[A](u), \tag{2.7}$$

and where prime denotes Frechet differentiation, i.e.,

$$A'[B](u)=\frac{\partial}{\partial\epsilon}A[u+\epsilon B(u)]\bigg|_{\epsilon=0}. \tag{2.8}$$

In the particular case when $\tilde{K}=0$, $\tilde{\tau}$ is given by the $O(\epsilon)$ terms of Eq. (2.2), and K is the right-hand side of the BO equation, i.e.,

$$K(v)=2vv_x+Hv_{xx}, \tag{2.9}$$

then Eq. (2.6) becomes Eq. (1.2) with its parameters given by Eq. (2.3). Eliminating the λ's from (2.3) we obtain Eqs. (1.5) and (1.6).

Proposition 2.2: Let v satisfy the integrable equation

$$v_t=K(v)+\epsilon\alpha K_1(v), \tag{2.10}$$

where $K(v)$ is given by Eq. (2.9), and $K_1(v)$ is defined by

$$K_1(v)=[v_{xx}-\tfrac{3}{2}(vHv_x+Hvv_x)-v^3]_x. \tag{2.11}$$

Define u by

$$u=v+\frac{\epsilon}{2}(\mu_1 v^2+\mu_2 Hv_x). \tag{2.12}$$

Then, u solves Eq. (1.2), where the coefficients satisfy the constraints given by Eqs. (1.5) and (1.6). Actually, these coefficients can be parametrized by

$$\beta_1=-\tfrac{3}{2}\alpha-\mu_2, \quad \beta_2=\mu_1, \quad \beta_3=-\tfrac{3}{2}\alpha-\mu_1+\mu_2, \quad \gamma=-3\alpha-\mu_1. \tag{2.13}$$

Proof: This result is a particular case of the more general result presented in the proof of Proposition 2.1.

Remark 2.1: (a) Equation (2.10) is integrable because K_1 is the first commuting flow of the BO equation, i.e.,

$$[K,K_1]_L=0. \tag{2.14}$$

(b) Let $\tau(u)\in\mathscr{H}$ be defined by

$$\tau=u^2+\tfrac{3}{2}Hu_x+x[2uu_x+Hu_{xx}]. \tag{2.15}$$

This function is the *mastersymmetry* of the BO equation.[6] It has the defining property that

$$K_1(u)=\tfrac{1}{2}[\tau,K]_L. \tag{2.16}$$

The terms of $K_1(u)$ differ from the $O(\epsilon)$ terms of Eq. (1.2) only in their numerical coefficients. Thus, in order to find the form of the transformation $\tilde{\tau}(u)$ in (2.5) it is natural to replace the numerical coefficients of $\tau(u)$ by arbitrary constants; in this way $\tau(u)$ becomes the $O(\epsilon)$ term of Eq. (2.2).

(c) If the coefficients of Eq. (1.2) are defined by Eq. (1.4), the constraints (1.5) and (1.6) are not satisfied. However, even in this case, Eq. (2.2) defines a three parameter group of infinitesimal transformations which maps Eq. (1.2) to itself. Using this group of transformations it is possible to show that the equations for the velocity amplitude, and for the fluid interface displacement, derived in Refs. 7 and 8 respectively, are equivalent.

Proposition 2.3: Let $a_j(t)$ and $X_j(t)$ be complex valued scalar functions of t, $a_j^*(t)$ and $X_j^*(t)$ denote their complex conjugation, $j=1,\dots,N$, and assume that $\operatorname{Im} X_j<0$. Equation (1.2) admits the pole-decomposition solution

$$u=\sum_{j=1}^{N}\left[\frac{ia_j(t)}{x-X_j(t)}-\frac{ia_j^*(t)}{x-X_j^*(t)}\right], \tag{2.17}$$

if and only if: (a) The coefficients of Eq. (1.2) satisfy Eqs. (1.5) and (1.6);

(b) a_j is given by

$$a_j=1-\frac{\epsilon\beta_2}{2}\dot{X}_j+O(\epsilon^2); \tag{2.18}$$

(c) X_j satisfy the perturbed Calogero–Moser dynamical system,

$$\ddot{X}_j=8\sum_k{}'\frac{1}{(X_j-X_k)^3}+12\epsilon\alpha\sum_k{}'\frac{\dot{X}_j+\dot{X}_k}{(X_j-X_k)^3}+O(\epsilon^2), \tag{2.19}$$

where $\dot{X}_j=dX_j/dt$, $\ddot{X}_j=d^2X_j/dt^2$, and the sign Σ_k' denotes summation over k from 1 to N excluding j.

Proof: Substituting the pole expansion (2.17) into (1.2) one finds a fourth-order polynomial in terms of $(x-X_j)^{-n}$. Equating the coefficients of the terms with $n=4$ and $n=1$ to zero it follows that

$$6\alpha+3\beta_1+2\beta_2+3\beta_3-\gamma=0, \tag{2.20}$$

and

$$[\beta_1+\beta_2+\beta_3-\gamma]\ddot{X}_j-8\beta_2\sum_k{}' \frac{1}{(X_j-X_k)^3}=0. \tag{2.21}$$

The $O(1)$ term of Eq. (1.2) is the BO equation, thus to the leading order $\ddot{X}_j = 8\Sigma_k'(X_j - X_k)^{-3}$.[11] Substituting this expression into Eq. (2.21), we find that Eqs. (2.20) and (2.21) yield (1.5) and (1.6). Using these two equations, the coefficient of the term $(x-X_j)^{-n}$ with $n=3$ implies Eq. (2.18), while that with $n=2$ implies

$$\begin{aligned} i\dot{X}_j=&2\sum_k{}' \frac{1}{X_j-X_k}-2\sum_k \frac{1}{X_j-X_k^*}+i\epsilon\Bigg[-3\alpha\Bigg(\sum_k\sum_l{}'' \frac{1}{(X_j-X_k)(X_j-X_l)}\\ &-2\sum_k{}'\sum_l \frac{1}{(X_j-X_k)(X_j-X_l^*)}+\sum_k\sum_l \frac{1}{(X_j-X_k^*)(X_j-X_l^*)}\Bigg)\\ &+(\beta_3-\beta_1-3\beta_2)\sum_k \frac{1}{(X_j-X_k^*)^2}\Bigg]+O(\epsilon^2); \end{aligned} \tag{2.22}$$

the sign $\Sigma_k\Sigma_l''$ denotes summation over all k and l from 1 to N which are not equal to j and to each other. Equations (2.22) are different from those derived by Case[11] from the integrable equation (2.10) where the last $O(\epsilon)$ term in Eq. (2.22) is absent. However, differentiating (2.22) with respect to t and using the pole-decomposition technique discussed in Ref. 12, it can be shown that this term cancels out to $O(\epsilon^2)$, and Eq. (2.22) reduces to (2.19).

Remark 2.2: (a) Let Y_j, $j=1,\ldots,N$, satisfy the integrable Calogero–Moser dynamical system,

$$\ddot{Y}_j=8\sum_k{}' \frac{1}{(Y_j-Y_k)^3}. \tag{2.23}$$

Define X_j by

$$X_j=Y_j+\frac{\epsilon\alpha}{2} Y_j\dot{Y}_j. \tag{2.24}$$

Then, X_j satisfy Eq. (2.19) to $O(\epsilon^2)$. We note that the transformation (2.24) also follows from Proposition 2.1. Indeed, since v satisfies the BO equation (2.1), it admits the pole decomposition,

$$v=\sum_{j=1}^N\left[\frac{i}{x-Y_j(t)}-\frac{i}{x-Y_j^*(t)}\right]. \tag{2.25}$$

Substituting this expansion and the corresponding one for u [see (2.17)] into (2.2) with $\lambda_1=\alpha+\beta_2$, $\lambda_2=-\beta_1$, and $\lambda_3=\alpha$ we find that Eq. (2.2) reduces to (2.24).

(b) The pole decomposition of Eq. (1.2) yields an integrable dynamical systems if the coefficients satisfy the constraints given by Eq. (1.5) and (1.6). This provides further evidence that these constraints are necessary and sufficient conditions for the asymptotic integrability of Eq. (1.2). Furthermore, explicit soliton and periodic wave solutions of this equation can be found in the integrable case by means of the pole-decomposition representation (2.17) (see Ref. 11).

III. RELATED REDUCTIONS

In this section we discuss the integrability of certain equations related to Eq. (1.2).

Reduction 3.1: Let u be analytic in the upper half plane of the complex extension of x. Then $Hu=iu$, and Eq. (1.2) reduces to the complex perturbed Burgers equation

$$u_t=2uu_x+iu_{xx}+\epsilon[\alpha u_{xxx}+i(\beta_1+\beta_3)u_x^2+i(\beta_1+\beta_2+\beta_3)uu_{xx}+\gamma u^2u_x]+O(\epsilon^2). \quad (3.1)$$

It can be shown that if v satisfies the complex Burgers equation

$$v_t=2vv_x+iv_{xx}, \quad (3.2)$$

and if u is defined by

$$u=v+\frac{\epsilon}{2}(\nu_1v^2+\nu_2v_x\partial^{-1}v+\nu_3x[2vv_x+iv_{xx}]), \quad (3.3)$$

then u satisfies Eq. (3.1), where the coefficients of Eq. (3.1) satisfy the single constraint

$$\nu=3\alpha+3\beta_1+\beta_2+3\beta_3-2\gamma=0. \quad (3.4)$$

We note that the constraint (3.4) does not coincide with either Eq. (1.5) or (1.6). Moreover, Eq. (3.3) contains a nonlocal term, which is absent in Eq. (2.2). However, if the coefficients of Eq. (1.2) satisfy Eqs. (1.5) and (1.6), then both $\nu_2=0$ and $\nu=0$ are valid.

Reduction 3.2: Let u be expanded in the asymptotic form

$$u=\sqrt{\epsilon}[\Psi(X,T)\exp[i(x-t)]+\text{c.c.}]+O(\epsilon), \quad X=\epsilon(x-2t), \quad T=\epsilon^2t, \quad (3.5)$$

where c.c. denotes complex conjugation. Then, it can be shown[7] that the function $\Psi(X,T)$ satisfies the equation

$$i\Psi_T+\Psi_{XX}+\Psi[i+H](|\Psi|^2)_X+\chi|\Psi|^2\Psi=0, \quad (3.6)$$

where

$$\chi=3\alpha+2\beta_1+\beta_2+2\beta_3-\gamma. \quad (3.7)$$

It was shown in Ref. 10 that if $\chi=0$ then Eq. (3.6) is integrable. [If one applies the ansatz (3.5) to the BO equation instead of Eq. (1.2), one finds Eq. (3.6) with $\chi=0$.] We note that if Eqs. (1.5) and (1.6) are valid, then $\chi=0$. Furthermore, Eqs. (3.4) and (3.7) are equivalent to Eqs. (1.5) and (1.6).

Reduction 3.3: Let the function u be represented asymptotically as $t\to\pm\infty$ by

$$u=u_0(\theta_1^\pm;a_1)+u_0(\theta_2^\pm;a_2)+O(\epsilon), \quad \theta_j^\pm=a_j(x+v_jt+\epsilon X_j^\pm), \quad j=1,2, \quad (3.8)$$

where $u_0(\theta_j^\pm;a_j)$ is the profile of the BO (algebraic) soliton solution. The parameters a_j, v_j, and $X_j^\pm$ describe the amplitude, the velocity, and the phase shift of the j soliton, respectively. Assume that $a_1<a_2$, which implies $v_1<v_2$. Then, the velocities of the individual BO solitons are expressed through their amplitudes by the equations

$$v_j=a_j-\frac{\epsilon a_j^2}{4}(6\alpha+7\beta_1+6\beta_2+5\beta_3-5\gamma)+O(\epsilon^2). \quad (3.9)$$

Furthermore, the total phase shifts of the BO soliton interactions, $\Delta X_j=X_j^+-X_j^-$, are given by

$$\Delta X_1 = -\frac{\pi}{(a_1+a_2)^2}\left[(3\alpha+\beta_1+\beta_2+\beta_3)(2a_1^2+4a_1a_2-2a_2^2) + (\beta_1+\beta_2-\gamma)(3a_1^2+6a_1a_2-a_2^2)\right], \tag{3.10}$$

$$\Delta X_2 = -\frac{\pi}{(a_1+a_2)^2}\left[(3\alpha+\beta_1+\beta_2+\beta_3)(-2a_1^2+4a_1a_2+2a_2^2) + (\beta_1+\beta_2-\gamma)(-a_1^2+6a_1a_2+3a_2^2)\right]. \tag{3.11}$$

When the coefficients of Eq. (1.2) are given by Eqs. (1.4), the total phase shifts (3.10) and (3.11) reduce to those found by Matsuno.[9] Here we have generalized his result to show that the total phase shifts exactly vanish only if the coefficients in (1.2) satisfy Eqs. (1.5) and (1.6). This is consistent with the fact that in this case, the interaction of the BO (algebraic) solitons is described by the integrable Calogero–Moser system (2.23) which does not produce any phase shifts of the algebraic soliton interactions.

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