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Rational solutions of the KP hierarchy and the dynamics of their poles. II. Construction of the degenerate polynomial solutions

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A general approach to constructing the polynomial solutions satisfying various reductions of the Kadomtsev–Petviashvili (KP) hierarchy is described. Within this approach, the reductions of the KP hierarchy are equivalent to certain differential equations imposed on the τ-function of the hierarchy. In particular, the l-reduction and the k-constraint as well as their generalized counterparts are considered. A general construction of the rational solutions to these reductions is found and the particular solutions are explicitly derived for some typical examples including the KdV and Gardner equations, the Boussinesq and classical Boussinesq systems, the NLS and Yajima–Oikawa equations. It is shown that the degenerate rational solutions of the KP hierarchy are related to stationary manifolds of the Calogero–Moser (CM) hierarchy of dynamical systems. The scattering dynamics of interacting particles in the CM systems may become complicated due to an anomalously slow fractional-power rate of the particle motion along the stationary manifolds. [S0022-2488(98)03110-7]

I. INTRODUCTION

It is generally believed that all known hierarchies of $(1+1)$-dimensional equations integrable by means of the inverse scattering method can be represented as certain reductions of a universal Kadomtsev–Petviashvili (KP) hierarchy of $(2+1)$-dimensional equations and/or of its extentions to modified and multicomponent cases. This hypothesis originates from a unifying Sato theory[1–3] that describes the KP hierarchy in terms of the pseudodifferential operator $\mathscr{L}=\partial_x+u_2\partial_x^{-1}+u_3\partial_x^{-2}+\cdots$. The infinite set of functions u_n for $n\geqslant 3$ can be expressed through the scalar function $u_2=u(t_1,t_2,\ldots)$ that depends on $t_1=x$ and on an infinite sequence of the time variables t_k, $k\geqslant 2$. Then, equations of the KP hierarchy arise from the generalized Lax equation,

$$\frac{\partial \mathscr{L}}{\partial t_k}=[B_k,\mathscr{L}],\quad k\geqslant 1, \tag{1.1}$$

where B_k is a differential part of $\mathscr{L}^k$. A typical example is the KP equation (see Ref. 3),

$$-4\frac{\partial^2 u}{\partial t_1 \partial t_3}+6\frac{\partial^2 u^2}{\partial t_1^2}+\frac{\partial^4 u}{\partial t_1^4}+3\frac{\partial^2 u}{\partial t_2^2}=0. \tag{1.2}$$

Equations of the KP hierarchy are associated with the isospectral deformation of the eigenvalue problems $\mathscr{L}\psi=\lambda\psi$ and $\mathscr{L}^*\bar{\psi}=\lambda\bar{\psi}$, where λ is a spectral parameter, $\mathscr{L}^*$ is a formal operator adjoint, and ψ and $\bar{\psi}$ are eigenfunctions satisfying the time evolution problems,

$$\frac{\partial \psi}{\partial t_k}=B_k\psi,\quad \frac{\partial \bar{\psi}}{\partial t_k}=-B_k^*\bar{\psi},\quad k\geqslant 1. \tag{1.3}$$

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The KP hierarchy is closely related to the modified KP hierarchy which starts with the modified KP equation,

$$-4\frac{\partial^2 v}{\partial t_1 \partial t_3} - 2\frac{\partial^2 v^3}{\partial t_1^2} + \frac{\partial^4 v}{\partial t_1^4} + 3\frac{\partial^2 v}{\partial t_2^2} + 6\frac{\partial}{\partial t_1}\left(\frac{\partial v}{\partial t_1}\,\partial_{t_1}^{-1}\,\frac{\partial v}{\partial t_2}\right) = 0. \tag{1.4}$$

Within the Sato approach, the modified KP hierarchy can be constructed by means of the gauge transformation of the KP hierarchy which results in modification of the pseudodifferential operator $\mathscr{L}$.[4] On the other hand, solutions of the modified KP hierarchy are formally expressed through the eigenfunction ψ satisfying Eqs. (1.3) according to the relation, $v = \psi^{-1}\partial\psi/\partial t_1$.

Besides its own meaning, the KP hierarchy is useful for generating numerous integrable hierarchies of $(1+1)$-dimensional equations by means of a certain reductive procedure. The conventional l-reduction arises under the condition imposed to the operator $\mathscr{L}$,

$$\mathscr{L}^l = B_l, \quad l \geqslant 2, \tag{1.5}$$

so that the whole set of functions u_n become independent on t_l as well as on t_{nl} (see, e.g., Ref. 3). For $l=2$ this reduction leads to the KdV hierarchy, for $l=3$ it produces the Boussinesq hierarchy, and so on.

Another reductive procedure was recently proposed through a symmetry constraint[5] which was also formulated as the k-constraint imposed on the operator $\mathscr{L}$.[6] The k-constraint occurs under the condition,

$$\mathscr{L}^k = B_k + \psi\partial_x^{-1}\overline{\psi}, \quad k \geqslant 1, \tag{1.6}$$

and leads to coupled nonlinear equations between the functions $u_2, u_3, \ldots, u_{k-1}$ of the KP hierarchy and the eigenfunctions ψ and $\overline{\psi}$. Typical examples of the k-constrained KP hierarchy include the NLS equation for $k=1$, the Yajima–Oikawa equations for $k=2$, and so on (see Refs. 5, 6).

The reductive procedure applied to the KP hierarchy typically gives an idea as to which exact solutions, such as soliton and rational solutions, remain invariant under the reduction and hence represent the exact solutions of the $(1+1)$-dimensional equations.[7–11] This search for the explicit solutions is especially straightforward if one deals with the functions u_n of the KP hierarchy and the eigenfunctions ψ and $\overline{\psi}$ under the zero boundary conditions at infinity. In contrast, a structure of explicit solutions with nonzero boundary conditions is not so obvious and is not usually under consideration. Some results concerned with multisoliton solutions of the k-constrained KP hierarchy under the nonzero boundary conditions for the fields ψ and $\overline{\psi}$ were obtained quite recently by Loris and Willox[12,13] while the existence of rational solutions for this reduction remains an open question. Therefore, a natural problem is how to detect those rational solutions of the KP hierarchy that satisfy a generalized reduction or constraint imposed to the operator $\mathscr{L}$.

The generalized l-reduction of the KP hierarchy can be formulated by means of a modification of the condition (1.5) given by

$$\mathscr{L}^l - \alpha\mathscr{L} = B_l - \alpha B_1, \quad l \geqslant 2, \tag{1.7}$$

where the parameter α is arbitrary. This reduction describes solutions of the KP hierarchy which depend on t_l according to the form, $u_m = u_m(t_1 + \alpha t_l, t_2, \ldots, t_{l-1}, t_{l+1}, \ldots)$. A typical example is the Boussinesq equation for acoustic waves[14] which arises from the condition (1.7) for $l=3$.

The generalized k-constraint was formulated by Loris and Willox[13] as the condition,

$$\mathscr{L}^k + \beta\mathscr{L}^{-1} = B_k + \psi\partial^{-1}\overline{\psi}, \quad k \geqslant 1, \tag{1.8}$$

where the constant β is related to the asymptotic values of the eigenfunctions ψ and $\overline{\psi}$ at infinity, i.e., $\psi\overline{\psi} \to \beta$ as $t_1 \to \infty$. In particular, the NLS equation at the continuous-wave background[15] arises from the condition (1.8) for $k=1$.

This paper is devoted to studies of the rational solutions satisfying the generalized reductions (1.7) and (1.8). The main ideas and methods of this paper follow those of the previous paper[16] referred to henceforth as paper I. The aims of both papers are twofold; (i) to analyze and classify various sets of rational solutions of the single-component KP hierarchy through the construction of the equivalent polynomial τ-function and (ii) to relate the different structure of the polynomials to different dynamics of particles in a many-body Calogero–Moser (CM) system and its higher commuting flows.[17,18]

The τ-function representation of solutions of the KP hierarchy appears naturally within the framework of the Hirota bilinear transformation (see, e.g., Ref. 3),

$$u=\frac{\partial^2}{\partial t_1^2}\log\tau,\quad \psi=c\,\frac{\tau^+}{\tau},\quad \bar{\psi}=\bar{c}\,\frac{\tau^-}{\tau},\quad v=\frac{\partial}{\partial t_1}\log\frac{\tau^+}{\tau}, \tag{1.9}$$

where c and $\bar{c}$ are arbitrary constants, the function $\tau=\tau(t_1,t_2,\ldots)$ is referred to as the τ-function of the KP hierarchy while the associated functions $\tau^{\pm}$ are expressed through τ as follows:

$$\tau^{\pm}=\tau\left(t_1\mp\frac{1}{\lambda},t_2\mp\frac{1}{2\lambda^2},\ldots\right)\exp\left(\pm\sum_{n=1}^{\infty}\lambda^n t_n\right). \tag{1.10}$$

It was recently proved by Shiota[19] (following the pioneer paper by Krichever[20]) that zeros of the polynomial τ-function of the KP hierarchy describe motion of particles in the whole hierarchy of the CM dynamical systems.

In paper I it was shown that a general polynomial τ-function of the KP hierarchy can be factorized through a set of partial polynomial solutions generated by derivatives of certain exponential functions with only one value of the spectral parameter λ. These partial solutions were referred to as the degenerate polynomials of the KP hierarchy. Different types of scattering of particles in the CM hierarchy were analyzed in paper I with the help of the factorizing identities and the degenerate polynomials were shown to be related to anomalous scattering of particles accompanying slow rates of the particle motion. The first example of such an anomalous scattering was presented by Gorshkov *et al.*[21] for the KP1 equation and later by Ward[22] and Ioannidou[23] for an integrable chiral model. Recently, it was shown by Ablowitz and Villarroel[24] that the degenerate rational solutions of the KP1 equation naturally appear in the inverse scattering formalism through a multiple pole expansion of the associated eigenfunction ψ (which is equivalent to the pole expansion of the Baker–Akhiezer function used in a geometric method).[20] However, in previous works a complete classification of the degenerate rational solutions of the KP hierarchy has not been proposed.

In this paper we develop a unifying approach to constructing the degenerate polynomial solutions of the KP hierarchy by using the theory of generalized Schur polynomials and vertex operators. This approach enables us to characterize all degenerate polynomials in terms of stationary manifolds of the CM hierarchy[25] and to find a simple representation of the rational solutions satisfying the generalized l-reduction and k-constraint of the KP hierarchy. A general scheme for constructing the degenerate polynomials is described and particular applications of the scheme to some physically meaningful nonlinear evolution equations are also given.

The paper is organized as follows. The basic notation and properties of the generalized Schur polynomials and the vertex operators acting on these polynomials are described in Sec. II. A connection of these polynomials with the stationary manifolds of the CM system is discussed in Sec. III. A general classification of the stationary manifolds and of the related rational solutions to the l-reduced KP hierarchy is given in Sec. IV. The rational solutions to the generalized l-reduction and k-constraint of the KP hierarchy are constructed in Sec. V and VI, respectively. A classification of different dynamics of the anomalous scattering of particles in the CM hierarchy is considered in Sec. VII. Finally, Sec. VIII concludes the paper.

II. THE GENERALIZED SCHUR POLYNOMIALS

Exact solutions of the KP hierarchy can be expressed through one of the conventional representations of the τ-function. Following paper I, only the Wronskian representation[26] is considered here but it enables us to generate all classes of rational solutions without loss of generality. Within the Wronskian representation, the τ-function is

$$\tau = W_N[\phi_1, \phi_2, \ldots, \phi_N] = \det \left| \frac{\partial^{k-1}\phi_n}{\partial t_1^{k-1}} \right|_{1 \leqslant n,k \leqslant N}, \tag{2.1}$$

where the entries of the Wronskian determinant $\phi_n = \phi_n(t_1, t_2, \ldots)$, $n = \overline{1,N}$ satisfy the differential equations

$$\frac{\partial \phi_n}{\partial t_k} = \frac{\partial^k \phi_n}{\partial t_1^k}, \quad k \geqslant 1. \tag{2.2}$$

A degenerate polynomial τ-function is generated by a partial solution of Eq. (2.2),

$$\phi_n = \left(\sum_{j=1}^{n} \alpha_j P_{m_j}(p) \right) e^{\Phi(p)}, \tag{2.3}$$

where α_j are arbitrary coefficients, p is a spectral parameter, the phase factor is given by

$$\Phi(p) = \sum_{k=1}^{\infty} p^k t_k, \tag{2.4}$$

and the functions $P_n(p)$ are defined by a generating exponential function,

$$P_n(p) = e^{-\Phi(p)} \frac{\partial^n}{\partial p^n} e^{\Phi(p)}. \tag{2.5}$$

These functions are quasipolynomials of the time variables t_k because they contain infinite series of t_k. However, the time variables t_k enter the quasipolynomials $P_n(p)$ only in the form of a finite number of the generalized time variables $\theta_k(p)$ given by

$$\theta_k(p) = \frac{1}{k!} \frac{\partial^k \Phi(p)}{\partial p^k} = \sum_{m=k}^{\infty} \binom{m}{k} p^{m-k} t_m. \tag{2.6}$$

Hence we refer to $P_n(p)$ as the generalized Schur polynomials. In the limit $p \to 0$ the polynomial $P_n(p)$ reduces to a conventional Schur polynomial $p_n(t_1, t_2, \ldots) = P_n(0)$ arising from the expansion $e^{\Phi(p)} = \Sigma_{n=1}^{\infty}(1/n!) p_n(t_1, t_2, \ldots) p^n$. The first few generalized polynomials have the form,

$$P_1 = \theta_1, \quad P_2 = \theta_1^2 + 2\theta_2, \quad P_3 = \theta_1^3 + 6\theta_1\theta_2 + 6\theta_3,$$

$$P_4 = \theta_1^4 + 12\theta_1^2\theta_2 + 12\theta_2^2 + 24\theta_1\theta_3 + 24\theta_4,$$

$$P_5 = \theta_1^5 + 20\theta_1^3\theta_2 + 60\theta_1\theta_2^2 + 60\theta_1^2\theta_3 + 120\theta_2\theta_3 + 120\theta_1\theta_4 + 120\theta_5.$$

The key property of the Schur polynomials $P_n(p)$ is that they satisfy the original system (2.2) rewritten in terms of the generalized variables θ_k and also the following relation:

$$\frac{\partial P_n}{\partial \theta_k} = \frac{\partial^k P_n}{\partial \theta_1^k} = \frac{n!}{(n-k)!} P_{n-k}. \tag{2.7}$$

Using this key property, Matveev[27] proved the following lemma.

Lemma 2.1: Suppose that $m_1 < m_2 < \cdots < m_n$ and $\alpha_n = 1$ in Eq. (2.3). Then, there exists constants θ_{k0}, $k = \overline{1,m_n}$ so that the superposition of the Schur polynomials in Eq. (2.3) is reducible to a polynomial of the highest degree, i.e.,

$$\sum_{j=1}^{n} \alpha_j P_{m_j}(p) = \tilde{P}_{m_n}(p) \equiv P_{m_n}(p; \theta_1 - \theta_{10}, \theta_2 - \theta_{20}, \ldots, \theta_{m_n} - \theta_{m_n 0}). \tag{2.8}$$

According to Lemma 2.1, the degenerate rational solutions of the KP hierarchy can be expressed through the leading-order polynomials $\tilde{P}_{m_n}(p)$ with different values of the phase constants θ_{k0}. Then, the rational solution of this type has the form,

$$\tau=\tau_p(\theta_1,\theta_2,\ldots)e^{N\Phi(p)}, \tag{2.9}$$

where $\tau_p(\theta_1,\theta_2,\ldots)$ is a polynomial (or, equivalently, quasipolynomial) with respect to the time variables $\theta_n\equiv\theta_n(p)$ or t_n, respectively. This polynomial τ-function is given by the Wronskian determinant,

$$\tau_p(\theta_1,\theta_2,\ldots)=W_N[\tilde{P}_{m_1}(p),\tilde{P}_{m_2}(p),\ldots,\tilde{P}_{m_N}(p)]. \tag{2.10}$$

Using Eqs. (1.10) and (2.10), we find the associated functions $\tau^{\pm}$ for the polynomial representation,

$$\tau^{\pm}=\left(\frac{\lambda-p}{\lambda}\right)^{\pm N}\tau_p\left(\theta_1\mp\frac{1}{\lambda-p},\theta_2\mp\frac{1}{2(\lambda-p)^2},\ldots\right)\exp(N\Phi(p)\pm\Phi(\lambda)), \tag{2.11}$$

where the spectral parameter λ may be arbitrary. However, the rational representations for the eigenfunctions ψ and $\bar{\psi}$ arise only in the limit $\lambda\to 0$ provided that the constants c and $\bar{c}$ are chosen appropriately. Then, the rational solutions to the KP hierarchy can be expressed through a scalar polynomial function $\tau_p=\tau_p(\theta_1,\theta_2,\ldots)$ as follows:

$$u=\frac{\partial^2}{\partial\theta_1^2}\log\,\tau_p,\quad \psi=\rho\,\frac{S(\mu)\tau_p}{\tau_p},\quad \bar{\psi}=\bar{\rho}\frac{S^{-1}(\mu)\tau_p}{\tau_p},\quad v=\frac{\partial}{\partial\theta_1}\log\frac{S(\mu)\tau_p}{\tau_p}. \tag{2.12}$$

Here the parameters ρ and $\bar{\rho}$ define the boundary conditions of the eigenfunctions ψ and $\bar{\psi}$ as $t_1\to\infty$, the parameter μ is $\mu=-p^{-1}$, and the generalized vertex operator $S(\mu)$ is introduced to characterize shifts of the phase variables $\theta_k\equiv\theta_k(p)$,

$$S(\mu)=\exp\left[-\sum_{k=1}^{\infty}\frac{1}{k}\,\mu^k\,\frac{\partial}{\partial\theta_k}\right]. \tag{2.13}$$

The action of the vertex operator on the generalized Schur polynomials $P_n(p)$ is described by the following lemma.

Lemma 2.2: If $P_n(p)$ satisfies Eq. (2.7) for any k and $S(\mu)$ is defined by Eq. (2.13), then the following identities are valid:

$$S^k(\mu)P_n(p)=\sum_{j=0}^{k}(-\mu)^j\binom{k}{j}\frac{\partial P_n(p)}{\partial\theta_j}, \tag{2.14}$$

$$S^{-k}(\mu)P_n(p)=\sum_{j=0}^{\infty}\mu^j\binom{j+k-1}{k-1}\frac{\partial P_n(p)}{\partial\theta_j}. \tag{2.15}$$

Lemma 2.2 has a classic analog in the theory of group characters (see, e.g., Ref. 28). Within the given context, the formulas (2.14) and (2.15) have already been used for the construction of integrable hierarchies[11] and that of the rational solutions.[14] Importantly, these formulas are not applicable to the action of vertex operators on the polynomial function $\tau_p(\theta_1,\theta_2,\ldots)$ because the polynomial function τ_p does not generally satisfy the key relations (2.7).

The multicomponent generalization of the vertex operator can be defined through the product of the vertex operators (2.13),

$$S(\nu_1,\ldots,\nu_k)=\prod_{i=1}^{k}S(\mu_i), \tag{2.16}$$

where the parameters ν_i are related to μ_j by the partial sums,

$$\nu_i=(-1)^{i-1}\sum_{z_1+\ldots+z_k=i}\mu_1^{z_1}\ldots\mu_k^{z_k}, \tag{2.17}$$

and the indices z_l are either zeros or unities. Using the formulas (2.14) and (2.16) one can prove the following lemma.

Lemma 2.3: If $P_n(p)$ satisfies Eq. (2.7) for any k and $S(\nu_1,\ldots,\nu_k)$ is defined by Eqs. (2.16) and (2.17), then the following identity is satisfied:

$$S(\nu_1,\ldots,\nu_k)P_n(p)=1-\sum_{i=1}^{k}\nu_i\frac{\partial P_n(p)}{\partial\theta_i}. \tag{2.18}$$

III. CONNECTION WITH THE CM HIERARCHY

Properties of the degenerate rational solutions of the KP hierarchy are described by the construction of the polynomial function $\tau_p(\theta_1,\theta_2,\ldots)$ within the form (2.10). The following theorem proved by Shiota[19] clarifies a relationship between this polynomial function and the dynamics of particles in the CM hierarchy of dynamical systems.

Theorem 3.1: Let the function $\tau_p(t_1,t_2,\ldots)$ be a monic polynomial in t_1 having R zeros $x_j(t_2,\ldots)$, $j=\overline{1,R}$ according to the representation,

$$\tau_p(t_1,t_2,\ldots)=\prod_{j=1}^{R}(t_1-x_j(t_2,\ldots)). \tag{3.1}$$

This function is the τ-function of the KP hierarchy if and only if the motion of zeros of τ_p is governed by the hierarchy of the CM dynamical systems,

$$\frac{\partial x_j}{\partial t_n}=\frac{\partial H_n}{\partial p_j},\quad \frac{\partial p_j}{\partial t_n}=-\frac{\partial H_n}{\partial x_j},\quad n\geqslant 2, \tag{3.2}$$

where $H_n=(-1)^n\mathrm{Sp}(L)^n$ and the elements of the matrix L are $L_{ij}=p_j\delta_{ij}+(1-\delta_{ij})(x_i-x_j)^{-1}$. In the particular case $n=2$, this system is just the Calogero–Moser system of particles,[17,18]

$$\frac{\partial^2 x_j}{\partial t_2^2}=8\sum_{i\neq j}\frac{1}{(x_i-x_j)^3}, \tag{3.3}$$

where x_j are coordinates of particles, while the velocities of particles are $v_j=\partial x_j/\partial t_2=2p_j$.

Collorary 3.2: The degree of the polynomial τ-function with respect to t_1 denoted by R coincides with the number of particles of the corresponding CM system.

Collorary 3.3: The number of linearly independent time variables $\theta_n(p)$ of the polynomial τ-function denoted by G coincides with the dimension of a dynamical manifold of the corresponding CM system.

A general solution describing normal scattering of R particles along a manifold of the dimension $G=2R$ is well known for the CM systems (3.2) and (3.3) (see, e.g., Ref. 19, and references therein). In paper I it was shown that besides a general solution there exists a wide set of degenerate solutions which describe anomalous scattering of particles with asymptotically equal velocities, i.e. when $v_j\to 2p$ as $t_2\to\pm\infty$. A complete picture of dynamical processes occurring in the CM systems can be explained as a superposition of elementary processes of either normal or anomalous scattering.[16] Here we study only the anomalous scattering in the CM hierarchy which is described by zeros of the degenerate polynomials $\tau_p(\theta_1,\theta_2,\ldots)$ according to Theorem 3.1.

Proposition 3.4: The degenerate polynomial $\tau_p(\theta_1,\theta_2,\ldots)$ given by Eq. (2.10) corresponds to the CM system of R particles, where

$$R=\sum_{j=1}^{N}m_j-\frac{N(N-1)}{2}. \tag{3.4}$$

Proof: Suppose that $0<m_1<m_2<\ldots<m_N$ in Eq. (2.10). The polynomials $P_n(p)$ at the leading order approximation with respect to θ_1 (or, equivalently, to t_1) are $P_n(p)\sim\theta_1^n+O(\theta_1^{n-2})$. Expanding the Wronskian $W_N[\tilde{P}_{m_1}(p),\ldots,\tilde{P}_{m_N}(p)]$ within this approximation, we find the result (3.4) for the degree of the polynomial $\tau_p(\theta_1,\theta_2,\ldots)$ which is equivalent to the number of particles R according to Collorary 3.2. □

The degenerate polynomials of the KP hierarchy and the corresponding dynamics of the CM systems can be classified into two different groups. The first group consists of the polynomials which can be reduced to the generalized Schur polynomials $P_R(p;\theta_1,\ldots,\theta_R)$. It is clear from the parametrization of these polynomials that a dynamical manifold of the CM system described by $P_R(p)$ has the dimension $G=R$ with the ''additional'' parameter p determining the asymptotic velocity of the particles as $t_2\to\infty$. The other group of the polynomial τ-functions which includes the rest of the polynomials τ_p describes the anomalous scattering occurring near one of the stationary manifolds of the CM hierarchy.[25]

Definition 3.5: A manifold of the CM hierarchy is stationary with respect to the time t_l if

$$\frac{\partial x_j}{\partial t_l}=0,\quad \text{for all}\quad j\quad\text{and}\quad l\geqslant 2. \tag{3.5}$$

Stationary manifolds of the CM hierarchy form certain embeddings into dynamical manifolds describing the anomalous dynamics of the CM particles. To be specific, the stationary manifolds have the dimension G which is less than the number of particles R, i.e., although all asymptotic velocities of particles are specified by a single parameter p, the asymptotic phases of the coordinates $x_j(t_2,\ldots)$ in the limit $t_2\to\infty$ are functionally dependent. The following result proves that all polynomial functions of the form (2.10) excepting those reducible to the polynomials $P_n(p)$ describe, within a particular limit, a certain stationary manifold of the CM hierarchy.

Proposition 3.6: Suppose that the polynomial τ-function of the KP hierarchy is given by

$$\tau_p(t_1,t_2,\ldots)=W_N[p_{m_1},p_{m_2},\ldots,p_{m_N}], \tag{3.6}$$

where $p_n\equiv p_n(t_1,t_2,\ldots)=P_n(0)$. Zeros of this polynomial function describe one of the stationary manifolds of the CM hierarchy if $N>1$ and the set $\{m_j\}_{j=1}^N$ is different from the ordered set $\{j\}_{j=1}^N$.

Proof: In the case $m_N<R$ it is clearly seen from Eq. (3.6) that the function $\tau_p(t_1,t_2,\ldots)$ is independent of at least the variables $t_{m_N+1},\ldots,t_R$ and hence, zeros of the polynomial function τ_p do not depend on these times.

In the case $m_N\geqslant R$ it follows from Eq. (3.4) that the only possible solution is $\tau_p(t_1,t_2,\ldots)=W_N[p_1,p_2,\ldots,p_{N-1},p_R]$ which is realized for $m_N=R$ and $R>N$. Then, it can be shown that the polynomial function τ_p does not depend on time t_R, i.e., corresponds to the stationary manifold with respect to t_R. In the exceptional case $R=N$ the condition of the proposition is not satisfied, and, indeed, the function τ_p is now reducible to the Schur polynomial $p_N(t_1,t_2,\ldots)$ (see paper I). □

In the next section we classify degenerate polynomial τ-functions of the KP hierarchy into different subgroups corresponding to different stationary manifolds of the CM hierarchy. This classification is directly related to the construction of rational solutions of the l-reduced KP hierarchy.

IV. THE *l*-REDUCED KP HIERARCHY

The following lemma can be readily proved by using the definition of the l-reduction (1.5) (see Ref. 3 for details).

Lemma 4.1: The polynomial τ-function of the l-reduced KP hierarchy satisfies the differential equation,

$$\frac{\partial\tau_p}{\partial t_l}=0,\qquad l\geqslant 2. \tag{4.1}$$

Collorary 4.2: Zeros of the function $\tau_p(t_1,t_2,\ldots)$ satisfying the l-reduced KP hierarchy define the stationary manifold of the CM hierarchy with respect to the time t_l.

Collorary 4.3: The τ-function of the l-reduced KP hierarchy does not depend on variables $t_l,t_{2l},\ldots,t_{nl}$.

Collorary 4.3 follows from the identity $L^{nl}=(B_l)^n=B_{nl}$ and leads to the fact that the τ-function of the l-reduced KP hierarchy also gives a subset of solutions of the nl-reduced hierarchies. In order to present a closed classification scheme for the rational solutions of the l-reduced KP hierarchy we assume that the function $\tau_p(t_1,t_2,\ldots)$ satisfies the complimentary constraints,

$$\frac{\partial \tau_p}{\partial t_j}\neq 0, \qquad j=\overline{1,l-1}. \tag{4.2}$$

Then, the following result defines a structural element of the polynomial τ-function of the l-reduced KP hierarchy.

Proposition 4.4: The differential equation (4.1) is satisfied by the $(l-1)$ families of the polynomial τ-functions,

$$\tau_p(t_1,t_2,\ldots)=W_{n_j}[p_j p_{j+l},\ldots,p_{j+(n_j-1)l}], \qquad j=\overline{1,l-1}, \tag{4.3}$$

where $p_n\equiv p_n(t_1,t_2,\ldots)=P_n(0)$ and n_j are positive integers enumerating the families of the polynomials.

A proof can be given by means of a direct differentiation of $\tau_p(t_1,t_2,\ldots)$ with respect to t_l, use the formula (2.7) for $p=0$, and elimination of two identical columns in the Wronskian determinants. This result has been widely used in previous literature (see, e.g., Ref. 11). A less known fact is that the $(l-1)$ families of the polynomials given by Eq. (4.3) do not cover all possible rational solutions of the l-reduced KP hierarchy except for the simple case $l=2$. Indeed, a more general solution can still be constructed through a combination of the particular solutions (4.3) in the following form:

$$\tau_p(t_1,t_2,\ldots)=\tau_l^{n_1n_2\cdots n_{l-1}}=W_N[p_1,p_{1+l},\ldots,p_{1+(n_1-1)l};\ldots;p_{l-1},p_{2l-1},\ldots,p_{n_{l-1}l-1}], \tag{4.4}$$

where $N=\Sigma_{j=1}^{l-1}n_j$. As follows from Collorary 4.2, zeros of this polynomial solution define a general construction of the stationary manifold of the CM hierarchy with respect to the time t_l. By using Eq. (3.4) we find that the number of particles forming a stationary configuration is not arbitrary but given by the expression,

$$R=\sum_{j=1}^{l-1} n_j\left(j+\frac{1}{2}(n_j-l+1)-\frac{1}{2}\sum_{i=1}^{l-1}(n_i-n_j)\right). \tag{4.5}$$

The dimension of the stationary manifold G is less than the number of particles R because the time variables t_l, t_{2l}, and so on, do not parametrize the function $\tau_l^{n_1n_2\cdots n_{l-1}}$. It seems to be difficult to find G for a general solution (4.4) (see, e.g., Ref. 14). Still for the particular solutions (4.3) we find the result,

$$R=\tfrac{1}{2}n_j(2j+(l-1)(n_j-1)), \quad G=j+(l-1)(n_j-1). \tag{4.6}$$

Example 4.5: The KdV hierarchy $(l=2)$.

The KdV hierarchy starts with the KdV equation,

$$-4\frac{\partial u}{\partial t_3}+12u\frac{\partial u}{\partial t_1}+\frac{\partial^3 u}{\partial t_1^3}=0. \tag{4.7}$$

A complete set of the polynomial τ-functions for the KdV hierarchy[29,30] is given by Eq. (4.4) for $l=2$ and $n_1=n$, i.e.,

$$\tau_p(t_1,t_3,\ldots)=\tau_2^n=W_n[p_1,p_3,\ldots,p_{2n-1}]. \tag{4.8}$$

Zeros of these polynomials define the stationary manifolds for the CM dynamical system (3.3) with respect to t_2.[25] It follows from Eq. (4.6) that these stationary manifolds exist only for $R=n(n+1)/2$ particles and have $G=n$ parameter subspace.[29] For reference, we reproduce here the first polynomials of this family, omitting a constant factor in front of τ_2^n,

$$\tau_2^1 \cong t_1,$$

$$\tau_2^2 \cong t_1^3 - 3t_3,$$

$$\tau_2^3 \cong t_1^6 - 15t_1^3 t_3 - 45t_3^2 + 45t_1 t_5,$$

$$\tau_2^4 \cong t_1^{10} - 45t_1^7 t_3 - 4725t_1 t_3^3 + 315t_1^5 t_5 + 4725t_1^2 t_3 t_5 - 4725t_5^2 - 1575t_1^3 t_7 + 4725t_3 t_7.$$

Example 4.6: The Boussinesq hierarchy $(l=3)$.

The Boussinesq hierarchy starts with the Boussinesq equation,

$$3\frac{\partial^2 u}{\partial t_2^2} + 6\frac{\partial^2 u^2}{\partial t_1^2} + \frac{\partial^4 u}{\partial t_1^4} = 0. \tag{4.9}$$

A general rational solution of the Boussinesq equation[14] is expressed through the polynomial τ-function (4.4) for $l=3$, $n_1=n$, and $n_2=m$, i.e.,

$$\tau_p(t_1,t_2,\ldots) = \tau_3^{nm} = W_{n+m}[p_1,p_4,\ldots,p_{3n-2};p_2,p_5,\ldots,p_{3m-1}]. \tag{4.10}$$

Zeros of these polynomials define the stationary manifold of the first higher-order commuting flow of the CM hierarchy (3.2), i.e., that with respect to t_3. It was found by Galkin *et al.*[14] that the number of particles R forming the stationary configuration and the dimension of the manifolds G are specified as follows:

$$R = n^2 + m(m+1) - nm, \tag{4.11}$$

$$G = \begin{cases} 2m-n & m \geqslant 2n \\ m+n & 2n > m \geqslant n/2 \\ 2n-m-1 & n/2 > m \end{cases}. \tag{4.12}$$

The first polynomials are

$$\tau_3^{1,0} \cong t_1, \qquad \tau_3^{0,1} \cong t_1^2 + 2t_2,$$

$$\tau_3^{2,0} \cong t_1^4 + 4t_1^2 t_2 - 4t_2^2 - 8t_4, \qquad \tau_3^{1,1} \cong t_1^2 - 2t_2,$$

$$\tau_3^{0,2} \cong t_1^6 + 10t_1^4 t_2 + 20t_1^2 t_2^2 + 40t_2^3 - 40t_1^2 t_4 + 80t_2 t_4 - 80t_1 t_5,$$

$$\begin{aligned}\tau_3^{3,0} \cong{}& t_1^9 + 16t_1^7 t_2 + 56t_1^5 t_2^2 - 560t_1 t_2^4 - 112t_1^5 t_4 - 2240t_1 t_2^2 t_4 - 2240t_1 t_4^2 - 280t_1^4 t_5 + 1120t_1^2 t_2 t_5 \\ &+ 1120t_2^2 t_5 - 2240t_4 t_5 + 1120t_1^2 t_7 + 2240t_2 t_7,\end{aligned}$$

$$\tau_3^{2,1} \cong t_1^4 - 4t_1^2 t_2 - 4t_2^2 + 8t_4, \qquad \tau_3^{1,2} \cong t_1^5 - 20t_1 t_2^2 + 20t_5,$$

$$\begin{aligned}\tau_3^{0,3} \cong{}& t_1^{12} + 28t_1^{10} t_2 + 260t_1^8 t_2^2 + 1120t_1^6 t_2^3 + 2800t_1^4 t_2^4 + 11200t_1^2 t_2^5 + 11200t_2^6 - 280t_1^8 t_4 - 2240t_1^6 t_2 t_4 \\ &- 11200t_1^4 t_2^2 t_4 + 44800t_1^2 t_2^3 t_4 + 67200t_2^4 t_4 - 11200t_1^4 t_4^2 + 134400t_1^2 t_2 t_4^2 + 44800t_2^2 t_4^2 - 89600t_4^3 \\ &- 960t_1^7 t_5 - 4480t_1^5 t_2 t_5 - 44800t_1^3 t_2^2 t_5 - 89600t_1 t_2^3 t_5 - 89600t_1^3 t_4 t_5 + 179200t_1 t_2 t_4 t_5 \\ &- 89600t_1^2 t_5^2 + 179200t_2 t_5^2 + 8960t_1^5 t_7 - 179200t_1 t_2^2 t_7 + 179200t_5 t_7 + 22400t_1^4 t_8 + 89600t_1^2 t_2 t_8 \\ &- 89600t_2^2 t_8 - 179200t_4 t_8.\end{aligned}$$

We notice from these explicit formulas that the polynomial solutions related to the symmetry of the KP hierarchy with respect to the transformation $t_k \to (-1)^{k-1} t_k$ appear naturally from the general formula (4.10) for m, $n \neq 0$. □

A straightforward generalization of the polynomial solutions (4.3) which satisfies the k-constraint (1.6) of the KP hierarchy was recently presented by Loris and Willox.[11] Unfortunately, the rational solutions of the l-reduced and k-constrained KP hierarchies usually turn out to be physically meaningless since they are either singular or complex. However, the results on the existence and dimension of stationary manifolds of the CM hierarchy serve as an useful tool for constructing and analyzing the degenerate rational solutions which satisfy the generalized reductions of the KP hierarchy. The latter solutions often describe physically interesting phenomena within the underlying nonlinear evolution equations. In the next two sections we construct the rational solutions satisfying the generalized l-reduction and k-constraint of the KP hierarchy and present a few examples where the rational solutions are physically meaningful.

V. THE GENERALIZED *l*-REDUCED KP HIERARCHY

Here we consider the generalized l-reduction of the KP hierarchy defined by the condition (1.7). Using a simple modification of the methods of Ref. 3 one can prove the following lemma.

Lemma 5.1: The polynomial τ-function of the generalized l-reduced KP hierarchy satisfies the differential equation,

$$\frac{\partial \tau_p}{\partial t_l} = \alpha \frac{\partial \tau_p}{\partial t_1}, \quad l \geqslant 2. \tag{5.1}$$

As in Lemma 4.1, we assume here that there are no differential relations similar to Eq. (5.1) for the evolution of $\tau_p(t_1, t_2, \ldots)$ with respect to the time variables t_j for $j = \overline{1, l-1}$. The differential equation (5.1) implies that functions of the KP hierarchy depend on t_l in the form of a stationary phase, i.e., $u_k = u_k(t_1 + \alpha t_l, t_2, \ldots, t_{l-1}, t_{l+1}, \ldots)$. It is important that no rational solutions to the generalized l-reduction could be constructed by means of the Schur polynomials $p_n(t_1, t_2, \ldots)$ and, therefore, we have to consider the generalized Schur polynomials $P_n(p)$ given by Eq. (2.5) at $p \neq 0$. Thus, a set of the rational solutions of the generalized l-reduced KP hierarchy is prescribed by the following result.

Proposition 5.2: The differential equation (5.1) is satisfied by the $(l-1)$ families of the polynomial τ-functions,

$$\tau_p(\theta_1, \theta_2, \ldots) = W_n[P_1(p), S^{-1}(\nu_1, \ldots, \nu_{l-2}) P_3(p), \ldots, S^{-(n-1)}(\nu_1, \ldots, \nu_{l-2}) P_{2n-1}(p)]. \tag{5.2}$$

Here n is a positive integer, $P_n(p)$ is the generalized Schur polynomial (2.5), and the generalized vertex operator $S(\nu_1, \ldots, \nu_{l-2})$ is defined by Eqs. (2.16) and (2.17). The parameters ν_i for $i = \overline{1, l-2}$ are given by

$$\nu_i = -\binom{l}{2}^{-1} \binom{l}{i+2} p^{-i} \tag{5.3}$$

and the parameter p is one of the $(l-1)$ roots of the algebraic equation,

$$l p^{l-1} = \alpha. \tag{5.4}$$

Proof: First, we find that the polynomial function $\tau_p(\theta_1, \theta_2, \ldots)$ defined by Eq. (5.2) satisfies a certain differential equation with respect to the generalized time variables $\theta_k \equiv \theta_k(p)$. This can be done by differentiating each column of the Wronskian (5.2) with respect to θ_2, by subsequent subtracting of the nearbouring column multiplied by a certain factor and by using the formulas (2.7) and (2.18). As a result, we find the following differential equation for $\tau_p(\theta_1, \theta_2, \ldots)$,

$$\frac{\partial \tau_p}{\partial \theta_2} = \sum_{i=3}^{l} \nu_{i-2} \frac{\partial \tau_p}{\partial \theta_i}. \tag{5.5}$$

This relation should be compared with the differential equation (5.1) To do this, we rewrite Eq. (5.1) in terms of the variables θ_k as follows:

$$(lp^{l-1}-\alpha)\frac{\partial\tau_p}{\partial\theta_1}+\sum_{i=2}^{l}\binom{l}{i}p^{l-i}\frac{\partial\tau_p}{\partial\theta_i}=0. \tag{5.6}$$

The leading-order term $\partial\tau_p/\partial\theta_1$ is always dominant in Eq. (5.6) because the time variable θ_1 always has a different weight in the polynomial function $\tau_p(\theta_1,\theta_2,\ldots)$ compared to the other time variables. Therefore, we have to remove this term and specify p as a root of the algebraic equation (5.4). Then, by comparing the coefficients in Eqs. (5.5) and (5.6) we find the parameters ν_i in the form (5.3). □

It is obvious that the expression (5.2) generalizes the polynomial solutions (4.8) for the 2-reduced KP hierarchy (the KdV hierarchy) (see Example 4.5). The polynomial function $\tau_p(\theta_1,\theta_2,\ldots)$ given by Eq. (5.2) has the degree $R=n(n+1)/2$ and is parametrized by $G=n$ linear superpositions of the time variables θ_k (see examples below). This polynomial function generates a rational solution of the generalized l-reduced KP hierarchy according to the representation (2.12). A more general rational solution can still be constructed within the framework of the original Wronskian representation (2.1) where the functions ϕ_n are specified according to the particular $(l-1)$ solutions (5.2) [recall that the parameter p may have $(l-1)$ values]. This general rational solution can always be analyzed by applying the superposition formula found in paper I.

Example 5.4: The Gardner equation $(l=2)$.

The reduction $v=v(t_1+\alpha t_2,t_3,\ldots)$ transforms the modified KP hierarchy to a hierarchy of the Gardner equation. The Gardner equation follows from (1.4) in the form,

$$-4\frac{\partial v}{\partial t_3}+3(\alpha^2+2\alpha v-2v^2)\frac{\partial v}{\partial t_1}+\frac{\partial^3 v}{\partial t_1^3}=0. \tag{5.7}$$

Equivalently, the Gardner equation can be thought as the modified KdV equation under the nonzero boundary conditions at infinity. Indeed, the modified KdV equation,

$$-4\frac{\partial w}{\partial t_3}+\frac{9}{2}\alpha^2\frac{\partial w}{\partial t_1}-6w^2\frac{\partial w}{\partial t_1}+\frac{\partial^3 w}{\partial t_1^3}=0, \tag{5.8}$$

reduces to the Gardner equation (5.7) by means of the transformation, $w=v-\alpha/2$. A single family of the rational solutions of the Gardner equation is expressed through the representation (2.12) with the polynomial τ-function given by Eq. (5.2) for $l=2$, i.e.,

$$\tau_p(\theta_1,\theta_2,\ldots)=W_n[P_1(p),P_3(p),\ldots,P_{2n-1}(p)], \tag{5.9}$$

where $p=\alpha/2$. The first polynomials are just the polynomials of the KdV hierarchy (see Example 4.5) with t_k replaced by θ_k, i.e., $\tau_p=\tau_2^n(\theta_1,\theta_3,\ldots)$. These polynomials can further be used to construct the real nonsingular solutions of the (focusing) Gardner equation which follows from Eq. (5.7) under the transformation $\alpha\to i\tilde{\alpha}$ and $v\to i\tilde{v}$. To do this, we just shift phases of the variables θ_k in the function $\tau_p(\theta_1,\theta_2,\ldots)$ and rewrite Eq. (2.12) for v in the equivalent form,

$$v=\frac{\partial}{\partial\theta_1}\log\frac{S^{1/2}(\mu)\tau_p}{S^{-1/2}(\mu)\tau_p}, \tag{5.10}$$

where $\mu=-p^{-1}=-2\alpha^{-1}$. Then, the transformation given above ensures the function $\tilde{v}$ be real and nonsingular. Within this context, the rational solutions (5.9) and (5.10) describe an algebraic soliton of the Gardner equation for $n=1$,[31] a structural instability of the algebraic soliton due to a resonance with a long wave shelf for $n=2$,[32] and the scattering dynamics of two and more algebraic solitons complicated by excitations of the wave shelfs for $n\geqslant 3$.

Example 5.5: The Boussinesq equation for acoustic waves $(l=3)$.

The reduction $u=u(t_1+\alpha t_3,t_2,t_4,\ldots)$ transforms the KP hierarchy to a hierarchy of the Boussinesq equation for acoustic waves. The latter equation follows from Eq. (1.2) in the form,

$$3\frac{\partial^2 u}{\partial t_2^2}-4\alpha\frac{\partial^2 u}{\partial t_1^2}+6\frac{\partial^2 u^2}{\partial t_1^2}+\frac{\partial^4 u}{\partial t_1^4}=0. \tag{5.11}$$

Equivalently, the Boussinesq equation for acoustic waves can be thought of as the Boussinesq equation (4.9) under the nonzero boundary conditions at infinity. Indeed, the transformation, $u \to u-\alpha/3$ reduces Eq. (4.9) to the form (5.11). Two particular families of the rational solutions of the Boussinesq equation (5.11) are expressed through the polynomial τ-function (5.2) for $l=3$, i.e.,

$$\tau_p(\theta_1,\theta_2,\ldots)=W_n[P_1(p),S^{-1}(\nu)P_3(p),\ldots,S^{-(n-1)}(\nu)P_{2n-1}(p)], \tag{5.12}$$

where $p=\pm\sqrt{\alpha/3}$, and $S(\nu)$ is given by Eq. (2.13) with the parameter $\nu=-(3p)^{-1}$.[14] The first polynomials (5.12) can be transformed by means of the phase translations to the polynomials of the KdV hierarchy (see Example 4.5) with the residue terms, i.e., $\tau_p=\tau_2^n(z_1,z_3,\ldots)+\Delta\tau_2^n(z_1,z_3,\ldots)$, where z_k are new variables, i.e.,

$$z_1=\theta_1,\quad z_3=\theta_3+\nu\theta_2+\tfrac{1}{4}\nu^2\theta_1,$$

$$z_5=\theta_5+2\nu\theta_4+\tfrac{3}{2}\nu^2\theta_3+\tfrac{3}{2}\nu^3\theta_2+\tfrac{9}{16}\nu^4\theta_1,$$

$$z_7=\theta_7+3\nu\theta_6+\tfrac{15}{4}\nu^2\theta_5+\tfrac{9}{2}\nu^3\theta_4+\tfrac{63}{16}\nu^4\theta_3+\tfrac{63}{16}\nu^5\theta_2+\tfrac{105}{64}\nu^6\theta_1,$$

and $\Delta\tau_2^n$ are the residue terms, i.e.,

$$\Delta\tau_2^1=0,\quad \Delta\tau_2^2=0,\quad \Delta\tau_2^3=\tfrac{45}{4}\nu^6,$$

$$\Delta\tau_2^4=-\frac{42525}{8}\nu^8 z_1^2+\frac{14175}{4}\nu^6 z_1 z_3+\frac{382725}{64}\nu^{10}.$$

These polynomials can further be used to construct the real nonsingular rational solutions to the (elliptic) Boussinesq equation which follows from Eq. (5.11) under the transformation $t_2\to iy$. This equation describes stationary wave processes in a medium with positive dispersion. The rational solutions to this equation correspond to multilump wave structures propagating with a constant velocity.[14,21]

VI. THE GENERALIZED *k*-CONSTRAINED KP HIERARCHY

Here we consider the generalized k-constraint of the KP hierarchy defined by the condition (1.8). Loris and Willox[13] proved the following result by using the technique based on a bilinear formulation and identities (see also Ref. 11).

Lemma 6.1: The τ-function of the generalized k-constrained KP hierarchy defined by Eq. (1.8) admits the Wronskian solutions in the form (2.1) with the functions $\phi_n(t_1,t_2,\ldots)$, $n=\overline{1,N}$ satisfying Eqs. (2.2) and the complimentary equation,

$$\frac{\partial\phi_n}{\partial t_k}+\beta\partial_{t_1}^{-1}\phi_n=c_n\phi_n, \tag{6.1}$$

where c_n is arbitrary and $\partial_{t_1}^{-1}$ is a formal operator of integration with respect to t_1.

This lemma is a modification of Theorem 3 by Loris and Willox.[13] Although they formulate the theorem for a particular exponential representation of the functions ϕ_n which is suitable for soliton solutions, the proof is based on the pseudodifferential equation (6.1) rather than on the explicit representation of the functions ϕ_n. In order to proceed with the rational solutions of the generalized k-constraint we apply a simple trick which essentially simplifies the analysis. Let us introduce an auxillary ''time'' variable t_{-1} according to the pseudo-differential equation,

$$\frac{\partial\phi_n}{\partial t_{-1}}=\partial_{t_1}^{-1}\phi_n. \tag{6.2}$$

Then, we can construct the polynomial function $\tau_p(t_1,t_2,\ldots;t_{-1})$ of the generalized k-constrained KP hierarchy according to the formulas (2.1)–(2.10) taking into account the auxillary equation (6.2). The only modification of this analysis is an extension of the generalized variables $\theta_k(p)$ to the form,

$$\theta_k(p)\rightarrow\tilde{\theta}_k(p)=\theta_k(p)+\frac{(-1)^k}{p(1+k)}\,t_{-1}. \tag{6.3}$$

As a result, the generalized k-constraint (1.8) can be reformulated for the polynomial solutions of the KP hierarchy as follows.

Lemma 6.2: The polynomial τ-function of the generalized k-constrained KP hierarchy satisfies the differential equation,

$$\frac{\partial\tau_p}{\partial t_k}+\beta\,\frac{\partial\tau_p}{\partial t_{-1}}=0,\quad k\geqslant 1. \tag{6.4}$$

For the particular case $k=1$ the differential equation (6.4) follows from the analysis of the nonlocal Boussinesq equation (see formulas (19) and (20) in Ref. 12) and also from the analysis of the Davey–Stewartson system at the continuous-wave background as a generalized reduction of the KP hierarchy (see formulas (19) and (31) in Ref. 15). The result of Lemma 6.2 enables us to construct a set of the rational solutions of the generalized k-constrained KP hierarchy.

Proposition 6.3: The differential equation (6.4) is satisfied by the $(k-1)$ families of the polynomial τ-functions,

$$\tau_p=W_n[P_1(p),S^{-1}(\nu_1,\ldots,\nu_{k-1})S(\mu)P_3(p),\ldots,S^{-(n-1)}(\nu_1,\ldots,\nu_{k-1})S^{n-1}(\mu)P_{2n-1}(p)], \tag{6.5}$$

where $S(\mu)$ and $S(\nu_1,\ldots,\nu_{k-1})$ are defined by Eqs. (2.13) and (2.16), respectively. The parameter μ is $\mu=-p^{-1}$, while the parameters ν_i for $i=\overline{1,k-1}$ are given by

$$\nu_i=-\binom{k+1}{2}^{-1}\binom{k+1}{i+2}p^{-i}. \tag{6.6}$$

Last, the parameter p is one of the $(k+1)$ roots of the algebraic equation,

$$kp^{k+1}=\beta. \tag{6.7}$$

Proof: First, we express Eq. (6.4) through the derivatives of $\tau_p(\tilde{\theta}_1,\tilde{\theta}_2,\ldots)$ with respect to the variables $\tilde{\theta}_k$ given by Eqs. (2.6) and (6.3). This leads to the differential equation for τ_p,

$$\left(kp^{k-1}-\frac{\beta}{p^2}\right)\frac{\partial\tau_p}{\partial\theta_1}+\sum_{i=2}^{k}\left[\binom{k}{i}p^{k-i}+\frac{(-1)^i\beta}{p^{i+1}}\right]\frac{\partial\tau_p}{\partial\theta_i}=\beta\sum_{i=k+1}^{\infty}\frac{(-1)^{i+1}}{p^{i+1}}\frac{\partial\tau_p}{\partial\theta_i}, \tag{6.8}$$

where we have omited the sign ‘‘tilde’’ for convenience. Moreover, after the transformation of the differential equation (6.4) to the form (6.8) we can treat the extra variable t_{-1} as a phase constant since the equations of the KP hierarchy are only expressed through the time variables t_k for $k\geqslant 1$.

The first term in Eq. (6.8) is removed if the parameter p is a root of the algebraic equation (6.7). The rest of terms in Eq. (6.8) disappear due to a special structure of the polynomial function $\tau(\theta_1,\theta_2,\ldots)$ given by (6.5). To show this, we apply a standard analysis described in the proof of Proposition 5.2 and find from Eq. (6.5) the differential relation for $\tau_p(\theta_1,\theta_2\ldots)$,

$$\frac{\partial\tau_p}{\partial\theta_2}=-\sum_{i=3}^{k}\mu^{i-2}\frac{\partial\tau_p}{\partial\theta_i}\left(1-\sum_{j=1}^{i-2}\frac{\nu_j}{\mu^j}\right)-\sum_{i=k+1}^{\infty}\mu^{i-2}\frac{\partial\tau_p}{\partial\theta_i}\left(1-\sum_{j=1}^{k-1}\frac{\nu_j}{\mu_j}\right). \tag{6.9}$$

Substituting Eq. (6.9) into Eq. (6.8) and taking into account Eq. (6.7) we find that the parameter p can be eliminated if μ is given $\mu=-p^{-1}$, while the left-hand side and right-hand-side of Eq. (6.8) reduce the system of linear equations for the coefficients $\nu_1,\ldots,\ \nu_{k-1}$,

$$1-\sum_{m=1}^{i-2}\ \frac{\nu_m}{\mu^m}=\frac{2}{k+1}\left[1+\frac{(k-1)!(-1)^i}{i!(k-i)!}\right] \tag{6.10}$$

for $3\leqslant i\leqslant k$ and

$$1-\sum_{m=1}^{k-1}\ \frac{\nu_m}{\mu^m}=\frac{2}{k+1}. \tag{6.11}$$

This linear algebraic system is not degenerate and the only solution is given by Eq. (6.6). Indeed, a direct substitution of Eq. (6.6) into Eqs. (6.10) and (6.11) produces the system of combinatorial identities,

$$\sum_{m=0}^{i}\ (-1)^m\binom{k+1}{m+2}=k+(-1)^i\binom{k}{i+2} \tag{6.12}$$

for $1\leqslant i\leqslant k-2$ and

$$\sum_{m=0}^{k-1}\ (-1)^m\binom{k+1}{m+2}=k. \tag{6.13}$$

For $i=1$ the equality (6.12) is obviously satisfied. Furthermore, the equality (6.12) is satisfied for any $(i+1)$ provided it is valid for i. This statement is a consequence of the Pascal' triangle identity,

$$\binom{k+1}{i+3}=\binom{k}{i+2}+\binom{k}{i+3}. \tag{6.14}$$

Finally, Eq. (6.13) is satisfied due to the combinatorial identity,

$$\sum_{m=-2}^{k-1}\ (-1)^m\binom{k+1}{m+2}=0. \qquad \square \tag{6.15}$$

By comparing the results of Propositions 5.2 and 6.3, we conclude that the only difference in the rational solutions of the generalized l-reduced and k-constrained KP hierarchies is the appearance of the vertex operator $S(\mu)$ in the columns of the Wronskian determinant (6.5). This additional operator provides an annihilation of the infinite sum in the right-hand side of Eq. (6.8) due to the remarkable property (2.15).

Example 6.4: The NLS hierarchy at the CW background $(k=1)$.

The NLS hierarchy at the continuous-wave (CW) background starts with the NLS equation written in the form,

$$-\frac{\partial\psi}{\partial t_2}+\frac{\partial^2\psi}{\partial t_1^2}+2(\psi\bar{\psi}-\beta)\psi=0,\quad \frac{\partial\bar{\psi}}{\partial t_2}+\frac{\partial^2\bar{\psi}}{\partial t_1^2}+2(\psi\bar{\psi}-\beta)\bar{\psi}=0, \tag{6.16}$$

where the functions ψ and $\bar{\psi}$ are supposed to have the nonzero boundary conditions at infinity, i.e., $\psi\bar{\psi}\rightarrow\beta$ as $t_1\rightarrow\infty$. Two particular families of rational solutions of the NLS hierarchy at the CW background are expressed by means of the formula (2.12) through the function τ_p given by Eq. (6.5) for $k=1$, i.e.,

$$\tau_p(\theta_1,\theta_2,\ldots)=W_n[P_1(p),S(\mu)P_3(p),\ldots,S^{n-1}(\mu)P_{2n-1}(p)], \tag{6.17}$$

where $p=\pm\sqrt{\beta}$ and $\mu=-p^{-1}$.[15] The first polynomials (6.17) can be transformed by means of the phase translations exactly to the polynomials of the KdV hierarchy (see Example 4.5), i.e., $\tau_p=\tau_2^n(z_1,z_3,\ldots)$, where z_k are new variables, i.e.,

$$z_1=\theta_1,\quad z_3=\theta_3-\mu\theta_2+\tfrac{1}{4}\mu^2\theta_1,$$

$$z_5=\theta_5-2\mu\theta_4+\tfrac{3}{2}\mu^2\theta_3-\tfrac{1}{2}\mu^3\theta_2+\tfrac{1}{16}\mu^4\theta_1,$$

$$z_7=\theta_7-3\mu\theta_6+\tfrac{15}{4}\mu^2\theta_5-\tfrac{5}{2}\mu^3\theta_4+\tfrac{15}{16}\mu^4\theta_3-\tfrac{3}{16}\mu^5\theta_2+\tfrac{1}{64}\mu^6\theta_1.$$

These polynomials can further be used to construct the real nonsingular rational solutions to the (focusing) NLS equation which arises under the transformation $t_2\to it$ and the reduction $\bar{\psi}=\psi^*$, where ψ^* is complex conjugate to ψ. These rational solutions describe the modulational instability of a CW background under a localized perturbation of the algebraic profile.[33,34]

In addition, the same set of the polynomial τ-function also gives the rational solutions of another physically important equation which is the Kaup equation (or, equivalently, the classical Boussinesq system).[35] The polynomials $\tau_p(\theta_1,\theta_2,\ldots)$ given by Eq. (6.17) generate the rational solutions of the Kaup equation written in the form,

$$\frac{\partial^2 v}{\partial t_2^2}-4\beta\frac{\partial^2 v}{\partial t_1^2}-\frac{\partial^4 v}{\partial t_1^4}-2\frac{\partial v}{\partial t_1}\frac{\partial^2 v}{\partial t_1\partial t_2}+2\frac{\partial}{\partial t_1}\left[\left(\frac{\partial v}{\partial t_1}\right)^3-\frac{\partial v}{\partial t_1}\frac{\partial v}{\partial t_2}\right]=0, \tag{6.18}$$

within an equivalent bilinear representation (5.10).[36]

Example 6.5: The Yajima–Oikawa hierarchy $(k=2)$.

The Yajima–Oikawa hierarchy starts with the system,

$$\frac{\partial u}{\partial t_2}=\frac{\partial\psi\bar{\psi}}{\partial t_1},\quad \frac{\partial\psi}{\partial t_2}=\frac{\partial^2\psi}{\partial t_1^2}+2u\psi,\quad -\frac{\partial\bar{\psi}}{\partial t_2}=\frac{\partial^2\bar{\psi}}{\partial t_1^2}+2u\bar{\psi}, \tag{6.19}$$

where we impose the boundary conditions, $u\to 0$ and $\psi\bar{\psi}\to\beta$ as $t_1\to\infty$. Three particular families of the rational solutions of the Yajima–Oikawa hierarchy are expressed by means of the formula (2.12) through the function τ_p given by Eq. (6.5) for $k=2$, i.e.,

$$\tau_p(\theta_1,\theta_2,\ldots)=W_n[P_1(p),S^{-1}(\nu)S(\mu)P_3(p),\ldots,S^{-(n-1)}(\nu)S^{n-1}(\mu)P_{2n-1}(p)], \tag{6.20}$$

where p may have three values, $p=(\beta/2)^{1/3}$ and $p=(\beta/2)^{1/3}(-1\pm\sqrt{3}i)/2$, $S(\mu)$ and $S(\nu)$ are both given by Eq. (2.13) with the parameters $\mu=-p^{-1}$ and $\nu=-(3p)^{-1}$. The first polynomials (6.20) can be transformed by means of the phase translations to the polynomials of the KdV hierarchy (see Example 4.5) with the residue terms, i.e., $\tau_p=\tau_2^n(z_1,z_3,\ldots)+\Delta\tau_2^n(z_1,z_3,\ldots)$, where z_k are new variables, i.e.,

$$z_1=\theta_1,\quad z_3=\theta_3+(\nu-\mu)\theta_2+\tfrac{1}{4}(\nu-\mu)^2\theta_1,$$

$$\begin{aligned}z_5=\theta_5+2(\nu-\mu)\theta_4+\tfrac{3}{2}(\nu-\mu)^2\theta_3+\tfrac{1}{2}(3\nu^3-5\nu^2\mu+3\nu\mu^2-\mu^3)\theta_2\\+\tfrac{1}{16}(9\nu^4-20\nu^3\mu+14\nu^2\mu^2-4\nu\mu^3+\mu^4)\theta_1,\end{aligned}$$

$$\begin{aligned}z_7=\theta_7+3(\nu-\mu)\theta_6+\tfrac{15}{4}(\nu-\mu)^2\theta_5+\tfrac{1}{2}(9\nu^3-19\nu^2\mu+15\nu\mu^2-5\mu^3)\theta_4+\tfrac{1}{16}(63\nu^4-156\nu^3\mu\\+138\nu^2\mu^2-60\nu\mu^3+15\mu^4)\theta_3+\tfrac{1}{16}(63\nu^5-147\nu^4\mu+130\nu^3\mu^2-58\nu^2\mu^3+15\nu\mu^4-3\mu^5)\theta_2\\+\tfrac{1}{64}(105\nu^6-294\nu^5\mu+303\nu^4\mu^2+359\nu^3\mu^3+39\nu^2\mu^4-6\nu\mu^5+\mu^6)\theta_1,\end{aligned}$$

and $\Delta\tau_2^n$ are the residue terms, i.e.,

$$\Delta\tau_2^1=0,\quad \Delta\tau_2^2=0,\quad \Delta\tau_2^3=\tfrac{45}{4}\nu^4(\nu-\mu)^2,$$

$$\Delta\tau_2^4=\frac{798525}{64}\,\nu^3\mu^3 z_1^4-\frac{4725}{8}\,\nu^4(9\nu^4-24\nu^3\mu+22\nu^2\mu^2-8\nu\mu^3+\mu^4)z_1^2$$
$$+\frac{14175}{64}\,\nu^3(16\nu^3-32\nu^2\mu+16\nu\mu^2-169\mu^3)z_1z_3$$
$$+\frac{4725}{64}\,\nu^4(81\nu^6-270\nu^5\mu+351\nu^4\mu^2-228\nu^3\mu^3+79\nu^2\mu^4-14\nu\mu^5+\mu^6).$$

Within a physical context, the Yajima–Oikawa system (6.19) transformed according to the reduction $t_2\to it$ and $\bar{\psi}=i\psi^*$ describes a resonance of a mean flow and short dispersive waves. The rational solutions to this system have not yet been analyzed. However, by an analogy with the NLS hierarchy at the CW background (see Example 6.4), these solutions seem to be relevant for description of the modulational instability of a CW background under a localized perturbation of the algebraic profile.

VII. ANOMALOUS SCATTERING IN THE CM HIERARCHY

We have shown that all degenerate rational solutions of the KP hierarchy excepting those reducible to the generalized Schur polynomials $P_n(p)$ describe, within a particular limit, the stationary manifolds of the CM hierarchy (see Proposition 3.6). Hence, in a general case, these solutions describe the scattering dynamics of interacting particles in a neighborhood of the stationary manifolds. The stationary manifolds of the CM system (3.3), i.e., those with respect to the time t_2, are especially important because they are relevant for the rational solutions satisfying the generalized reductions of the KP hierarchy. Here we study the dynamical processes of scattering of the CM particles associated with this family of the rational solutions.

Proposition 7.1: Suppose that the τ-function of the KP hierarchy is given by

$$\tau_p(\theta_1,\theta_2,\ldots)=W_N[\tilde{P}_1(p),\tilde{P}_3(p),\ldots,\tilde{P}_{2N-1}(p)],\tag{7.1}$$

where the tilde means that the translations of the time variables $\theta_k=\theta_k(p)$ may be arbitrary. Define the index s for the scattering rate of R particles according to the asymptotic representation $x_j-2pt_2\sim t_2^s$ as $t_2\to\infty$ for $j=\overline{1,R}$, where $x_j(t_2,\ldots)$ are zeros of the τ-function given by Eq. (3.1). Then, the scattering dynamics described by the polynomials (7.1) occur near a stationary manifold of the CM system (3.3) of $R=N(N+1)/2$ particles, and the scattering rate s may only have the particular values,

$$s=s_{k,m}=\frac{k}{2k+m},\tag{7.2}$$

where for each integer $k=\overline{1,N-1}$ there exists a set of integers $m=\{2l-1\}_{l=1}^{N-k}$.

Proof: Noting that the τ-function given by Eq. (7.1) can be reduced, within a particular limit, to the form $\tau(t_1,t_3,\ldots)=W_N[p_1,p_3,\ldots,p_{2N-1}]$ which describes a stationary manifold of the CM system (3.3) of $R=N(N+1)/2$ particles according to Collorary (4.2) and Proposition 4.4. Therefore, a more general form (7.1) of the τ-function describes the scattering dynamics of R particles occurring near this stationary manifold. In order to find the scattering rate of the particle dynamics, we need to show that zeros of the polynomial function $\tau_p(\theta_1,\theta_2,\ldots)$ have the asymptotic representation $\theta_1\sim\theta_2^s$ as $\theta_1,\ \theta_2\to\infty$, where the index s is given by Eq. (7.2).

The polynomial solution (7.1) has generally $G=R=N(N+1)/2$ parameters. Among these parameters, $G_1=N$ parameters correspond to the phases $\theta_1,\ \theta_3,\ldots,\theta_{2N-1}$ which define the configuration of the stationary manifold of the CM system while $G_2=N(N-1)/2$ parameters can be regarded as amplitudes of elementary excitations of particles near the stationary manifold. We show that an individual elementary excitation can be described by the particular polynomial function τ_p following from Eq. (7.1),

$$\tau_p(\theta_1,\theta_2,\ldots)=\tau_N^{m,k}=W_N[P_1(p),\ldots,S_m^{-1}(a)P_{1+2k}(p),\ldots,S_m^{-2}(a)P_{1+4k}(p),\ldots],\tag{7.3}$$

where k and m are the same integers as below Eq. (7.2), while the operator $S_m(a)$ is given by a modification of Eq. (2.13) in the form,

$$S_m(a)=\exp\left[-\sum_{i=1}^{\infty}\frac{1}{i}a^i\frac{\partial}{\partial\theta_{mi}}\right]. \tag{7.4}$$

Using the standard analysis, one can readily show that the polynomial function $\tau_p(\theta_1,\theta_2,\ldots)$ given by Eq. (7.3) depends on the phases θ_{2k+m} and θ_{2k} only in the form of a linear superposition $(\theta_{2k+m}+a\theta_{2k})$. Next, because the time variables θ_{2k+m} define the stationary manifold of the CM system, zeros of the polynomial $\tau_N^{m,k}$ have the asymptotic representation $\theta_1\sim\theta_{2k+m}^{1/(2k+m)}$ as θ_1, $\theta_{2k+m}\to\infty$. Substituting the linear superposition in this asymptotic representation and recalling that $\theta_{2k}\sim\theta_2^k$ as θ_2, $\theta_{2k}\to\infty$, we conclude that the (m,k) elementary excitation of the stationary manifold of the CM system prescribed by Eq. (7.4) displays the fractional scattering rate s defined by Eq. (7.2). Finally, it is clear that there exists exactly $G_2=N(N-1)/2$ particular representations (7.4) for each polynomial function (7.1). Therefore, all possible asymptotical scattering rates are listed in the formula (7.2). □

Example 7.2: The stationary manifold of 6 particles $(N=3)$.

The following polynomial τ-function of the KP hierarchy follows from Eq. (7.1) at $N=3$ and describes the scattering dynamics of $R=6$ particles occurring near the corresponding stationary manifold of the CM system (3.3),

$$\tau_p=z_1^6-15z_1^3z_3-45z_3^2+45z_1z_5+\Delta, \tag{7.5}$$

where

$$z_1=\theta_1,\quad z_3=\theta_3+b\theta_2+\tfrac{1}{64}(16b^2+9a^2)\theta_1,$$

$$z_5=\theta_5+2b\theta_4+\frac{3}{32}(16b^2-a^2)\theta_3+a\theta_2^2+c\theta_2-\frac{3}{512}a(16b^2+9a^2)\theta_1^2-\frac{3}{8}\alpha\theta_1\theta_3$$
$$+\frac{5}{8}ab\theta_1\theta_2-\frac{1}{2048}(81a^4+1360a^2b^2+384b^4-1024bc)\theta_1,$$

and

$$\Delta=-\frac{225}{8}az_5-\frac{135}{512}a(19a^2+256b^2)z_3+45a^2\theta_2^2-\frac{45}{32}a(9a^2b+16b^3-32c)\theta_2$$
$$-\frac{45}{262144}(2781a^6+32704a^4b^2+129024a^2b^4-16384b^6$$
$$+36864a^2bc+65536b^3c-65536c^2).$$

Here the variables z_1, z_3, and z_5 define the $G_1=3$ parameters of the stationary manifold, while the constants a, b, and c correspond to the $G_2=3$ elementary excitations of the particles near the manifold. It follows from zeros of the polynomial function (7.5) that if $a\neq 0$ the scattering rate s is $s_{2,1}=2/5$ according to Eq. (7.2). If $a=0$ but $b\neq 0$, then the scattering rate is slower, i.e., $s_{1,1}=1/3$. At last, if $a=b=0$ but $c\neq 0$ then the rate is still slower, i.e., $s_{1,2}=1/5$. For a, b, and c all equal to zero, the particles are not excited and lie on the stationary manifold of the CM system.□

Thus, we conclude that the existence of the stationary manifolds of the CM system leads to a slowing down of the anomalous scattering of interacting particles according to a hierarchy of characteristic scattering rates (7.2). It is worthwhile to notice that the rates of anomalous scattering occurring far from the stationary manifolds of the CM systems are higher than those given by Eq. (7.2). Indeed, zeros of the degenerate polynomials of the KP hierarchy including the generalized Schur polynomials $P_n(p)$ have generally an asymptotic representation $\theta_1\sim\theta_2^{1/2}$ as θ_1, $\theta_2\to\infty$, i.e., the scattering rate is generally $s=1/2$. Only if the degenerate polynomials reduce to the form (7.1) the scattering rate becomes slower with respect to the time t_2, i.e., $s_{k,m}<1/2$. A first example of the anomalous scattering occurring near the stationary manifolds of the CM system was presented by Gorshkov *et al.*[21] and recently reproduced by Ablowitz and Villarroel for the lump solutions in the KP1 equation.[24] Although we do not consider here the rational solutions of the KP1 equation, the results presented can further be generalized to describe the same scattering dynamics of the lump solitons as the particles in the CM system (see also Ref. 21).

VIII. CONCLUSION

In this paper we have developed a general approach to construct and analyze the degenerate rational solutions of equations of the KP hierarchy and its reductions. In particular, we have presented the rational solutions to the generalized l-reduction and k-constraint of the KP hierarchy which correspond to the reductions of the KP hierarchy under the nonzero boundary conditions at infinity. We have also shown that these solutions describe anomalously slow dynamical processes of scattering occurring near the stationary manifolds of the CM dynamical system.

Since the paper does not cover all possible generalized reductions of the KP hierarchy (see, e.g., Refs. 5, 6 for other reductions) it is worthwhile to summarize our approach and present a general scheme of a search for the polynomial τ-function satisfying a generalized reduction of the KP hierarchy. Suppose that the reduction is formulated in terms of a differential equation imposed on the τ-function of the KP hierarchy. Then, in order to find the particular form of the polynomial function $\tau_p(\theta_1,\theta_2,\ldots)$ (2.10) satisfying this reduction, one needs

- (i) to rewrite the differential equation in terms of the generalized time variables $\theta_k(p)$;
- (ii) to choose p such that the leading-order term $\partial\tau_p/\partial\theta_1$ is removed;
- (iii) to find the first nonzero coefficient in front of the derivative $\partial\tau_p/\partial\theta_m$, then the polynomials τ_p generalize the polynomials of the m-reduced KP hierarchy given by (4.4);
- (iv) to introduce the generalized vertex operators (2.13) and (2.16) which displace phases of the polynomials $P_n(p)$ in each subsequent column of the Wronskian determinant (2.10);
- (v) to find the parameters of the vertex operators by comparing the differential equation derived for the polynomial function (2.10) with that of the given reduction.

This scheme and the approaches used in both papers I and II can be developed for analysis of the rational solutions of multicomponent KP hierarchies which include such an important example as the Davey–Stewartson system (see, e.g., Ref. 15). The τ-function of the multicomponent KP hierarchies can be expressed through a multicomponent Wronskian[26] which can be treated in the same manner as that in a single-component case. The classification and construction of rational solutions of the multicomponent KP hierarchies remain open for further studies.

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