Rational solutions of the Kadomtsev–Petviashvili hierarchy and the dynamics of their poles. I. New form of a general rational solution

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A new approach to the construction of the rational solutions to the hierarchy of the Kadomtsev–Petviashvili equation is presented. The generalization of the "superposition formula" is found to permit the construction of a new general solution from some partial ones. The features of the general polynomial factorization and the types of scattering in the many-body Calogero–Moser problem are investigated.

I. INTRODUCTION

In the past twenty years, rational solutions were found in the majority of the equations of mathematical physics that possess a rich algebraic structure: an infinite set of symmetries and related commuting flows.[1,2] It was ascertained that the rational solutions of such equations are special limiting forms of exponential ("multisoliton") solutions and can be obtained from the latter ones by long-wave degeneration.[3–7] This approach is productive for some simple problems, for example, finding solutions describing the scattering of algebraic solitons ("lumps") in the Kadomtsev–Petviashvili (KP)[3–5] and Benjamin–Ono[6] equations or the investigation of the unique family of the polynomial τ function in the Korteweg–de Vries equation.[7] However, for other, more complicated cases this approach does not give explicit rational solutions.

Therefore, direct methods for the construction of rational solutions have great importance in mathematical physics. For instance, such methods are based on the transformation of various forms of the τ function to the equation of interest,[8–10] or on the group theory,[11–13] or on the analytical properties of the Baker–Akhiezer function in the framework of a linear system related to the nonlinear equation.[1,14,15] In spite of certain successes in this field, the structure and the features of a general rational solution even for the best investigated hierarchy of the KP equation have not been studied deep enough until now. For instance, recent works[16–18] revealed new classes of rational solutions and new types of related dynamic phenomena in the classical KP equation.

In the series of articles the rational solutions to the KP hierarchy are considered in the context of dynamics of particles in the Hamiltonian many-body problem which is described by the poles of the rational solutions (zeros of the corresponding τ function). As is well known,[19–22] for the KP hierarchy, such a problem is generated by the flows of the complete integrable Calogero–Moser (CM) system. The investigation of the general rational solution allows us to classify the features of particle scattering in the CM system.

This article is devoted to the construction and investigation of the new form of a general solution to the KP hierarchy. It is obtained from the Wronskian form of the τ function in Sec. II and is a tensor of N rank that is generated by a characteristic function. The features of the general polynomial factorization are found in Sec. III. The various types of particle dynamics in the CM system are considered in Sec. IV. They are also compared with the known classes of the rational solutions to the KP hierarchy. The consequences of the new approach are discussed in Sec. V. The formula of the relationship between different rational solutions is proven in the Appendix.

II. THE CHARACTERISTIC FUNCTION FOR RATIONAL SOLUTIONS

Let us consider the function $w(t_1, t_2, t_3, \ldots)$ that depends on infinite sequences of the variables t_n, $n \geqslant 1$ and satisfies the hierarchy of the equations (the KP hierarchy)

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$$-4\partial_{t_1}\partial_{t_3}w+6\partial_{t_1}(\partial_{t_1}w\cdot\partial_{t_1}w)+\partial_{t_1}^4 w+3\partial_{t_2}^2 w=0,$$

$$-3\partial_{t_1}\partial_{t_4}w+6\partial_{t_1}(\partial_{t_1}w\cdot\partial_{t_2}w)+\partial_{t_1}^3\partial_{t_2}w+2\partial_{t_2}\partial_{t_3}w=0, \tag{2.1}$$

$$\cdots$$

The procedure for finding all members of the KP hierarchy in the bilinear form was considered, for example, in Refs. 11–13. A bilinear form appears as a result of replacing the dependent variable

$$w(t_1,t_2,t_3,\ldots)=\partial_{t_1}\ln \tau(t_1,t_2,t_3,\ldots). \tag{2.2}$$

For broad classes of partial solutions to Eqs. (2.1), the τ function (2.2) can be expressed by the Wronskian[8,13]

$$\tau=\det|J_{nk}|\equiv W[\phi_1,\phi_2,\ldots,\phi_N], \tag{2.3}$$

where $J_{nk} = \partial_{t_1}^{k-1}\phi_n$, $1\leqslant n,\ k\leqslant N$, and the functions ϕ_n are arbitrary solutions of the system of linear partial differential equations with constant coefficients

$$\partial_{t_k}\phi_n=\partial_{t_1}^k\phi_n,\quad k\geqslant 1. \tag{2.4}$$

The rational solutions to the KP hierarchy for which the τ function is a polynomial (it is possible to multiply this polynomial to an arbitrary exponential factor depending linearly on t_1), are generated by the following partial solutions of the system (2.4):

$$\phi_n=\partial_{p_n}^{m_n}\exp[\Phi(p_n)]\equiv P_{m_n}(p_n)\cdot\exp[\Phi(p_n)], \tag{2.5}$$

where $\Phi(p_n)=\Sigma_{j=1}^{\infty} p_n^j(t_j+\tilde{t}_{nj})$, p_n is a spectral parameter, and $\tilde{t}_{nj}$ are arbitrary phase constants. The polynomials P_{m_n}, which are defined by the formula (2.5), generalize the Schur polynomials[11–13] and can be written in an explicit form (see, for example, Refs. 8 and 13)

$$P_{m_n}(p_n)=m_n!\cdot\sum_{(i_1,i_2,\ldots,i_{m_n})}\prod_{k=1}^{m_n}\frac{(\theta_k(p_n))^{i_k}}{i_k!}. \tag{2.6}$$

Here the sum consists of all possible combinations of integer non-negative numbers $(i_1,i_2,\ldots,i_{m_n})$ which satisfy the condition $i_1+2i_2+\cdots+m_n i_{m_n}=m_n$ and the new variables are introduced

$$\theta_k(p_n)=\frac{1}{k!}\,\partial_{p_n}^k\Phi(p_n)=\sum_{j=k}^{\infty}\binom{j}{k}\cdot p_n^{j-k}\cdot t_j+\tilde{\theta}_{nk},$$

where $\tilde{\theta}_{nk}=\Sigma_{j=k}^{\infty}(j/k)\cdot p_n^{j-k}\cdot\tilde{t}_{nj}$ and $(j/k)=j!/(k!(j-k)!)$.

For further analysis, the key property of the polynomials P_{m_n} is the relationship between the polynomials of different degrees

$$\partial_{\theta_k}P_{m_n}=\partial_{\theta_1}^k P_{m_n}=\frac{m_n!}{(m_n-k)!}\cdot P_{m_n-k} \tag{2.7}$$

that follows from the explicit expression (2.6).

First, let there be no identical values in the set $\{p_n\}_{n=1}^N$ so that $p_n \neq p_m$ for $n \neq m$. We shall call such solutions (2.3) with functions ϕ_n (2.5) nondegenerate rational solutions. Since the operators ∂_{t_1} and ∂_{p_n} commute, the formal transformation of the nondegenerate solution is correct

$$\tau=\partial_{p_1}^{m_1}\partial_{p_2}^{m_2}\cdots\partial_{p_N}^{m_N}W[\exp[\Phi(p_1)],\exp[\Phi(p_2)],\ldots,\exp[\Phi(p_N)]]$$

$$=\partial_{p_1}^{m_1}\partial_{p_2}^{m_2}\cdots\partial_{p_N}^{m_N}V[p_1,p_2,\ldots,p_N]\cdot\exp\left(\sum_{n=1}^{N}\Phi(p_n)\right),$$

where $V[p_1,p_2,\ldots,p_N]=\Pi\Pi_{N\geq n>k\geq 1}(p_n-p_k)$ is the Vandermond determinant.

Let us introduce the characteristic function

$$f=\exp\left(\sum_{n=1}^{N}\Phi(p_n)+\sum\sum_{N\geq n>k\geq 1}\ln(p_n-p_k)\right)\equiv\exp[\Psi(p_1,p_2,\ldots,p_N)]. \tag{2.8}$$

Then an arbitrary nondegenerate rational solution is a tensor of N rank which is generated by the characteristic function (2.8)

$$\tau_{1\ 2\ \cdots N}^{m_1 m_2\cdots m_N}=f^{-1}\cdot\partial_{p_1}^{m_1}\partial_{p_2}^{m_2}\cdots\partial_{p_N}^{m_N}f. \tag{2.9}$$

The tensor of N rank (2.9) is well known in the mathematical methods of the probability theory[23] as the tensor of mixed moments of the degree $(m_1,m_2,\ldots,m_N)$ for some vector of probable values. As was found earlier,[23] the expression (2.9) can be rewritten in an explicit form over the derivatives of the function $\Psi(p_1,p_2,\ldots,p_N)$, which are referred to in the theory as the semi-invariants of the vector of probable values

$$\tau_{1\ 2\ \cdots N}^{m_1 m_2\cdots m_N}=m_1!m_2!\cdots m_N!\cdot\sum_{(i_1,i_2,\ldots,i_M)}\prod_{k=1}^{M}\frac{(s_{1\ 2\ \cdots N}^{\mu_{k1}\mu_{k2}\cdots\mu_{kN}})^{i_k}}{i_k!}, \tag{2.10}$$

where the sum consists of all possible nonordered combinations of different integer non-negative numbers $\{\mu_{k1},\mu_{k2},\ldots,\mu_{kN}\}_{k=1}^M$ and of all ordered combinations of integer positive numbers $(i_1,i_2,\ldots,i_M)$ which satisfy the condition $i_1\mu_{1n}+i_2\mu_{2n}+\cdots+i_M\mu_{Mn}=m_n$ for $1\leq n\leq N$, and

$$s_{1\ 2\ \cdots N}^{\mu_{k1}\mu_{k1}\cdots\mu_{kN}}=\frac{1}{\mu_{k1}!\mu_{k2}!\cdots\mu_{kN}!}\partial_{p_1}^{\mu_{k1}}\partial_{p_2}^{\mu_{k2}}\cdots\partial_{p_N}^{\mu_{kN}}\Psi(p_1,p_2,\ldots,p_N). \tag{2.11}$$

At $N=1$, the tensor (2.10) is a sequence of the polynomials $P_{m_1}(p_1)$ (2.6), and the semi-invariants $s_1^{\mu_{k1}}=\theta_{\mu_{k1}}(p_1)$.

Now, let all the values of the spectral parameter in the set $\{p_n\}_{n=1}^N$ be identical so that $p_n=p_m$ for all n,m. Such a rational solution can be referred to as a degenerate one. The operator representation (2.8), (2.9) is no longer correct for this solution. However, using the features of a determinant it is readily found that the degenerate polynomials can be written in a Wronskian form

$$\tau_1^{\mathbf{M}_1^N}=\exp(-N\Phi(p_1))\cdot W[\phi_1,\phi_2,\ldots,\phi_N]=W[P_{m_1}(p_1),P_{m_2}(p_1),\ldots,P_{m_N}(p_1)], \tag{2.12}$$

where the vector $\mathbf{M}_1^N=(m_1,m_2,\ldots,m_N)$.

A general rational solution $\tau_{1\;\;2\;\;\cdots N}^{\mathbf{M}_1^{K_1}\mathbf{M}_2^{K_2}\ldots\mathbf{M}_N^{K_N}}$ combines the forms (2.9) and (2.12) and is expressed by the determinant (2.3) of order $K=\Sigma_{n=1}^N K_n$ with the functions (2.5) depending only on $N<K$ values of the spectral parameter. Such a solution can be classified as a partially degenerate polynomial.

Further investigation of the rational solutions to the KP hierarchy is based on the remarkable fact that mixed second-order semi-invariants (2.11) do not depend on the variables t_k and all mixed higher-order semi-invariants are identically equal to zero. As a result, it is not difficult to prove from the formula (2.10) that a nondegenerate rational solution as a whole preserves the property of the polynomials P_{m_n}

$$\partial_{\eta_1}^{k_1}\partial_{\eta_2}^{k_2}\cdots\partial_{\eta_N}^{k_N}\tau_{1\;\;2\;\;\cdots N}^{m_1m_2\;\;m_N}=\left(\prod_{n=1}^{N}\frac{m_n!}{(m_n-k_n)!}\right)\cdot\tau_{1\;\;\;\;\;\;2\;\;\;\;\;\;\cdots N}^{m_1-k_1m_2-k_2\;\;m_N-k_N}, \tag{2.13}$$

where $\eta_n=\theta_1(p_n)$.

The expression (2.13) means that there exists some differential operator $\mathbf{F}$ mapping the set of N polynomials $\tau_n^{m_n} = P_{m_n}(p_n)$ into the tensor of N rank (2.9)

$$\tau_{1\;\;2\;\;\cdots N}^{m_1m_2\;\;m_N}=\mathbf{F}*\tau_1^{m_1}\times\tau_2^{m_2}\times\cdots\times\tau_N^{m_N}. \tag{2.14}$$

We shall call the relations similar to (2.14) factorizing relations of the polynomial solutions. The explicit forms of the operator $\mathbf{F}$ for nondegenerate and partially degenerate solutions are found in the next section.

III. FACTORIZATION OF THE POLYNOMIALS

In order to investigate the features of factorization of the nondegenerate polynomials (2.9) which are generated by the characteristic function (2.8), we shall derive the formula for the relationship between the polynomials $\tau_{1\;\;2\;\;\cdots N}^{m_1m_2\cdots m_N}$, $\tau_{N+1}^{m_{N+1}}$, and $\tau_{1\;\;2\;\;\cdots N\;\;N+1}^{m_1m_2\cdots m_Nm_{N+1}}$.

First, we set $m_{N+1}=0$. Then it is obvious from Eqs. (2.8) and (2.9) that the solution $\tau_{1\;\;2\;\;\cdots N\;\;N+1}^{m_1m_2\cdots m_N0}$ differs from the solution $\tau_{1\;\;2\;\;\cdots N}^{m_1m_2\cdots m_N}$ only by a shift of all the variables $\theta_k(p_n)$

$$\theta_k(p_n)\;\rightarrow\;\theta_k(p_n)-\frac{1}{k\cdot(p_{N+1}-p_n)^k},\quad 1\leqslant n\leqslant N.$$

Let us introduce the vertex operator making the shift of $\theta_k(p_n)$

$$\mathbf{S}(\lambda_{m,n})=\exp\left(-\sum_{k=1}^{\infty}\frac{\lambda_{m,n}^k}{k}\cdot\frac{\partial}{\partial\theta_k(p_n)}\right),\quad \lambda_{m,n}=\frac{1}{p_m-p_n}. \tag{3.1}$$

Using this operator, the relationship between the polynomials can be written in a compact form

$$\tau_{1\;\;2\;\;\cdots N\;\;N+1}^{m_1m_2\cdots m_N0}=\left(\prod_{n=1}^{N}\mathbf{S}(\lambda_{N+1,n})\right)*\tau_{1\;\;2\;\;\cdots N}^{m_1m_2\cdots m_N}. \tag{3.2a}$$

On the other hand, similar calculations lead to the formula

$$\tau_{12\cdots NN+1}^{00\cdots0\,m_{N+1}}=\left(\prod_{n=1}^{N}\mathbf{S}(\lambda_{n,N+1})\right)*\tau_{N+1}^{m_{N+1}}. \tag{3.2b}$$

In a general case, when $m_{N+1} \neq 0$, the polynomial $\tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N m_{N+1}}$ is related to the polynomials (3.2a) and (3.2b) as follows:

$$\tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N m_{N+1}} = \tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N 0} \cdot \tau_{12 \cdots N N+1}^{00 \cdots 0\, m_{N+1}} + \sum_{n=1}^{N} \frac{1}{(p_n - p_{N+1})^2} \cdot \frac{\partial \tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N 0}}{\partial \lambda_{N+1,n}}$$
$$\cdot \frac{\partial \tau_{12 \cdots N N+1}^{00 \cdots 0\, m_{N+1}}}{\partial \lambda_{n,N+1}} + \cdots + \frac{1}{N!} \sum\sum_{n_1, n_2, \ldots, n_N} \prod_{k=1}^{N} \frac{1}{(p_{n_k} - p_{N+1})^2}$$
$$\cdot \frac{\partial^N \tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N 0}}{\partial \lambda_{N+1,n_1} \partial \lambda_{N+1,n_2} \cdots \partial \lambda_{N+1,n_N}} \cdot \frac{\partial^N \tau_{12 \cdots N N+1}^{00 \cdots 0\, m_{N+1}}}{\partial \lambda_{n_1,N+1} \partial \lambda_{n_2,N+1} \cdots \partial \lambda_{n_N,N+1}}. \tag{3.3}$$

The proof of this equality is presented in the Appendix.

The formula (3.3), together with (3.2a) and (3.2b), represents the "superposition formula" for the nondegenerate rational solutions to the KP hierarchy since it permits to construct a new, more general solution $\tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N m_{N+1}}$ from two known solutions $\tau_{1\ 2\ \cdots N}^{m_1 m_2 \cdots m_N}$ and $\tau_{N+1}^{m_{N+1}}$. On the other hand, taking into account the leading term of the equality, we reveal the factorizing relationship between the KP polynomials. Moreover, the formula (3.3) allows us to write a nondegenerate rational solution in the form (2.14).

We shall express (3.3) in a compact operator form. For this purpose we note that the terms in the right-hand side of Eq. (3.3) correspond to the first N terms in the Taylor expansion of the following operator exponent:

$$\prod_{n=1}^{N} \mathbf{Q}(p_n, p_{N+1}) = \prod_{n=1}^{N} \exp\left(\frac{1}{(p_n - p_{N+1})^2} \cdot \frac{\partial^2}{\partial \lambda_{N+1,n} \partial \lambda_{n,N+1}} \right). \tag{3.4}$$

Here the operator $\mathbf{Q}(p_n, p_{N+1})$ influences only the operator $\mathbf{S}(\lambda_{N+1,n})$ in the expression (3.2a) for $\tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N 0}$ and the operator $\mathbf{S}(\lambda_{n,N+1})$ in the expression (3.2b) for $\tau_{12 \cdots N N+1}^{00 \cdots 0 m_{N+1}}$. Since the vertex operators $\mathbf{S}(\lambda_{m,n})$ are linear on the variables $\lambda_{m,n}$ for the nondegenerate polynomials [see the formula (A3) in Appendix], the higher-order terms in the Taylor expansion of the operator (3.4), which differ from the terms in the right-hand side of (3.3), are identically equal to zero. So, the equality (3.3) is equivalent to the compact formula

$$\tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N m_{N+1}} = \left(\prod_{n=1}^{N} \mathbf{Q}(p_n, p_{N+1}) \mathbf{S}(\lambda_{N+1,n}) \mathbf{S}(\lambda_{n,N+1}) \right) * \tau_{1\ 2\ \cdots N}^{m_1 m_2 \cdots m_N} \times \tau_{N+1}^{m_{N+1}}. \tag{3.5}$$

Using subsequently $N-1$ transformations (3.5) from the solution $\tau_1^{m_1}$ to the solution $\tau_{1\ 2\ \cdots N}^{m_1 m_2 \cdots m_N}$, we obtain the factorizing relation (2.14) for a nondegenerate polynomial of the KP hierarchy in an explicit form

$$\tau_{1\ 2\ \cdots N}^{m_1 m_2 \cdots m_N} = \left(\prod_{1 \leq n < m \leq N} \prod \mathbf{Q}(p_n, p_m) \mathbf{S}(\lambda_{m,n}) \mathbf{S}(\lambda_{n,m}) \right) * \prod_{n=1}^{N} \tau_n^{m_n}. \tag{3.6}$$

The factorizing relation depends nonsmoothly on the spectral parameters p_n and becomes incorrect when $p_m = p_n$ for some $m \neq n$ since $\lambda_{m,n} = \infty$ in this case.

Nevertheless, we succeed in generalizing the relation (3.6) to partially degenerate polynomials if we rearrange the factor by combining the subsets of the polynomials $\tau_n^{m_n} = P_{m_n}(p_n)$ with identical values of the spectral parameter p_n into the generate polynomials $\tau_n^{\mathbf{M}_n^{K_n}}$ which have the form (2.12). The result of the corresponding calculations is presented by the expression

$$\tau_{1\ 2\ \cdots N}^{\mathbf{M}_1^{K_1}\mathbf{M}_2^{K_2}\cdots\mathbf{M}_N^{K_N}} = \left(\prod_{1\leqslant n<m\leqslant N}\prod \mathbf{Q}(p_n,p_m)\mathbf{S}^{K_m}(\lambda_{m,n})\mathbf{S}^{K_n}(\lambda_{n,m}) \right) * \prod_{n=1}^{N} \tau_n^{\mathbf{M}_n^{K_n}}. \tag{3.7}$$

Here the operator $\mathbf{Q}(p_n,p_m)$ must be written in the following form:

$$\mathbf{Q}(p_n,p_m)=\exp\left(\frac{1}{(p_n-p_m)^2} \cdot \mathbf{S}^{-1}(\lambda_{m,n})\mathbf{S}^{-1}(\lambda_{n,m}) \frac{\partial^2}{\partial\eta_n\partial\eta_m} \right), \tag{3.8}$$

which is equivalent to Eq. (3.4) for the nondegenerate polynomials $\tau_n^{m_n}$.

So, the polynomial τ function for the KP hierarchy can be presented in a factorizing form by an arbitrary set of the degenerate polynomials $\tau_n^{\mathbf{M}_n^{K_n}}$ of lower degrees which can be regarded as structural elements of a general rational solution. We shall show in the next section that the main features of particle dynamics in the many-body CM problem are related to this property of the polynomial solutions. As a result, an arbitrary ensemble of particles decays after some collision process into individual partial ensembles that weakly interact with each other.

IV. GENERAL FEATURES OF THE PARTICLE DYNAMICS RELATED TO THE RATIONAL SOLUTIONS

It is known that the motion of zeros $X_n(t_2,t_3,\ldots)$ of the polynomial τ function of the KP hierarchy with respect to the variable t_1 in a complex plane is related to the integrable many-body CM problem

$$\partial_{t_2}^2 X_n = 8\sum_{m\neq n} \frac{1}{(X_m-X_n)^3} \tag{4.1}$$

and to its higher commuting Hamiltonian flows[19–22]

$$\partial_{t_k}X_n=\partial_{V_n}H_k, \quad \partial_{t_k}V_n=-\partial_{X_n}H_k, \quad k>2, \tag{4.2}$$

where

$$H_k=\frac{(-1)^k}{2^{k-1}}\,\mathrm{Sp}\,L^k, \quad L_{i,j}=V_i\delta_{i,j}+\frac{2(1-\delta_{i,j})}{X_i-X_j}, \quad 1\leqslant i,j\leqslant R,$$

and R is a degree of the KP polynomial.

The scattering of the particles having different asymptotic velocities was investigated for the CM system by Calogero.[20] The main feature of the dynamics is the absence of any tracks of interaction between the particles after their scattering (even the phase shift along the trajectory is equal to zero). If we designate the particle coordinates and velocities at $t_2\to-\infty$ by X_{n0} and V_{n0}, where all V_{n0} are different, and at $t_2\to+\infty$ by X'_{n0} and V'_{n0}, respectively, then the sets $\{X_{n0},V_{n0}\}_{n=1}^R$ and $\{X'_{n0},V'_{n0}\}_{n=1}^R$ are identical. However, it was discovered a little later by Airault *et al.*[21] that stationary manifolds exist in the phase space of the CM system containing a certain amount of particles $R=M(M+1)/2$, where M is a natural number. On these manifolds, all the particles are in equilibrium states: $V_n=0$ for $1\leqslant n\leqslant R$. Still later, Matveev[8] constructed the solutions describing the particle motion in the CM system which is related to the reduction of the system (4.1) to the subsystem of first-order equations. Only recently it was pointed out by Gorshkov *et al.*[17] that this reduction produces new features of particle scattering in the CM system. These features are characterized by an appearance of an infinite phase shift of the particles as a result of the scattering and anomalously slow dynamics of their interaction.

The aim of this section is to use the general formula (3.7) for the description of the main features of the particle dynamics in the CM system and its higher commuting flows (4.2).

A. Normal scattering of particles

The best known feature of the CM system is the absence of any tracks of interaction between the particles having different asymptotic velocities on their scattering. The corresponding solutions follow from the general formula (3.7) at $K_n=1$ and $m_n=1$ for any n. Designating

$$\eta_n=\theta_1(p_n)+\sum_{m\neq n}\frac{1}{p_n-p_m},\quad 1\leqslant n\leqslant N$$

and expanding the exponential operators $\mathbf{Q}(p_n,p_m)$ in the Taylor series, we obtain the well-known formula describing the scattering of the particles having different asymptotic velocities[5]

$$\tau_{12\cdots N}^{11\cdots 1}=\prod_{n=1}^{N}\eta_n+\frac{1}{2}\sum_{(k,l)}\frac{1}{(p_k-p_l)^2}\prod_{n\neq k,l}\eta_n+\cdots+\frac{1}{M!2^M}$$

$$\times\sum_{(k_m,l_m)}\prod_{m=1}^{M}\frac{1}{(p_{k_m}-p_{l_m})^2}\prod_{n\neq k_m,l_m}\eta_n+\cdots. \tag{4.3}$$

The summation is performed over all possible combinations of $k_1,l_1;k_2,l_2;\ldots;k_M,l_M;\ldots$ from integer numbers $1,2,\ldots,N$, all numbers being different. A simple analysis shows that the expansion (4.3) contains nonvanishing terms for each polynomial of N degree up to the $M=[N/2]$ term.

Since at $t_k\to\infty$ for $k\geqslant2$; $\eta_n\to\infty$ for $1\leqslant n\leqslant N$, the factorization of the leading term of the expansion (4.3) accounts for the fact that all N particles move rectilinearly after interaction and have the same asymptotic velocity $V'_{n0}=V_{n0}=2p_n$ and the same phase $X'_{n0}=X_{n0}=\tilde{\theta}_{n1}+\Sigma_{m\neq n}(p_n-p_m)^{-1}$, as before interaction. Obviously, the trajectories described by this class of solutions completely fill the $2N$-dimensional phase space of the system (4.1), (4.2) of N particles, except possible manifolds of lower dimension. In Ref. 17 the dynamics of N particles with different asymptotic velocities is referred to as normal scattering.

For the case $p_k=p_l$ at $k\neq l$, the formula (4.3) is no longer valid. It is known that the interaction of certain particles (or solitons) may not occur when they move with identical asymptotic velocities at infinite distances from each other (for example, see Ref. 6 for the Benjamin—Ono equation). Therefore, such equations have no rational solutions other than Eq. (4.3). However, for the KP equation it was ascertained in Ref. 17 that the limit transition $p_k\to p_l$ leads to a new class of the rational solutions at certain renormalization of the phase constants X_{n0}. These solutions describe dynamics of the particles having identical asymptotic velocities. The Wronskian form used in this article is convenient because the necessary phase renormalization has already been produced by the action of the vertex operator (3.1). So, it directly follows from the solution (4.3) that

$$\tau_{12\cdots N}^{11\cdots 1}\xrightarrow[p_1\to p_2\to\cdots\to p_N]{}\tilde{P}_N(-p_N),$$

where the polynomials $\tilde{P}_n(p_n)$ can be obtained from the polynomials $P_n(p_n)$ by replacing $t_{2k} \to -t_{2k}$ for any k which is admitted by the system (4.1) and its even flows (4.2).

The polynomial $P_N(p_N)$ of $R=N$ degree is parametrized by N phase constants $\tilde{\theta}_{Nk}$, $1 \leq k \leq N$, and the unique spectral parameter p_N. Therefore, the motion described by the zeros of this polynomial is contained in the $(N+1)$-dimensional manifold of a common $2N$-dimensional phase space. Moreover, the motion of the particles on this manifold corresponds to the first-order system that is a reduction of the system (4.1) (see, for example, Refs. 8 and 22)

$$\partial_{t_2} X_n = 2p_N + 2 \sum_{m \neq n} \frac{1}{X_m - X_n}, \quad 1 \leq n \leq N. \tag{4.4}$$

It is obvious from this system that, first, all particles move with the same average velocity $V_N = 2p_N$ and, second, their relative motion in the reference frame propagating with the velocity V_N and described by the coordinate $\Delta X_n = X_n - V_N t_2$ occurs by a slower law: $\Delta X_n \sim t_2^{1/2}$ at $t_2 \to \pm\infty$, and $\Delta X_{n|t_2 \to +\infty} \to \pm i \Delta X_{n|t_2 \to -\infty}$. Such a slow interaction gives rise to an infinite phase shift along the trajectories of the particles. Following Ref. 17, we call the scattering of N particles with identical asymptotic velocities anomalous scattering.

B. Anomalous scattering of particles

There exist two groups of the solutions corresponding to anomalous interactions of R particles in the framework of the CM system. The first group is described by the solution (3.7) for all $K_n = 1$ but with $m_n > 1$ for some n. The main structural elements of these solutions are the polynomials $P_{m_n}(p_n)$, $1 \leq n \leq N$. As it follows from the factorization properties, after some collision processes the common ensemble of $R = \Sigma_{n=1}^N m_n$ particles decays into N partial ensembles of $m_1, m_2, \ldots, m_N$ particles which move, as a whole, along the same trajectories as before interaction. However, inside each partial ensemble, the particles interact in the manner described above and the result of such an interaction is the irreversible shift of their trajectories.

Another type of particle dynamics is described by zeros of the polynomials (3.7) at the degeneration of the spectral parameter values: $K_n > 1$ for some n. In this case the structural elements of the solution are the degenerate polynomials $\tau_n^{\mathbf{M}_n^{K_n}}$, $1 \leq n \leq N$. We designate their degrees through r_n. From the same factorization properties, it follows that a common ensemble of R particles ($R = \Sigma_{n=1}^N r_n$) also decays into N partial weakly interacting ensembles. However, the motion of the particles inside each partial ensemble concentrates near stationary manifolds of some commuting flows (4.2) of the CM system.[17,18] Unlike the first group of anomalous processes, stationary manifolds and the related slow scattering of the particles have a dimension lower than $(R+1)$ and exist only for a certain amount of interacting particles. On the other hand, no phase shift of the particle trajectories occurs. A more detailed investigation of the degenerate polynomials of the KP hierarchy and the second type of anomalous scattering in the CM system is carried out in the next article.

V. CONCLUSION

The new form of rational solutions to the KP hierarchy presented in this article generalizes the formula (4.3) found earlier and allows us to obtain any rational solution in an explicit form for a given set of spectral parameter values $\{p_n\}_{n=1}^N$ without additional degeneration. The construction and investigation of a general rational solution reduce, within this approach, to the analysis of a certain set of polynomials of lower degrees which factorize the general solution in the operator form (3.7).

The factorization properties of the rational solution determine the characteristic features of the dynamics of particle scattering in the many-body CM problem. In a general case, the scattering of particles consists of two phases, a fast and a slow one. In the first phase which is referred to as the normal one, the ensemble of particles decays into partial weakly interacting ensembles containing a smaller amount of particles. Moreover, each partial ensemble moves along the same trajectory along which it would move without interaction with other ensembles. In the second anomalous phase which proceeds by slower laws, particle interaction and scattering are observed inside each partial ensemble. We revealed that there are two different types of anomalous phases of scattering. The first type is accompanied by changing trajectories of the particles and the appearance of an infinite phase shift. The other type is related to stationary manifolds of some flows of the CM system.

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APPENDIX: PROOF OF THE FORMULA (33)

We shall prove that the polynomial $\tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N m_{N+1}}$ can be expressed through two polynomials $\tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N 0}$ and $\tau_{12\cdots NN+1}^{00\cdots 0 m_{N+1}}$ by the formula (3.3).

Using the Leibniz differentiation formula to an arbitrary rational solution of $(N+1)$ rank (2.9) we obtain

$$\begin{aligned}
\tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N m_{N+1}} &= f^{-1} \cdot \partial_{p_1}^{m_1} \partial_{p_2}^{m_2} \cdots \partial_{p_N}^{m_N} (\tau_{12\cdots NN+1}^{00\cdots 0 m_{N+1}} \cdot f) \\
&= \sum_{k_1,k_2,\ldots,k_N} \left(\prod_{n=1}^{N} \binom{m_n}{k_n} \right) \cdot \tau_{1\ \ \ \ \ \ \ \ 2\ \ \ \ \ \ \ \ \cdots N\ \ \ \ \ \ \ N+1}^{m_1-k_1 m_2-k_2 \cdots m_N-k_N 0} \cdot \partial_{p_1}^{k_1} \partial_{p_2}^{k_2} \cdots \partial_{p_N}^{k_N} \tau_{12\cdots NN+1}^{00\cdots 0 m_{N+1}} \\
&= \sum_{k_1,k_2,\ldots,k_N} \left(\prod_{n=1}^{N} \frac{1}{k_n!} \right) \cdot \partial_{\eta_1}^{k_1} \partial_{\eta_2}^{k_2} \cdots \partial_{\eta_N}^{k_N} \tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N 0} \cdot \partial_{p_1}^{k_1} \partial_{p_2}^{k_2} \cdots \partial_{p_N}^{k_N} \tau_{12\cdots NN+1}^{00\cdots 0 m_{N+1}},
\end{aligned} \tag{A1}$$

where $\eta_n = \theta_1(p_n)$.

The last equality in the expression (A1) follows directly from the formula (2.13). The summation over k_n can be extended from zero to infinity because all derivatives of $\tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N 0}$ with respect to η_n that are higher than the (m_n+1)th one are identically equal to zero. We rearrange the expansion (A1) extracting the terms with an equal number of the variables p_n with respect to which the differentiation is performed

$$\begin{aligned}
\tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N m_{N+1}} &= \tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N 0} \cdot \tau_{12\cdots NN+1}^{00\cdots 0 m_{N+1}} + \sum_{n=1}^{N} \sum_{k} \frac{1}{k!} \cdot \partial_{\eta_n}^{k} \tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N 0} \cdot \partial_{p_n}^{k} \tau_{12\cdots NN+1}^{00\cdots 0 m_{N+1}} \\
&\quad + \cdots + \frac{1}{N!} \sum_{n_1,n_2,\ldots,n_N} \sum_{k_1,k_2,\ldots,k_N} \frac{1}{k_1! k_2! \cdots k_N!} \cdot \partial_{\eta_{n_1}}^{k_1} \partial_{\eta_{n_2}}^{k_2} \cdots \partial_{\eta_{n_N}}^{k_N} \tau_{1\ 2\ \cdots N\ N+1}^{m_1 m_2 \cdots m_N 0} \\
&\quad \cdot \partial_{p_{n_1}}^{k_1} \partial_{p_{n_2}}^{k_2} \cdots \partial_{p_{n_N}}^{k_N} \tau_{12\cdots NN+1}^{00\cdots 0 m_{N+1}}.
\end{aligned} \tag{A2}$$

Here the summation over $n_1,n_2,\ldots,n_N$ is performed from 1 to N. Using the expression (3.2b) for the solution through the operators $\mathbf{S}(\lambda_{n,N+1})$ and the following property of the vertex operator acting on the class of the polynomials $P_{m_n}(p_n)$[13,18]

$$\mathbf{S}(\lambda_{m,n})P_{m_n}(p_n)=(1-\lambda_{m,n}\partial_{\eta_n})P_{m_n}(p_n) \tag{A3}$$

it is not difficult to calculate the derivatives of the solution (3.2b) on p_n

$$\partial^{k_1}_{p_{n_1}}\partial^{k_2}_{p_{n_2}}\cdots\partial^{k_N}_{p_{n_N}}\tau^{00\cdots 0m_{N+1}}_{12\cdots NN+1}=\left(\prod_{m=1}^{N}\partial^{k_m}_{p_{n_m}}\lambda_{n_m,N+1}\right)\cdot\frac{\partial^N\tau^{00\cdots 0m_{N+1}}_{12\cdots NN+1}}{\partial\lambda_{n_1,N+1}\partial\lambda_{n_2,N+1}\cdots\partial\lambda_{n_N,N+1}} \tag{A4}$$

and

$$\partial^k_{p_n}\lambda_{n,N+1}=-\frac{k!}{(p_{N+1}-p_n)^{k+1}}. \tag{A5}$$

Substituting Eqs. (A4) and (A5) into the formula (A2) we find

$$\begin{aligned}\tau^{m_1m_2\cdots m_Nm_{N+1}}_{1\ 2\ \cdots N\ N+1}={}&\tau^{m_1m_2\cdots m_N0}_{1\ 2\ \cdots N\ N+1}\cdot\tau^{00\cdots 0m_{N+1}}_{12\cdots NN+1}+\sum_{n=1}^{N}\frac{1}{(p_{N+1}-p_n)^2}\cdot T_n\cdot\frac{\partial\tau^{00\cdots 0m_{N+1}}_{12\cdots NN+1}}{\partial\lambda_{n,N+1}}+\cdots\\&+\frac{1}{N!}\sum_{n_1,n_2,\cdots,n_N}\sum\prod_{k=1}^{N}\frac{1}{(p_{n_k}-p_{N+1})^2}\\&\cdot T_{n_1n_2\cdots n_N}\cdot\frac{\partial^N\tau^{00\cdots 0m_{N+1}}_{12\cdots NN+1}}{\partial\lambda_{n_1,N+1}\partial\lambda_{n_2,N+1}\cdots\partial\lambda_{n_N,N+1}},\end{aligned} \tag{A6}$$

where we designate

$$T_{n_1n_2\cdots n_N}\equiv(-1)^N\sum_{k_1,k_2,\cdots,k_N}\sum\prod_{m=1}^{N}(\lambda_{N+1,n_m})^{k_m}\cdot\partial^{k_1+1}_{\eta_{n_1}}\partial^{k_2+1}_{\eta_{n_2}}\cdots\partial^{k_N+1}_{\eta_{n_N}}\tau^{m_1m_2\cdots m_N0}_{1\ 2\ \cdots N\ N+1}.$$

Then, using the formula (3.2a), the property of the vertex operator (A3) and the expression

$$\mathbf{S}^{-1}(\lambda_{N+1,n})\tau^{m_1m_2\cdots m_N0}_{1\ 2\ \cdots N\ N+1}=\sum_k(\lambda_{N+1,n})^k\cdot\partial^k_{\eta_n}\tau^{m_1m_2\cdots m_N0}_{1\ 2\ \cdots N\ N+1} \tag{A7}$$

we have the identity

$$\begin{aligned}T_{n_1n_2\cdots n_N}&=(-1)^N\partial_{\eta_{n_1}}\partial_{\eta_{n_2}}\cdots\partial_{\eta_{n_N}}\prod_{m=1}^{N}\mathbf{S}^{-1}(\lambda_{N+1,n_m})\tau^{m_1m_2\cdots m_N0}_{1\ 2\ \cdots N\ N+1}\\&=\frac{\partial^N\tau^{m_1m_2\cdots m_N0}_{1\ 2\ \cdots N\ N+1}}{\partial\lambda_{N+1,n_1}\partial\lambda_{N+1,n_2}\cdots\partial\lambda_{N+1,n_N}}.\end{aligned} \tag{A8}$$

The formulas (A6) and (A8) prove the validity of the formula (3.3).

[1] A. C. Newell, *Solitons in Mathematics and Physics* (SIAM Philadelphia, 1985).
[2] K. De Vos, Theor. Math. Phys. **96**, 163 (1993).
[3] S. V. Manakov, V. E. Zakharov, L. A. Bordag, A. R. Its, and V. B. Matveev, Phys. Lett. A **63**, 205 (1977).
[4] M. J. Ablowitz and J. Satsuma, J. Math. Phys. **19**, 2180 (1978).
[5] J. Satsuma and M. J. Ablowitz, J. Math. Phys. **20**, 1496 (1979).
[6] J. Satsuma and Y. Ishimori, J. Phys. Soc. Jpn. **46**, 681 (1979).
[7] J. J. C. Nimmo and N. C. Freeman, Phys. Lett. A **96**, 443 (1983).
[8] V. B. Matveev, Lett. Math. Phys. **3**, 503 (1979).
[9] C. R. Gilson and J. J. C. Nimmo, Phys. Lett. A **147**, 472 (1990).
[10] A. Nakamura, J. Phys. Soc. Jpn. **58**, 412 (1989).
[11] M. Sato and Y. Sato, RIMS Kokyaroku **388**, 183 (1980); **414**, 181 (1981).
[12] E. Date, M. Kashiwara, and T. Miwa, Proc. Jpn. Acad. Ser. A **57**, 342 (1981); **57**, 387 (1981).
[13] Y. Ohta, J. Satsuma, D. Takahashi, and T. Tokihiro, Prog. Theor. Phys. Suppl. **94**, 210 (1988).
[14] I. M. Krichever, Funkt. Analis i Priloghen. **12**, 76 (1978) (in Russian).
[15] I. M. Krichever, in *Boundary Value Problems of Mathematical Physics and Related Problems in the Theory of Functions*, edited by O. A. Ladyzhenskaya (Nauka, Moscow, 1979), pp. 117–130 (in Russian).
[16] D. E. Pelinovsky and Yu. A. Stepanyants, JETP Lett. **57**, 24 (1993).
[17] K. A. Gorshkov, D. E. Pelinovsky, and Yu. A. Stepanyants, JETP **77**, 237 (1993).
[18] D. E. Pelinovsky, "Degenerate Rational Solutions of the Hierarchy of the Zakharov–Shabat Equations," Preprint N 336, IAP, Nizhny Novgorod, 1993 (in Russian).
[19] J. Moser, Adv. Math. **16**, 197 (1975).
[20] F. Calogero, Lett. Nuovo Cimento **16**, 35 (1976).
[21] H. Airault, H. P. McKean, and J. Moser, Commun. Pure Appl. Math. **30**, 95 (1977).
[22] D. V. Choodnovsky and G. V. Choodnovsky, Nuovo Cimento B **40**, 339 (1977).
[23] A. N. Schiryaev, *Probability* (Nauka, Moscow, 1989), p. 308 (in Russian).