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Eigenvalues of zero energy in the linearized NLS problem

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We study a pair of neutrally stable eigenvalues of zero energy in the linearized NLS equation. We prove that the pair of isolated eigenvalues, where each eigenvalue has geometric multiplicity one and algebraic multiplicity N, is associated with $2P$ negative eigenvalues of the energy operator, where $P=N/2$ if N is even and $P=(N-1)/2$ or $P=(N+1)/2$ if N is odd. When the potential of the linearized NLS problem is perturbed due to parameter continuations, we compute the exact number of unstable eigenvalues that bifurcate from the neutrally stable eigenvalues of zero energy. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2203233]

I. INTRODUCTION

Spectral stability of solitary waves is defined by the number of unstable eigenvalues in a linearized problem associated with the underlying nonlinear equation. Unstable eigenvalues for solitary waves of the nonlinear Schrödinger (NLS) equation have been recently studied within the inertia law,[1] the constrained variational problems and wave operators,[2] and the Grillakis projection method.[3]

The count of unstable eigenvalues is simplified in Refs. 1 and 2, with the technical assumption that the unstable and potentially unstable eigenvalues are structurally stable to parameter continuations. In particular, the count of eigenvalues is modified if the linearized problem admits an isolated eigenvalue of zero Krein signature (see Ref. 3 for definitions), which we refer to here as *an eigenvalue of zero energy*. The simplest instability bifurcation, called *the Hamiltonian-Hopf bifurcation*, occurs when the eigenvalue of zero Krein signature arises due to coalescence of two simple eigenvalues with positive and negative Krein signatures.[4,5]

In our present paper, we focus on a general Hamiltonian-Hopf bifurcation within the framework of a scalar NLS equation. We study properties of this general bifurcation and the number of unstable eigenvalues that are generated due to parameter continuations. Specifically, we use the notations from Ref. 2 and consider the linearized operator $\sigma_3\mathcal{H}$, where $\mathcal{H}$ is the energy operator,

$$\mathcal{H}=\begin{pmatrix} -\Delta+\omega+f(x) & g(x) \\ g(x) & -\Delta+\omega+f(x)\end{pmatrix}, \tag{1.1}$$

the standard Pauli matrices are used,

$$\sigma_1=\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \quad \sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}, \tag{1.2}$$

$x\in\mathbb{R}^n$, $n\geq 1$, $\omega>0$, and $f,g:\mathbb{R}^n\to\mathbb{R}$ are exponentially decaying C^∞ functions. The spectral problem for the operator $\sigma_3\mathcal{H}$ is considered on $L^2(\mathbb{R}^n,\mathbb{C}^2)$:

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$$\sigma_3 \mathcal{H} \boldsymbol{\psi} = z \boldsymbol{\psi}, \tag{1.3}$$

where $\boldsymbol{\psi} = (\psi_1, \psi_2)^T$. Eigenvalues z of the spectral problem (1.3) are called *unstable* if $\mathrm{Im}(z) > 0$, *neutrally stable* if $\mathrm{Im}(z) = 0$ and *stable* if $\mathrm{Im}(z) < 0$ (see Ref. 2). The following notations are used for inner products: $\langle \mathbf{c}, \mathbf{d} \rangle_{\mathbb{C}^N}$ for $\mathbf{c}, \mathbf{d} \in \mathbb{C}^N$, $\langle \mathbf{f}, \mathbf{g} \rangle$ for $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^n, \mathbb{C}^2)$ and (f, g) for $f, g \in L^2(\mathbb{R}^n)$.

It is shown in the previous paper[2] that the spectrum of the linearized operator $\sigma_3 \mathcal{H}$ is defined in part by the sign of the energy quadratic form in $H^1(\mathbb{R}^3, \mathbb{C}^2)$:

$$h = \langle \boldsymbol{\psi}, \mathcal{H} \boldsymbol{\psi} \rangle. \tag{1.4}$$

In particular, the number of unstable eigenvalues in the point spectrum of $\sigma_3 \mathcal{H}$ in the upper half-plane is bounded by the number of negative eigenvalues of $\mathcal{H}$, while the energy quadratic form (1.4) is positive definite on the nonsingular part of the essential spectrum of $\sigma_3 \mathcal{H}$. The singular part of the essential spectrum of $\sigma_3 \mathcal{H}$ (which only includes the embedded eigenvalues but not the embedded resonances) is studied with the Fermi Golden rule, from which it follows that the embedded eigenvalue z of positive energy $h > 0$ disappears under generic perturbation, while that of negative energy $h < 0$ bifurcates into isolated complex eigenvalues of the point spectrum of $\sigma_3 \mathcal{H}$.[2,5]

One of the technical assumptions in Ref. 2 postulates that the real eigenvalues z of the point spectrum of $\sigma_3 \mathcal{H}$ have nonzero energy $h \neq 0$. In the present work we remove this assumption and study the case when there exists a pair of isolated eigenvalues $z = \pm z_0$, where $0 < z_0 < \omega$, which corresponds to the zero energy $h = 0$. We show that a pair of isolated eigenvalues of zero energy has a higher algebraic multiplicity in the spectrum of $\sigma_3 \mathcal{H}$. Our precise result is formulated as follows.

Theorem 1: *Let $z = \pm z_0$ be a pair of isolated eigenvalues of $\sigma_3 \mathcal{H}$, where each eigenvalue has geometric multiplicity one and algebraic multiplicity N. Let $\mathcal{X}_g$ be a subspace spanned by a set of generalized eigenvectors $\{\boldsymbol{\psi}_j, \sigma_1 \boldsymbol{\psi}_j\}_{j=0}^{N-1}$, such that $s_0 = \frac{1}{2} \langle \sigma_3 \boldsymbol{\psi}_{N-1}, \boldsymbol{\psi}_0 \rangle \neq 0$. Then,*

$$n(\mathcal{H}) \Big|_{\mathcal{X}_g} = 2P,$$

where $P = N/2$, when N is even, $P = (N-1)/2$, when N is odd, and $s_0 > 0$, and $P = (N+1)/2$, when N is odd, and $s_0 < 0$. Therefore, $n(\mathcal{H}) \geq 2P$.

When the potentials of the linearized operator $\sigma_3 \mathcal{H}$ are perturbed by a continuous deformation, we compute the exact number of unstable eigenvalues with $\mathrm{Im}(z) > 0$ that bifurcate from the eigenvalue z_0 of zero energy $h = 0$. In particular, we study the perturbed spectral problem:

$$\sigma_3 (\mathcal{H} + \varepsilon \mathcal{V}_p(x)) \boldsymbol{\psi} = z \boldsymbol{\psi}, \tag{1.5}$$

where the perturbation matrix $\mathcal{V}_p(x)$ is C^∞, real-valued and bounded, such that $\mathcal{V}_p \in L^\infty(\mathbb{R}^n)$. We assume that the perturbation matrix $\mathcal{V}_p$ is *generic* in the sense that $\langle \boldsymbol{\psi}_0, \mathcal{V}_p \boldsymbol{\psi}_0 \rangle \neq 0$, where $\boldsymbol{\psi}_0$ is the eigenvector of $\sigma_3 \mathcal{H}$ for the eigenvalue $z = z_0$. Our precise result is formulated as follows.

Theorem 2: *Let z_0 be an isolated eigenvalue of $\sigma_3 \mathcal{H}$ with geometric multiplicity one and algebraic multiplicity N. Let $\boldsymbol{\psi}_0$ be the corresponding eigenvector of $\sigma_3 \mathcal{H}$, such that $s_0 = \frac{1}{2} \langle \sigma_3 \boldsymbol{\psi}_{N-1}, \boldsymbol{\psi}_0 \rangle \neq 0$. Then, there exists a small $\varepsilon_0 > 0$, such that the problem (1.5) with $0 < |\varepsilon| < \varepsilon_0$ has N simple eigenvalues near $z = z_0$, which are approximated to the leading order by roots of*

$$(z_k - z_0)^N = \frac{\varepsilon}{2 s_0} \langle \boldsymbol{\psi}_0, \mathcal{V}_p \boldsymbol{\psi}_0 \rangle + O(\varepsilon^{(N+1)/N}), \quad k = 1, \ldots, N. \tag{1.6}$$

We note that results of Theorems 1 and 2 are no longer restricted to the case $n = 3$, unlike the previous work.[2] Theorem 1 is proved in Sec. II, while Theorem 2 is proved in Sec. III.

A typical example of the spectral problem (1.3) with operators (1.1)–(1.2) arises in the linearization of the nonlinear Schrödinger (NLS) equation,

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$$i\psi_t = -\Delta\psi + U(x)\psi + F(|\psi|^2)\psi, \tag{1.7}$$

where $F(0)=0$ and $U(x)$ decays to zero, at the solitary wave solution $\psi=\phi(x)e^{i\omega t}$, where $\phi(x)$ is a real-valued function and $\omega>0$ is a parameter. Linearization of the NLS equation (1.7) with the ansatz,

$$\psi = (\phi(x) + \varphi(x)e^{-izt} + \bar{\theta}(x)e^{i\bar{z}t})e^{i\omega t}, \tag{1.8}$$

leads to the spectral problem (1.3) with $\boldsymbol{\psi}=(\varphi,\theta)^T$, $f(x)=U(x)+F(\phi^2)+F'(\phi^2)\phi^2$, and $g(x)=F'(\phi^2)\phi^2$. When $F(\phi^2)$ is C^∞ and $U(x)$ and $\phi(x)$ are exponentially decaying C^∞ functions, then the assumptions on $f(x)$, $g(x)$ are satisfied.

The Hamiltonian-Hopf bifurcation is typical with $N=2$ when the real isolated eigenvalue $z=z_0$ has a geometric multiplicity *one* and an algebraic multiplicity *two*.[3,4] If (ω,ϕ) is a pair for the solitary wave solution that corresponds to the bifurcation case and $\varepsilon=\delta\omega$ is a variation of parameter ω while $\varepsilon\,\delta\phi$ is a variation of the solution $\phi(x)$ along the solution family, then the perturbation matrix $\mathcal{V}_p(x)$ takes the form:

$$\mathcal{V}_p = \begin{pmatrix} 1+\delta f(x) & \delta g(x) \\ \delta g(x) & 1+\delta f(x) \end{pmatrix},$$

where

$$\varepsilon\,\delta f(x) \equiv F((\phi+\varepsilon\delta\phi)^2) - F(\phi^2) + F'((\phi+\varepsilon\delta\phi)^2)(\phi+\varepsilon\delta\phi)^2 - F'(\phi^2)\phi^2,$$

$$\varepsilon\,\delta g(x) \equiv F'((\phi+\varepsilon\delta\phi)^2)(\phi+\varepsilon\delta\phi)^2 - F'(\phi^2)\phi^2,$$

are variations of the potentials $f(x)$ and $g(x)$ along the solution family. The particular perturbation matrix $\mathcal{V}_p(x)$ satisfies the assumption that $\mathcal{V}_p(x)$ is C^∞ bounded matrix-valued function. An analysis of a general Hamiltonian–Hopf bifurcation is described in Sec. IV.

II. PROPERTIES OF AN EIGENVALUE OF ZERO ENERGY

We rewrite the system (1.3) in new variables $\boldsymbol{\psi}=(u+w,u-w)^T$:

$$\sigma_1 \mathrm{H}\mathbf{u} = z\mathbf{u}, \tag{2.1}$$

where $\mathbf{u}=(u,w)^T$ and H is the new energy operator:

$$H = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \tag{2.2}$$

with $L_\pm=-\Delta+\omega+f(x)\pm g(x)$. Let $n(H)$ denote the negative index of the energy operator H, which is the number of negative eigenvalues of H in $L^2(\mathbb{R}^n,\mathbb{C}^2)$ counting their multiplicity. Let $n(H)|_X$ be the negative index of H restricted to some subspace $X\subset L^2(\mathbb{R}^n,\mathbb{C}^2)$. We assume that there exists an isolated eigenvalue $0<z_0<\omega$ and eigenvector $\mathbf{u}_0=(u_0,w_0)^T$ of the spectral problem (2.1), such that

$$\sigma_1 \mathrm{H}\mathbf{u}_0 = z_0\mathbf{u}_0. \tag{2.3}$$

The adjoint problem has then the same eigenvalue z_0 with the eigenvector $\mathbf{u}_{0,\mathbf{a}}$, such that

$$\mathrm{H}\sigma_1\mathbf{u}_{0,a} = z_0\mathbf{u}_{0,a}, \quad \mathbf{u}_{0,a} = \sigma_1\mathbf{u}_0. \tag{2.4}$$

We assume that the eigenvalue z_0 has zero energy $h=\langle\mathbf{u}_0,H\mathbf{u}_0\rangle=0$, such that the algebraic multiplicity of the eigenvalue z_0 exceeds its geometric multiplicity. We consider the situation when the geometric multiplicity of the eigenvalue z_0 is one, while its algebraic multiplicity is N, such that $\ker(\sigma_1 H - z_0)=\{\mathbf{u}_0\}$ and $N_g(\sigma_1 H - z_0)=\{\mathbf{u}_0,\mathbf{u}_1,\ldots,\mathbf{u}_{N-1}\}$. The generalized eigenvectors $\mathbf{u}_j=(u_j,w_j)^T$, $1\le j\le N-1$ are defined from the Jordan chain equations:

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$$\sigma_1 H \mathbf{u}_j = z_0 \mathbf{u}_j + \mathbf{u}_{j-1}. \tag{2.5}$$

The adjoint problem has $N-1$ generalized eigenvectors $\mathbf{u}_{j,a}=\sigma_1\mathbf{u}_j$, $1\le j\le N-1$ that solve the inhomogeneous equations:

$$H\sigma_1 \mathbf{u}_{j,a} = z_0 \mathbf{u}_{j,a} + \mathbf{u}_{j-1,a}. \tag{2.6}$$

If z_0 is an eigenvalue of $\sigma_1 H$, then $-z_0$ is also an eigenvalue of both $\sigma_1 H$ and its adjoint $H\sigma_1$, such that $\dim(N_g(\sigma_1\mathrm{H}+z_0))=\dim(N_g(\mathrm{H}\sigma_1+z_0))=N$. Explicitly, we have

$$\sigma_1 H \mathbf{u}_0^- = -z_0 \mathbf{u}_0^-, \tag{2.7}$$

$$\sigma_1 H \mathbf{u}_j^- = -z_0 \mathbf{u}_j^- + \mathbf{u}_{j-1}^-, \tag{2.8}$$

where $\mathbf{u}_0^-=\sigma_3\mathbf{u}_0$ and $\mathbf{u}_j^-=(-1)^j\sigma_3\mathbf{u}_j$ and similarly

$$\mathrm{H}\sigma_1 \mathbf{u}_{0,a}^- = -z_0 \mathbf{u}_{0,a}^-, \tag{2.9}$$

$$\mathrm{H}\sigma_1 \mathbf{u}_{j,a}^- = -z_0 \mathbf{u}_{j,a}^- + \mathbf{u}_{j-1,a}^-, \tag{2.10}$$

where $\mathbf{u}_{0,a}^-=\sigma_1\mathbf{u}_0^-$ and $\mathbf{u}_{j,a}^-=\sigma_1\mathbf{u}_j^-$. Let $X_g=N_g(\sigma_1H-z_0)\oplus N_g(\sigma_1 H+z_0)$ be the subspace spanned by the generalized eigenvectors $\{\mathbf{u}_j,\sigma_3\mathbf{u}_j\}_{j=0}^{N-1}$. Let L_+ and L_- be diagonal compositions of $(L_+,0)$ and $(0,L_-)$. The exact number of negative eigenvalues of H restricted to the subspace X_g is given by Theorem 1 in notations of the equivalent system (1.3). In order to prove Theorem 1, we establish some useful relations between eigenvectors of the generalized subspace X_g.

Lemma 2.1: Let $2N=\dim(X_g)$, *where* $2\le N<\infty$. *Then, it is true that*

$$\langle\sigma_1\mathbf{u}_0,\mathbf{u}_j\rangle = 0, \quad 0\le j\le N-2, \quad \langle\sigma_1\mathbf{u}_0,\mathbf{u}_{N-1}\rangle \neq 0. \tag{2.11}$$

Proof: By the Fredholm Alternative Theorem, it follows from (2.4) and (2.5) that $\langle\mathbf{u}_{j-1},\mathbf{u}_{0,a}\rangle=0$, $1\le j\le N-1$ and $\langle\mathbf{u}_{N-1},\mathbf{u}_{0,a}\rangle\neq 0$. Since $\mathbf{u}_{0,a}=\sigma_1\mathbf{u}_0$, these conditions are equivalent to the statement (2.11). ■

Let U be an auxiliary matrix with the elements:

$$U_{i,j} = \langle\sigma_1\mathbf{u}_{i-1},\mathbf{u}_{j-1}\rangle, \quad 1\le i,\ j\le N. \tag{2.12}$$

We study the structure of the matrix U.

Definition 2.2: *Suppose M is a square matrix of size N. The subset of its elements $M_{i,j}$, $1\le i,j\le N$, such that $i+j=k+1$ is said to be the kth antidiagonal of the matrix M, where $1\le k\le 2N-1$. The kth antidiagonal of M is said to be constant if all its elements are equal.*

Lemma 2.3: *Each kth antidiagonal of U is constant. There exists a basis $\{\hat{\mathbf{u}}_j,\sigma_3\hat{\mathbf{u}}_j\}_{j=0}^{N-1}$ in the subspace X_g, such that*

$$U = s_0 \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}, \tag{2.13}$$

where $s_0=\langle\sigma_1\hat{\mathbf{u}}_{N-1},\hat{\mathbf{u}}_0\rangle$. *Moreover,*

$$\langle\sigma_1\sigma_3\hat{\mathbf{u}}_i,\hat{\mathbf{u}}_j\rangle = 0, \quad 0\le i,\ j\le N-1. \tag{2.14}$$

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Proof: By Lemma 2.1, the first antidiagonal of U is zero. It follows from (2.5) and (2.6) that

$$\langle \mathbf{u}_{i-1,a}, \mathbf{u}_j \rangle = \langle \mathbf{u}_{i,a}, \mathbf{u}_{j-1} \rangle, \quad 1 \le i,\ j \le N-1. \tag{2.15}$$

Therefore, the relation $\langle \sigma_1 \mathbf{u}_i, \mathbf{u}_{j-1}\rangle = \langle \sigma_1 \mathbf{u}_{i-1}, \mathbf{u}_j \rangle$ holds for $2 \le i+j \le 2N-2$. Since $\langle \sigma_1 \mathbf{u}_0, \mathbf{u}_j \rangle = 0$ by Lemma 2.1 for any $0 \le j \le N-2$, then all antidiagonals of U are zero for $1 \le k \le N-1$, such that $\langle \sigma_1 \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ for $0 \le i+j \le N-2$. We show that by a proper choice of the generalized eigenvectors all but the main antidiagonals of U are zero for $N \le k \le 2N-1$. We introduce a sequence of transformations for $1 \le j \le N-1$:

$$\hat{\mathbf{u}}_k = \mathbf{u}_k, \quad 0 \le k \le j-1,$$

$$\hat{\mathbf{u}}_k = \mathbf{u}_k - \alpha_j \mathbf{u}_{k-j}, \quad j \le k \le N-1, \tag{2.16}$$

where the constants α_j are chosen from the condition that

$$\langle \sigma_1 \hat{\mathbf{u}}_j, \hat{\mathbf{u}}_{N-1} \rangle = 0, \quad 1 \le j \le N-1. \tag{2.17}$$

It is clear from (2.5) that the modified generalized eigenvectors $\{\hat{\mathbf{u}}_k\}_{k=0}^{N-1}$ satisfy the same inhomogeneous problems for any $1 \le j \le N-1$. By using recurrently the relations (2.16), we find that

$$\langle \sigma_1 \hat{\mathbf{u}}_j, \hat{\mathbf{u}}_{N-1} \rangle = \langle \sigma_1 \mathbf{u}_j, \mathbf{u}_{N-1} \rangle - 2\alpha_j \langle \sigma_1 \mathbf{u}_0, \mathbf{u}_{N-1} \rangle,$$

such that

$$\alpha_j = \frac{\langle \sigma_1 \mathbf{u}_j, \mathbf{u}_{N-1} \rangle}{2\langle \sigma_1 \mathbf{u}_0, \mathbf{u}_{N-1} \rangle}.$$

Since all antidiagonals of U are constants, the previous orthogonalization implies that $\langle \sigma_1 \hat{\mathbf{u}}_i, \hat{\mathbf{u}}_j \rangle = 0$, $N \le i+j \le 2N-2$. Furthermore, we have the normalization $\langle \sigma_1 \hat{\mathbf{u}}_{N-1}, \hat{\mathbf{u}}_0 \rangle = s_0$ on the main antidiagonal. It follows from (2.5) and (2.10) that

$$2z_0 \langle \mathbf{u}_{i,a}^-, \mathbf{u}_j \rangle + \langle \mathbf{u}_{i,a}^-, \mathbf{u}_{j-1} \rangle - \langle \mathbf{u}_{i-1,a}^-, \mathbf{u}_j \rangle = 0, \quad 1 \le i,\ j \le N-1. \tag{2.18}$$

Using the relation (2.18), we prove (2.14) by induction. ■

Corollary 2.4: *There exists a basis of generalized eigenvectors* $\{\hat{\mathbf{u}}_j, \sigma_3 \hat{\mathbf{u}}_j\}_{j=0}^{N-1}$ *in the subspace* X_g, *such that it satisfies the skew-orthogonality relations*,

$$\langle \sigma_1 \hat{\mathbf{u}}_i, \hat{\mathbf{u}}_j \rangle = s_0 \delta_{i,N-1-j}, \quad 0 \le i,\ j \le N-1, \tag{2.19}$$

and any $\mathbf{f} \in L^2(\mathbb{R}^n, \mathbb{C}^2)$ *can be decomposed as follows:*

$$\mathbf{f} = \sum_{j=0}^{N-1} (c_j \hat{\mathbf{u}}_j + d_j \sigma_3 \hat{\mathbf{u}}_j) + \mathbf{f}_c, \tag{2.20}$$

where

$$c_j = \frac{1}{s_0} \langle \sigma_1 \hat{\mathbf{u}}_{N-1-j}, \mathbf{f} \rangle, \quad d_j = -\frac{1}{s_0} \langle \sigma_1 \sigma_3 \hat{\mathbf{u}}_{N-1-j}, \mathbf{f} \rangle, \quad 0 \le j \le N-1$$

and

$$\langle \sigma_1 \hat{\mathbf{u}}_j, \mathbf{f}_c \rangle = \langle \sigma_1 \sigma_3 \hat{\mathbf{u}}_j, \mathbf{f}_c \rangle = 0, \quad 0 \le j \le N-1.$$

Remark 2.5: Let P_g be the skew-orthogonal projection operator to the subspace X_g. We have $P_g^2 = P_g$ and $(I-P_g)^2 = I - P_g$ on $L^2(\mathbb{R}^n, \mathbb{C}^2)$. Operators P_g and $(I-P_g)$ are bounded, such that

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$$\|P_g\| \leq \frac{2}{|s_0|} \sum_{k=0}^{N-1} \|\hat{\mathbf{u}}_{N-1-k}\|_{L^2} \|\hat{\mathbf{u}}_k\|_{L^2}, \quad \|I - P_g\| \leq 1 + \|P_g\|,$$

where the operator norm $\|\cdot\|_{L^2 \to L^2}$ is denoted as $\|\cdot\|$.

Remark 2.6: In what follows, the hats over the basis elements of the subspace X_g will be omitted for the simplicity of notations. Let $H|_{X_g} = P_g^* H P_g$ be the restriction of the energy operator to the generalized subspace X_g. The restriction of the energy operator is defined in terms of the matrices M_+ and M_-, where

$$(M_+)_{i,j} = (L_+ u_{i-1}, u_{j-1}), \quad 1 \leq i, j \leq N,$$

$$(M_-)_{i,j} = (L_- w_{i-1}, w_{j-1}), \quad 1 \leq i, j \leq N. \tag{2.21}$$

The structure of the matrices M_+ and M_- follows from that of the matrix U.

Lemma 2.7: *The matrices M_+ and M_- are equal and have the structure*:

$$M_+ = M_- = \frac{s_0}{2} \begin{pmatrix} 0 & 0 & \dots & 0 & z_0 \\ 0 & 0 & \dots & z_0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_0 & 1 & \dots & \dots & 0 \end{pmatrix}. \tag{2.22}$$

Proof: Using the definitions (2.12) and (2.21), the systems (2.3) and (2.5), and the identity (2.14), we derive the relations

$$2(M_+)_{1,j} = z_0 U_{j,1}, \quad 2(M_-)_{1,j} = z_0 U_{1,j}, \quad 1 \leq j \leq N,$$

and

$$2(M_+)_{i,j} = z_0 U_{j,i} + U_{j,i-1}, \quad 2(M_-)_{i,j} = z_0 U_{i,j} + U_{i-1,j}, \quad 2 \leq i \leq N, \quad 1 \leq j \leq N.$$

By Lemma 2.3, these formulas imply the representation (2.22). ■

Remark 2.8: The structure of matrices U and $M_\pm$ enables us to estimate the number of their positive and negative eigenvalues. Since $s_0 \neq 0$ and $z_0 \neq 0$, the main anti-diagonal of the matrices is nonzero, such that their determinant is nonzero and no zero eigenvalue exists. We denote the number of positive and negative eigenvalues of an abstract symmetric non-singular matrix M by $p(M)$ and $n(M)$ respectively, such that $p(M)+n(M)=N$.

Lemma 2.9: *Let M be a symmetric N-by-N matrix, such that each kth antidiagonal of M is zero for $1 \leq k \leq N-1$ and the Nth antidiagonal of M is $a \neq 0$. If N is even, then $n(M)=p(M)=N/2$. If N is odd, then $n(M)=(N-1)/2$ and $p(M)=(N+1)/2$ for $a>0$ and $n(M)=(N+1)/2$ and $p(M)=(N-1)/2$ for $a<0$.*

Proof: Let $\{\lambda_k\}_{k=0}^{N-1}$ be eigenvalues of M ordered by $\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{N-1}$. It follows from the structure of M that $\det M = (-1)^{N(N-1)/2} a^N$. We introduce the auxiliary subspace $V_d \subset \mathbb{R}^N$ spanned by the unit vectors $\{\mathbf{e}_k\}_{k=1}^{d}$ in $\mathbb{R}^N$, where $d=N/2$ when N is even and $d=(N-1)/2$ when N is odd. Let P_d be the orthogonal projection from $\mathbb{R}^N$ onto V_d, such that P_d has the matrix elements:

$$(P_d)_{i,j} = \begin{cases} 1, & 1 \leq i = j \leq d, \\ 0, & d+1 \leq i = j \leq N, \\ 0, & i \neq j. \end{cases} \tag{2.23}$$

A trivial computation gives that the elements of $M|_{V_d}$ are

$$(P_d M P_d)_{i,j} = (P_d)_{i,i} M_{i,j} (P_d)_{j,j}, \quad 1 \leq i,j \leq N,$$

which are nonzero only if $i,j \leq d$ and $i+j \geq N+1$. However, the intersection of these two sets is empty due to the definition of d. Therefore, the matrix $P_d M P_d = 0$ and all d eigenvalues of $M|_{V_d}$ are

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zero. By the Rayleigh-Ritz theorem (see Theorem XIII.3 in Ref. 6), d eigenvalues of M are nonpositive due to the upper bound by the eigenvalues of $M|_{V_d}$. Therefore, $n(M)\geq d$. The same argument for $-M$ shows that $p(M)\geq d$. This proves that $n(M)=p(M)=N/2$ if N is even. If N is odd, there are two possibilities: either $n(M)=(N-1)/2$ and $p(M)=(N+1)/2$ or $n(M)=(N+1)/2$ and $p(M)=(N-1)/2$. Since

$$\operatorname{sign}(\det(M)) = (-1)^{N(N-1)/2}(\operatorname{sign}(a))^N = (-1)^{n(M)},$$

we find that the first case occurs for $a>0$ and the second case occurs for $a<0$. ■

Proof of Theorem 1: We recall that $\mathrm{H}|_{X_g}=P_g^*HP_g$ denotes the restriction of the energy operator to the generalized subspace X_g. By Lemma 2.7, a trivial calculation shows that the quadratic form of $H|_{X_g}$ can be rewritten as

$$\left\langle H\big|_{X_g}\mathbf{u},\mathbf{u}\right\rangle = 2\langle M_+\mathbf{c},\mathbf{c}\rangle_{\mathbb{C}^N} + 2\langle M_+\mathbf{d},\mathbf{d}\rangle_{\mathbb{C}^N}, \tag{2.24}$$

where $\mathbf{c}=(c_0,c_1,\ldots,c_{N-1})^T\in\mathbb{C}^N$, $\mathbf{d}=(d_0,d_1,\ldots,d_{N-1})^T\in\mathbb{C}^N$, and the matrices $M_\pm$ are defined by (2.21). Let $\{\mathbf{v}_s\}_{s=1}^{n(M_+)}$ be an orthonormal set of eigenvectors corresponding to negative eigenvalues of the matrix M_+. Then we can construct a subspace of X_g, spanned by the vectors $\tilde{\mathbf{u}}_s=\Sigma_{k=0}^{N-1}(\mathbf{v}_s)_k\mathbf{u}_\mathbf{k}$ and $\sigma_3\tilde{\mathbf{u}}_s$ for $1\leq s\leq n(M_+)$. It follows from (2.24) that the operator $H|_{X_g}$ is negative definite on this subspace, i.e., via the Rayleigh-Ritz theorem $n(H)|_{X_g}\geq 2n(M_+)$. Let $\mathcal{M}_+=M_+\otimes M_+$ be the block matrix on $\mathbb{C}^{2N}$. To obtain the inequality reverse to the one above we consider the orthonormal set of eigenvectors $\{\mathbf{w}_s\}_{s=1}^{n(H)|_{X_g}}$, which spans the negative subspace of $H|_{X_g}$. Applying the projection operator, we have $P_g\mathbf{w}_\mathbf{s}=\Sigma_{k=0}^{N-1}(\boldsymbol{\alpha}_s)_k\mathbf{u}_k+\Sigma_{k=0}^{N-1}(\boldsymbol{\beta}_s)_k\sigma_3\mathbf{u}_\mathbf{k}$. Therefore, there is a subspace of $\mathbb{C}^{2N}$ spanned by the vectors $(\boldsymbol{\alpha}_s,\boldsymbol{\beta}_s)^T$, $1\leq s\leq n(H)|_{X_g}$. It follows from (2.24) that the matrix $\mathcal{M}_+$ is negative definite on this subspace, which yields $n(\mathcal{M}_+)=2n(M_+)\geq n(H)|_{X_g}$. We have $a=\frac{1}{2}z_0s_0$ and $z_0>0$, such that it follows by Lemmas 2.7 and 2.9 that $n(H)|_{X_g}=2n(M_+)=2P$. By the Rayleigh–Ritz Theorem,[6] we then have $n(H)\geq n(H)|_{X_g}$. ■

III. SPLITTING OF AN EIGENVALUE OF ZERO ENERGY

When the perturbation is applied to the spectral problem (2.1), one can expect that the multiple isolated eigenvalue $z=z_0$ is destroyed and N simple eigenvalues bifurcate in the neighborhood of $z=z_0$. This splitting of the multiple eigenvalue $z=z_0$ may result in the instability bifurcations if $\mathrm{Im}(z)>0$ for some of the simple eigenvalues. The location of N simple eigenvalues in the neighborhood of $z=z_0$ for small nonzero ε is given by Theorem 2 in notations of the perturbed spectral problem (1.5). For convenience, we rewrite (1.5) in the equivalent form

$$\sigma_1(H+\varepsilon V_p)\mathbf{u} = z\mathbf{u}, \tag{3.1}$$

where $V_p(x)$ is C^∞, real-valued, and a bounded perturbation matrix given by $V_p=U^{-1}\mathcal{V}_pU$, with

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = U^{-1}.$$

In order to prove Theorem 2, we use the resolvent $R(z)=(\sigma_1\mathrm{H}-z)^{-1}$ defined for the values of z away from the spectrum of σ_1H. The perturbed problem (3.1) is rewritten in the resolvent form

$$(I+\varepsilon R(z)\sigma_1V_p)\mathbf{u} = 0. \tag{3.2}$$

We shall consider the subspace $X_g^+=N_g(\sigma_1\mathrm{H}-z_0)$ with the projection operator P_g^+ onto this subspace. In what follows, the superscript for the subspace X_g^+ and the projection operator P_g^+ will be omitted for the simplicity of notations. Our plan is to use the projection operator P_g and to reduce the perturbed problem (3.1) to the finite-dimensional equations on the subspace X_g, where a regular perturbation theory can be applied.[7] For this purpose, we study properties of the resolvent $R(z)$ and the related projection operator onto X_g.

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Lemma 3.1: Let the subspace X_g *be spanned by the set* $\{\mathbf{u}_j\}_{j=0}^{N-1}$. *Then,*

$$R(z)\mathbf{u}_j = \sum_{k=0}^{j} \frac{(-1)}{(z-z_0)^{j-k+1}}\mathbf{u}_k, \tag{3.3}$$

for $z \neq z_0$ *and* $0 \leq j \leq N-1$.

Proof: The formula (3.3) with $j=0$ follows from the problem (2.3). We use the induction method and assume that the formula (3.3) holds for some $j=s$ for $0 \leq s \leq N-2$. It follows from (2.5) that

$$R(z)\mathbf{u}_{s+1} = \frac{1}{z-z_0}R(z)\mathbf{u}_s - \frac{1}{z-z_0}\mathbf{u}_{s+1},$$

such that

$$R(z)\mathbf{u}_{s+1} = \sum_{k=0}^{s+1} \frac{(-1)}{(z-z_0)^{s-k+2}}\mathbf{u}_k,$$

and the formula (3.3) remains true for $j=s+1$. ■

Corollary 3.2: Let z be not in the spectrum of $\sigma_1 H$. *Then,*

$$(I-P_g)R(z)P_g = 0, \quad P_g R(z)(I-P_g) = 0.$$

Proof: Let $\mathbf{f}$ be an arbitrary function in $L^2(\mathbb{R}^n, \mathbb{C}^2)$. By Lemma 3.1, we have $R(z)P_g\mathbf{f} \in X_g$ for $z \neq z_0$, and therefore $(I-P_g)R(z)P_g\mathbf{f}=0$. On the other hand, for any $k=0,\ldots,N-1$ we have

$$\langle R(z)(I-P_g)\mathbf{f}, \sigma_1\mathbf{u}_k\rangle = \langle (I-P_g)\mathbf{f}, (H\sigma_1 - \bar{z})^{-1}\sigma_1\mathbf{u}_k\rangle = \langle (I-P_g)\mathbf{f}, \sigma_1 R(\bar{z})\mathbf{u}_k\rangle = 0$$

such that $P_g R(z)(I-P_g)\mathbf{f}=0$. ■

Remark 3.3: Let $\bar{D}(z_0,\delta)=\{z \in \mathbb{C} : |z-z_0| \leq \delta\}$ be a closed disk that contains no eigenvalues of the operator $\sigma_1 H$ other than $z=z_0$. The resolvent $R(z)$ has the only singularity in $\bar{D}(z_0,\delta)$ at $z=z_0$.

Lemma 3.4: Let $T(z)=(I-P_g)R(z)(I-P_g)\sigma_1 V_p$. *The operator* $T(z)$ *is uniformly bounded in* $\bar{D}(z_0,r) \subseteq \bar{D}(z_0,\delta)$, *such that*

$$\|T(z)\| \leq C < \infty, \tag{3.4}$$

where $r>0$ and $C>0$.

Proof: The proof is a standard argument, which uses the analyticity of the resolvent outside the spectrum, the Corollary 3.3, the boundedness of the operator $I-P_g$, and the fact that $V_p \in L^\infty$. ■

Lemma 3.5: The problem (3.2) *projected onto the subspace* X_g *is equivalent to the problem:*

$$[I + \varepsilon R(z)P_g\sigma_1 V_p(I + S(\varepsilon,z))]P_g\mathbf{u} = 0, \tag{3.5}$$

where the operator $S(\varepsilon,z)$ *is analytic in* ε *and* $\|S(\varepsilon,z)\| \leq 2C|\varepsilon|$, *while C is defined in* (3.4), $z \in \bar{D}(z_0,r)$, *and* $\varepsilon \neq 0$ *is small, such that* $|\varepsilon| \leq 1/2C$.

Proof: Applying P_g and $(I-P_g)$ to the problem (3.2), we obtain

$$P_g\mathbf{u} + \varepsilon P_g R(z)\sigma_1 V_p P_g\mathbf{u} + \varepsilon P_g R(z)\sigma_1 V_p(I-P_g)\mathbf{u} = 0$$

and

$$[I + \varepsilon(I-P_g)R(z)\sigma_1 V_p](I-P_g)\mathbf{u} = -\varepsilon(I-P_g)R(z)\sigma_1 V_p P_g\mathbf{u}.$$

By Corollary 3.2, we have

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$$[I+\varepsilon P_g R(z) P_g \sigma_1 V_p] P_g \mathbf{u} + \varepsilon P_g R(z) P_g \sigma_1 V_p (I-P_g)\mathbf{u} = 0 \tag{3.6}$$

and

$$[I+\varepsilon T(z)](I-P_g)\mathbf{u} = -\varepsilon T(z) P_g \mathbf{u}. \tag{3.7}$$

By Lemma 3.1, we have $P_g R(z) P_g = R(z) P_g$ for $z \neq z_0$, such that we need to derive the equation expressing $(I-P_g)\mathbf{u}$ via $P_g\mathbf{u}$. It follows from (3.7) that

$$(I-P_g)\mathbf{u} = [(I+\varepsilon T(z))^{-1} - I] P_g \mathbf{u}, \tag{3.8}$$

such that the equation (3.6) yields

$$[I+\varepsilon R(z) P_g \sigma_1 V_p [I+\varepsilon T(z)]^{-1}] P_g \mathbf{u} = 0.$$

The operator $[I+\varepsilon T(z)]^{-1}$ can be written as $I+S(\varepsilon,z)$, where

$$S(\varepsilon,z) = \sum_{k=1}^{\infty} [-\varepsilon T(z)]^k. \tag{3.9}$$

By Lemma 3.4 and a comparison with the geometric series, the series (3.9) converges absolutely in the $\|.\|$ norm for $z \in \bar{D}(z_0,r)$. When $|\varepsilon| \leq 1/2C$, we have $\|S(\varepsilon,z)\| \leq 2C|\varepsilon|$. ■

Remark 3.6: By Lemma 3.5, the finite-rank operator on the left-hand side of the projection equations (3.5) is analytic in ε for small ε. The determinant of the left-hand side is the Weinstein–Aronszajn determinant, properties of which and its relation to the spectrum of the perturbed operator are rigorously derived in Ref. 7, pp. 244–250. We truncate the projection equations (3.5) by linear terms in ε and obtain the finite rank operator $W(\varepsilon,z) = I + \varepsilon R(z) P_g \sigma_1 V_p$ on the subspace X_g. The matrix elements of $W(\varepsilon,z)$ are expressed in terms of the elements $\{\mathbf{u}_j\}_{j=0}^{N-1}$ of the subspace X_g.

Lemma 3.7: For all $i,j=1,\ldots,N$, *we have*

$$W_{i,j}(\varepsilon,z) = \delta_{i,j} - \frac{\varepsilon}{s_0} \sum_{s=i-1}^{N-1} \frac{\langle \mathbf{u}_{N-1-s}, V_p \mathbf{u}_{j-1}\rangle}{(z-z_0)^{s-i+2}}. \tag{3.10}$$

Proof: Due to the orthogonality relations (2.19), the matrix elements of the finite rank operator W are equal to

$$W_{i,j}(\varepsilon,z) = \frac{1}{s_0} \langle \sigma_1 \mathbf{u}_{N-i}, W \mathbf{u}_{j-1}\rangle, \quad 1 \leq i,j \leq N. \tag{3.11}$$

Corollary 2.4 and Lemma 3.1 yield

$$\varepsilon R(z) P_g \sigma_1 V_p \mathbf{u}_j = -\frac{\varepsilon}{s_0} \sum_{s=0}^{N-1} \sum_{q=0}^{s} \frac{\langle \mathbf{u}_{N-1-s}, V_p \mathbf{u}_j\rangle}{(z-z_0)^{s-q+1}} \mathbf{u}_q.$$

Substituting this relation into the representation (3.11) and using the orthogonality conditions (2.19) we derive the identity (3.10). ■

Proof of Theorem 2: According to Ref. 7, the Weinstein-Aronszajn determinant is a meromorphic function, which has the pole of the Nth order at $z=z_0$. The eigenvalues of the perturbed problem (3.1) are given by the zeros $z=z_k$ of the Weinstein–Aronszajn determinant. We define the operator $F(\varepsilon,z) = (z-z_0) W(\varepsilon,z)$, such that $\det F(\varepsilon,z)$ is a polynomial in $z-z_0$ and ε. By Lemma 3.7, the matrix elements of $F(\varepsilon,z)$ are equal to

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$$F_{i,j}(\varepsilon,z) = (z - z_0)\,\delta_{i,j} - \frac{\varepsilon}{s_0}\sum_{s=i-1}^{N-1}\frac{\langle \mathbf{u}_{N-1-s}, V_p\mathbf{u}_{j-1}\rangle}{(z-z_0)^{s-i+1}}, \quad i,j = 1,\ldots,N.$$

Since $\det F(\varepsilon,z)$ is a polynomial in ε, we have the expansion:

$$\det F(\varepsilon,z) = \det F(0,z) + \varepsilon\frac{\partial}{\partial\varepsilon}\det F(0,z) + O(\varepsilon^2, z - z_0), \tag{3.12}$$

where $O(\varepsilon^2, z-z_0)$ is a polynomial in ε and $z-z_0$. The expansion (3.12) allows us to obtain the estimate on the zeros $z=z_k$ of the Weinstein–Aronszajn determinant. We find that

$$\det F(0,z) = (z - z_0)^N$$

and

$$\frac{\partial}{\partial\varepsilon}\det F(0,z) = \sum_{i=1}^{N}\det D^i(z),$$

where

$$D^i_{q,j}(z) = \begin{cases} (z - z_0)\,\delta_{q,j}, & q \neq i, \\ D^i_{i,j}(z), & q = i, \end{cases}$$

such that

$$D^i_{i,j}(z) = -\frac{1}{s_0}\sum_{s=i-1}^{N-1}\frac{\langle \mathbf{u}_{N-1-s}, V_p\mathbf{u}_{j-1}\rangle}{(z-z_0)^{s-i+1}},$$

or explicitly

$$D^i(z) = \begin{pmatrix} z-z_0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & z-z_0 & 0 & 0 & \ldots & 0 \\ D^i_{i,1}(z) & D^i_{i,2}(z) & \ldots & D^i_{i,i}(z) & \ldots & \ldots & D^i_{i,N}(z) \\ 0 & \ldots & 0 & 0 & z-z_0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & z-z_0 \end{pmatrix}.$$

A straightforward computation shows that

$$\det D^i(z) = (z - z_0)^{N-1}D^i_{i,i}(z) = -\frac{1}{s_0}\sum_{s=i-1}^{N-1}(z-z_0)^{N-2-(s-i)}\langle \mathbf{u}_{N-1-s}, V_p\mathbf{u}_{i-1}\rangle.$$

The expansion (3.12) implies that the zeros of the Weinstein–Aronszajn determinant are given by the algebraic equation

$$(z - z_0)^N - \frac{\varepsilon}{s_0}\sum_{i=1}^{N}\sum_{s=i-1}^{N-1}(z-z_0)^{N-2-(s-i)}\langle \mathbf{u}_{N-1-s}, V_p\mathbf{u}_{i-1}\rangle + \varepsilon^2 F_2(z - z_0, \varepsilon) = 0, \tag{3.13}$$

where $\lim_{\varepsilon\to 0}F_2(z-z_0,\varepsilon)$ exists. By the Implicit Function Theorem, roots of the algebraic equation (3.13) exist and satisfy the estimate $z-z_0=O(\varepsilon^{1/N})$. Therefore, the zero energy eigenvalue of the unperturbed problem splits into N simple eigenvalues, which are given asymptotically by roots of

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$$(z_k - z_0)^N = \frac{\varepsilon}{s_0}\langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle + O(\varepsilon^{(N+1)/N}), \quad k = 1, \ldots, N. \tag{3.14}$$

Equation (3.14) coincides with (1.6) in notations of (1.5). ■

IV. A GENERAL HAMILTONIAN–HOPF INSTABILITY BIFURCATION

The pair of eigenvalues of zero energy $z = \pm z_0$ is neutrally stable in the linearized spectral problem (2.1), since $\mathrm{Im}(z_0) = 0$. If there are no other isolated eigenvalues with $\mathrm{Im}(z) > 0$, it would imply that the spectral problem (2.1) is weakly spectrally stable. However, the multiple eigenvalue of zero energy is structurally unstable and splits into simple eigenvalues, when the spectral problem (2.1) is perturbed with a bounded potential V_p in the form (3.1). When simple eigenvalues from the roots of (3.14) satisfy $\mathrm{Im}(z_k) > 0$ for some k, the spectral problem (3.1) undertakes an instability bifurcation, referred to as the *general Hamiltonian–Hopf bifurcation*. We show that there is only one bifurcation that gives a transition to instability, such that neutrally stable eigenvalues for one sign of ε split into stable and unstable eigenvalues for the other sign of ε. This bifurcation occurs for $N=2$. Other bifurcations with $N \geq 3$ lead to unstable eigenvalues for either sign of $\varepsilon \neq 0$. We compute the exact number of unstable eigenvalues for each N, considering separately the cases $N=2$ and $N \geq 3$.

Proposition 4.1: Let $\langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle \neq 0$ *and* $0 < |\varepsilon| < \varepsilon_0$. *When* $N=2$, *there exist two neutrally stable eigenvalues of the spectral problem* (3.1) *in a local neighborhood of* $z=z_0$ *for* $\mathrm{sign}(\varepsilon s_0 \langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle) = 1$, *one of positive and the other one of negative energy, and two (stable and unstable) eigenvalues for* $\mathrm{sign}(\varepsilon s_0 \langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle) = -1$.

Proof: It follows from (3.14) with $N=2$ that

$$z_{1,2}^{+} = z_0 \pm \sqrt{\frac{\varepsilon}{s_0}\langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle} + O(\varepsilon), \tag{4.1}$$

when $\mathrm{sign}(\varepsilon s_0 \langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle) = 1$ and

$$z_{1,2}^{-} = z_0 \pm i\sqrt{\left|\frac{\varepsilon}{s_0}\langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle\right|} + O(\varepsilon),$$

when $\mathrm{sign}(\varepsilon s_0 \langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle) = -1$. Since the total multiplicity of eigenvalues near $z=z_0$ is continuous in ε and complex eigenvalues occur in pairs, the eigenvalues $z_{1,2}^{+}$ are real and neutrally stable. The eigenvalues $z_{1,2}^{-}$ are complex, such that z_1^{+} is stable and z_1^{-} is unstable. Let $\mathbf{v}_{1,2}^{+}$ be the eigenvectors of the problem (3.1) that correspond to $z_{1,2}^{+}$. We need to show that the energy operator is positive on one of the eigenvectors and negative on the other eigenvector. It follows from (3.1) that

$$\langle (H + \varepsilon V_p)\mathbf{v}_j^{+}, \mathbf{v}_j^{+}\rangle = z_j^{+}\langle \sigma_1 \mathbf{v}_j^{+}, \mathbf{v}_j^{+}\rangle, \quad j = 1,2. \tag{4.2}$$

Using the projection onto the subspace X_g, we rewrite (4.2) as follows:

$$\langle \sigma_1 \mathbf{v}_j^{+}, \mathbf{v}_j^{+}\rangle = \langle \sigma_1 P_g \mathbf{v}_j^{+}, P_g \mathbf{v}_j^{+}\rangle + \langle \sigma_1 (I - P_g)\mathbf{v}_j^{+}, (I - P_g)\mathbf{v}_j^{+}\rangle + 2\langle \sigma_1 P_g \mathbf{v}_j^{+}, (I - P_g)\mathbf{v}_j^{+}\rangle. \tag{4.3}$$

Let us estimate each of these three terms and show that only the first one plays the leading role. We have

$$P_g \mathbf{v}_j^{+} = c_0^{+}\mathbf{u}_0 + c_1^{+}\mathbf{u}_1, \tag{4.4}$$

where the constants c_0^{+} and c_1^{+} satisfy

$$W_{2,1}(\varepsilon, z_j^{+})c_0^{+} + W_{2,2}(\varepsilon, z_j^{+})c_1^{+} + O(\varepsilon^2) = 0. \tag{4.5}$$

Using the explicit expression (3.10), we have

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$$W_{2,1}(\varepsilon,z_j^+) = -\frac{\varepsilon}{s_0}\frac{\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle}{z_j^+ - z_0}, \quad W_{2,2}(\varepsilon,z_j^+) = 1 - \frac{\varepsilon}{s_0}\frac{\langle \mathbf{u}_0, V_p\mathbf{u}_1\rangle}{z_j^+ - z_0}. \tag{4.6}$$

This expression can be rewritten by using (4.1), or, explicitly,

$$\frac{|\varepsilon|}{z_j^+ - z_0} = (-1)^{j+1}\sqrt{\frac{s_0\varepsilon}{\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle}} + O(\varepsilon),$$

such that

$$W_{2,1}(\varepsilon,z_j^+) = (-1)^j\sqrt{\frac{\varepsilon}{s_0}\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle} + O(\varepsilon)$$

and

$$W_{2,2}(\varepsilon,z_j^+) = 1 + (-1)^j \operatorname{sign}(\varepsilon s_0)\sqrt{\frac{\varepsilon}{s_0\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle}}\langle \mathbf{u}_0, V_p\mathbf{u}_1\rangle + O(\varepsilon).$$

We let $c_0^+=1$ by scaling and obtain from (4.5) that

$$c_1^+ = (-1)^{j+1}\sqrt{\frac{\varepsilon}{s_0}\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle} + O(\varepsilon).$$

By the triangle inequality applied to (4.4), we have the norm estimate

$$\|P_g\mathbf{v}_j^+\|_{L^2} \le \|\mathbf{u}_0\|_{L^2} + O(\sqrt{\varepsilon})\|\mathbf{u}_1\|_{L^2}. \tag{4.7}$$

For the first term on the right side of (4.3) using (4.4), we derive

$$\langle \sigma_1 P_g\mathbf{v}_j^+, P_g\mathbf{v}_j^+\rangle = 2(-1)^{j+1}\operatorname{sign}(s_0)\sqrt{\varepsilon s_0\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle} + O(\varepsilon). \tag{4.8}$$

For the second and the third terms on the right side of (4.3), we get bounds via the identity (3.8), the Schwarz inequality, the estimate on the norm of the operator $S(\varepsilon,z)$ given in Lemma 3.5, and the bound (4.7). Thus

$$|\langle \sigma_1(I-P_g)\mathbf{v}_j^+, (I-P_g)\mathbf{v}_j^+\rangle| \le \|S(\varepsilon,z_j^+)P_g\mathbf{v}_j^+\|_{L^2}^2 \le \tilde{C}^2\varepsilon^2(\|\mathbf{u}_0\|_{L^2} + O(\sqrt{\varepsilon})\|\mathbf{u}_1\|_{L^2})^2$$

and

$$|2\langle \sigma_1 P_g\mathbf{v}_j^+, (I-P_g)\mathbf{v}_j^+\rangle| \le 2\|P_g\mathbf{v}_j^+\|_{L^2}\|S(\varepsilon,z_j^+)P_g\mathbf{v}_j^+\|_{L^2} \le 2\tilde{C}|\varepsilon|(\|\mathbf{u}_0\|_{L^2} + O(\sqrt{\varepsilon})\|\mathbf{u}_1\|_{L^2})^2.$$

Using the inequalities above along with (4.8) in the identity (4.3) and substituting into (4.2), we arrive at

$$\langle (H+\varepsilon V_p)\mathbf{v}_1^+, \mathbf{v}_1^+\rangle = 2z_0\operatorname{sign}(s_0)\sqrt{\varepsilon s_0\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle} + O(\varepsilon), \tag{4.9}$$

$$\langle (H+\varepsilon V_p)\mathbf{v}_2^+, \mathbf{v}_2^+\rangle = -2z_0\operatorname{sign}(s_0)\sqrt{\varepsilon s_0\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle} + O(\varepsilon), \tag{4.10}$$

for ε sufficiently small and $\operatorname{sign}(\varepsilon s_0\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle)=1$. Thus, the quadratic forms (4.9) and (4.10) have opposite signs. ■

Proposition 4.2: Let $\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle \neq 0$ *and* $0<|\varepsilon|<\varepsilon_0$. *When* $N\ge 3$, *there exist unstable eigenvalues of the spectral problem* (3.1) *in a local neighborhood of* $z=z_0$ *for either sign of* $\varepsilon\neq 0$. *When* N *is odd, there are* $(N-1)/2$ *unstable eigenvalues. When* N *is even, there are* $N/2-1$ *unstable eigenvalues for* $\operatorname{sign}(\varepsilon s_0\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle)=1$ *and* $N/2$ *unstable eigenvalues for* $\operatorname{sign}(\varepsilon s_0\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle)=-1$.

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Proof: We have

$$z_k^+ = z_0 + \sqrt[N]{\frac{\varepsilon}{s_0}\langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle} e^{i[2\pi(k-1)/N]} + O(\varepsilon^{2/N}), \quad k = 1, \dots, N, \tag{4.11}$$

for $\text{sign}(\varepsilon s_0 \langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle) = 1$ and

$$z_k^- = z_0 + \sqrt[N]{\left|\frac{\varepsilon}{s_0}\langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle\right|} e^{i[(-\pi+2\pi k)/N]} + O(\varepsilon^{2/N}), \quad k = 1, \dots, N, \tag{4.12}$$

for $\text{sign}(\varepsilon s_0 \langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle) = -1$. Unstable eigenvalues with $\text{Im}(z_k) > 0$ exist among both $\{z_k^+\}_{k=1}^N$ and $\{z_k^-\}_{k=1}^N$, e.g., $\text{Im}(z_2^+) > 0$ and $\text{Im}(z_1^-) > 0$. The count of unstable eigenvalues follows from the explicit expressions (4.11) and (4.12). ■

Remark 4.3: In practical situations, the NLS equation (1.7) has limits when the eigenvalues of the spectral problem (1.3) are all neutrally stable. When the NLS equation (1.7) deviates from the stable limit due to parameter continuations, real eigenvalues z start to move, which may lead to coalescence. The practical outcome of the above analysis shows that the coalescence of N neutrally stable eigenvalues with $N \geq 3$ cannot lead to a multiple eigenvalue $z = z_0$ of geometric multiplicity one and algebraic multiplicity N. If the resulting eigenvalue $z = z_0$ has zero energy, then it corresponds to several Jordan blocks, where each block splits according to our analysis in Propositions 4.1 and 4.2.

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