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On modified NLS, Kaup and NLBq equations: differential transformations and bilinearization

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Abstract. New transformations between the nonlinear Schrödinger, Kaup and non-local Boussinesq equations as well as their modified counterparts are found and analysed. The bilinear representations of these equations, including an alternative bilinear form of the Chen–Lee–Liu equation, are obtained by a direct method based on the Bell's exponential polynomials. Explicit Wronskian solutions to these equations are also presented.

1. Introduction

Direct methods based on bilinearization and differential transformations of nonlinear partial differential equations (NLPDEs) are commonly used in soliton theory [1, 2] to disclose 'hidden' algebraic structures of equations solvable by the inverse scattering transform technique. The Hirota bilinear representation is known to produce exact (quasiperiodic, soliton and rational) solutions, Bäcklund transformations (BTs) and associated linear systems for integrable NLPDEs, and to reveal links with other integrable equations and their commuting flows [3].

A unifying approach to direct bilinearization has recently been proposed on the basis of a generalization of Bell's exponential polynomials (the so-called $\mathcal{Y}$ and P polynomials) and related combinatorial identities [3]. This technique has been applied to the study of a non-local alternative to the Boussinesq (NLBq) equation [4] and the related hierarchy [5, 6], which can be interpreted as a constrained KP hierarchy [7].

The NLBq equation

$$\mathrm{NLBq}(u) = u_{tt} + u_{xxxx} + \left(2u_x^2 - \frac{u_t^2 + u_{xx}^2}{u_x}\right)_x = 0 \qquad (1.1)$$

arises as a compatibility condition for a linear system associated with the bilinear representation (see [4]) of Kaup's higher-order wave equation [8]:

$$\mathrm{Kaup}(v) = v_{tt} + v_{xxxx} + 2v_x v_{xt} + 2(v_x v_t + v_x^3)_x = 0. \qquad (1.2)$$

From a different point of view, the NLBq equation (1.1) was obtained by Boiti, Laddomada and Pempinelli [9] as a governing equation for the potential of the square modulus of a nonlinear Schrödinger (NLS) field. Since it was noticed by Hirota [10] that the Kaup equation (1.2) can be transformed into the NLS equation, it is not a surprise that the above equations are closely connected with NLS. The NLBq equation turned out [4] to be related

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to the Kaup equation by means of a differential transformation of Miura type (in the same way as the KdV equation is related to the modified KdV equation [1]). In view of this result it seemed natural to interpret the Kaup equation as a modified version of the NLBq equation.

More recently it was found that the transformation which maps N-soliton solutions of equation (1.1) onto $N+1$-soliton solutions does not produce the Kaup equation [6]. This suggests that the Kaup equation should not be regarded as a proper modified version of the NLBq equation and that one needs to go further in the analysis of the above equations.

Starting with a study of the coupled NLS system (in which ψ and ψ' are generally not complex conjugate)

$$\mathrm{NLS}(\psi,\psi') = \begin{cases} \mathrm{i}\psi_t + \psi_{xx} + 2\psi^2\psi' = 0 \\ -\mathrm{i}\psi'_t + \psi'_{xx} + 2\psi\psi'^2 = 0 \end{cases} \tag{1.3}$$

it is the purpose of this paper to produce a complete analysis of the relations between equations (1.1)–(1.3) and their modified counterparts. Within this analysis we derive a new equation

$$\mathrm{mNLBq}(w) = w_{tt} + w_{xxxx} + 2w_x w_{xt} + \left(w_x^3 - \frac{w_t^2 + w_{xx}^2}{w_x}\right)_x = 0 \tag{1.4}$$

which may be regarded as a modified version of both the NLBq and the Kaup equation.

A crucial element in the analysis is the Chen–Lee–Liu (CLL) system [11] (also called the derivative NLS system):

$$\mathrm{CLL}(\phi,\phi') = \begin{cases} \mathrm{i}\phi_t + \phi_{xx} + 2\mathrm{i}\phi\phi'\phi_x = 0 \\ -\mathrm{i}\phi'_t + \phi'_{xx} - 2\mathrm{i}\phi\phi'\phi'_x = 0 \end{cases} \tag{1.5}$$

which is known as a modified version of the NLS system (1.3) [12]. The CLL system is also related by means of a gauge transformation to a generalized derivative NLS equation (see [13–15]) which has found applications in the description of Alfvén waves in an electromagnetic plasma (see [16] and references therein). The various transformations of the NLS and CLL systems as well as a complete classification of integrable systems of this type were considered by Mikhailov *et al* [17].

A four-field bilinearization of the CLL system can be given in close correspondence with that found by Nakamura and Chen [18] for the CLL equation ($\phi' = \phi^*$), which leads to the construction of multisoliton solutions for the latter. Recently, a six-field bilinear system was also obtained by Kakei *et al* for a generalized derivative NLS equation [19]. Here, we derive an alternative two-field bilinearization of the CLL system (1.5) through bilinearization of an equivalent system, the compatibility condition of which produces the mNLBq equation (1.4). This new bilinear representation enables one to obtain explicit Wronskian-type solutions to the CLL system and to the mNLBq equation (see [20, 21]).

The paper is organized as follows. In section 2 we review the differential links between equations (1.1)–(1.3) and show that the NLBq and Kaup equations are related by reversible differential transformations of Miura type. A modified version of these equations is derived and analysed in section 3 where the proper Miura transformations between solutions to (1.1), (1.2) and (1.4) are found. Bilinear representations of the CLL system and of the mNLBq equation are reported in section 4. Finally, the explicit Wronskian solutions to these bilinear equations are presented in section 5.

2. Links between NLS, Kaup and NLBq equations

In order to clarify the main ideas of our approach, we start our discussion by analysing the differential relationships between three integrable equations of physical interest: the NLS equation, the Kaup equation and the non-local Boussinesq equation. Introducing the (complex-valued) variables ρ and θ into the NLS system (1.3) by the substitution

$$\psi = \sqrt{\rho}\,\mathrm{e}^{\mathrm{i}\theta} \qquad \psi' = \sqrt{\rho}\,\mathrm{e}^{-\mathrm{i}\theta} \tag{2.1}$$

one finds that ρ and θ are subject to the following coupled equations:

$$\rho_t + 2\,(\rho\theta_x)_x = 0 \tag{2.2}$$

$$\theta_t + \theta_x^2 - 2\rho - \frac{\rho_{xx}}{2\rho} + \frac{\rho_x^2}{4\rho^2} = 0. \tag{2.3}$$

It is easy to establish the link between these equations and the above equations (1.1) and (1.2). It suffices to introduce new variables u and v through the formulae

$$\rho = u_x \qquad \theta = v \pm \frac{\mathrm{i}}{2}\ln u_x \tag{2.4}$$

which transform the system (2.2) and (2.3) into the fractionless system:

$$\widetilde{\mathrm{NLS}}_\pm(u, v) = \begin{cases} u_t + 2u_x v_x \pm \mathrm{i}u_{xx} = 0 & (2.5)\\ v_t + v_x^2 \mp \mathrm{i}v_{xx} - 2u_x = 0. & (2.6)\end{cases}$$

Note that equation (2.5) was obtained after integration with respect to x on account of the fact that the variable u is only determined by ρ up to an arbitrary function of t.

By differentiating the system (2.5) and (2.6) with respect to x and by setting $v_x = -\frac{1}{2}V$ and $u_x = -\frac{1}{4}(U \mp \mathrm{i}V_x + 1)$, one recovers the classical Boussinesq system for the fields V and U as reported in [10] (see equations (1.6a, b) in [10] for $a^2 = -1$).

Eliminating u from the equations (2.5) and (2.6) one finds the Kaup equation (1.2) as a compatibility condition for the system

$$u_x = \tfrac{1}{2}(v_t + v_x^2 \mp \mathrm{i}v_{xx}) \tag{2.7}$$

$$u_t = -\tfrac{1}{2}(v_{xxx} \pm \mathrm{i}v_{xt} + 2v_x v_t + 2v_x^3). \tag{2.8}$$

Similarly, the NLBq equation (1.1) can be obtained from the $\widetilde{\mathrm{NLS}}_\pm(u, v)$ system, by elimination of v, as a compatibility condition for the system

$$v_x = -\frac{1}{2u_x}\,(u_t \pm \mathrm{i}u_{xx}) \tag{2.9}$$

$$v_t = \frac{1}{2u_x}\left(\mp\mathrm{i}u_{xt} + u_{xxx} + 4u_x^2 - \frac{u_t^2 + u_{xx}^2}{2u_x}\right). \tag{2.10}$$

It is clear from formulae (2.1) and (2.4) that the NLBq equation (1.1) governs the evolution of the potential of $u_x = \psi\psi'$, whereas the Kaup equation (1.2) governs the evolution of $v = -\mathrm{i}\ln\psi$ or, equivalently, of $v = \mathrm{i}\ln\psi'$ (see also [9, 10]).

Using the above results, it is easy to verify the map

$$\mathrm{NLBq}(u) = -\frac{1}{2}\left(\pm\mathrm{i}\frac{\partial}{\partial x} + 2v_x\right)\mathrm{Kaup}(v) \tag{2.11}$$

subject to relations (2.7) and (2.8) between u and v, which may be interpreted as a differential Miura transformation between the NLBq and the Kaup equation. Yet, one also obtains a reversed 'Miura'-type transformation

$$\mathrm{Kaup}(v) = -\frac{1}{2}\left(\pm\frac{\mathrm{i}}{u_x}\frac{\partial}{\partial x} + \frac{u_t}{u_x^2}\right)\mathrm{NLBq}(u) \tag{2.12}$$

subject to relations (2.9) and (2.10) between v and u.

Hence, the NLBq equation (1.1) and the Kaup equation (1.2) are found to be linked by a reversible transformation. However, their bilinear representations are quite different [4, 5]. The bilinear representation of the NLS system (1.3) is similar to the customary one for the NLS equation [22]:

$$(\mathrm{i}D_t + D_x^2) f \cdot g = 0 \tag{2.13}$$

$$(-\mathrm{i}D_t + D_x^2) f' \cdot g = 0 \tag{2.14}$$

$$D_x^2 g \cdot g = 2ff' \tag{2.15}$$

with the bilinearizing transformation

$$\psi = \frac{f}{g} \qquad \psi' = \frac{f'}{g}. \tag{2.16}$$

The first two bilinear equations are equivalent with the first equation of the modified KP hierarchy, whereas the last one is referred to as a 1-constraint on the KP hierarchy [7, 23].

The bilinearization of the Kaup equation can be found by using the simple partitional structure of the generalized Bell polynomials. Thus, the $\widetilde{\mathrm{NLS}}_\pm(u, v)$ systems (2.5) and (2.6) can be written in the form of homogeneous linear combinations of these polynomials (see appendix). It follows from this analysis that the transformation

$$\widetilde{\mathrm{NLS}}_+(u, v): \qquad v = -\mathrm{i} \ln \frac{F}{G} \qquad u = \frac{\partial}{\partial x} \ln G \tag{2.17}$$

$$\widetilde{\mathrm{NLS}}_-(u, v): \qquad v = -\mathrm{i} \ln \frac{F}{G} \qquad u = \frac{\partial}{\partial x} \ln F \tag{2.18}$$

reduces (2.5) and (2.6) into the system

$$(\mathrm{i}D_t + D_x^2) F \cdot G = 0 \tag{2.19}$$

$$(\mathrm{i}D_x D_t + D_x^3) F \cdot G = 0 \tag{2.20}$$

which is known [10, 24] to provide a bilinear representation of the Kaup equation (1.2). Moreover, this system can also be regarded as an alternative bilinear representation of the NLS system indicating that the NLS equation is a reduction of bilinear equations of the modified KP hierarchy written in terms of only two fields F and G [10]. More specifically, each solution (F, G) of the bilinear system (2.19) and (2.20) generates a pair of solutions $(\psi_\pm, \psi'_\pm)$ to the NLS system according to the relations (2.1) and (2.4):

$$\psi_+ = \frac{F}{G} \qquad \psi'_+ = \frac{G_{xx}G - G_x^2}{FG} \tag{2.21}$$

$$\psi_- = \frac{F_{xx}F - F_x^2}{FG} \qquad \psi'_- = \frac{G}{F}. \tag{2.22}$$

These two different solutions of the NLS system (1.3) are actually connected by the canonical symmetry transformation presented by Leznov and Razumov [12]:

$$\psi_- = \psi_{+,xx} - \frac{(\psi_{+,x})^2}{\psi_+} + \psi_+^2 \psi'_+ \qquad \psi'_- = \frac{1}{\psi_+}. \tag{2.23}$$

The role of the alternative bilinear representation (2.19) and (2.20), with its particular structure, is crucial in establishing the connection of the NLS equation, as a reduction of the modified KP hierarchy, with both the Kaup equation (or, equivalently, the classical Boussinesq system) and the NLBq equation. We note from (2.17) and (2.18) that this representation provides a bilinear transformation between two different solutions of the NLBq equation (1.1). Indeed, the system (2.19) and (2.20) is symmetric with respect to the interchange of F and G and the replacement of i by $-$i. Thus, this system can be transformed, by the standard substitution $F = \psi G,\ G_x = uG$, into a linear system,

$$\mathrm{i}\psi_t + \psi_{xx} + 2u_x\psi = 0 \tag{2.24}$$

$$2u_x\psi_x + (\mathrm{i}u_t - u_{xx})\psi = 0 \tag{2.25}$$

the compatibility of which (elimination of ψ) is subject to the condition $\mathrm{NLBq}(u) = 0$. On the other hand, the substitution $G = \psi' F,\ F_x = u'F$ leads to another system,

$$-\mathrm{i}\psi'_t + \psi'_{xx} + 2u'_x\psi' = 0 \tag{2.26}$$

$$2u'_x\psi'_x - (\mathrm{i}u'_t + u'_{xx})\psi' = 0 \tag{2.27}$$

the compatibility of which produces the condition $\mathrm{NLBq}(u') = 0$. We conclude that both u and u' satisfy the NLBq equation (1.1).

As one compares the two bilinear representations (2.13)–(2.15) and (2.19) and (2.20) of the NLS system it is worth noticing the functional similarity between the bilinearizing transformation (2.21) and the transformation which is obtained from the formula (2.16) by eliminating f' through equation (2.15):

$$\psi = \frac{f}{g} \qquad \psi' = \frac{f'}{g} \equiv \frac{f'f}{fg} = \frac{1}{2}\frac{D_x^2 g\cdot g}{fg} \equiv \frac{g_{xx}g - g_x^2}{fg}. \tag{2.28}$$

It suggests a direct connection between the two bilinear representations. Indeed, one may verify, by direct calculation, that the system obtained by elimination of f' from the system (2.13)–(2.15):

$$(\mathrm{i}D_t + D_x^2) f\cdot g = 0 \tag{2.29}$$

$$(-\mathrm{i}D_t + D_x^2)\left(\frac{D_x^2 g\cdot g}{f}\right)\cdot g = 0 \tag{2.30}$$

is identically satisfied if f and g satisfy the system (2.19) and (2.20).

The connection between the two bilinear representations becomes more striking if one specializes to the 'physical' NLS equation (NLS system with $\psi' = \psi^*$). Consider a solution $\{\psi = F/G, \psi' = (FG)^{-1}(G_{xx}G - G_x^2)\}$ of the NLS system, generated by the alternative representation (2.19) and (2.20), with G real (non-vanishing). It produces a solution to the NLS equation if $\psi' = \psi^*$, i.e. if F and G are linked by the customary bilinear condition (cf equation (2.15)):

$$D_x^2 G\cdot G = 2|F|^2. \tag{2.31}$$

This condition happens to be satisfied, up to a real constant, as a consequence of the equations (2.19) and (2.20). This can be seen by considering their linear equivalent (2.24) and (2.25) in terms of $\psi = F/G$ and $u = \partial_x \ln G$. It follows from equation (2.25) and from the reality of u that ψ and u are linked by the relation

$$\left(\frac{u_x}{|\psi|^2}\right)_x \equiv |\psi|^{-2}u_{xx} - |\psi|^{-4}u_x(|\psi|^2)_x = 0 \tag{2.32}$$

indicating that

$$u_x = |\psi|^2 \tag{2.33}$$

where an integration constant is specified to be equal to unity according to equations (2.1) and (2.4). Hence, it is clear from equation (2.24) and from the formula (2.33) that $\psi = F/G$ must solve the 'physical' NLS equation:

$$\mathrm{i}\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0. \tag{2.34}$$

In contrast to the direct bilinearizability of the system (2.7) and (2.8)—and of the Kaup equation [4]—the bilinearization of the NLBq equation (1.1) is not straightforward as it involves only one Hirota field. A $(1+2)$-dimensional single-field bilinear representation was obtained by means of higher-order flows of the NLBq hierarchy [5]:

$$(-4D_x D_\tau + D_x^4 - 3D_t^2) G \cdot G = 0 \tag{2.35}$$

$$(-4D_x^3 D_\tau + D_x^6 - 3D_x^2 D_t^2) G \cdot G = 0 \tag{2.36}$$

where the time variable τ corresponds to a weight 3 flow of the KP hierarchy.

This bilinear system was used to compute multisoliton solutions to the NLBq equation, from which it became clear that the transformation mapping a N-soliton solution onto a $N+1$-soliton solution does not correspond to the bilinear representation (2.19) and (2.20) of the Kaup equation [6]. Thus, the Kaup equation cannot be considered as a genuine modified version of the NLBq equation. In order to find the modified NLBq equation we take advantage of the link between the NLS system (1.3) and the CLL system (1.5) and apply the above analysis to the latter.

3. Modified versions of NLS and NLBq equations

The CLL system (1.5) has been shown to generate solutions to the NLS system (1.3) through gauge transformations of the following two types [13–15]:

$$\psi_+ = \mathrm{i}\phi_x \exp\left[\mathrm{i}\int \phi\phi' \,\mathrm{d}x\right] \qquad \psi'_+ = \phi' \exp\left[-\mathrm{i}\int \phi\phi' \,\mathrm{d}x\right] \tag{3.1}$$

$$\psi_- = \phi \exp\left[\mathrm{i}\int \phi\phi' \,\mathrm{d}x\right] \qquad \psi'_- = -\mathrm{i}\phi'_x \exp\left[-\mathrm{i}\int \phi\phi' \,\mathrm{d}x\right]. \tag{3.2}$$

A substitution similar to formulae (2.1) and (2.4)

$$\phi = \sqrt{n_x}\,\mathrm{e}^{\mathrm{i}\Theta} \qquad \phi' = \sqrt{n_x}\,\mathrm{e}^{-\mathrm{i}\Theta} \qquad \Theta = w \pm \frac{\mathrm{i}}{2}\log n_x \tag{3.3}$$

transforms the CLL system into either of the systems

$$\widetilde{\mathrm{CLL}}_\pm(w,n) = \begin{cases} n_t + n_x^2 + 2n_x w_x \pm \mathrm{i} n_{xx} = 0 & (3.4) \\ w_t + w_x^2 + 2w_x n_x \mp \mathrm{i} w_{xx} = 0 & (3.5) \end{cases}$$

where equation (3.4) was obtained after integration with respect to x. On account of the above transformation (2.1) and (2.4) and the formulae (3.1)–(3.3) one can find several different relationships between the variables u, v satisfying (1.1) and (1.2) and the variables w, n. It is sufficient here to reproduce only the two simplest relationships,

$$u_x = -w_x n_x \qquad v = n + w. \tag{3.6}$$

Using these relations, one can eliminate n from the systems (3.4) and (3.5) so as to close the system for u, w or v, w. In the first case, one obtains the system:

$$u_x = \frac{1}{2}(w_t + w_x^2 \mp \mathrm{i} w_{xx}) \tag{3.7}$$

$$u_t = -\frac{1}{2}\left(\pm \mathrm{i} w_{xt} + w_{xxx} + w_x^3 - \frac{w_t^2 + w_{xx}^2}{w_x}\right) \tag{3.8}$$

the compatibility of which is the modified NLBq equation (1.4). It is easy to verify the following map:

$$\mathrm{NLBq}(u) = \frac{1}{2}\left(\mp \mathrm{i}\frac{\partial}{\partial x} + \frac{2w_t}{w_x}\right)\mathrm{mNLBq}(w) \tag{3.9}$$

subject to relations (3.7) and (3.8) between u and w. This map is not reversible and produces a proper Miura transformation converting solutions of the mNLBq equation (1.4) into solutions of the NLBq equation (1.1).

Furthermore, the other relation (3.6) leads to a connection of (1.4) with the Kaup equation (1.2). Indeed, one can close the systems (3.4) and (3.5) for v, w as follows:

$$v_x = -\frac{1}{2w_x}(w_t - w_x^2 \mp \mathrm{i}w_{xx}) \tag{3.10}$$

$$v_t = \frac{1}{2w_x}\left(\pm\mathrm{i}w_{xt} + w_{xxx} + 3w_t w_x + \frac{3}{2}w_x^3 - \frac{w_t^2 + w_{xx}^2}{2w_x}\right) \tag{3.11}$$

the compatibility of which is again the modified NLBq equation (1.4). Finally, one finds that equation (1.4) is mapped onto (1.2) as follows:

$$\mathrm{Kaup}(v) = \frac{1}{2}\left(\pm\frac{\mathrm{i}}{w_x}\frac{\partial}{\partial x} - \frac{w_t}{w_x^2} + 3\right)\mathrm{mNLBq}(w) \tag{3.12}$$

subject to relations (3.10) and (3.11).

4. Bilinearization of the modified equations

The CLL system (1.5) can be bilinearized through the use of the following transformation

$$\phi = \frac{f}{g} \qquad \phi' = \frac{f'}{g'} \tag{4.1}$$

resulting in the bilinear system

$$(\mathrm{i}D_t + D_x^2)f \cdot g = 0 \tag{4.2}$$

$$(-\mathrm{i}D_t + D_x^2)f' \cdot g' = 0 \tag{4.3}$$

$$D_x^2 g \cdot g' = \mathrm{i}D_x f \cdot f' \tag{4.4}$$

$$D_x g \cdot g' = \mathrm{i}ff' \tag{4.5}$$

which is closely related to that found by Nakamura and Chen [18] (for the CLL equation $\phi' = \phi^*$). A recent generalization in terms of six fields was given by Kakei *et al* [19]. It is clear again that the bilinear equations (4.2) and (4.3) are the same as the first equation of the modified KP hierarchy, while the other two equations (4.4) and (4.5) represent a certain constraint imposed on the modified KP hierarchy [25].

It is easy to bilinearize the systems (3.7) and (3.8) by applying the direct bilinearization procedure (see the appendix). One finds that both systems can be transformed into

$$(\mathrm{i}D_t + D_x^2)F \cdot G = 0 \tag{4.6}$$

$$(\mathrm{i}D_x^2 D_t + D_x^4)F \cdot G = 0 \tag{4.7}$$

by means of the following transformations:

$$\widetilde{\mathrm{CLL}}_+: \qquad w = -\mathrm{i}\ln\frac{F}{G} \qquad u = \frac{\partial}{\partial x}\ln G \tag{4.8}$$

$$\widetilde{\mathrm{CLL}}_-: \qquad w = -\mathrm{i}\ln\frac{F}{G} \qquad u = \frac{\partial}{\partial x}\ln F. \tag{4.9}$$

This shows that the system (4.6) and (4.7) can be regarded as an alternative bilinear representation of the CLL system which produces solutions by pairs. To each solution

(F, G) of the system (4.6) and (4.7) there corresponds a pair of solutions $(\phi_\pm, \phi'_\pm)$ to system (1.5):

$$\phi_+ = \frac{F}{G} \qquad \phi'_+ = \mathrm{i}\left(\frac{GG_{xx} - G_x^2}{FG_x - GF_x}\right) \tag{4.10}$$

$$\phi_- = \mathrm{i}\left(\frac{FF_{xx} - F_x^2}{FG_x - GF_x}\right) \qquad \phi'_- = \frac{G}{F}. \tag{4.11}$$

These solutions are related by a canonical symmetry transformation according to [12]

$$\phi_- = -\mathrm{i}\left(\frac{\phi_+\phi_{+,xx}}{\phi_{+,x}} - \phi_{+,x}\right) + \phi_+^2\phi'_+ \qquad \phi'_- = \frac{1}{\phi_+}. \tag{4.12}$$

As a further analogy with the NLS case, we note that the system (4.6) and (4.7) displays the same symmetry property as the system (2.19) and (2.20) with respect to the interchange of F and G, and that it linearizes under the same transformation $F = \phi G, \quad G_x = uG$, being equivalent with the system

$$\mathrm{i}\phi_t + \phi_{xx} + 2u_x\phi = 0 \tag{4.13}$$

$$2u_x\phi_{xx} + (\mathrm{i}u_t - u_{xx})\phi_x + 2u_x^2\phi = 0 \tag{4.14}$$

the compatibility of which (elimination of ϕ) is subject to the condition $\mathrm{NLBq}(u) = 0$. On the other hand, the transformation $G = \phi' F$, $F_x = u' F$ produces the system

$$-\mathrm{i}\phi'_t + \phi'_{xx} + 2u'_x\phi' = 0 \tag{4.15}$$

$$2u'_x\phi'_{xx} + (-\mathrm{i}u'_t - u'_{xx})\phi'_x + 2u'^2_x\phi' = 0 \tag{4.16}$$

the compatibility of which leads to the condition $\mathrm{NLBq}(u') = 0$. Thus, both u and u' are again two different solutions of the NLBq equation (1.1).

Similarly to the NLS case, there is a connection between the above two bilinear representations of the CLL system. Indeed, inspection of formula (4.10) shows that the alternative bilinear representation (4.6) and (4.7) produces solutions of the 'physical' CLL equation (CLL system with $\phi' = \phi^*$) if F and G are linked by the additional relation

$$F^* D_x G \cdot F = \tfrac{1}{2}\mathrm{i}G^* D_x^2 G \cdot G \tag{4.17}$$

which is easily seen to be satisfied when F and G satisfy the customary bilinear conditions

$$D_x^2 G \cdot G^* = \mathrm{i}D_x F \cdot F^* \tag{4.18}$$

$$D_x G \cdot G^* = \mathrm{i}|F|^2. \tag{4.19}$$

Hence, the customary bilinear representation (4.2)–(4.5) seems to be the adequate tool to generate solutions to the 'physical' CLL equation. On the other hand, the alternative representation (4.6) and (4.7) is suited to produce solutions to the physical NLS equation. Indeed, let

$$\phi = \frac{F}{G} \qquad \phi' = \mathrm{i}\left(\frac{GG_{xx} - G_x^2}{FG_x - GF_x}\right) \tag{4.20}$$

be a solution of the CLL system generated by the bilinear representation (4.6) and (4.7), with G real (non-vanishing). The corresponding solution of the NLS system is produced according to the gauge transformation (3.1). It then follows from the relation

$$u_x \equiv \partial_x^2 \ln G = \mathrm{i}\phi_x\phi' = \psi\psi' \tag{4.21}$$

and from the reality of u_x that the constraint $\psi' = \psi^*$ for the corresponding solution of the NLS system is subject to the condition

$$\partial_x \ln\left(\frac{u_x}{|\phi_x|^2}\right) = \frac{u_x}{|\phi_x|^2}(|\phi|^2)_x. \tag{4.22}$$

One can show that condition (4.22) is satisfied as a consequence of equation (4.14) (and its complex conjugate). Thus, real solutions of the NLBq equation given by $u = \partial_x \ln G$, with G real, produce solutions of the 'physical' NLS equation (2.34).

It is also worth mentioning that the system (4.6) and (4.7) can be obtained by direct bilinearization of the mNLBq equation (1.4) by using the method of the appendix. We therefore conclude that the bilinearization (4.6) and (4.7) of the mNLBq equation provides an alternative bilinearization of the CLL system as well as a bilinear transformation between two different solutions of the NLBq equation (1.1). In fact, it turns out that the bilinear equations (4.6) and (4.7) are identical with those derived by Loris and Willox from the knowledge of the N-soliton solutions to the NLBq equation [6].

5. Wronskian solutions

Here we use an alternative form of the bilinear equations to find the explicit Wronskian solutions of the NLS and CLL systems as reductions of the corresponding solutions of the modified KP equation. These Wronskian solutions are particularly convenient for representation of rational solutions to these integrable equations (see, e.g., [20]).

It is well known [21] that the first bilinear equation of the modified KP hierarchy (i.e. (2.19) or (4.6)) admits two sets of Wronskian solutions given by

$$\begin{aligned} F &= W[\varphi_1, \varphi_2, \ldots, \varphi_N, \varphi_{N+1}] \\ G &= W[\varphi_1, \varphi_2, \ldots, \varphi_N] \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} F &= W[\varphi_{1,x}, \varphi_{2,x}, \ldots, \varphi_{N,x}] \\ G &= W[\varphi_1, \varphi_2, \ldots, \varphi_N] \end{aligned} \tag{5.2}$$

where the Wronskians are

$$W[\varphi_1, \varphi_2, \ldots, \varphi_N] = \det\left|\frac{\partial^{i-1}\varphi_j}{\partial x^{i-1}}\right|_{1\leqslant i,j\leqslant N}$$

and the set of functions φ_n are arbitrary solutions of linear equations

$$\mathrm{i}\varphi_{n,t} + \varphi_{n,xx} = 0. \tag{5.3}$$

Thus, general Wronskian solutions to the first bilinear equations (2.19) and (4.6) of the alternative representation of the NLS and CLL systems are known. The remaining problem is to find a particular choice of the functions φ_n which ensures that the Wronskian solution satisfies the other bilinear equation, i.e. the constraint of the modified KP hierarchy. For the NLS system, this bilinear constraint is given by equation (2.20). According to Loris and Willox [6] we rewrite this bilinear equation in an equivalent form (on account of equation (2.19))

$$(\mathrm{i}D_xD_t + D_x^3)F\cdot G = 2(\mathrm{i}D_t + D_x^2)F_x\cdot G = 0. \tag{5.4}$$

Then it follows from (5.4) that the only form of the Wronskian solutions (5.1) and (5.2) which satisfies both (2.19) and (5.4) is

$$\begin{aligned} F &= W[\varphi, \varphi_x, \ldots, \varphi_{(N-1)x}, \varphi_{Nx}] \\ G &= W[\varphi, \varphi_x, \ldots, \varphi_{(N-1)x}] \end{aligned} \tag{5.5}$$

where φ satisfies equation (5.3) and where the notation φ_{px} stands for the pth x-derivative of φ. This pair of Wronskians generates explicit solutions to the NLBq and Kaup equations by means of (2.17) and (2.18) as well as to the NLS system by means of (2.21) and (2.22). The Wronskian solutions of the form (5.5) were first found by Hirota [20] with the help of a different and lengthy analysis, and recently recovered and generalized by Loris and Willox [6]. In particular, the rational solutions follow from the Wronskian representation (5.5) for a simple choice of an explicit solution to equation (5.3),

$$\varphi = \left(\frac{\partial^n}{\partial p^n} \mathrm{e}^{px+\mathrm{i}p^2 t} \right) \Bigg|_{p=0} \tag{5.6}$$

where the nth derivative with respect to the parameter p produces a polynomial for the function φ with respect to x of degree n. Along with the order of the Wronskian determinant N, the index n defines the degree of the polynomial solutions for the bilinear fields F and G. It is clear that the Wronskian representation provides a wide set of rational solutions to the NLBq, Kaup equations as well as to the NLS system (see also [21]).

Similarly, we analyse the bilinear constraint (4.7) for the CLL system and rewrite it subject to equation (4.6) in the equivalent form

$$(\mathrm{i}D_x^2 D_t + D_x^4) F \cdot G = -4(\mathrm{i}D_t + D_x^2) F_x \cdot G_x = 0. \tag{5.7}$$

There are two possible sets of the Wronskian solutions (5.1) and (5.2) satisfying both (4.6) and (5.7). They are given by

$$\begin{aligned} F &= W[\partial_x^{-1}\varphi, \varphi, \ldots, \varphi_{(N-2)x}, \varphi_{(N-1)x}] \\ G &= W[\varphi, \varphi_x, \ldots, \varphi_{(N-1)x}] \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} F &= W[\varphi_x, \varphi_{2x}, \ldots, \varphi_{Nx}] \\ G &= W[\varphi, \varphi_x, \ldots, \varphi_{(N-1)x}]. \end{aligned} \tag{5.9}$$

These two sets of Wronskian solutions for the bilinear fields (F, G) generate two different solutions of the mNLBq equation (1.4) according to the representation (4.8) and (4.9) as well as of the CLL system according to equations (4.10) and (4.11).

6. Conclusions

In conclusion, we have presented new transformations between the NLS, Kaup and NLBq equations as well as their modified versions and found some additional bilinearizations. The bilinearization of the modified NLBq equation can be regarded as an alternative to the usual bilinear representation of the CLL equation, which simplifies the reduction procedure used for finding exact solutions of these equations, for example Wronskian solutions. The results and methods presented in our paper can be further extended to mixed NLS and CLL equations as well as to higher-order members of the hierarchy of integrable equations related to the NLS, Kaup and NLBq equations.

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Appendix

The exponential polynomials introduced by Bell [26] are defined in terms of the derivatives of a $\mathcal{C}^\infty$ function $f(x)$:

$$Y_{nx}(f) \equiv Y_n(f_x, \ldots, f_{nx}) = \mathrm{e}^{-f(x)} \left(\frac{\partial}{\partial x}\right)^n \mathrm{e}^{f(x)}. \tag{A.1}$$

They are homogeneous of weight n (f being of weight 0 and x of weight -1) and obey a simple partitional recipe:

$$Y_x(f) = f_x \qquad Y_{2x}(f) = f_{2x} + f_x^2 \qquad Y_{3x}(f) = f_{3x} + 3f_x f_{2x} + f_x^3, \ldots. \tag{A.2}$$

We consider the following generalizations.

(i) *'Binary' Bell polynomials* (defined in terms of the odd-order derivatives of a function V and the even-order derivatives of another function U):

$$\mathcal{Y}_{nx}(V, U) = Y_n(f_x, \ldots, f_{nx})\Big|_{f_{px} = \begin{cases} V_{px} & \text{if } p = \text{odd} \\ U_{px} & \text{if } p = \text{even} \end{cases}} \tag{A.3}$$

which are linked to the Hirota D-operator through the identity [3]

$$(FG)^{-1} D_x^n F \cdot G = \mathcal{Y}_{nx}\left(V = \ln\frac{F}{G}, U = \ln FG\right). \tag{A.4}$$

They are homogeneous of weight n (V and U being of weight 0) and obey the partitional recipe inherited from the definition (A.1).

(ii) *Two-dimensional (binary) Bell polynomials*:

$$\mathcal{Y}_{px,qt}\left(V = \ln\frac{F}{G}, U = \ln FG\right) = (FG)^{-1} D_x^p D_t^q F \cdot G \tag{A.5}$$

which are homogeneous of weight $p + qr$ (t being of weight $-r$) and obey a similar partitional recipe extended to the two variables x and t:

$$\begin{aligned} &\mathcal{Y}_{xt}(V, U) = U_{xt} + V_x V_t \\ &\mathcal{Y}_{2x,t}(V, U) = V_{2x,t} + U_{2x} V_t + 2U_{xt} V_x + V_x^2 V_t \\ &\cdots. \end{aligned} \tag{A.6}$$

On account of the formulae (A.4) and (A.5) we may obtain a direct bilinearization of the system (2.5) and (2.6) by looking for a transformation which expresses v and u in terms of new (weightless) variables V and U which are such that the equations (2.5) and (2.6) are expressible as homogeneous linear combinations of polynomials $\mathcal{Y}_{px,qt}(V, U)$ set equal to zero. It is clear from the form of the polynomials of weight 2 (one sees in equations (2.5) and (2.6) that t has twice the weight of x),

$$\mathcal{Y}_t(V) = V_t \qquad \mathcal{Y}_{2x}(V, U) = U_{2x} + V_x^2 \tag{A.7}$$

that equation (2.6) can be transformed into

$$\mathrm{i}\mathcal{Y}_t(V) + \mathcal{Y}_{2x}(V, U) = 0 \qquad \Longleftrightarrow \qquad (\mathrm{i}D_t + D_x^2) F \cdot G = 0 \tag{A.8}$$

by noting that v has weight 0 whereas u has weight -1 and by setting

$$v = -\mathrm{i}V \qquad u = \tfrac{1}{2}(U_x \mp V_x). \tag{A.9}$$

It is easy to verify that the same map transforms equation (2.5) into an equation which can be expressed as a linear combination of the weight 3 polynomials $\mathcal{Y}_{xt}(V, U)$ and $\mathcal{Y}_{3x}(V, U) = V_{3x} + 3U_{2x} V_x + V_x^3$, set equal to zero:

$$\mathrm{i}\mathcal{Y}_{xt}(V, U) + \mathcal{Y}_{3x}(V, U) = 0 \qquad \Longleftrightarrow \qquad (\mathrm{i}D_x D_t + D_x^3) F \cdot G = 0. \tag{A.10}$$

Thus we obtain the bilinear system (2.21) and (2.22), the bilinearizing transformations (2.19) and (2.20) being given by formula (A.9) and the above relations:

$$V = \ln \frac{F}{G} \qquad U = \ln FG. \tag{A.11}$$

A direct bilinearization of the systems (3.7) and (3.8) can be obtained in the same manner, as equations (2.7) and (3.7) are of the same form. By introducing a similar transformation,

$$w = -\mathrm{i}V \qquad u = \tfrac{1}{2}(U_x \mp V_x) \tag{A.12}$$

one maps equation (3.7) onto (A.8). By using equation (A.8) to eliminate V_t one finds on account of the explicit form of the polynomials $\mathcal{Y}_{2x,t}(V, U)$ and

$$\mathcal{Y}_{4x}(V, U) = U_{4x} + 4V_x V_{3x} + 3U_{2x}^2 + 6V_x^2 U_{2x} + V_x^4 \tag{A.13}$$

that the transformation (A.12) transforms (3.8) into

$$\frac{-\mathrm{i}}{2V_x}[\mathrm{i}\mathcal{Y}_{2x,t}(V, U) + \mathcal{Y}_{4x}(V, U)] = 0. \tag{A.14}$$

Hence, it is clear through formula (A.5) that the system (4.6) and (4.7) is a bilinear representation of the system (3.7) and (3.8), the bilinearizing transformations (4.8) and (4.9) being given by the formulae (A.11) and (A.12).

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