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NONLINEAR INSTABILITY OF A CRITICAL TRAVELING WAVE IN THE GENERALIZED KORTEWEG–DE VRIES EQUATION*

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Abstract. We prove the instability of a "critical" solitary wave of the generalized Korteweg–de Vries equation, the one with the speed at the border between the stability and instability regions. The instability mechanism involved is "purely nonlinear" in the sense that the linearization at a critical soliton does not have eigenvalues with positive real part. We prove that critical solitons correspond generally to the saddle-node bifurcation of two branches of solitons.

Key words. Korteweg–de Vries equation, critical soliton, dynamic instability, orbital stability

AMS subject classifications. 35Q51, 35Q53, 70K50

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1. Introduction and main results.

We consider the generalized Korteweg–de Vries (KdV) equation in one dimension,

$$\partial_t \boldsymbol{u} = \partial_x \left(-\partial_x^2 \boldsymbol{u} + f(\boldsymbol{u})\right), \qquad \boldsymbol{u} = \boldsymbol{u}(x,t) \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{1.1}$$

where $f \in C^\infty(\mathbb{R})$ is a real-valued function that satisfies

$$f(0) = f'(0) = 0. \tag{1.2}$$

Depending on the nonlinearity f, (1.1) may admit solitary wave solutions, or solitons, of the form $\boldsymbol{u}(x,t) = \boldsymbol{\phi}_c(x - ct)$. Generically, solitons exist for speeds c from (finite or infinite) intervals of a real line. For a particular nonlinearity f, solitons with certain speeds are (orbitally) stable with respect to the perturbations of the initial data, while others are linearly (and also dynamically) unstable. We will study the stability of the critical solitons, the ones with speeds c on the border of stability and instability regions. These solitons are no longer linearly unstable. Still, we will prove their instability, which is the consequence of the higher algebraic multiplicity of the zero eigenvalue of the linearized system.

When $f(\boldsymbol{u}) = -3\boldsymbol{u}^2$, (1.1) turns into the classical KdV equation

$$\partial_t \boldsymbol{u} + \partial_x^3 \boldsymbol{u} + 6\boldsymbol{u}\partial_x \boldsymbol{u} = 0, \tag{1.3}$$

which is well known to have solitons

$$\boldsymbol{u}_c(x,t) = \boldsymbol{\phi}_c(x - ct) = \frac{c}{2\cosh^2\left(\frac{\sqrt{c}}{2}(x - ct)\right)}, \qquad c > 0.$$

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For $f(\boldsymbol{u}) = -\boldsymbol{u}^p$, $p > 1$, we obtain the family of generalized KdV equations (also known as gKdV-k with $k = p - 1$) that have the form

$$\partial_t \boldsymbol{u} + \partial_x^3 \boldsymbol{u} + \partial_x(\boldsymbol{u}^p) = 0. \tag{1.4}$$

They also have solitary wave solutions. All solitary waves of the classical KdV equation and of the subcritical generalized KdV equations ($1 < p < 5$) are orbitally stable; see [Ben72, Bon75, Wei87, ABH87]. Orbital stability is defined in the following sense.

DEFINITION 1.1. *The traveling wave $\boldsymbol{\phi}_c(x - ct)$ is said to be orbitally stable if for any $\epsilon > 0$ there exists $\delta > 0$ so that for any $\boldsymbol{u}_0$ with $\|\boldsymbol{u}_0 - \boldsymbol{\phi}_c\|_{H^1} \le \delta$ there is a solution $\boldsymbol{u}(t)$ with $\boldsymbol{u}(0) = \boldsymbol{u}_0$, defined for all $t \ge 0$, such that*

$$\sup_{t\ge 0} \inf_{s\in\mathbb{R}} \|\boldsymbol{u}(x,t) - \boldsymbol{\phi}(x - s)\|_{H^1} < \epsilon,$$

where $H^1 = H^1(\mathbb{R})$ is the standard Sobolev space. Otherwise the traveling wave is said to be unstable.

Equation (1.1) is a Hamiltonian system, with the Hamiltonian functional

$$E(\boldsymbol{u}) = \int_{\mathbb{R}} \left(\frac{1}{2}(\partial_x \boldsymbol{u})^2 + F(\boldsymbol{u}) \right) dx, \tag{1.5}$$

with $F(\boldsymbol{u})$ the antiderivative of $f(\boldsymbol{u})$ such that $F(0) = 0$. There are two more invariants of motion: the mass

$$I(\boldsymbol{u}) = \int_{\mathbb{R}} \boldsymbol{u}\, dx \tag{1.6}$$

and the momentum

$$\mathcal{N}(\boldsymbol{u}) = \int_{\mathbb{R}} \frac{1}{2} \boldsymbol{u}^2\, dx. \tag{1.7}$$

Assumption 1. There is an open set $\Sigma \subset \mathbb{R}_+$ so that for $c \in \Sigma$ the equation $-c\boldsymbol{\phi}_c = -\boldsymbol{\phi}_c'' + f(\boldsymbol{\phi}_c)$ has a unique solution $\boldsymbol{\phi}_c(x) \in H^\infty(\mathbb{R})$ such that $\boldsymbol{\phi}_c(x) > 0$, $\boldsymbol{\phi}_c(-x) = \boldsymbol{\phi}_c(x)$, $\lim_{|x|\to\infty} \boldsymbol{\phi}_c(x) = 0$. The map $c \mapsto \boldsymbol{\phi}_c \in H^s(\mathbb{R})$ is C^∞ for $c \in \Sigma$ and for any s. Consequently, (1.1) admits traveling wave solutions

$$\boldsymbol{u}(x,t) = \boldsymbol{\phi}_c(x - ct), \qquad c \in \Sigma. \tag{1.8}$$

In Appendix A we specify conditions under which Assumption 1 is satisfied.

Let $\mathcal{N}_c$ and I_c denote $\mathcal{N}(\boldsymbol{\phi}_c)$ and $I(\boldsymbol{\phi}_c)$, respectively. By Assumption 1, $\mathcal{N}_c$ and I_c are C^∞ functions of $c \in \Sigma$. For the general KdV equation (1.1) with smooth $f(\boldsymbol{u})$, Bona, Souganidis, and Strauss [BSS87] show that the traveling wave $\boldsymbol{\phi}_c(x - ct)$ is orbitally stable if

$$\mathcal{N}_c' = \frac{d}{dc}\mathcal{N}_c = \frac{d}{dc}\mathcal{N}(\boldsymbol{\phi}_c) > 0 \tag{1.9}$$

and unstable if instead $\mathcal{N}_c' < 0$. See Figure 1. The criterion (1.9) coincides with the stability condition obtained in [GSS87] in the context of abstract Hamiltonian systems with $\mathbf{U}(1)$ symmetry (the theory developed there does not apply to the generalized KdV equation).

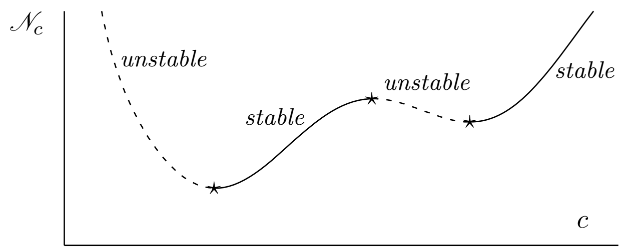

FIG. 1. *Stable and unstable regions on a possible graph of $\mathcal{N}_c$ vs. c. Three critical solitary waves are denoted by stars.*

Remark 1.2. Note that, as one can readily show, the amplitude of solitary waves is monotonically increasing with their speed c, while the momentum $\mathcal{N}_c$ does not have to.

Remark 1.3. For the generalized KdV equations (1.4), the soliton profiles satisfy the scaling relation $\boldsymbol{\phi}_c(x) = c^{\frac{1}{p-1}}\boldsymbol{\phi}_1(c^{\frac{1}{2}}x)$. The values of the momentum functional that correspond to solitons with different speeds c are given by $\mathcal{N}(\boldsymbol{\phi}_c) = \mathrm{const}\, c^{\frac{2}{p-1}-\frac{1}{2}} = \mathrm{const}\, c^{\frac{5-p}{2(p-1)}}$, so that $\frac{d}{dc}\mathcal{N}(\boldsymbol{\phi}_c) > 0$ for $p < 5$, in agreement with the stability criterion (1.9) derived in [BSS87].

In [BSS87] it is stated that critical traveling waves $\boldsymbol{\phi}_{c_\star}(x)$, that is, $c_\star$ such that $\mathcal{N}_{c_\star}' = 0$, are unstable as a consequence of the claim that the set $\{c\colon \boldsymbol{\phi}_c \text{ is stable}\}$ is open. This claim, however, is left unproved in [BSS87]. Moreover, this is not true in general. (This is demonstrated by the dynamical system in $\mathbb{R}^2$ described in the polar coordinates by $\dot\theta = \sin\theta$, $\dot r = 0$. The set of stationary states is the line $y = 0$; the subset of stable stationary points, $x \le 0$, is closed.) The question of stability of critical traveling waves has been left open. We address this question in this paper, proving the instability under certain rather generic assumptions. This result is the analogue of [CP03] for the generalized KdV equation (1.1).

Remark 1.4. We will not consider the L^2-critical KdV equation given by (1.4) with $p = 5$, when $\mathcal{N}_c = \mathrm{const}$. In this case, the solitons are not only unstable but also exhibit a blow-up behavior. This blow-up is considered in a series of papers by Martel and Merle [Mer01, MM01b, MM02a, MM02b].

The analysis of the instability of critical solitary waves (with no linear instability) requires better control of the growth of a particular perturbation. We achieve this by employing the asymptotic stability methods. Pego and Weinstein [PW94] proved that the traveling wave solutions to (1.4) for the subcritical values $p = 2, 3, 4$, and also $p \in (2, 5)\backslash E$ with E a finite and possibly empty set, are asymptotically stable in the weighted spaces. Their approach was extended in [Miz01]. For other deep results of stability see [MM01a, MM05]. The proofs extend, under certain spectral hypotheses, to solitary solutions to a generalized KdV equation (1.1) with c such that $\mathcal{N}_c' > 0$.

Substituting $\boldsymbol{u}(x,t) = \boldsymbol{\phi}_c(x-ct) + \boldsymbol{\rho}(x-ct,t)$ into (1.1) and discarding terms nonlinear in $\boldsymbol{\rho}$, we get the linearization at $\boldsymbol{\phi}_c$:

$$\partial_t\boldsymbol{\rho} = \partial_x\big(-\partial_x^2\boldsymbol{\rho} + f'(\boldsymbol{\phi}_c)\boldsymbol{\rho} + c\boldsymbol{\rho}\big) \equiv J\mathcal{H}_c\boldsymbol{\rho}, \tag{1.10}$$

where

$$J = \partial_x, \qquad \mathcal{H}_c = -\partial_x^2 + f'(\boldsymbol{\phi}_c) + c. \tag{1.11}$$

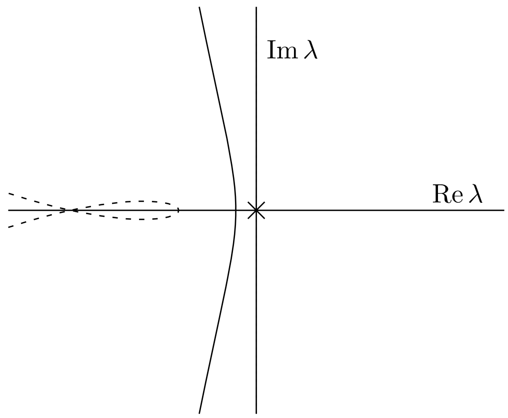

FIG. 2. *Essential spectrum of $J\mathcal{H}_c$, $c = 1$, in the exponentially weighted space $L^2_\mu(\mathbb{R})$ for $\mu = 0.1 < \sqrt{c/3}$ (solid line) and $\mu = 0.65 > \sqrt{c/3}$ (dashed line).*

In (1.10), both $\boldsymbol{\phi}_c(\cdot)$ and $\boldsymbol{\rho}(\cdot, t)$ are evaluated at $x - ct$, but we change the variable and write x instead.

The essential spectrum of $J\mathcal{H}_c$ in $L^2(\mathbb{R})$ coincides with the imaginary axis. $\lambda = 0$ is an eigenvalue (with $\partial_x\boldsymbol{\phi}_c$ being the corresponding eigenvector). To use the asymptotic stability methods from [PW94], we will consider the action of $J\mathcal{H}_c$ in the exponentially weighted spaces. For $s \in \mathbb{R}$ and $\mu \geq 0$, we define

$$H^s_\mu(\mathbb{R}) = \{\boldsymbol{\psi} \in H^s_{loc}(\mathbb{R})\colon e^{\mu x}\boldsymbol{\psi}(x) \in H^s(\mathbb{R})\}, \qquad \mu \geq 0, \tag{1.12}$$

where $H^s(\mathbb{R})$ is the standard Sobolev space of order s. We also denote $L^2_\mu(\mathbb{R}) = H^0_\mu(\mathbb{R})$. We define the operator $A^\mu_c = e^{\mu x} \circ J\mathcal{H}_c \circ e^{-\mu x}$, where $e^{\pm\mu x}$ are understood as the operators of multiplication by the corresponding functions, so that the action of $J\mathcal{H}_c$ in $L^2_\mu(\mathbb{R})$ corresponds to the action of A^μ_c in $L^2(\mathbb{R})$. The explicit form of A^μ_c is

$$A^\mu_c = e^{\mu x} \circ J\mathcal{H}_c \circ e^{-\mu x} = (\partial_x - \mu)\big[-(\partial_x - \mu)^2 + c - f'(\boldsymbol{\phi}_c)\big]. \tag{1.13}$$

The domain of A^μ_c is given by $D(A^\mu_c) = H^3(\mathbb{R})$. Since the operator $[\partial_x - \mu]f'(\boldsymbol{\phi}_c)$ is relatively compact with respect to $\mathscr{A}^\mu_c = -(\partial_x - \mu)^3 + c(\partial_x - \mu)$, the essential spectrum of A^μ_c coincides with that of $\mathscr{A}^\mu_c$ and thus is given by

$$\sigma_{\mathrm{e}}(A^\mu_c) = \big\{\lambda \in \mathbb{C}\colon \lambda = \lambda_{\mathrm{cont}}(k) = (\mu - ik)^3 - c(\mu - ik), \quad k \in \mathbb{R}\big\}. \tag{1.14}$$

The essential spectrum of $\mathscr{A}^\mu_c$ is located in the left half-plane for $0 < \mu < \sqrt{c}$ and is simply connected for $0 < \mu < \sqrt{c/3}$; see Figure 2.

We need assumptions about the existence and properties of a critical wave.

Assumption 2. There exists $c_\star \in \Sigma\backslash\partial\Sigma$, $c_\star > 0$, such that $\mathscr{N}'_{c_\star} = 0$.

Remark 1.5. Let us give examples of the nonlinearities that lead to the existence of critical solitary waves. Take $f_-(z) = -Az^p + Bz^q$, with $2 < p < q$, $A > 0$, $B > 0$, or $f_+(z) = Az^p - Bz^q + Cz^r$, with $2 < p < q < r$, $A > 0$, $B > 0$, $C > 0$. In the case of f_+, we require that B be sufficiently large so that $f_+(z)$ takes negative values on a nonempty interval $I \subset \mathbb{R}_+$. Then there will be traveling wave solutions $\boldsymbol{\phi}_c(x - ct)$ with $c \in (0, c_1)$ (also with $c = 0$ in the case of f_+) for some $c_1 > 0$.[1] Elementary

[1]The value of c_1 is determined from the system $f(z_1) + c_1 z_1 = 0$, $F(z_1) + c_1 z_1^2/2 = 0$, with F the primitive of f such that $F(0) = 0$. See Appendix A or [BL83] for more details.

computations show that the value of the momentum $\mathscr{N}_c$ goes to infinity as $c \nearrow c_1$. It also goes to infinity as $c \searrow 0$ if $p > 5$ (also if $p = 5$ in the case of f_+), so that there is a global minimum of $\mathscr{N}_c$ at some point $c_\star \in (0, c_1)$.

Assumption 3. There exists $\mu_0 \in (0, \sqrt{c_\star}/2)$ such that for $0 \le \mu \le \mu_0$ the operator $A^\mu_{c_\star}$ has no L^2-eigenvalues except $\lambda = 0$.

Remark 1.6. We require that $\mu \le \sqrt{c_\star}/2$ so that the inequality $\mu < \sqrt{c/3}$ (needed in the condition of Lemma 4.2) is satisfied for c from an open neighborhood of $c_\star$.

Assumption 4. At the critical value $c_\star$, the nondegeneracy condition $I'_{c_\star} \neq 0$ is satisfied. Here $I_c = I(\boldsymbol{\phi}_c)$ is the value of the mass functional (1.6) on the traveling wave $\boldsymbol{\phi}_c$.

Remark 1.7. If $I'_{c_\star} = 0$, then the eigenvalue $\lambda = 0$ of $J\mathcal{H}_{c_\star}$ corresponds to a Jordan block larger than 3×3. We will not consider this situation.

Our main result is that the critical traveling wave $\boldsymbol{\phi}_{c_\star}(x)$ of the generalized KdV equation (1.1) is (nonlinearly) unstable.

THEOREM 1 (main theorem). *Let Assumptions* 1*,* 2*,* 3*, and* 4 *be satisfied, and assume that $\boldsymbol{\phi}_{c_\star}$ is a critical soliton. Assume that there exists an open neighborhood $\mathcal{O}(c_\star) \subset \Sigma$ of $c_\star$ so that $\mathscr{N}'_c$ is strictly negative and nonincreasing for $c \in \mathcal{O}(c_\star)$, $c > c_\star$ (or strictly negative and nondecreasing for $c < c_\star$, or both). Then the critical traveling wave $\boldsymbol{\phi}_{c_\star}(x)$ is orbitally unstable. More precisely, there exists $\epsilon > 0$ such that for any $\delta > 0$ there exists $\boldsymbol{u}_0 \in H^1(\mathbb{R})$ with $\|\boldsymbol{u}_0 - \boldsymbol{\phi}_{c_\star}\|_{H^1} < \delta$ and $t > 0$ so that*

$$\inf_{s\in\mathbb{R}} \|\boldsymbol{u}(\cdot, t) - \boldsymbol{\phi}_{c_\star}(\cdot - s)\|_{H^1} = \epsilon. \tag{1.15}$$

Remark 1.8. For definiteness, we consider the case when $\mathscr{N}'_c$ is strictly negative and nonincreasing for $c > c_\star$, $c \in \mathcal{O}(c_\star)$. The proof for the case when $\mathscr{N}'_c$ is strictly negative and nondecreasing for $c < c_\star$, $c \in \mathcal{O}(c_\star)$, is the same.

Thus, we assume that there exists $\eta_1 > 0$ such that

$$[c_\star, c_\star + \eta_1] \subset \Sigma, \qquad \mathscr{N}'_c < 0 \quad \text{for } c \in (c_\star, c_\star + \eta_1] \subset \Sigma. \tag{1.16}$$

Strategy of the proof and the structure of the paper. In our proof, we develop the method of Pego and Weinstein [PW94] and derive the nonlinear bounds relating the energy estimate and the dissipative estimate. We follow a center manifold approach; that is, we reduce the infinite-dimensional Hamiltonian system to a finite-dimensional system which contains the main features of the dynamics. Specifically, we consider the spectral decomposition near the zero eigenvalue in section 2, and a center manifold reduction is considered in section 3, this part being similar to the approach in [CP03]. Estimates in the energy space and in the weighted space for the error terms are in sections 4 and 5. In this part of our argument we develop the approach of [PW94]. In section 6, we complete the proof of Theorem 1. In section 7, we give an alternative approach to the instability of the critical traveling wave $\boldsymbol{\phi}_{c_\star}(x)$ by a normal form argument [Car81, IA98], under the additional hypothesis that the critical point $c_\star$ of $\mathscr{N}_c$ is nondegenerate:

$$\mathscr{N}''_{c_\star} = \frac{d^2\,\mathscr{N}(\boldsymbol{\phi}_c)}{dc^2}\Big|_{c=c_\star} \neq 0. \tag{1.17}$$

The construction of traveling waves is considered in Appendix A. The details on the Fredholm alternative for $\mathcal{H}_c$ are in Appendix B. An auxiliary technical result is proved in Appendix C.

2. Spectral decomposition in $L^2_\mu(\mathbb{R})$ near $\lambda = 0$. First, we observe that for any $c \in \Sigma$ (see Assumption 1), the linearization operator $J\mathcal{H}_c$ given by (1.11) satisfies the following relations:

$$\mathcal{H}_c \boldsymbol{e}_{1,c} = 0, \qquad \text{where } \boldsymbol{e}_{1,c} = -\partial_x \boldsymbol{\phi}_c(x), \tag{2.1}$$

$$J\mathcal{H}_c \boldsymbol{e}_{2,c} = \boldsymbol{e}_{1,c}, \qquad \text{where } \boldsymbol{e}_{2,c} = \partial_c \boldsymbol{\phi}_c(x). \tag{2.2}$$

Let $\mathscr{S}(\mathbb{R})$ denote the Schwarz space of functions.

Definition 2.1. *Let $\chi_+ \in C^\infty(\mathbb{R})$ be such that $0 \le \chi_+ \le 1$, $\chi_+|_{(-\infty,-1]} = 0$, $\chi_+|_{[0,\infty)} \equiv 1$. Define $\mathscr{S}_{+,m}(\mathbb{R})$, $m \ge 0$, to be the set of functions $u \in C^\infty(\mathbb{R})$ such that $\chi_+ u \in \mathscr{S}(\mathbb{R})$ and for any $N \in \mathbb{Z}$, $N \ge 0$, there exists $C_N > 0$ such that*

$$|u^{(N)}(x)| \le C_N (1 + |x|)^m.$$

Note that for any $m \ge 0$, $\mathrm{Image}(J\mathcal{H}_c|_{\mathbb{S}_{+,m}(\mathbb{R})}) \subset \mathscr{S}_{+,m}(\mathbb{R})$. The algebraic multiplicity of zero eigenvalue of the operator $J\mathcal{H}_c$ considered in $\mathscr{S}_{+,m}(\mathbb{R})$ depends on the values of $\mathscr{N}_c'$ and I_c' as follows.

Proposition 2.2. *Fix $m \ge 0$, and consider the operator $J\mathcal{H}_c$ in $\mathscr{S}_{+,m}(\mathbb{R})$.*

(i) *The eigenvalue $\lambda = 0$ is of geometric multiplicity one, with the kernel generated by $\boldsymbol{e}_{1,c}$.*

(ii) *Assume that $c \in \Sigma$ is such that $\mathscr{N}_c' \ne 0$. Then the eigenvalue $\lambda = 0$ is of algebraic multiplicity two.*

(iii) *Assume that $c_\star \in \Sigma$ is such that $\mathscr{N}_{c_\star}' = 0$, $I_{c_\star}' \ne 0$. Then the eigenvalue $\lambda = 0$ is of algebraic multiplicity three.*

Proof. First of all we claim that in $\mathscr{S}_{+,m}(\mathbb{R})$ we have $\dim\ker J\mathcal{H}_c = 1$.

The differential equation $\mathcal{H}_c \boldsymbol{\psi} = 0$ has two linearly independent solutions. According to (2.1), one of them is $\boldsymbol{e}_{1,c}$, which is odd and exponentially decaying at infinity. The other solution is even and exponentially growing as $|x| \to \infty$ and hence does not belong to $\mathscr{S}_{+,m}(\mathbb{R})$; we denote this solution by $\boldsymbol{\Xi}_c(x)$.

Observe that if $\boldsymbol{v} \in \ker J\mathcal{H}_c$, then $\mathcal{H}_c \boldsymbol{v} = K$, $\boldsymbol{v} \in C^\infty(\mathbb{R})$. Set $\boldsymbol{v} = \frac{K}{c} + \boldsymbol{w}$. Then $\mathcal{H}_c \boldsymbol{w} = -\frac{K}{c} f'(\boldsymbol{\phi}_c)$. Since $\langle f'(\boldsymbol{\phi}_c), \boldsymbol{e}_{1,c}\rangle = 0$, by Lemma B.1 there exists a function $\boldsymbol{w}_0 \in \mathscr{S}_{+,m}(\mathbb{R})$ such that $\mathcal{H}_c \boldsymbol{w}_0 = -\frac{K}{c} f'(\boldsymbol{\phi}_c)$. So $\boldsymbol{w} = \boldsymbol{w}_0 + A\partial_x \boldsymbol{\phi}_c + B\boldsymbol{\Xi}_c$, with A and B constants. Since

$$\boldsymbol{v} = \frac{K}{c} + \boldsymbol{w} = \frac{K}{c} + \boldsymbol{w}_0 + A\partial_x \boldsymbol{\phi}_c + B\boldsymbol{\Xi}_c \in \mathscr{S}_{+,m}(\mathbb{R}),$$

we need $\boldsymbol{v}(x) \to 0$ for $x \to +\infty$, and therefore $B = 0$ and $K = 0$. Hence, $\boldsymbol{v} \in \ker \mathcal{H}_c$, proving that $\ker J\mathcal{H}_c = \ker \mathcal{H}_c$. This proves Proposition 2.2 (i).

Let us introduce the function

$$\boldsymbol{\Theta}_c(x) = \int_{+\infty}^x \partial_c \boldsymbol{\phi}_c(y)\, dy. \tag{2.3}$$

Then $\partial_x \boldsymbol{\Theta}_c(x) = \partial_c \boldsymbol{\phi}_c(x)$, $\lim_{x\to-\infty} \boldsymbol{\Theta}_c(x) = -I_c'$; hence $\Theta_c \in \mathscr{S}_{+,0}(\mathbb{R})$. If $\boldsymbol{v}$ satisfies

$$J\mathcal{H}_c \boldsymbol{v} = \partial_c \boldsymbol{\phi}_c(x), \qquad \lim_{x\to+\infty} \boldsymbol{v}(x) = 0, \tag{2.4}$$

then $\boldsymbol{v}(x)$ is the only solution to the problem

$$\mathcal{H}_c \boldsymbol{v} = \boldsymbol{\Theta}_c(x), \qquad \lim_{x\to+\infty} \boldsymbol{v}(x) = 0. \tag{2.5}$$

According to Lemma B.1 (see Appendix B), if $\langle \boldsymbol{e}_{1,c}, \boldsymbol{\Theta}_c\rangle = \langle \boldsymbol{\phi}_c, \partial_c\boldsymbol{\phi}_c\rangle = \mathscr{N}_c' \neq 0$, then $\boldsymbol{v}(x)$ has exponential growth as $x \to -\infty$,

$$\boldsymbol{v}(x) \propto e^{\sqrt{c}|x|}, \qquad x \to -\infty, \tag{2.6}$$

and therefore does not belong to $\mathscr{S}_{+,m}(\mathbb{R})$. This finishes the proof of Proposition 2.2 (ii).

Let us now assume that $\mathscr{N}_{c_\star}' = 0$ for some $c_\star \in \Sigma$. Then, again by Lemma B.1 with $m = 0$, there exists $\boldsymbol{e}_{3,c_\star}(x) \in \mathscr{S}_{+,0}(\mathbb{R})$ such that

$$\mathcal{H}_{c_\star}\boldsymbol{e}_{3,c_\star} = \boldsymbol{\Theta}_{c_\star}(x), \qquad \lim_{x\to+\infty} \boldsymbol{e}_{3,c_\star}(x) = 0. \tag{2.7}$$

Now let us consider $\boldsymbol{w} \in C^\infty(\mathbb{R})$ such that

$$J\mathcal{H}_{c_\star}\boldsymbol{w} = \boldsymbol{e}_{3,c_\star}, \qquad \lim_{x\to+\infty} \boldsymbol{w}(x) = 0. \tag{2.8}$$

Let $\boldsymbol{E}(x) = \int_{+\infty}^{x} \boldsymbol{e}_{3,c_\star}(y)\,dy$; the function $\boldsymbol{w}(x)$ satisfies $\mathcal{H}_{c_\star}\boldsymbol{w} = \boldsymbol{E}$. Taking the pairing of $\boldsymbol{E}$ with $\boldsymbol{e}_{1,c_\star}$, we get

$$\begin{aligned}\langle \boldsymbol{e}_{1,c_\star}, \boldsymbol{E}\rangle &= -\langle \boldsymbol{\phi}_{c_\star}, \boldsymbol{e}_{3,c_\star}\rangle = \langle \mathcal{H}_{c_\star}\partial_c\boldsymbol{\phi}_{c_\star}, \boldsymbol{e}_{3,c_\star}\rangle = \langle \partial_c\boldsymbol{\phi}_{c_\star}, \mathcal{H}_{c_\star}\boldsymbol{e}_{3,c_\star}\rangle \\ &= \langle \partial_x\boldsymbol{\Theta}_{c_\star}, \boldsymbol{\Theta}_{c_\star}\rangle = \frac{\boldsymbol{\Theta}_{c_\star}^2}{2}\bigg|_{-\infty}^{+\infty} = -\lim_{x\to-\infty}\frac{\boldsymbol{\Theta}_{c_\star}^2(x)}{2} = -\frac{(I_{c_\star}')^2}{2} < 0.\end{aligned} \tag{2.9}$$

(In the first equality, the boundary term does not appear because when $x \to \pm\infty$ the function $\boldsymbol{E}(x)$ grows at most algebraically while $\boldsymbol{\phi}_c$ decays exponentially.) By Lemma B.1, since $\langle \boldsymbol{e}_{1,c_\star}, \boldsymbol{E}\rangle$ is nonzero, $\boldsymbol{w}(x)$ grows exponentially as $x \to -\infty$. This proves that the algebraic multiplicity of the eigenvalue $\lambda = 0$ is exactly three. ◻

Now we would like to consider $J\mathcal{H}_c$ in the weighted space $L^2_\mu(\mathbb{R})$, $\mu > 0$. This is equivalent to considering $A_c^\mu = e^{\mu x} \circ J\mathcal{H}_c \circ e^{-\mu x}$ in $L^2(\mathbb{R})$. In what follows, we always require that

$$0 < \mu < \min(\mu_0, \mu_1), \tag{2.10}$$

with μ_0 from Assumption 3 and μ_1 from Lemma C.1.

We define

$$\boldsymbol{e}_{j,c}^\mu = e^{\mu x}\boldsymbol{e}_{j,c}, \quad j = 1, 2; \qquad \boldsymbol{e}_{3,c_\star}^\mu = e^{\mu x}\boldsymbol{e}_{3,c_\star}. \tag{2.11}$$

From Proposition 2.2, we obtain the following statement.

Corollary 2.3.

(i) *If $\mathscr{N}_c' \neq 0$, then the basis for the generalized kernel of A_c^μ in $L^2(\mathbb{R})$ is formed by the generalized eigenvectors $\{\boldsymbol{e}_{1,c}^\mu, \boldsymbol{e}_{2,c}^\mu\}$.*

(ii) *At $c_\star$ where $\mathscr{N}_{c_\star}' = 0$, $I_{c_\star}' \neq 0$, the basis for the generalized kernel of $A_{c_\star}^\mu$ in $L^2(\mathbb{R})$ is formed by the generalized eigenvectors $\{\boldsymbol{e}_{1,c_\star}^\mu, \boldsymbol{e}_{2,c_\star}^\mu, \boldsymbol{e}_{3,c_\star}^\mu\}$.*

Proof. As follows from Lemma A.1 in Appendix A,

$$|\boldsymbol{e}_{1,c}(x)| \le \mathrm{const}\, e^{-\sqrt{c}|x|}, \qquad x \in \mathbb{R}. \tag{2.12}$$

Applying Lemma A.2 to (2.2) (for both $x \ge 0$ and $x \le 0$), we also see that

$$|\boldsymbol{e}_{2,c}(x)| \le \mathrm{const}(1 + |x|)e^{-\sqrt{c}|x|}, \qquad x \in \mathbb{R}. \tag{2.13}$$

It follows that $\boldsymbol{e}^{\mu}_{1,c}$, $\boldsymbol{e}^{\mu}_{2,c} \in L^2(\mathbb{R})$.

If $\mathscr{N}_c' \neq 0$, then by (2.6) $e^{\mu x}\boldsymbol{v}(x) \neq L^2(\mathbb{R})$.

If $\mathscr{N}_c' = 0$ at $c = c_\star$, then $\boldsymbol{e}_{3,c_\star} \in \mathscr{S}_{+,0}(\mathbb{R})$ (belongs to $\mathscr{S}$ for $x \geq 0$ and remains bounded for $x \leq 0$). Moreover, applying Lemma A.2 to (2.7), we see that

$$|\boldsymbol{e}_{3,c_\star}(x)| \leq \mathrm{const}(1+|x|)e^{-\sqrt{c_\star}\,x}, \qquad x \geq 0. \tag{2.14}$$

It follows that $\boldsymbol{e}^{\mu}_{3,c_\star} \in L^2(\mathbb{R})$. As follows from Proposition 2.2, the function $e^{\mu x}\boldsymbol{w}(x)$ in (2.8) does not belong to $L^2(\mathbb{R})$, so the algebraic multiplicity of $\lambda = 0$ is precisely three. □

LEMMA 2.4.

(i) *Let $c \in (c_\star, c_\star + \eta_1]$. Then there exists a simple positive eigenvalue λ_c of A^{μ}_c. This eigenvalue does not depend on μ.*

(ii) *λ_c is a simple eigenvalue of the operator $J\mathcal{H}_c$ considered in $L^2(\mathbb{R})$.*

(iii) *There exists a C^∞ extension of $\boldsymbol{e}_{3,c_\star}$ into an interval $[c_\star, c_\star + \eta_1]$,*

$$c \mapsto \boldsymbol{e}_{3,c} \in H^{\infty}_{\mu}(\mathbb{R}), \qquad c \in [c_\star, c_\star + \eta_1],$$

so that the frame

$$\{\boldsymbol{e}^{\mu}_{j,c} = e^{\mu x}\boldsymbol{e}_{j,c} \in H^{\infty}(\mathbb{R})\colon j = 1,2,3\}, \qquad c \in [c_\star, c_\star + \eta_1],$$

depends smoothly on c (in L^2), $X^{\mu}_c = \mathrm{span}\langle \boldsymbol{e}^{\mu}_{1,c}, \boldsymbol{e}^{\mu}_{2,c}, \boldsymbol{e}^{\mu}_{3,c}\rangle$ is the invariant subspace of A^{μ}_c, and $A^{\mu}_c|_{X^{\mu}_c}$ is represented in the frame $\{\boldsymbol{e}^{\mu}_{j,c}\}$ by the following matrix:

$$A^{\mu}_c|_{X^{\mu}_c} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \lambda_c \end{bmatrix}, \tag{2.15}$$

where λ_c equals

$$\lambda_c = -\frac{\mathscr{N}_c'}{\langle \boldsymbol{\phi}_c, \boldsymbol{e}_{3,c}\rangle}, \tag{2.16}$$

with $\langle \boldsymbol{\phi}_c, \boldsymbol{e}_{3,c}\rangle > 0$ for $c \in [c_\star, c_\star + \eta_1]$.

Proof. Due to the restriction (2.10) on μ, the essential spectrum of A^{μ}_c for $c \geq c_\star$ is given by (1.14) and is located strictly to the left of the imaginary axis. By Assumption 3, the discrete spectrum of $A^{\mu}_{c_\star}$ consists of the isolated eigenvalue $\lambda = 0$, which is of algebraic multiplicity three by Corollary 2.3. We choose a closed contour $\gamma \subset \rho(A^{\mu}_{c_\star})$ in $\mathbb{C}^1$ so that the interval $[0, \Lambda]$ of the real axis is strictly inside γ, where

$$\Lambda = \sup_{c \in \Sigma}\sup_{x \in \mathbb{R}} |f''(\boldsymbol{\phi}_c(x))\boldsymbol{\phi}_c'(x)|. \tag{2.17}$$

Remark 2.5. The value of Λ is chosen so that all point eigenvalues of the operator $J\mathcal{H}_c$, $c \in \Sigma$, are bounded by Λ. Indeed, if $\boldsymbol{\psi}$ satisfies $J\mathcal{H}_c\boldsymbol{\psi} = \lambda\boldsymbol{\psi}$ with $\lambda \in \mathbb{R}$, then $\boldsymbol{\psi} \in H^{\infty}(\mathbb{R})$ and can be assumed to be real-valued. Therefore, we have

$$\begin{aligned}\lambda\langle\boldsymbol{\psi},\boldsymbol{\psi}\rangle &= \langle \boldsymbol{\psi}, \partial_x(-\partial_x^2 + f'(\boldsymbol{\phi}_c) + c)\boldsymbol{\psi}\rangle \\ &= -\langle \boldsymbol{\psi}', f'(\boldsymbol{\phi}_c)\boldsymbol{\psi}\rangle = -\langle \boldsymbol{\psi}\boldsymbol{\psi}', f'(\boldsymbol{\phi}_c)\rangle = \frac{1}{2}\int_{\mathbb{R}} \boldsymbol{\psi}^2 \partial_x f'(\boldsymbol{\phi}_c)\,dx,\end{aligned}$$

so that $|\lambda| \le \sup_{x\in\mathbb{R}} |f''(\boldsymbol{\phi}_c(x))\boldsymbol{\phi}_c'(x)|/2$.

We notice that for c from an open neighborhood of $c_\star$, γ belongs to the resolvent set $\rho(A_c^\mu)$. Indeed, we have

$$\frac{1}{A_c^\mu - z} = \frac{1}{A_{c_\star}^\mu - z + (A_c^\mu - A_{c_\star}^\mu)} = \frac{1}{(A_{c_\star}^\mu - z)}\frac{1}{(1 + (A_{c_\star}^\mu - z)^{-1}(A_c^\mu - A_{c_\star}^\mu))}. \tag{2.18}$$

Since $A_{c_\star}^\mu - z$, $z \in \gamma$, is invertible in L^2 and is smoothing of order three, while $A_c^\mu - A_{c_\star}^\mu$ depends continuously on c as a differential operator of order 1, the operator $(A_{c_\star}^\mu - z)^{-1}(A_c^\mu - A_{c_\star}^\mu)$ is bounded by $1/2$ as an operator in L^2 for all $z \in \gamma$ and for all c sufficiently close to $c_\star$. We assume that $\eta_1 > 0$ is small enough so that

$$\gamma \in \rho(A_c^\mu) \quad \text{for} \quad c \in [c_\star, c_\star + \eta_1]. \tag{2.19}$$

Integrating $(A_c^\mu - z)^{-1}$ along γ, we get a projection

$$P_c^\mu = -\frac{1}{2\pi i}\oint_\gamma \frac{dz}{A_c^\mu - z}, \qquad c \in [c_\star, c_\star + \eta_1]. \tag{2.20}$$

Since $\operatorname{rank} P_{c_\star}^\mu = 3$, we also have

$$\operatorname{rank} P_c^\mu = 3, \qquad c \in [c_\star, c_\star + \eta_1].$$

The three-dimensional spectral subspace $\operatorname{Range} P_{c_\star}^\mu$ corresponds to the eigenvalue $\lambda = 0$ that has algebraic multiplicity three. According to Corollary 2.3, when $\mathscr{N}_c' \ne 0$, $\lambda = 0$ is of algebraic multiplicity two, and therefore $X_c^\mu \equiv \operatorname{Range} P_c^\mu$ splits into a two-dimensional spectral subspace of A_c^μ corresponding to $\lambda = 0$ (it is spanned by $\{\boldsymbol{e}_{1,c}^\mu, \boldsymbol{e}_{2,c}^\mu\}$) and a one-dimensional subspace that corresponds to a nonzero eigenvalue.

For $c \in [c_\star, c_\star + \eta_1]$, we define

$$\tilde{\boldsymbol{e}}_{3,c}^\mu = P_c^\mu \boldsymbol{e}_{3,c_\star}^\mu, \qquad c \in [c_\star, c_\star + \eta_1]. \tag{2.21}$$

Note that $\tilde{\boldsymbol{e}}_{3,c}^\mu \in L^2(\mathbb{R})$ since P_c^μ is continuous in L^2. In the frame $\{\boldsymbol{e}_{1,c}^\mu, \boldsymbol{e}_{2,c}^\mu, \tilde{\boldsymbol{e}}_{3,c}^\mu\}$ we can write

$$A_c^\mu \tilde{\boldsymbol{e}}_{3,c}^\mu = a_c \boldsymbol{e}_{1,c}^\mu + b_c \boldsymbol{e}_{2,c}^\mu + \lambda_c \tilde{\boldsymbol{e}}_{3,c}^\mu. \tag{2.22}$$

Since the frame $\{\boldsymbol{e}_{1,c}^\mu, \boldsymbol{e}_{2,c}^\mu, \tilde{\boldsymbol{e}}_{3,c}^\mu\}$ and also $A_c^\mu \tilde{\boldsymbol{e}}_{3,c}^\mu$ depend smoothly on c (as functions from $[c_\star, c_\star + \eta_1]$ to $L^2(\mathbb{R})$; recall that f is smooth), the coefficients a_c, b_c, and λ_c are smooth functions of c for $c \in [c_\star, c_\star + \eta_1]$. It is also important to point out that a_c, b_c, and λ_c do not depend on $\mu > 0$, since if the relation (2.22) holds for certain values of a_c, b_c, and λ_c for a particular value $\mu > 0$, then, by the definition of A_c^μ, $\boldsymbol{e}_{1,c}^\mu$, $\boldsymbol{e}_{2,c}^\mu$, and $\tilde{\boldsymbol{e}}_{3,c}^\mu$, the relation (2.22) also holds for μ' from an open neighborhood of μ.

According to the construction of $\boldsymbol{e}_{3,c_\star}$ in Proposition 2.2, $a_{c_\star} = \lambda_{c_\star} = 0$ and $b_{c_\star} = 1$. We define

$$\boldsymbol{e}_{3,c}^\mu = \frac{1}{b_c + a_c\lambda_c}(\tilde{\boldsymbol{e}}_{3,c}^\mu - a_c \boldsymbol{e}_{2,c}^\mu).$$

Then $\boldsymbol{e}_{3,c}^\mu \in L^2(\mathbb{R})$ for $c \in [c_\star, c_\star + \eta_1]$. We compute

$$A_c^\mu \boldsymbol{e}_{3,c}^\mu = \boldsymbol{e}_{2,c}^\mu + \lambda_c \boldsymbol{e}_{3,c}^\mu. \tag{2.23}$$

Thus, in the frame $\{\boldsymbol{e}^\mu_{j,c}\colon\ j=1,2,3\}$ the operator $A^\mu_c|_{\mathrm{Range}\,P^\mu_c}$ has the desired matrix form (2.15). Conjugating by means of $e^{\mu x}$ we get a corresponding frame $\{\boldsymbol{e}_{j,c}\colon\ j=1,2,3\}$ in L^2_μ, with $\boldsymbol{e}_{3,c}$ satisfying

$$J\mathcal{H}_c\boldsymbol{e}_{3,c}=\boldsymbol{e}_{2,c}+\lambda_c\boldsymbol{e}_{3,c},\qquad \boldsymbol{e}_{3,c}\in L^2_\mu(\mathbb{R}). \tag{2.24}$$

For $c\in[c_\star,c_\star+\eta_1]$ and $z\notin\sigma(A^\mu_c)$, $R^\mu_c(z)=(A^\mu_c-z)^{-1}$ is a pseudodifferential operator of order -3, and hence P^μ_c is smoothing of order three in the Sobolev spaces $H^s(\mathbb{R})$. The bootstrapping argument applied to the relations $\boldsymbol{e}^\mu_{j,c}=P^\mu_c\boldsymbol{e}^\mu_{j,c}$ shows that $\boldsymbol{e}^\mu_{j,c}\in H^\infty(\mathbb{R})$. By definition (1.12), this means that

$$\boldsymbol{e}_{j,c}\in H^\infty_\mu(\mathbb{R}),\qquad j=1,2,3,\quad c\in[c_\star,c_\star+\eta_1]. \tag{2.25}$$

Using (2.24), we compute

$$\begin{aligned}0&=\langle\mathcal{H}_c\boldsymbol{e}_{1,c},\boldsymbol{e}_{3,c}\rangle=-\langle\mathcal{H}_cJ\boldsymbol{\phi}_c,\boldsymbol{e}_{3,c}\rangle\\&=\langle\boldsymbol{\phi}_c,J\mathcal{H}_c\boldsymbol{e}_{3,c}\rangle=\langle\boldsymbol{\phi}_c,\boldsymbol{e}_{2,c}\rangle+\lambda_c\langle\boldsymbol{\phi}_c,\boldsymbol{e}_{3,c}\rangle,\quad c\in[c_\star,c_\star+\eta_1].\end{aligned}$$

We conclude that

$$\lambda_c=-\frac{\langle\boldsymbol{\phi}_c,\boldsymbol{e}_{2,c}\rangle}{\langle\boldsymbol{\phi}_c,\boldsymbol{e}_{3,c}\rangle},\qquad c\in[c_\star,c_\star+\eta_1],$$

where $\langle\boldsymbol{\phi}_c,\boldsymbol{e}_{2,c}\rangle=\langle\boldsymbol{\phi}_c,\partial_c\boldsymbol{\phi}_c\rangle=\mathscr{N}_c'<0$. Note that $\langle\boldsymbol{\phi}_c,\boldsymbol{e}_{3,c}\rangle>0$ for $c_\star<c\le c_\star+\eta_1$, since $\langle\boldsymbol{\phi}_{c_\star},\boldsymbol{e}_{3,c_\star}\rangle>0$ by (2.9) and $\langle\boldsymbol{\phi}_c,\boldsymbol{e}_{3,c}\rangle$ does not change sign for $c_\star<c\le c_\star+\eta_1$ (this follows from the inequality $|\langle\boldsymbol{\phi}_c,\boldsymbol{e}_{3,c}\rangle|>|\mathscr{N}_c'|/\Lambda>0$; see Remark 2.5). This finishes the proof of the lemma. ∎

Remark 2.6. According to Assumption 3, we may assume that η_1 is small enough so that for $c\in[c_\star,c_\star+\eta_1]$ and $0\le\mu\le\mu_0$ there is no discrete spectrum of A^μ_c except $\lambda=0$ and $\lambda=\lambda_c$. It follows that P^μ_c is the spectral projector that corresponds to the discrete spectrum of A^μ_c.

LEMMA 2.7. *If* $\lambda_c>0$, *then* $\boldsymbol{e}_{3,c}\in H^\infty(\mathbb{R})$.

Proof. By Lemma 2.4, $\lambda_c>0$ is a simple eigenvalue of $J\mathcal{H}_c$ considered in $L^2(\mathbb{R})$. By (2.1), (2.2), and (2.24),

$$\boldsymbol{\psi}_c=\boldsymbol{e}_{c,1}+\lambda_c\boldsymbol{e}_{c,2}+\lambda_c^2\boldsymbol{e}_{c,3}\in C^\infty(\mathbb{R}) \tag{2.26}$$

satisfies $J\mathcal{H}_c\boldsymbol{\psi}_c=\lambda_c\boldsymbol{\psi}_c$, and also $\lim_{x\to+\infty}\boldsymbol{\psi}_c(x)=0$. Thus, $\boldsymbol{\psi}_c$ coincides with an L^2 eigenvector of $J\mathcal{H}_c\boldsymbol{\psi}_c$ that corresponds to λ_c. Therefore, $\boldsymbol{\psi}_c\in H^\infty(\mathbb{R})$. Since $\boldsymbol{e}_{c,1},\boldsymbol{e}_{c,2}\in H^1(\mathbb{R})$ and $\lambda_c\ne0$, the statement of the lemma follows from the relation (2.26). ∎

Let us also introduce the dual basis that consists of eigenvectors of the adjoint operator $(J\mathcal{H}_c)^*=-\mathcal{H}_cJ=-\mathcal{H}_c\partial_x$ which we consider in the weighted space

$$L^2_{-\mu}(\mathbb{R})=\left\{\boldsymbol{\psi}\in L^2_{loc}(\mathbb{R})\colon e^{-\mu x}\boldsymbol{\psi}(x)\in L^2(\mathbb{R})\right\},\quad \mu>0. \tag{2.27}$$

For any $c\in\Sigma$, the generalized kernel of $(J\mathcal{H}_c)^*$ contains at least two linearly independent vectors:

$$-\mathcal{H}_c\partial_x\boldsymbol{g}_{1,c}=0,\qquad -\mathcal{H}_c\partial_x\boldsymbol{g}_{2,c}=\boldsymbol{g}_{1,c}, \tag{2.28}$$

where

$$\boldsymbol{g}_{1,c}(x) = -\int_{-\infty}^{x} \boldsymbol{e}_{1,c}(y,c)\,dy = \boldsymbol{\phi}_c(x), \tag{2.29}$$

$$\boldsymbol{g}_{2,c}(x) = \int_{-\infty}^{x} \boldsymbol{e}_{2,c}(y,c)\,dy = \int_{-\infty}^{x} \partial_c \boldsymbol{\phi}_c(y)\,dy. \tag{2.30}$$

The lower limit of integration ensures that $\lim_{x\to-\infty} \boldsymbol{g}_{2,c}(x) = 0$, so that $\boldsymbol{g}_{2,c} \in L^2_{-\mu}(\mathbb{R})$.

Proposition 2.8. *Assume that $c_\star \in \Sigma$ is such that $\mathscr{N}'_{c_\star} = 0$, $I'_{c_\star} \neq 0$. The eigenvalue $\lambda = 0$ of the operator $-\mathcal{H}_{c_\star}\partial_x$ is of algebraic multiplicity three in $L^2_{-\mu}(\mathbb{R})$, and there exists $\boldsymbol{g}_{3,c_\star} \in H^\infty_{-\mu}(\mathbb{R})$ such that*

$$-\mathcal{H}_{c_\star}\partial_x \boldsymbol{g}_{3,c_\star} = \boldsymbol{g}_{2,c_\star}.$$

Proof. The argument repeats the steps of the proof of Proposition 2.2. The function $\boldsymbol{g}_{3,c_\star}$ is given by

$$\boldsymbol{g}_{3,c_\star}(x) = -\int_{-\infty}^{x} \tilde{\boldsymbol{e}}_{3,c_\star}(y)dy, \tag{2.31}$$

where $\tilde{\boldsymbol{e}}_{3,c_\star}(x)$ satisfies

$$\mathcal{H}_{c_\star}\tilde{\boldsymbol{e}}_{3,c_\star} = \int_{-\infty}^{x} \boldsymbol{e}_{2,c_\star}(y)\,dy, \qquad \lim_{x\to-\infty} \tilde{\boldsymbol{e}}_{3,c_\star}(x) = 0. \tag{2.32}$$

Since $\int_{-\infty}^{x} \boldsymbol{e}_{2,c_\star}(y)\,dy$ remains bounded as $x \to +\infty$, while $\langle \boldsymbol{g}_{2,c_\star}, \boldsymbol{\phi}_{c_\star}\rangle = 0$, the function $\tilde{\boldsymbol{e}}_{3,c_\star}(x)$ remains bounded as $x \to +\infty$. This follows from Lemma B.1 of Appendix B (after the reflection $x \to -x$). Therefore, $\boldsymbol{g}_{3,c_\star}(x)$ has a linear growth as $x \to +\infty$; $\boldsymbol{g}_{3,c_\star} \in \mathscr{S}_{-,1}(\mathbb{R})$ (defined similarly to $\mathscr{S}_{+,1}$ in Definition 2.1). ◻

As in Lemma 2.4, one can show that there is an extension of $\boldsymbol{g}_{3,c_\star}$ into an interval $[c_\star, c_\star + \eta_1]$,

$$c \mapsto \boldsymbol{g}_{3,c} \in H^\infty_{-\mu}(\mathbb{R}), \qquad c \in [c_\star, c_\star + \eta_1],$$

so that, similarly to (2.24) and (2.25),

$$-\mathcal{H}_c\partial_x \boldsymbol{g}_{3,c} = \boldsymbol{g}_{2,c}(x) + \lambda_c \boldsymbol{g}_{3,c}, \qquad \boldsymbol{g}_{3,c} \in H^\infty_{-\mu}(\mathbb{R}), \quad c \in [c_\star, c_\star + \eta_1]. \tag{2.33}$$

Using the bases $\{\boldsymbol{e}_{j,c} \in H^\infty_{\mu}(\mathbb{R}) \colon j = 1,2,3\}$, $\{\boldsymbol{g}_{j,c} \in H^\infty_{-\mu}(\mathbb{R}) \colon j = 1,2,3\}$, we can write the projection operator $e^{-\mu x} \circ P^\mu_c \circ e^{\mu x}$ that corresponds to the discrete spectrum of $J\mathcal{H}_c$ in the form

$$(e^{-\mu x} \circ P^\mu_c \circ e^{\mu x})\boldsymbol{\psi} = \sum_{j,k=1}^{3} \mathcal{T}^{jk}_c \langle \boldsymbol{g}_{k,c}, \boldsymbol{\psi}\rangle \boldsymbol{e}_{j,c}, \tag{2.34}$$

with $\mathcal{T}^{jk}_c$ being the inverse of the matrix

$$\mathcal{T}_c = \{\mathcal{T}_{jk,c}\}, \quad \mathcal{T}_{jk,c} = \langle \boldsymbol{g}_{j,c}, \boldsymbol{e}_{k,c}\rangle, \quad c \in [c_\star, c_\star + \eta_1], \quad 1 \le j,k \le 3. \tag{2.35}$$

Let us introduce the functions

$$\alpha_c = \langle \boldsymbol{g}_{1,c}, \boldsymbol{e}_{3,c}\rangle, \qquad \beta_c = \langle \boldsymbol{g}_{2,c}, \boldsymbol{e}_{3,c}\rangle, \qquad \gamma_c = \langle \boldsymbol{g}_{3,c}, \boldsymbol{e}_{3,c}\rangle. \tag{2.36}$$

Since $\boldsymbol{e}_{j,c} \in L^2_\mu(\mathbb{R})$ and $\boldsymbol{g}_{j,c} \in L^2_{-\mu}(\mathbb{R})$, α_c, β_c, and γ_c are continuous functions of c for $c \in [c_\star, c_\star + \eta_1]$. Recalling that $\langle \boldsymbol{g}_{2,c}, \boldsymbol{e}_{1,c}\rangle = \langle \boldsymbol{g}_{2,c}, J\mathcal{H}_c \boldsymbol{e}_{2,c}\rangle = -\langle \mathcal{H}_c J \boldsymbol{g}_{2,c}, \boldsymbol{e}_{2,c}\rangle = \langle \boldsymbol{g}_{1,c}, \boldsymbol{e}_{2,c}\rangle = \langle \boldsymbol{\phi}_c, \partial_c \boldsymbol{\phi}_c\rangle = \mathcal{N}_c'$, $\langle \boldsymbol{g}_{1,c}, \boldsymbol{e}_{1,c}\rangle = -\langle \boldsymbol{\phi}_c, \partial_x \boldsymbol{\phi}_c\rangle = 0$, we may write the matrix $\mathcal{T}$ in the following form:

$$\mathcal{T}_c = \begin{bmatrix} 0 & \mathcal{N}_c' & \alpha_c \\ \mathcal{N}_c' & \frac{1}{2}(I_c')^2 & \beta_c \\ \alpha_c & \beta_c & \gamma_c \end{bmatrix}. \tag{2.37}$$

Note that $\mathcal{T}_{c_\star}$ is nondegenerate, because $\mathcal{N}_{c_\star}' = 0$ by the choice of $c_\star$, while $\alpha_{c_\star} = \langle \boldsymbol{g}_{1,c_\star}, \boldsymbol{e}_{3,c_\star}\rangle = \langle \boldsymbol{\phi}_{c_\star}, \boldsymbol{e}_{3,c_\star}\rangle = \frac{1}{2}(I_{c_\star}')^2 > 0$ by (2.9).

3. Center manifold reduction. We first discuss the existence of a solution $\boldsymbol{u}(t)$ that corresponds to perturbed initial data. We will rely on the well-posedness results due to T. Kato.

LEMMA 3.1. *For any $\mu > 0$, $s \ge 2$, and $\boldsymbol{u}_0 \in H^s(\mathbb{R}) \cap L^2_{2\mu}(\mathbb{R})$ with $\|\boldsymbol{u}_0\|_{H^1} < 2\|\boldsymbol{\phi}_{c_\star}\|_{H^1}$, there exists a function*

$$\boldsymbol{u}(t) \in C([0,\infty), H^s(\mathbb{R}) \cap L^2_{2\mu}(\mathbb{R})), \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{3.1}$$

which solves (1.1) *for $0 \le t < t_1$, where t_1 is finite or infinite, defined by*

$$t_1 = \sup\{t \ge 0 : \ \|\boldsymbol{u}(t)\|_{H^1} < 2\|\boldsymbol{\phi}_{c_\star}\|_{H^1}\}. \tag{3.2}$$

Proof. According to [Kat83, Theorem 10.1], equation (1.1) is globally well-posed in $H^s(\mathbb{R}) \cap L^2_{2\mu}(\mathbb{R})$ for any $s \ge 2$, $\mu > 0$ (for the initial data with arbitrarily large norm) if f satisfies

$$\lim_{|z|\to\infty} |z|^{-4} f'(z) \ge 0. \tag{3.3}$$

We modify the nonlinearity $f(z)$ for $|z| > 2\|\boldsymbol{\phi}_{c_\star}\|_{H^1}$ so that (3.3) is satisfied; let us call this modified nonlinearity $\tilde{f}(z)$. Thus, for any $\boldsymbol{u}_0 \in H^s(\mathbb{R}) \cap L^2_{2\mu}(\mathbb{R})$ with $\|\boldsymbol{u}_0\|_{H^1} < 2\|\boldsymbol{\phi}_{c_\star}\|_{H^1}$, there exists a function

$$\boldsymbol{u}(t) \in C([0,\infty), H^s(\mathbb{R}) \cap L^2_{2\mu}(\mathbb{R})), \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{3.4}$$

that solves the equation with the modified nonlinearity

$$\partial_t \boldsymbol{u} = \partial_x\big(-\partial_x^2 \boldsymbol{u} + \tilde{f}(\boldsymbol{u})\big). \tag{3.5}$$

For $0 \le t < t_1$, with t_1 defined by (3.2), one has $\|\boldsymbol{u}(t)\|_{L^\infty} \le \|\boldsymbol{u}(t)\|_{H^1} < 2\|\boldsymbol{\phi}_{c_\star}\|_{H^1}$. Therefore, for $0 \le t < t_1$, $\boldsymbol{u}(t)$ solves both (3.5) and (1.1) since $\tilde{f}(z) = f(z)$ for $|z| \le 2\|\boldsymbol{\phi}_{c_\star}\|_{H^1}$. $\square$

We fix μ satisfying (2.10). For the initial data $\boldsymbol{u}_0 \in H^2(\mathbb{R}) \cap L^2_{2\mu}(\mathbb{R})$ with $\|\boldsymbol{u}_0\|_{H^1} < 2\|\boldsymbol{\phi}_{c_\star}\|_{H^1}$ there is a function $\boldsymbol{u} \in C([0,\infty), H^2(\mathbb{R}) \cap L^2_{2\mu}(\mathbb{R}))$ that solves (1.1) for $0 \le t < t_1$, with t_1 from (3.2). We will approximate the solution $\boldsymbol{u}(x,t)$ by a traveling wave $\boldsymbol{\phi}_c$ moving with the variable speed $c = c(t)$. Thus, we decompose the solution $\boldsymbol{u}(x,t)$ into the traveling wave $\boldsymbol{\phi}_c(x)$ and the perturbation $\boldsymbol{\rho}(x,t)$ as follows:

$$\boldsymbol{u}(x,t) = \boldsymbol{\phi}_{c(t)}\bigg(x - \xi(t) - \int_0^t c(t')\,dt'\bigg) + \boldsymbol{\rho}\bigg(x - \xi(t) - \int_0^t c(t')\,dt',\, t\bigg). \tag{3.6}$$

The functions $\xi(t)$ and $c(t)$ are yet to be chosen.

Using (3.6), we rewrite the generalized KdV equation (1.1) as an equation on $\boldsymbol{\rho}$:

$$\dot{\boldsymbol{\rho}} - J\mathcal{H}_c\boldsymbol{\rho} = -\dot{\xi}\boldsymbol{e}_{1,c} - \dot{c}\boldsymbol{e}_{2,c} + \dot{\xi}\partial_x\boldsymbol{\rho} + J\boldsymbol{N}, \tag{3.7}$$

with $\mathcal{H}_c$ given by (1.11) and with $J\boldsymbol{N}$ given by

$$J\boldsymbol{N} = \partial_x\left[f(\boldsymbol{\phi}_c + \boldsymbol{\rho}) - f(\boldsymbol{\phi}_c) - \boldsymbol{\rho}f'(\boldsymbol{\phi}_c)\right], \tag{3.8}$$

where we changed coordinates, denoting $y = x - \xi(t) - \int_0^t c(t')\,dt'$ by x. By Proposition 2.2 (iii), the eigenvalue $\lambda = 0$ of operator $J\mathcal{H}_{c_\star}$ in $L^2_\mu(\mathbb{R})$ has algebraic multiplicity three. We decompose the perturbation $\boldsymbol{\rho}(x,t)$ as follows:

$$\boldsymbol{\rho}(x,t) = \zeta(t)\boldsymbol{e}_{3,c(t)}(x) + \boldsymbol{v}(x,t), \tag{3.9}$$

where $\boldsymbol{e}_{3,c}$ is constructed in Lemma 2.4. The inclusions $\boldsymbol{\phi}_c \in H^2(\mathbb{R}) \cap L^2_{2\mu}(\mathbb{R}) \subset H^1_\mu(\mathbb{R})$ and $\boldsymbol{e}_{3,c} \in H^1_\mu(\mathbb{R})$ show that $\boldsymbol{v}(\cdot,t) \in H^1_\mu(\mathbb{R})$.

We would like to choose $\xi(t)$, $c(t) = c_\star + \eta(t)$, and $\zeta(t)$ so that

(3.10)
$$\boldsymbol{v}(x,t) = \boldsymbol{u}\left(x + \xi(t) + \int_0^t (c_\star + \eta(t'))\,dt',\, t\right) - \boldsymbol{\phi}_{c_\star+\eta(t)}(x) - \zeta(t)\boldsymbol{e}_{3,c_\star+\eta(t)}(x)$$

represents the part of the perturbation that corresponds to the continuous spectrum of $J\mathcal{H}_c$.

Proposition 3.2. *There exist $\eta_1 > 0$, $\zeta_1 > 0$, and $\delta_1 > 0$ such that if η_0 and ζ_0 satisfy*

$$|\eta_0| < \eta_1, \qquad |\zeta_0| < \zeta_1, \qquad \|\boldsymbol{\phi}_{c_\star+\eta_0} + \zeta_0\boldsymbol{e}_{3,c_\star+\eta_0} - \boldsymbol{\phi}_{c_\star}\|_{H^1} < \|\boldsymbol{\phi}_{c_\star}\|_{H^1}, \tag{3.11}$$

then there is $T_1 \in \mathbb{R}_+ \cup \{+\infty\}$ such that the following hold:

(i) *There exists $\boldsymbol{u} \in C([0,\infty), H^2(\mathbb{R}) \cap L^2_{2\mu}(\mathbb{R}))$ so that*

$$\boldsymbol{u}(0) = \boldsymbol{\phi}_{c_\star+\eta_0} + \zeta_0\boldsymbol{e}_{3,c_\star+\eta_0} \tag{3.12}$$

and $\boldsymbol{u}(t)$ solves (1.1) for $0 \le t < T_1$.

(ii) *There exist functions*

$$\xi,\eta,\zeta \in C([0,\infty)), \qquad \xi(0) = 0, \quad \eta(0) = \eta_0, \quad \zeta(0) = \zeta_0, \tag{3.13}$$

such that the function $\boldsymbol{v}(t)$ defined by (3.10) satisfies

$$e^{\mu x}\boldsymbol{v}(x,t) \in \ker P^\mu_{c_\star+\eta(t)}, \qquad 0 \le t < T_1. \tag{3.14}$$

(iii) *The following inequalities hold for $0 \le t < T_1$:*

$$\|\boldsymbol{u}(t)\|_{H^1} < 2\|\boldsymbol{\phi}_{c_\star}\|_{H^1}, \quad |\eta(t)| < \eta_1, \quad |\zeta(t)| < \zeta_1, \quad \|\boldsymbol{v}(t)\|_{H^1_\mu} < \delta_1. \tag{3.15}$$

(iv) *If one cannot choose $T_1 = \infty$, then at least one of the inequalities in (3.15) turns into an equality at $t = T_1$.*

Proof. Since $\boldsymbol{u}_0 = \boldsymbol{\phi}_{c_\star+\eta_0} + \zeta_0\boldsymbol{e}_{3,c_\star+\eta_0} \in H^2(\mathbb{R}) \cap L^2_{2\mu}(\mathbb{R})$ and the conditions (3.11) are satisfied, by Lemma 3.1, there is a function $\boldsymbol{u}(t) \in C([0,\infty), H^2(\mathbb{R}) \cap L^2_{2\mu}(\mathbb{R}))$ and $t_1 \in \mathbb{R}_+ \cup \{+\infty\}$ such that $\boldsymbol{u}(t)$ solves (1.1) for $0 \le t < t_1$ and, if $t_1 < \infty$, then

$\|\boldsymbol{u}(t_1)\|_{H^1} = 2\|\boldsymbol{\phi}_{c_\star}\|_{H^1}$. We thus need to construct $\xi(t)$, $\eta(t)$, and $\zeta(t)$ so that $\boldsymbol{v}(x,t)$ defined by (3.10) satisfies the constraints

$$\langle \boldsymbol{g}_{1,c_\star+\eta(t)}, \boldsymbol{v}(t)\rangle = \langle \boldsymbol{g}_{2,c_\star+\eta(t)}, \boldsymbol{v}(t)\rangle = \langle \boldsymbol{g}_{3,c_\star+\eta(t)}, \boldsymbol{v}(t)\rangle = 0. \tag{3.16}$$

Let us note that $\boldsymbol{v}(0) = 0$ by (3.10), (3.12), and (3.13). Since $J\mathcal{H}_c\boldsymbol{e}_{3,c} = \lambda_c\boldsymbol{e}_{3,c} + \boldsymbol{e}_{2,c}$,

$$\partial_t(\zeta\boldsymbol{e}_{3,c}) - J\mathcal{H}_c(\zeta\boldsymbol{e}_{3,c}) = \dot\zeta\boldsymbol{e}_{3,c} + \dot\eta\zeta\partial_c\boldsymbol{e}_{3,c} - \zeta(\lambda_c\boldsymbol{e}_{3,c} + \boldsymbol{e}_{2,c}). \tag{3.17}$$

Therefore, (3.7) can be written as the following equation on $\boldsymbol{v}(t) = \boldsymbol{\rho} - \zeta\boldsymbol{e}_{3,c}$:

$$\dot{\boldsymbol{v}} - J\mathcal{H}_c\boldsymbol{v} = -\dot\xi\boldsymbol{e}_{1,c} - (\dot\eta - \zeta)\,\boldsymbol{e}_{2,c} - (\dot\zeta - \lambda_c\zeta)\boldsymbol{e}_{3,c} - \dot\eta\zeta\partial_c\boldsymbol{e}_{3,c} + \dot\xi\partial_x\boldsymbol{\rho} + J\boldsymbol{N}. \tag{3.18}$$

Differentiating the constraints (3.16) and using the evolution equation (3.18), we derive the center manifold reduction,

$$\mathcal{T}_c\begin{bmatrix}\dot\xi\\ \dot\eta-\zeta\\ \dot\zeta-\lambda_c\zeta\end{bmatrix} - \dot\eta\begin{bmatrix}\langle\partial_c\boldsymbol{g}_{1,c},\boldsymbol{v}\rangle\\ \langle\partial_c\boldsymbol{g}_{2,c},\boldsymbol{v}\rangle\\ \langle\partial_c\boldsymbol{g}_{3,c},\boldsymbol{v}\rangle\end{bmatrix} = -\dot\eta\zeta\begin{bmatrix}\langle\boldsymbol{g}_{1,c},\partial_c\boldsymbol{e}_{3,c}\rangle\\ \langle\boldsymbol{g}_{2,c},\partial_c\boldsymbol{e}_{3,c}\rangle\\ \langle\boldsymbol{g}_{3,c},\partial_c\boldsymbol{e}_{3,c}\rangle\end{bmatrix} + \dot\xi\begin{bmatrix}\langle\boldsymbol{g}_{1,c},\partial_x\boldsymbol{\rho}\rangle\\ \langle\boldsymbol{g}_{2,c},\partial_x\boldsymbol{\rho}\rangle\\ \langle\boldsymbol{g}_{3,c},\partial_x\boldsymbol{\rho}\rangle\end{bmatrix} + \begin{bmatrix}\langle\boldsymbol{g}_{1,c},J\boldsymbol{N}\rangle\\ \langle\boldsymbol{g}_{2,c},J\boldsymbol{N}\rangle\\ \langle\boldsymbol{g}_{3,c},J\boldsymbol{N}\rangle\end{bmatrix}, \tag{3.19}$$

where the matrix $\mathcal{T}_c$ is given by (2.35). The above can be rewritten as

$$\mathcal{S}\begin{bmatrix}\dot\xi\\ \dot\eta-\zeta\\ \dot\zeta-\lambda_c\zeta\end{bmatrix} = \begin{bmatrix}-\zeta^2\langle\boldsymbol{g}_{1,c},\partial_c\boldsymbol{e}_{3,c}\rangle + \zeta\langle\partial_c\boldsymbol{g}_{1,c},\boldsymbol{v}\rangle + \langle\boldsymbol{g}_{1,c},J\boldsymbol{N}\rangle\\ -\zeta^2\langle\boldsymbol{g}_{2,c},\partial_c\boldsymbol{e}_{3,c}\rangle + \zeta\langle\partial_c\boldsymbol{g}_{2,c},\boldsymbol{v}\rangle + \langle\boldsymbol{g}_{2,c},J\boldsymbol{N}\rangle\\ -\zeta^2\langle\boldsymbol{g}_{3,c},\partial_c\boldsymbol{e}_{3,c}\rangle + \zeta\langle\partial_c\boldsymbol{g}_{3,c},\boldsymbol{v}\rangle + \langle\boldsymbol{g}_{3,c},J\boldsymbol{N}\rangle\end{bmatrix}, \tag{3.20}$$

where $c = c_\star + \eta$ and

$$\mathcal{S}(\eta,\zeta,\boldsymbol{v}) = \mathcal{T}_c + \begin{bmatrix}-\langle\boldsymbol{g}_{1,c},\partial_x(\zeta\boldsymbol{e}_{3,c}+\boldsymbol{v})\rangle & \zeta\langle\boldsymbol{g}_{1,c},\partial_c\boldsymbol{e}_{3,c}\rangle - \langle\partial_c\boldsymbol{g}_{1,c},\boldsymbol{v}\rangle & 0\\ -\langle\boldsymbol{g}_{2,c},\partial_x(\zeta\boldsymbol{e}_{3,c}+\boldsymbol{v})\rangle & \zeta\langle\boldsymbol{g}_{2,c},\partial_c\boldsymbol{e}_{3,c}\rangle - \langle\partial_c\boldsymbol{g}_{2,c},\boldsymbol{v}\rangle & 0\\ -\langle\boldsymbol{g}_{3,c},\partial_x(\zeta\boldsymbol{e}_{3,c}+\boldsymbol{v})\rangle & \zeta\langle\boldsymbol{g}_{3,c},\partial_c\boldsymbol{e}_{3,c}\rangle - \langle\partial_c\boldsymbol{g}_{3,c},\boldsymbol{v}\rangle & 0\end{bmatrix}. \tag{3.21}$$

Note that the matrix $\mathcal{S}(\eta,\zeta,\boldsymbol{v})$ depends continuously on $(\eta,\zeta,\boldsymbol{v}) \in \mathbb{R}^2 \times H^1_\mu(\mathbb{R})$. Since the matrix $\mathcal{T}_{c_\star}$ is nonsingular (see (2.37)), the matrix $\mathcal{S}(\eta,\zeta,\boldsymbol{v})$ is invertible for sufficiently small values of $|\eta|$, $|\zeta|$, and $\|\boldsymbol{v}\|_{H^1_\mu}$.

Thus, there exist $\eta_1 > 0$, $\zeta_1 > 0$, and $\delta_1 > 0$ so that the matrix $\mathcal{S}(\eta,\zeta,\boldsymbol{v})$ is invertible if

$$|\eta| \le 2\eta_1, \qquad |\zeta| \le 2\zeta_1, \qquad \|\boldsymbol{v}\|_{H^1_\mu} \le 2\delta_1. \tag{3.22}$$

For such η, ζ, and $\boldsymbol{v}$, we can write

$$\begin{bmatrix}\dot\xi\\ \dot\eta-\zeta\\ \dot\zeta-\lambda_c\zeta\end{bmatrix} = \begin{bmatrix}R_1(\eta,\zeta,\boldsymbol{v})\\ R_2(\eta,\zeta,\boldsymbol{v})\\ R_3(\eta,\zeta,\boldsymbol{v})\end{bmatrix}, \tag{3.23}$$

where the right-hand side is given by

$$\begin{bmatrix}R_1(\eta,\zeta,\boldsymbol{v})\\ R_2(\eta,\zeta,\boldsymbol{v})\\ R_3(\eta,\zeta,\boldsymbol{v})\end{bmatrix} = \mathcal{S}(\eta,\zeta,\boldsymbol{v})^{-1}\begin{bmatrix}-\zeta^2\langle\boldsymbol{g}_{1,c},\partial_c\boldsymbol{e}_{3,c}\rangle + \zeta\langle\partial_c\boldsymbol{g}_{1,c},\boldsymbol{v}\rangle + \langle\boldsymbol{g}_{1,c},J\boldsymbol{N}\rangle\\ -\zeta^2\langle\boldsymbol{g}_{2,c},\partial_c\boldsymbol{e}_{3,c}\rangle + \zeta\langle\partial_c\boldsymbol{g}_{2,c},\boldsymbol{v}\rangle + \langle\boldsymbol{g}_{2,c},J\boldsymbol{N}\rangle\\ -\zeta^2\langle\boldsymbol{g}_{3,c},\partial_c\boldsymbol{e}_{3,c}\rangle + \zeta\langle\partial_c\boldsymbol{g}_{3,c},\boldsymbol{v}\rangle + \langle\boldsymbol{g}_{3,c},J\boldsymbol{N}\rangle\end{bmatrix}. \tag{3.24}$$

Assume that η_0 and ζ_0 are such that the conditions (3.11) are satisfied. Let $\varrho_0 \in C^\infty_{comp}(\mathbb{R})$ be such that $0 \le \varrho_0(s) \le 1$, $\varrho_0(s) \equiv 1$ for $|s| \le 1$, and $\varrho_0(s) \equiv 0$ for $|s| \ge 2$. Define a continuous matrix-valued function $\tilde{\mathcal{S}} : \mathbb{R}^2 \times H^1_\mu \to GL(3)$ by

$$\tilde{\mathcal{S}}(\eta, \zeta, \boldsymbol{v}) = \mathcal{S}(\varrho\eta, \varrho\zeta, \varrho\boldsymbol{v}), \qquad \text{where} \quad \varrho = \varrho_0(\eta/\eta_1)\varrho_0(\zeta/\zeta_1)\varrho_0(\|\boldsymbol{v}\|_{H^1_\mu}/\delta_1).$$

This function coincides with $\mathcal{S}$ (defined in (3.21)) for $|\eta| < \eta_1$, $|\zeta| < \zeta_1$, and $\|\boldsymbol{v}\|_{H^1_\mu} < \delta_1$, and has uniformly bounded inverse. The system (3.23) with the right-hand side as in (3.24) but with $\tilde{\mathcal{S}}$ instead of $\mathcal{S}$, and with $\boldsymbol{v}$ given by the ansatz (3.10), defines differentiable functions $\xi(t)$, $\eta(t)$, and $\zeta(t)$ for all $t \ge 0$. Note that $\boldsymbol{v}(t)$ defined by (3.10) is a continuous function of time and is valued in $H^1_\mu(\mathbb{R})$, since so are $\boldsymbol{u}$, $\boldsymbol{\phi}_c$, and $\boldsymbol{e}_{3,c}$. Define $t_2 \in \mathbb{R}_+ \cup \{+\infty\}$ by

$$t_2 = \sup\{t \ge 0\colon |\eta(t)| < \eta_1,\, |\zeta(t)| < \zeta_1,\, \|\boldsymbol{v}(\cdot,t)\|_{H^1_\mu} < \delta_1\}. \tag{3.25}$$

For $t \in (0, t_2)$, the solution $(\xi(t), \eta(t), \zeta(t))$ also solves (3.23), since the inequalities $|\eta(t)| < \eta_1$, $|\zeta(t)| < \zeta_1$, and $\|\boldsymbol{v}(\cdot,t)\|_{H^1_\mu} < \delta_1$ ensure that $\tilde{\mathcal{S}}$ coincides with $\mathcal{S}$. Thus, Proposition 3.2 is proved with

$$T_1 = \min(t_1, t_2) \in \mathbb{R}_+ \cup \{+\infty\}, \tag{3.26}$$

where t_1, t_2 are from (3.2) and (3.25). $\square$

4. Energy and dissipative estimates. We adapt the analysis from [PW94]. In this section, we formulate two lemmas that are the analogue of [PW94, Proposition 6.1]. Lemma 4.1 is based on the energy conservation and allows us to control $\|\boldsymbol{\rho}\|_{H^1}$ in terms of $\|\boldsymbol{v}\|_{H^1_\mu}$. Lemma 4.3 bounds $\|\boldsymbol{v}\|_{H^1_\mu}$ in terms of $\|\boldsymbol{\rho}\|_{H^1}$ and is based on dissipative estimates on the semigroup generated by A^μ_c (see Lemma 4.2).

Let $\eta_1 > 0$, $\zeta_1 > 0$, and $\delta_1 > 0$ not be larger than in Proposition 3.2, and assume that δ_1 satisfies

$$\delta_1 < \frac{\min(1, c_\star)}{4 \sup_{|z| \le 2\|\boldsymbol{\phi}_{c_\star}\|_{H^1}} |f''(z)|}. \tag{4.1}$$

Let $\eta_0 > 0$ and ζ_0 be such that the conditions (3.11) are satisfied. According to Proposition 3.2, there exists $T_1 \in \mathbb{R}_+ \cup \{+\infty\}$ such that there is a solution $\boldsymbol{u} \in C((0, T_1), H^2(\mathbb{R}) \cap L^2_{2\mu}(\mathbb{R}))$ to (1.1) with the initial data

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 := \boldsymbol{\phi}_{c_\star + \eta_0} + \zeta_0 \boldsymbol{e}_{3, c_\star + \eta_0},$$

and functions $\xi(t)$, $\eta(t)$, and $\zeta(t)$ and $\boldsymbol{v}(t)$ (given by (3.10)), defined for $0 \le t < T_1$, such that (3.14) and (3.15) are satisfied. For given η_0 and ζ_0, define the following function of η:

$$\mathscr{Y}(\eta) = \|\boldsymbol{\rho}_0\|_{H^1} + \|\boldsymbol{\rho}_0\|_{H^1}^{1/2} |\eta - \eta_0|^{1/2} + |\mathscr{N}_{c_\star + \eta} - \mathscr{N}_{c_\star + \eta_0}|^{1/2}, \tag{4.2}$$

where $\boldsymbol{\rho}_0 \equiv \zeta_0 \boldsymbol{e}_{3, c_\star + \eta_0}$.

Lemma 4.1. *There exists $C_1 > 0$ such that if at some moment $0 \le t < T_1$,*

$$\|\boldsymbol{\rho}(t)\|_{H^1} \le \delta_1,$$

then

$$\|\boldsymbol{\rho}(t)\|_{H^1} \le C_1 \left(\mathscr{Y}(\eta(t)) + |\zeta(t)| + \|\boldsymbol{v}(t)\|_{H^1_\mu} \right), \tag{4.3}$$

where $\mathscr{Y}(\eta)$ is given by (4.2)*.*

Proof. Let us introduce the effective Hamiltonian $\mathscr{L}_c$:

$$\mathscr{L}_c(\boldsymbol{u}) = E(\boldsymbol{u}) + c\mathscr{N}(\boldsymbol{u}), \quad \mathscr{L}_c'(\boldsymbol{\phi}_c) = E'(\boldsymbol{\phi}_c) + c\mathscr{N}'(\boldsymbol{\phi}_c) = 0, \quad \mathscr{L}_c''(\boldsymbol{\phi}_c) = \mathcal{H}_c, \tag{4.4}$$

where E and $\mathscr{N}$ are the energy and momentum functionals defined in (1.5) and (1.7). Using the Taylor series expansion for $\mathscr{L}_c$ at $\boldsymbol{\phi}_c$, we have

$$\begin{aligned} \mathscr{L}_c(\boldsymbol{u}(t)) &= \mathscr{L}_c(\boldsymbol{\phi}_c) + \frac{1}{2}\langle\boldsymbol{\rho}, \mathcal{H}_c\boldsymbol{\rho}\rangle + \int_{\mathbb{R}} g(\boldsymbol{\phi}_c, \boldsymbol{\rho})\boldsymbol{\rho}^3\,dx \\ &= \mathscr{L}_c(\boldsymbol{\phi}_c) + \frac{1}{2}\langle\boldsymbol{\rho}, (-\partial_x^2 + c)\boldsymbol{\rho}\rangle + \frac{1}{2}\langle\boldsymbol{\rho}, f'(\boldsymbol{\phi}_c)\boldsymbol{\rho}\rangle + \int_{\mathbb{R}} g(\boldsymbol{\phi}_c, \boldsymbol{\rho})\boldsymbol{\rho}^3\,dx, \end{aligned} \tag{4.5}$$

where

$$g(\boldsymbol{\phi}_c, \boldsymbol{\rho}) = \frac{1}{2}\int_0^1 (1-s)^2 f''(\boldsymbol{\phi}_c + s\boldsymbol{\rho})\,ds. \tag{4.6}$$

For the second term in (4.5), there is the following bound from below:

$$\frac{1}{2}\int_{\mathbb{R}} \left((\partial_x\boldsymbol{\rho})^2 + c\boldsymbol{\rho}^2\right)dx \ge m\|\boldsymbol{\rho}\|_{H^1}^2, \qquad m = \frac{1}{2}\min(1, c_\star) > 0. \tag{4.7}$$

The bound for the third term in the right-hand side of (4.5) follows from the inequalities

$$\int_{\mathbb{R}} |f'(\boldsymbol{\phi}_c)|\boldsymbol{\rho}^2\,dx \le \|e^{-2\mu x} f'(\boldsymbol{\phi}_c)\|_{L^\infty}\|\boldsymbol{\rho}\|_{L^2_\mu}^2 \le b\left[|\zeta|\|\boldsymbol{e}_{3,c}\|_{L^2_\mu} + \|\boldsymbol{v}(t)\|_{L^2_\mu}\right]^2, \tag{4.8}$$

where $b = \sup_{c\in[c_\star, c_\star+\eta_1]} \|e^{-2\mu x} f'(\boldsymbol{\phi}_c)\|_{L^\infty} < \infty$ due to (2.10), the assumption (1.2) that $f'(0) = 0$, and due to Lemma A.1 from Appendix A. We bound the last term in (4.5) by

$$\int_{\mathbb{R}} |g(\boldsymbol{\phi}_c, \boldsymbol{\rho})\boldsymbol{\rho}^3|\,dx \le \|g(\boldsymbol{\phi}_c, \boldsymbol{\rho})\|_{L^\infty}\|\boldsymbol{\rho}\|_{H^1}^3 \le \delta_1\|g(\boldsymbol{\phi}_c, \boldsymbol{\rho})\|_{L^\infty}\|\boldsymbol{\rho}\|_{H^1}^2. \tag{4.9}$$

According to (4.1), g from (4.6) satisfies $\delta_1\|g(\boldsymbol{\phi}_c, \boldsymbol{\rho})\|_{L^\infty} < \frac{\min(1,c_\star)}{4} = \frac{m}{2}$, and this leads to

$$\int_{\mathbb{R}} |g(\boldsymbol{\phi}_c, \boldsymbol{\rho})\boldsymbol{\rho}^3|\,dx \le \frac{m}{2}\|\boldsymbol{\rho}\|_{H^1}^2. \tag{4.10}$$

Combining (4.5) with the bounds (4.7), (4.8), and (4.10), we obtain

$$\frac{m}{2}\|\boldsymbol{\rho}\|_{H^1}^2 \le |\mathscr{L}_c(\boldsymbol{u}) - \mathscr{L}_c(\boldsymbol{\phi}_c)| + \frac{b}{2}\left[|\zeta|\|\boldsymbol{e}_{3,c}\|_{L^2_\mu} + \|\boldsymbol{v}\|_{H^1_\mu}\right]^2,$$

so that, for some $C > 0$,

$$\|\boldsymbol{\rho}\|_{H^1} \le C\left[|\mathscr{L}_c(\boldsymbol{u}) - \mathscr{L}_c(\boldsymbol{\phi}_c)|^{1/2} + |\zeta| + \|\boldsymbol{v}\|_{H^1_\mu}\right]. \tag{4.11}$$

Now let us estimate $|\mathscr{L}_c(\boldsymbol{u}(t)) - \mathscr{L}_c(\boldsymbol{\phi}_c)|$. Note that $\mathscr{L}_c(\boldsymbol{u}(t)) = \mathscr{L}_c(\boldsymbol{u}_0)$ since the value of the energy functional E given by (1.5) and the value of the momentum functional $\mathscr{N}$ given by (1.7) are conserved along the trajectories of (1.1). Thus, we can write

$$|\mathscr{L}_c(\boldsymbol{u}(t)) - \mathscr{L}_c(\boldsymbol{\phi}_c)| \le |\mathscr{L}_c(\boldsymbol{u}_0) - \mathscr{L}_c(\boldsymbol{\phi}_{c_0})| + |\mathscr{L}_c(\boldsymbol{\phi}_c) - \mathscr{L}_c(\boldsymbol{\phi}_{c_0})|. \tag{4.12}$$

Using the definition (4.4) of the functional $\mathscr{L}_c$, we express the first term in the right-hand side of (4.12) as

$$\mathscr{L}_c(\boldsymbol{u}_0) - \mathscr{L}_c(\boldsymbol{\phi}_{c_0}) = \mathscr{L}_{c_0}(\boldsymbol{u}_0) - \mathscr{L}_{c_0}(\boldsymbol{\phi}_{c_0}) + (\eta - \eta_0)(\mathscr{N}(\boldsymbol{u}_0) - \mathscr{N}(\boldsymbol{\phi}_{c_0})). \tag{4.13}$$

Since $\mathscr{L}_{c_0}'(\boldsymbol{\phi}_{c_0}) = 0$, there exists $k > 0$ such that $|\mathscr{L}_{c_0}(\boldsymbol{u}_0) - \mathscr{L}_{c_0}(\boldsymbol{\phi}_{c_0})| \le k\|\boldsymbol{\rho}_0\|_{H^1}^2$, where $\boldsymbol{\rho}_0 = \boldsymbol{u}_0 - \boldsymbol{\phi}_{c_0}$; this allows us to bound (4.13) by

$$|\mathscr{L}_c(\boldsymbol{u}_0) - \mathscr{L}_c(\boldsymbol{\phi}_{c_0})| \le \mathrm{const}(\|\boldsymbol{\rho}_0\|_{H^1}^2 + |\eta - \eta_0|\|\boldsymbol{\rho}_0\|_{H^1}). \tag{4.14}$$

For the second term in the right-hand side of (4.12), we have

$$|\mathscr{L}_c(\boldsymbol{\phi}_c) - \mathscr{L}_c(\boldsymbol{\phi}_{c_0})| \le |E_c - E_{c_0}| + c|\mathscr{N}_c - \mathscr{N}_{c_0}|.$$

From the relation

$$\frac{d}{dc}E_c = -c\frac{d}{dc}\mathscr{N}_c$$

we conclude that $|E_c - E_{c_0}| \le \max(c, c_0)|\mathscr{N}_c - \mathscr{N}_{c_0}|$, since $\mathscr{N}_c'$ is sign-definite for $c_\star < c \le c_\star + \eta_1$ by (1.16). Therefore, there is the following bound for the second term in the right-hand side of (4.12):

$$|\mathscr{L}_c(\boldsymbol{\phi}_c) - \mathscr{L}_c(\boldsymbol{\phi}_{c_0})| \le 2\max(c, c_0)|\mathscr{N}_c - \mathscr{N}_{c_0}|. \tag{4.15}$$

Using the bounds (4.14) and (4.15) in (4.12), we obtain

$$|\mathscr{L}_c(\boldsymbol{u}(t)) - \mathscr{L}_c(\boldsymbol{\phi}_c)| \le \mathrm{const}\left(\|\boldsymbol{\rho}_0\|_{H^1}^2 + |\eta - \eta_0|\|\boldsymbol{\rho}_0\|_{H^1} + |\mathscr{N}_c - \mathscr{N}_{c_0}|\right).$$

Substituting this result into (4.11), we obtain the bound (4.3). ◻

Lemma 4.2 (see [PW94]). *Let Assumption 3 be satisfied, and pick $\mu \in (0, \sqrt{c/3})$. Let $Q_c^\mu = I - P_c^\mu$, where P_c^μ introduced in (2.20) is the spectral projection that corresponds to the discrete spectrum of A_c^μ (see Remark 2.6). Then A_c^μ is the generator of a strongly continuous linear semigroup on $H^s(\mathbb{R})$ for any real s, and there exist constants $a > 0$ and $b > 0$ such that for all $\boldsymbol{v} \in L^2(\mathbb{R})$ and $t > 0$ the following estimate is satisfied:*

$$\|e^{A_c^\mu t} Q_c^\mu \boldsymbol{v}\|_{H^1} \le a t^{-1/2} e^{-bt} \|\boldsymbol{v}\|_{L^2}. \tag{4.16}$$

We require that η_1 be small enough, so that

$$\eta_1 \sup_{c \in [c_\star, c_\star + \eta_1]} \|\partial_c Q_c^\mu\|_{H^1 \to H^1} \le \frac{1}{2}. \tag{4.17}$$

Lemma 4.3. *There exists $C_2 > 0$ such that if*

$$\eta_1 + \zeta_1 + \delta_1 < C_2 \tag{4.18}$$

and

(4.19)
$$\sup_{s\in[0,t]}|\eta(s)|\le\eta_1,\quad \sup_{s\in[0,t]}|\zeta(s)|\le\zeta_1,\quad \sup_{s\in[0,t]}\|\boldsymbol{\rho}(s)\|_{H^1}\le\delta_1,\quad \sup_{s\in[0,t]}\|\boldsymbol{v}(s)\|_{H^1_\mu}\le\delta_1,$$

then

$$\|\boldsymbol{v}(t)\|_{H^1_\mu}\le C_2\sup_{s\in[0,t]}\left[\zeta^2(s)+|\zeta(s)|\|\boldsymbol{\rho}(s)\|_{H^1}\right]. \tag{4.20}$$

Proof. Using the center manifold reduction (3.23), we rewrite the evolution equation (3.18) in the following form:

$$\dot{\boldsymbol{v}}-J\mathcal{H}_c\boldsymbol{v}=-\sum_{j=1}^{3}R_j\boldsymbol{e}_{j,c}-\zeta(\zeta+R_2)\partial_c\boldsymbol{e}_{3,c}+R_1\partial_x(\zeta\boldsymbol{e}_{3,c}+\boldsymbol{v})+J\boldsymbol{N}, \tag{4.21}$$

where $c=c(t)=c_\star+\eta(t)$, $\zeta=\zeta(t)$, and the nonlinear terms $R_j(t)$ are given by (3.24). We set

$$\boldsymbol{\omega}(x,t)=e^{\mu x}\boldsymbol{v}(x,t),\qquad \boldsymbol{e}^\mu_{j,c}(x)=e^{\mu x}\boldsymbol{e}_{j,c}(x),\qquad c\in[c_\star,c_\star+\eta_1],\quad j=1,2,3,$$

and consider A^μ_c given by (1.13). Equation (4.21) takes the following form:

$$\dot{\boldsymbol{\omega}}-A^\mu_c\boldsymbol{\omega}=\boldsymbol{G}, \tag{4.22}$$

where

$$\boldsymbol{G}(x,t)=-\sum_{j=1}^{3}R_j\boldsymbol{e}^\mu_{j,c}-\zeta(\zeta+R_2)\partial_c\boldsymbol{e}^\mu_{3,c}+R_1(\partial_x-\mu)(\zeta\boldsymbol{e}^\mu_{3,c}+\boldsymbol{\omega})+e^{\mu x}J\boldsymbol{N}. \tag{4.23}$$

As follows from (4.22),

$$\partial_t(Q^\mu_{c_\star}\boldsymbol{\omega})=Q^\mu_{c_\star}\dot{\boldsymbol{\omega}}=Q^\mu_{c_\star}(A^\mu_c\boldsymbol{\omega}+\boldsymbol{G})=A^\mu_{c_\star}Q^\mu_{c_\star}\boldsymbol{\omega}+Q^\mu_{c_\star}(A^\mu_c-A^\mu_{c_\star})\boldsymbol{\omega}+Q^\mu_{c_\star}\boldsymbol{G}.$$

We may write $Q^\mu_{c_\star}\boldsymbol{\omega}$ as follows:

$$Q^\mu_{c_\star}\boldsymbol{\omega}(t)=\int_0^t e^{A^\mu_{c_\star}(t-s)}\mathfrak{G}(s)\,ds, \tag{4.24}$$

where

$$\mathfrak{G}(x,t)=Q^\mu_{c_\star}(A^\mu_c-A^\mu_{c_\star})\boldsymbol{\omega}(x,t)+Q^\mu_{c_\star}\boldsymbol{G}(x,t). \tag{4.25}$$

Using the dissipative estimate given by (4.16), we get

$$\|Q^\mu_{c_\star}\boldsymbol{\omega}(t)\|_{H^1}\le C\int_0^t(t-s)^{-1/2}e^{-b(t-s)}\|\mathfrak{G}(s)\|_{L^2}\,ds \tag{4.26}$$

$$\le Ce^{-bt/2}\sup_{s\in[0,t]}e^{bs/2}\|\mathfrak{G}(s)\|_{L^2}\int_0^t(t-s)^{-1/2}e^{-b(t-s)/2}\,ds \tag{4.27}$$

$$\le C\sup_{s\in[0,t]}e^{bs/2}\|\mathfrak{G}(s)\|_{L^2}. \tag{4.28}$$

Since $\boldsymbol{\omega} = Q_c^\mu \boldsymbol{\omega} = Q_{c_\star}^\mu \boldsymbol{\omega} + (Q_c^\mu - Q_{c_\star}^\mu)\boldsymbol{\omega}$, we have

$$\|\boldsymbol{\omega}\|_{H^1} \le \|Q_{c_\star}^\mu \boldsymbol{\omega}\|_{H^1} + |\eta| \sup_{c\in[c_\star, c_\star+\eta_1]} \|\partial_c Q_c^\mu\|_{H^1\to H^1} \|\boldsymbol{\omega}\|_{H^1} \le \|Q_{c_\star}^\mu \boldsymbol{\omega}\|_{H^1} + \frac{1}{2}\|\boldsymbol{\omega}\|_{H^1},$$

where we used (4.17). It follows that $\|\boldsymbol{\omega}\|_{H^1} \le 2\|Q_{c_\star}^\mu \boldsymbol{\omega}\|_{H^1}$. Hence, we have

$$\|\boldsymbol{\omega}(t)\|_{H^1} \le C e^{-bt/2} \sup_{s\in[0,t]} e^{bs/2} \|\mathfrak{G}(s)\|_{L^2}. \tag{4.29}$$

We now need the bound on $\|\mathfrak{G}\|_{L^2}$. We start with

$$\|\mathfrak{G}\|_{L^2} \le \|Q_{c_\star}^\mu (A_c^\mu - A_{c_\star}^\mu)\boldsymbol{\omega}\|_{L^2} + \|Q_{c_\star}^\mu \boldsymbol{G}\|_{L^2}. \tag{4.30}$$

We estimate the first term in the right-hand side of (4.30) as follows:

$$\|Q_{c_\star}^\mu (A_c^\mu - A_{c_\star}^\mu)\boldsymbol{\omega}(t)\|_{L^2} \le \|Q_{c_\star}^\mu (A_c^\mu - A_{c_\star}^\mu)\|_{H^1\text{-}L^2} \|\boldsymbol{\omega}(t)\|_{H^1} \le C|\eta| \|\boldsymbol{\omega}(t)\|_{H^1}. \tag{4.31}$$

Since $\boldsymbol{e}_{j,c}^\mu$, $1\le j\le 3$, depend continuously on c while $Q_{c_\star}^\mu \boldsymbol{e}_{j,c_\star}^\mu = 0$, there are bounds $\|Q_{c_\star}^\mu \boldsymbol{e}_{j,c}^\mu\|_{H^1} \le C|\eta|$. This allows us to derive the following bound for the second term in the right-hand side of (4.30):

$$\begin{aligned}\|Q_{c_\star}^\mu \boldsymbol{G}\|_{L^2} &\le C\left(|\eta| \sup_{1\le j\le 3} |R_j| + |\zeta||\zeta + R_2| + |R_1|(|\zeta| + \|\boldsymbol{\omega}\|_{H^1}) + \|J\boldsymbol{N}\|_{L^2_\mu}\right) \\ &\le C\left(\zeta^2 + (|\eta| + |\zeta| + \|\boldsymbol{\omega}\|_{H^1}) \sup_{1\le j\le 3} |R_j| + \|J\boldsymbol{N}\|_{L^2_\mu}\right).\end{aligned}$$

Using the representation (3.24) and the inclusions $\partial_c \boldsymbol{e}_{3,c} \in H_\mu^\infty(\mathbb{R})$, $\boldsymbol{g}_i \in H_{-\mu}^\infty(\mathbb{R})$, $\partial_c \boldsymbol{g}_i \in H_{-\mu}^\infty(\mathbb{R})$, we obtain the following estimates on R_j:

$$|R_j(\eta,\zeta,\boldsymbol{v})| \le C\left(\zeta^2 + |\zeta|\|\boldsymbol{v}\|_{H^1_\mu} + \|J\boldsymbol{N}\|_{L^2_\mu}\right), \qquad j = 1,2,3. \tag{4.32}$$

Taking into account (4.32), we get

$$\begin{aligned}\|Q_{c_\star}^\mu \boldsymbol{G}\|_{L^2_\mu} &\le C\left(\zeta^2 + (|\eta| + |\zeta| + \|\boldsymbol{\omega}\|_{H^1})(\zeta^2 + |\zeta|\|\boldsymbol{\omega}\|_{H^1} + \|J\boldsymbol{N}\|_{L^2_\mu}) + \|J\boldsymbol{N}\|_{L^2_\mu}\right) \\ &\le C\left(\zeta^2 + (|\eta| + \|\boldsymbol{\omega}\|_{H^1})|\zeta|\|\boldsymbol{\omega}\|_{H^1} + \|J\boldsymbol{N}\|_{L^2_\mu}\right).\end{aligned} \tag{4.33}$$

In the last inequality, we used the uniform boundedness of $|\eta|$, $|\zeta|$, and $\|\boldsymbol{\omega}\|_{H^1}$ that follows from (4.19).

Summing up (4.31) and (4.33), we obtain the following bound on $\|\mathfrak{G}\|_{L^2_\mu}$:

$$\|\mathfrak{G}\|_{L^2_\mu} \le C\left[\zeta^2 + (|\eta| + |\zeta|)\|\boldsymbol{\omega}\|_{H^1} + \|J\boldsymbol{N}\|_{L^2_\mu}\right]. \tag{4.34}$$

Using the integral representation for the nonlinearity (3.8),

$$J\boldsymbol{N} = \partial_x[f(\boldsymbol{\phi}_c + \boldsymbol{\rho}) - f(\boldsymbol{\phi}_c) - f'(\boldsymbol{\phi}_c)\boldsymbol{\rho}] = \partial_x\left[\frac{\boldsymbol{\rho}^2}{2}\int_0^1 (1-s)^2 f''(\boldsymbol{\phi}_c + s\boldsymbol{\rho})\,ds\right], \tag{4.35}$$

we obtain the bound

$$\|J\boldsymbol{N}\|_{L^2_\mu} \le C\|\boldsymbol{\rho}\|_{H^1_\mu}\|\boldsymbol{\rho}\|_{H^1} \le C\left(|\zeta|\|\boldsymbol{e}_{3,c}\|_{H^1_\mu} + \|\boldsymbol{v}\|_{H^1_\mu}\right)\|\boldsymbol{\rho}\|_{H^1},$$

with the constant C that depends on $\|\boldsymbol{\phi}_c\|_{H^1}$ and on the bounds on $f''(z)$ and $f'''(z)$ for $|z| \le \|\boldsymbol{u}\|_{L^\infty}$, which is bounded by $2\|\boldsymbol{\phi}_{c_\star}\|_{H^1}$. This bound allows us to rewrite (4.34) as

$$\|\mathfrak{G}\|_{L^2_\mu} \le C\left[\zeta^2 + (|\eta| + |\zeta| + \|\boldsymbol{\rho}\|_{H^1})\|\boldsymbol{\omega}\|_{H^1} + |\zeta|\|\boldsymbol{\rho}\|_{H^1}\right] \le C\left[g_0 + g_1\|\boldsymbol{\omega}\|_{H^1}\right], \tag{4.36}$$

where

$$g_0(t) = \zeta^2(t) + |\zeta(t)|\|\boldsymbol{\rho}(t)\|_{H^1}, \qquad g_1(t) = |\eta(t)| + |\zeta(t)| + \|\boldsymbol{\rho}(t)\|_{H^1}. \tag{4.37}$$

Thus, (4.29) could be written as

$$2e^{bt/2}\|\boldsymbol{\omega}(t)\|_{H^1} \le C_2 \sup_{s\in[0,t]} e^{bs/2}\left[g_0(s) + g_1(s)\|\boldsymbol{\omega}(s)\|_{H^1}\right] \tag{4.38}$$

for some $C_2 > 0$. Since the right-hand side is monotonically increasing with t, we also have

$$\sup_{s\in[0,t]} 2e^{bs/2}\|\boldsymbol{\omega}(s)\|_{H^1} \le C_2 \sup_{s\in[0,t]} e^{bs/2}\left[g_0(s) + g_1(s)\|\boldsymbol{\omega}(s)\|_{H^1}\right]. \tag{4.39}$$

The function g_1 from (4.37) satisfies $C_2 \sup_{s\in[0,t]} g_1(s) < 1$ (this follows from the assumptions (4.18) and (4.19)), and therefore

$$\|\boldsymbol{\omega}(t)\|_{H^1} \le C_2 e^{-bt/2} \sup_{s\in[0,t]} e^{bs/2} g_0(s) \le C_2 \sup_{s\in[0,t]} \left[\zeta^2(s) + |\zeta(s)|\|\boldsymbol{\rho}(s)\|_{H^1}\right].$$

Since $\boldsymbol{\omega} = e^{\mu x}\boldsymbol{v}$, the last inequality yields (4.20). ◻

5. Nonlinear estimates. Now we close the estimates using the bounds on $\|\boldsymbol{\rho}\|_{H^1}$ (Lemma 4.1) and on $\|\boldsymbol{v}\|_{H^1_\mu}$ (Lemma 4.3) from the previous section.

We assume that $\eta_1 > 0$, $\zeta_1 > 0$, and $\delta_1 > 0$ are sufficiently small: not larger than in Proposition 3.2, satisfy the bounds (4.1), (4.17), and (4.18), and also that ζ_1 satisfies

$$\zeta_1 < \frac{1}{3\max(1, C_1)C_2}. \tag{5.1}$$

Define

$$C_3 = 2C_1, \qquad C_4 = 2C_2 \max(1, C_3), \tag{5.2}$$

with C_1 and C_2 as in Lemmas 4.1 and 4.3. Choosing smaller values of η_1 and ζ_1 if necessary, we may assume that

$$C_3\left(\zeta_1 + 2\eta_1 + (\mathscr{N}_{c_\star+\eta_1} - \mathscr{N}_{c_\star})^{1/2}\right) < \delta_1, \tag{5.3}$$

$$C_4\left(\zeta_1^2 + 2\eta_1\zeta_1 + \zeta_1(\mathscr{N}_{c_\star+\eta_1} - \mathscr{N}_{c_\star})^{1/2}\right) < \delta_1. \tag{5.4}$$

Define

$$\eta_M(t) = \sup_{0\le s\le t} \eta(s), \tag{5.5}$$

$$\zeta_M(t) = \sup_{0\le s\le t} |\zeta(s)|. \tag{5.6}$$

PROPOSITION 5.1. *Assume that the initial data $\eta_0 > 0$ and ζ_0 are such that the following inequalities are satisfied:*

$$\eta_0 \in (0, \eta_1), \qquad |\zeta_0| < \zeta_1, \qquad \|\boldsymbol{\rho}_0\|_{H^1} < \min(\eta_1, \delta_1). \tag{5.7}$$

Then for $0 \le t < T_1$ the functions $\boldsymbol{\rho}(t)$, $\boldsymbol{v}(t)$ satisfy the bounds

$$\|\boldsymbol{\rho}(t)\|_{H^1} \le C_3 \left[\zeta_M(t) + \mathscr{Y}(\eta_M(t))\right], \tag{5.8}$$

$$\|\boldsymbol{v}(t)\|_{H^1_\mu} \le C_4 \left[\zeta_M(t)^2 + \zeta_M(t)\mathscr{Y}(\eta_M(t))\right], \tag{5.9}$$

where C_3, C_4 are defined by (5.2), $\mathscr{Y}(\eta) = \|\boldsymbol{\rho}_0\|_{H^1} + \|\boldsymbol{\rho}_0\|_{H^1}^{1/2}|\eta - \eta_0|^{1/2} + |\mathscr{N}_{c_\star+\eta} - \mathscr{N}_{c_\star+\eta_0}|^{1/2}$ *is introduced in* (4.2), *and η_M, ζ_M are defined in* (5.5) *and* (5.6).

Proof. Let

$$S = \{t \in [0, T_1) \colon \|\boldsymbol{\rho}(t)\|_{H^1} < \delta_1\}.$$

S is nonempty since $\|\boldsymbol{\rho}(0)\|_{H^1} < \delta_1$ by (5.7). According to Proposition 3.2 and representation (3.6), $\|\boldsymbol{\rho}(t)\|_{H^1}$ is a continuous function of t. Since the inequality in the definition of S is sharp, S is an open subset of $[0, T_1)$. Let us assume that $T_2 \in (0, T_1)$ is such that

$$\|\boldsymbol{\rho}(t)\|_{H^1} < \delta_1, \qquad 0 \le t < T_2. \tag{5.10}$$

It is enough to prove that $T_2 \in S$ (then the connected subset of S that contains $t = 0$ is both open and closed in $[0, T_1)$ and hence coincides with $[0, T_1)$). Since $\|\boldsymbol{v}(t)\|_{H^1_\mu} < \delta_1$ for $0 \le t < T_1$, Lemmas 4.1 and 4.3 are both applicable for $t \le T_2$. The estimate (4.3) on $\|\boldsymbol{\rho}(t)\|_{H^1}$ together with the estimate (4.20) on $\|\boldsymbol{v}(t)\|_{H^1_\mu}$ give

$$\begin{aligned}\|\boldsymbol{\rho}(t)\|_{H^1} &\le C_1\Big(\mathscr{Y}(\eta(t)) + |\zeta(t)| + \|\boldsymbol{v}(t)\|_{H^1_\mu}\Big) \\ &\le C_1\Big(\mathscr{Y}(t) + |\zeta(t)| + C_2 \sup_{s\in[0,t]} \left[\zeta^2 + |\zeta|\|\boldsymbol{\rho}\|_{H^1}\right]\Big).\end{aligned}$$

For $0 \le t \le T_2$, define $M(t) = \sup_{s\in[0,t]} \|\boldsymbol{\rho}(s)\|_{H^1}$. We have

$$M(t) \le C_1\Big(\sup_{s\in[0,t]} (\mathscr{Y}(\eta(s)) + |\zeta(s)|) + C_2 \sup_{s\in[0,t]} \left[\zeta^2(s) + |\zeta(s)|M(t)\right]\Big).$$

We carry the term $C_1C_2|\zeta|M(t)$ to the left-hand side of the inequality, taking into account that $C_1C_2|\zeta(t)| \le C_1C_2\zeta_1 \le \frac{1}{3}$ for all $0 \le t < T_1$ by (5.1). This results in the following relation:

$$\|\boldsymbol{\rho}(t)\|_{H^1} \le M(t) \le \frac{3}{2}C_1\Big(\sup_{s\in[0,t]} (\mathscr{Y}(\eta(s)) + |\zeta(s)|) + C_2 \sup_{s\in[0,t]} \zeta^2(s)\Big).$$

Since $C_2\zeta^2 \le C_2\zeta_1|\zeta| \le |\zeta|/3$ by (5.1), we obtain

$$\|\boldsymbol{\rho}(t)\|_{H^1} \le \frac{3}{2}C_1 \sup_{s\in[0,t]} \Big(\mathscr{Y}(\eta(s)) + \frac{4}{3}|\zeta(s)|\Big) \le C_3 \sup_{s\in[0,t]} (\mathscr{Y}(\eta(s)) + |\zeta(s)|), \quad t \in [0, T_2],$$

with $C_3 = 2C_1$. This proves (5.8) for $t \in [0, T_2]$. It then follows that

$$\|\boldsymbol{\rho}(T_2)\|_{H^1} \le C_3\left[\zeta_1 + \mathscr{Y}(\eta_1)\right] \le C_3\Big[\zeta_1 + 2\eta_1 + (\mathscr{N}_{c_\star+\eta_1} - \mathscr{N}_{c_\star})^{1/2}\Big] < \delta_1,$$

where we took into account the definition of $\mathscr{Y}(\eta)$ in (4.2), the bound $\|\boldsymbol{\rho}_0\|_{H^1} < \eta_1$ from (5.7), and inequality (5.3). Hence, $T_2 \in S$. It follows that S coincides with $[0, T_1)$.

Using the bound (5.8) in (4.20) and recalling the definition of C_4 in (5.2), we derive the bound (5.9) on $\|\boldsymbol{v}(t)\|_{H^1_\mu}$. ◻

Corollary 5.2. *Assume that the conditions of Proposition* 5.1 *are satisfied. If* $\eta_1 > 0$ *and* $\zeta_1 > 0$ *were chosen sufficiently small, then there exists a constant* $C_5 > 0$ *so that for* $0 \le t < T_1$ *the function* $\boldsymbol{v}(t)$ *satisfies the bound*

$$\|\boldsymbol{v}(t)\|_{H^1_{\mu/2}} \le C_5 \left[\zeta_M^2(t) + \zeta_M(t)\mathscr{Y}(\eta_M(t))\right], \tag{5.11}$$

where η_M, ζ_M *are defined in* (5.5), (5.6).

Proof. The bound (5.11) is proved in the same way as (5.9). We may need to take smaller values of η_1 and ζ_1 so that Lemmas 4.1 and 4.3 become applicable for the new exponential weight. Note that the exponential weight does not enter the definition (4.2) of the function $\mathscr{Y}(\eta)$. ◻

Lemma 5.3. *Assume that the bounds* (5.9) *and* (5.11) *are satisfied for* $0 \le t < T_1$. *Then there exists* $C_6 > 0$ *so that the terms* R_2 *and* R_3 *defined in* (3.24) *satisfy for* $0 \le t < T_1$ *the bounds*

$$|R_j(\eta, \zeta, \boldsymbol{v})| \le C_6 \zeta_M^2, \qquad j = 2, 3. \tag{5.12}$$

Proof. By (4.32),

$$|R_j(\eta, \zeta, \boldsymbol{v})| \le C \left(\zeta^2 + |\zeta| \|\boldsymbol{v}\|_{H^1_\mu} + \|J\boldsymbol{N}\|_{L^2_\mu}\right), \qquad j = 2, 3. \tag{5.13}$$

According to (5.9), the second term in the right-hand side of (5.13) is bounded by $C\zeta^2$ as long as $\eta \in (0, \eta_1)$ and $|\zeta| \le \zeta_1$. We now need a bound on $\|J\boldsymbol{N}\|_{L^2_\mu}$. Using the representation (4.35) for the nonlinearity, we obtain the bounds

$$\|J\boldsymbol{N}\|_{L^2_\mu} \le C\|\boldsymbol{\rho}\|^2_{H^1_{\mu/2}} \le C\left(\zeta^2 \|\boldsymbol{e}_{3,c}\|^2_{H^1_{\mu/2}} + \|\boldsymbol{v}\|^2_{H^1_{\mu/2}}\right). \tag{5.14}$$

The constant depends on $\|\boldsymbol{\phi}_c\|_{H^1}$ and on the bounds on $f''(z)$ and $f'''(z)$ for $|z| \le \|\boldsymbol{u}\|_{L^\infty}$, which is bounded by $2\|\boldsymbol{\phi}_{c_\star}\|_{H^1}$. As follows from (5.11),

$$\|\boldsymbol{v}(t)\|_{H^1_{\mu/2}} \le C_5(\zeta_1 + \mathscr{Y}(\eta_1))\zeta_M(t). \tag{5.15}$$

Using this bound in (5.14), we get $\|J\boldsymbol{N}\|_{L^2_\mu} \le C\zeta_M^2$. The bound (5.12) follows. ◻

6. Choosing the initial perturbation. In this section, we show how to choose the initial perturbation that indeed leads to the instability and conclude the proof of Theorem 1.

We choose $\eta_1 > 0$, $\zeta_1 > 0$, and $\delta_1 > 0$ small enough so that (4.1), (4.17), (4.18) are satisfied, and so that Lemmas 4.1 and 4.3 apply to both exponential weights μ and $\mu/2$. Taking $\eta_1 > 0$, $\zeta_1 > 0$ smaller if necessary, we may assume that the conditions (5.1), (5.3), and (5.4) are satisfied, and moreover that

$$C_6 \zeta_1 < 1/2, \tag{6.1}$$

where $C_6 > 0$ is from Lemma 5.3.

Let

$$\lambda(\eta) = \lambda_{c_\star+\eta}, \qquad \Lambda(\eta) = \int_0^\eta \lambda(\eta')\,d\eta'. \tag{6.2}$$

Let us recall that, according to (1.16), we assume that there exists $\eta_1 > 0$ so that $\mathscr{N}_c' < 0$ and is nonincreasing for $c_\star < c \le c_\star + \eta_1$. Thus, we assume that $\lambda(\eta) > 0$ for $0 < \eta \le \eta_1$ (according to (2.16), $\mathscr{N}_c'$ and λ_c are of opposite sign).

Lemma 6.1. *One can choose $\eta_1 > 0$ sufficiently small so that for $0 < \eta \le \eta_1$ one has*

$$3C_6 e^{2C_6\eta}\Lambda(\eta) < \lambda(\eta). \tag{6.3}$$

Proof. By (2.16), $\lambda_c = -\frac{\mathrm{N}_c'}{B_c}$, where

$$B_c = \langle \boldsymbol{\phi}_c, \boldsymbol{e}_{3,c}\rangle. \tag{6.4}$$

Since $B_{c_\star} > 0$ by (2.9), we may assume that $\eta_1 > 0$ is small enough so that

$$B_{c_\star}/2 \le B_c \le 2B_{c_\star}, \qquad c \in [c_\star, c_\star + \eta_1]. \tag{6.5}$$

According to Theorem 1, $\mathscr{N}_c' < 0$ and is nonincreasing for $c \in (c_\star, c_\star + \eta_1)$. Therefore, using inequalities (6.5), we obtain

$$\Lambda(\eta) = \int_{c_\star}^{c_\star+\eta} \lambda_c\,dc = \int_{c_\star}^{c_\star+\eta} \frac{-\mathscr{N}_c'}{B_c}\,dc \le -\frac{2\eta\mathscr{N}_{c_\star+\eta}'}{B_{c_\star}} \le 4\eta\lambda(\eta), \qquad 0 \le \eta \le \eta_1,$$

where $\lambda(\eta) > 0$ for $0 < \eta \le \eta_1$. We take $\eta_1 > 0$ so small that $12\,\eta_1 C_6 e^{2C_6\eta_1} < 1$; then (6.3) is satisfied. ∎

Taking $\eta_1 > 0$ smaller if necessary, we may assume that Lemma C.1 is satisfied and that

$$\lambda(\eta)/C_6 < \zeta_1. \tag{6.6}$$

Remark 6.2. Inequality (6.6) ensures that $\eta(t)$ reaches η_1 prior to $\zeta(t)$ reaching ζ_1 (see Lemma 6.4 and Figure 3).

Since $\Lambda(\eta) = \mathrm{o}(\eta)$, we may also assume that $\eta_1 > 0$ is small enough so that

$$K_1\Lambda(\eta_1) \le \kappa\eta_1/2, \tag{6.7}$$

where $K_1 = K_1(\eta_1, \zeta_1)$ is defined below in (6.26) and $\kappa > 0$ is from Lemma C.1.

Lemma 6.3. *For any $\delta \in (0, \min(\eta_1, \delta_1))$, one can choose the initial data $\eta_0 \in (0, \eta_1)$, $\zeta_0 \in (0, \zeta_1)$ so that the following estimates are satisfied:*

$$\|\zeta_0 \boldsymbol{e}_{3,c_\star+\eta_0}\|_{H^1} < \min(\eta_1, \delta_1), \tag{6.8}$$

$$\|(\boldsymbol{\phi}_{c_\star+\eta_0} + \zeta_0\boldsymbol{e}_{3,c_\star+\eta_0}) - \boldsymbol{\phi}_{c_\star}\|_{H^1\cap H^1_\mu} < \delta < \min(\eta_1, \delta_1), \tag{6.9}$$

$$\zeta_0 < \Lambda(\eta_0). \tag{6.10}$$

Proof. Pick $\eta_0 \in (0, \eta_1)$ so that

$$\|\boldsymbol{\phi}_{c_\star+\eta_0} - \boldsymbol{\phi}_{c_\star}\|_{H^1\cap H^1_\mu} < \delta/2. \tag{6.11}$$

For given $\eta_0 > 0$, we take $\zeta_0 \in (0, \zeta_1)$ small enough so that

$$\zeta_0 \| \boldsymbol{e}_{3,c_\star+\eta_0} \|_{H^1 \cap H^1_\mu} < \delta/2. \tag{6.12}$$

Note that $\| \boldsymbol{e}_{3,c_\star+\eta_0} \|_{H^1}$ for $\eta_0 > 0$ is finite by Lemma 2.7. Inequality (6.12) implies that (6.8) is satisfied. Together with (6.11), it also guarantees that (6.9) holds. We then require that $\zeta_0 > 0$ be small enough so that (6.10) takes place. ∎

We rewrite the last two equations from the system (3.23):

$$\begin{cases} \dot{\eta} = \zeta + R_2(\eta, \zeta, \boldsymbol{v}), \\ \dot{\zeta} = \lambda(\eta)\zeta + R_3(\eta, \zeta, \boldsymbol{v}). \end{cases} \tag{6.13}$$

LEMMA 6.4. *For $0 \le t < T_1$, with $T_1 > 0$ as in Proposition 3.2,*

$$\dot{\eta} \ge \zeta_0/2, \qquad \dot{\zeta} \ge 0, \tag{6.14}$$

$$\zeta_0 \le \zeta(t) < 3e^{2C_6\eta(t)}\Lambda(\eta(t)). \tag{6.15}$$

Proof. According to Proposition 3.2, the trajectory $(\eta(t), \zeta(t))$ that starts at (η_0, ζ_0) satisfies the inequalities $\eta(t) < \eta_1$ and $\zeta(t) < \zeta_1$ for $0 \le t < T_1$. We define the region $\Omega \subset \mathbb{R}_+ \times \mathbb{R}_+$ by

$$\Omega = \{(\eta, \zeta) \colon \zeta_0 \le \zeta \le \lambda(\eta)/C_6,\ \eta_0 \le \eta \le \eta_1\}. \tag{6.16}$$

Define $T_\Omega \in \mathbb{R}_+ \cup \{+\infty\}$ by

$$T_\Omega = \sup\{\, t \in [0, T_1) \colon\ (\eta(t), \zeta(t)) \in \Omega, \quad \dot{\zeta}(t) \ge 0 \,\}. \tag{6.17}$$

Let us argue that $T_\Omega > 0$. At $t = 0$, $(\eta(0), \zeta(0)) = (\eta_0, \zeta_0) \in \Omega$. From (6.13), we compute $\dot{\eta}(0) \ge \zeta_0 - C_6\zeta_0^2 > 0$, where we applied the bounds (5.12) and the inequality $C_6\zeta_0 < 1/2$ that follows from (6.1) and the choice $\zeta_0 < \zeta_1$. Similarly, $\dot{\zeta}(0) \ge \lambda(\eta_0)\zeta_0 - C_6\zeta_0^2 > 0$ due to the inequality $C_6\zeta_0 < \lambda(\eta_0)$ that follows from (6.10) and (6.3). Therefore, $(\eta(t), \zeta(t)) \in \Omega$ and $\dot{\zeta}(t) > 0$ for times $t > 0$ from a certain open neighborhood of $t = 0$, proving that $T_\Omega > 0$.

The monotonicity of $\zeta(t)$ for $t < T_\Omega$ implies that $\zeta_M(t) := \sup_{s\in(0,t)} |\zeta(s)| = \zeta(t)$ for $0 \le t < T_\Omega$, and (5.12) takes the form

$$|R_j(\eta, \zeta, \boldsymbol{v})| \le C_6\zeta^2, \qquad j = 2, 3, \qquad 0 \le t < T_\Omega. \tag{6.18}$$

Using (6.13) and (6.18), and taking into account (6.1) and monotonicity of $\zeta(t)$ for $0 \le t < T_\Omega$, we compute

$$\dot{\eta}(t) = \zeta(t) + R_2 \ge \zeta(t) - C_6\zeta^2(t) = \zeta(t)(1 - C_6\zeta(t)) > \zeta_0/2, \tag{6.19}$$

which is valid for $0 \le t < T_\Omega$. This allows us to consider ζ as a function of η (as long as $0 \le t < T_\Omega$). By (6.13), (6.18), and (6.1),

$$\frac{d\zeta}{d\eta} = \frac{\lambda(\eta)\zeta + R_3}{\zeta + R_2} \le \frac{\lambda(\eta)\zeta + C_6\zeta^2}{\zeta - C_6\zeta^2} = \frac{\lambda(\eta) + C_6\zeta}{1 - C_6\zeta} \le 2(\lambda(\eta) + C_6\zeta), \tag{6.20}$$

which is valid for $0 \le t < T_\Omega$. Thus, $\frac{d\zeta}{d\eta} - 2C_6\zeta < 2\lambda(\eta)$ for $0 \le t < T_\Omega$. Multiplying both sides of this relation by $e^{-2C_6\eta}$ and integrating, we get Gronwall's inequality:

$$\int_{\eta_0}^{\eta} \frac{d}{d\eta'}\left(e^{-2C_6\eta'}\zeta(\eta')\right) d\eta' < 2\int_{\eta_0}^{\eta} e^{-2C_6\eta'}\lambda(\eta')\, d\eta' \le 2e^{-2C_6\eta_0}\Lambda(\eta), \tag{6.21}$$

$$\zeta < e^{2C_6\eta}\Big(2e^{-2C_6\eta_0}\Lambda(\eta) + e^{-2C_6\eta_0}\zeta_0\Big) < 3e^{2C_6\eta}\Lambda(\eta), \qquad 0 \le t < T_\Omega. \tag{6.22}$$

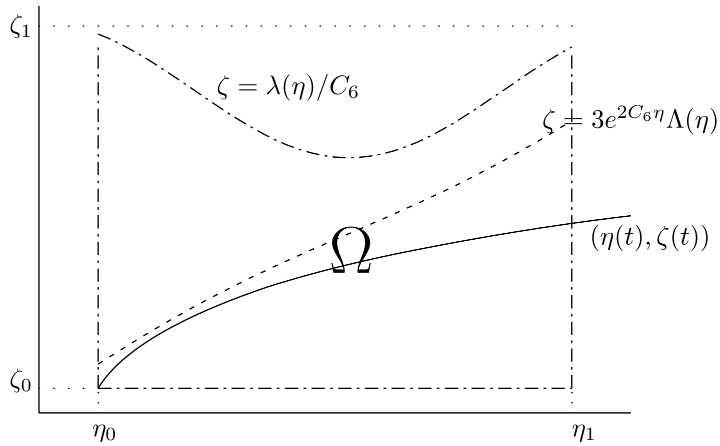

FIG. 3. *The trajectory $(\eta(t), \zeta(t))$ (the solid line) stays in the part of the region Ω below the dashed line $\zeta = 3e^{2C_6\eta}\Lambda(\eta)$.*

See Figure 3. We used the inequality $\zeta_0 < \Lambda(\eta_0) \le \Lambda(\eta)$ that follows from (6.10) and monotonicity of $\Lambda(\eta)$.

Now let us argue that $T_\Omega = T_1$. If $T_\Omega = \infty$, we are done; therefore we only need to consider the case $T_\Omega < \infty$. By (6.17), the moment T_Ω is characterized by

$$\text{either} \quad T_\Omega = T_1 \quad \text{or} \quad (\eta(T_\Omega), \zeta(T_\Omega)) \in \partial\Omega \quad \text{or} \quad \dot\eta(T_\Omega) = 0, \tag{6.23}$$

or any combination of these three conditions. By continuity, the bound (6.22) is also valid at T_Ω (the last inequality in (6.22) remains strict); therefore,

$$\zeta(T_\Omega) < 3e^{2C_6\eta(T_\Omega)}\Lambda(\eta(T_\Omega)) < \lambda(\eta(T_\Omega))/C_6. \tag{6.24}$$

In the last inequality, we used Lemma 6.1. Inequality (6.24) also leads to

$$\dot\zeta = \lambda(\eta)\zeta + R_3 \ge \zeta(\lambda(\eta) - C_6\zeta) > 0, \qquad 0 \le t \le T_\Omega. \tag{6.25}$$

Using (6.24) and (6.25) in (6.23), we conclude that either $T_\Omega = T_1$ or $\eta(T_\Omega) = \eta_1$, and hence again $T_\Omega = T_1$ (by (3.15), $\eta(t) < \eta_1$ for $0 \le t < T_1$). The bounds (6.14) and (6.15) for $0 \le t < T_\Omega = T_1$ follow from (6.19) and (6.22) (note that $\dot\zeta \ge 0$ for $0 \le t < T_\Omega = T_1$ by (6.17)). □

LEMMA 6.5. *Assume that $\|\boldsymbol\rho_0\|_{H^1} < \eta_1$. There exists $C_7 > 0$ so that*

$$\|\boldsymbol\rho(t)\|_{L^2_\mu} \le C_7\Lambda(\eta), \qquad 0 \le t < T_1.$$

Proof. Using the estimate (6.15) from Lemma 6.4 and the estimate (5.9) from Proposition 5.1 (where $\eta_M(t) = \eta(t)$ and $\zeta_M(t) = \zeta(t)$ due to (6.14) and positivity of η_0 and ζ_0), we obtain

$$\|\boldsymbol\rho(t)\|_{L^2_\mu} \le |\zeta|\|\boldsymbol e_{3,c}\|_{L^2_\mu} + \|\boldsymbol v\|_{L^2_\mu} \le |\zeta| \left(\|\boldsymbol e_{3,c}\|_{L^2_\mu} + C_4[\zeta + \mathscr{Y}(\eta)]\right).$$

Now the statement of the lemma follows from the bound (6.15). The value of C_7 could be taken equal to $K_1 = K_1(\eta_1, \zeta_1)$, which we define by

$$K_1 = 3e^{2C_6\eta_1}\left(\sup_{c\in[c_\star, c_\star+\eta_1]} \|\boldsymbol e_{3,c}\|_{L^2_\mu} + C_4\big[\zeta_1 + \big\{2\eta_1 + |\mathscr{N}_{c_\star+\eta_1} - \mathscr{N}_{c_\star}|^{\frac12}\big\}\big]\right). \tag{6.26}$$

Note that the term in the braces dominates $\mathscr{Y}(\eta)$, which was defined in (4.2) (when estimating $\mathscr{Y}(\eta)$, we used the bound $\|\boldsymbol{\rho}_0\|_{H^1} < \eta_1$). ◻

Conclusion of the proof of Theorem 1. In Theorem 1, let us take

$$\epsilon = \min(\kappa\eta_1/2, \|\boldsymbol{\phi}_{c_\star}\|_{H^1}) > 0. \tag{6.27}$$

Pick $\delta > 0$ arbitrarily small. To comply with the requirements of Lemmas 6.3 and 6.5, we may assume that δ is smaller than $\min(\eta_1, \delta_1)$. Fix $\mu \in (0, \min(\mu_0, \mu_1))$, with μ_0 from Assumption 3 and μ_1 as in Lemma C.1. Let η_0 and ζ_0 satisfy all the inequalities in Lemma 6.3; then the conditions (3.11) of Proposition 3.2 are satisfied. Let

$$\boldsymbol{u}_0 = \boldsymbol{\phi}_{c_\star+\eta_0} + \zeta_0 \boldsymbol{e}_{3,c_\star+\eta_0},$$

so that $\boldsymbol{u}_0 \in H^2(\mathbb{R}) \cap L^2_{2\mu}(\mathbb{R})$ by (2.25) and $\|\boldsymbol{u}_0 - \boldsymbol{\phi}_{c_\star}\|_{H^1} < \delta$ by (6.9). Proposition 3.2 states that there exist $T_1 \in \mathbb{R}_+ \cup \{+\infty\}$ and a function $\boldsymbol{u}(t) \in C([0,\infty), H^2(\mathbb{R}) \cap L^2_{2\mu}(\mathbb{R}))$, $\boldsymbol{u}(0) = \boldsymbol{u}_0$, so that for $0 \le t < T_1$ the function $\boldsymbol{u}(t)$ solves (1.1) and all the inequalities (3.15) are satisfied.

LEMMA 6.6. *In Proposition* 3.2*, one can only take* $T_1 < \infty$.

Proof. If we had $T_1 = +\infty$, then $\dot{\eta} \ge \zeta_0/2$ for $t \in \mathbb{R}_+$ by Lemma 6.4; hence $\eta(t)$ would reach η_1 in finite time, contradicting the bound $\eta(t) < \eta_1$ for $0 \le t < T_1$ from Proposition 3.2 (iii). ◻

Since $T_1 < \infty$, Proposition 3.2 (iv) states that at least one of the inequalities in (3.15) turns into an equality at $t = T_1$. As follows from the bound (5.9) and inequality (5.4), $\|\boldsymbol{v}(T_1)\|_{H^1_\mu} < \delta_1$. Also, by (6.15) (where the bound from above does not have to be strict at T_1),

$$\zeta(T_1) \le 3e^{2C_6\eta_1}\Lambda(\eta(T_1)) \le 3e^{2C_6\eta_1}\Lambda(\eta_1) < \lambda(\eta)/C_6 < \zeta_1. \tag{6.28}$$

We took into account the monotonicity of $\Lambda(\eta)$ and inequalities (6.3) and (6.6). Therefore, either $\|\boldsymbol{u}(T_1)\|_{H^1} = 2\|\boldsymbol{\phi}_{c_\star}\|_{H^1}$ or $\eta(T_1) = \eta_1$ (or both). In the first case,

$$\inf_{s\in\mathbb{R}} \|\boldsymbol{u}(\cdot, T_1) - \boldsymbol{\phi}_{c_\star}(\cdot - s)\|_{H^1} \ge \|\boldsymbol{u}(\cdot, T_1)\|_{H^1} - \|\boldsymbol{\phi}_{c_\star}\|_{H^1} \ge \|\boldsymbol{\phi}_{c_\star}\|_{H^1} \ge \epsilon; \tag{6.29}$$

hence the instability of $\boldsymbol{\phi}_{c_\star}$ follows. We are left to consider the case $\eta(T_1) = \eta_1$. According to (3.6),

$$\begin{aligned}\inf_{s\in\mathbb{R}} \|\boldsymbol{u}(\cdot, t) - \boldsymbol{\phi}_{c_\star}(\cdot - s)\|_{L^2} &\ge \inf_{s\in\mathbb{R}} \|\boldsymbol{u}(\cdot, t) - \boldsymbol{\phi}_{c_\star}(\cdot - s)\|_{L^2(\mathbb{R}, \min(1, e^{\mu x})\,dx)} \\ &\ge \inf_{s\in\mathbb{R}} \|\boldsymbol{\phi}_{c(t)}(\cdot) - \boldsymbol{\phi}_{c_\star}(\cdot - s)\|_{L^2(\mathbb{R}, \min(1, e^{\mu x})\,dx)} - \|\boldsymbol{\rho}(t)\|_{L^2_\mu}.\end{aligned} \tag{6.30}$$

Applying Lemmas C.1 and 6.5 to the two terms in the right-hand side of (6.30), we see that

$$\inf_{s\in\mathbb{R}} \|\boldsymbol{u}(\cdot, t) - \boldsymbol{\phi}_{c_\star}(\cdot - s)\|_{L^2} \ge \kappa\eta - C_7\Lambda(\eta), \qquad 0 \le t < T_1, \quad \kappa > 0. \tag{6.31}$$

Since $C_7\Lambda(\eta_1) \le \kappa\eta_1/2$ by (6.7),

$$\inf_{s\in\mathbb{R}} \|\boldsymbol{u}(\cdot, T_1) - \boldsymbol{\phi}_{c_\star}(\cdot - s)\|_{L^2} \ge \kappa\eta_1/2 \ge \epsilon, \tag{6.32}$$

and again the instability of $\boldsymbol{\phi}_{c_\star}$ follows.

This completes the proof of Theorem 1.

7. Nondegenerate case: Normal form. In this section, we prove that the critical soliton with speed $c_\star$ generally corresponds to the saddle-node bifurcation of two branches of noncritical solitons. We assume for simplicity that $c_\star$ is a nondegenerate critical point of $\mathscr{N}_c$, in the sense that

$$\mathscr{N}'_{c_\star} = 0, \qquad \mathscr{N}''_{c_\star} \neq 0. \tag{7.1}$$

We rewrite the last two equations from system (3.23):

$$\begin{bmatrix} \dot\eta \\ \dot\zeta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \lambda_c \end{bmatrix} \begin{bmatrix} \eta \\ \zeta \end{bmatrix} + \begin{bmatrix} R_2(\eta,\zeta,\boldsymbol{v}) \\ R_3(\eta,\zeta,\boldsymbol{v}) \end{bmatrix}. \tag{7.2}$$

As follows from (2.9) and (2.16),

$$\lambda_c = \lambda_{c_\star+\eta} = \lambda'_{c_\star}\eta + \mathrm{O}(\eta^2), \qquad \lambda'_{c_\star} = -\frac{2\mathscr{N}''_{c_\star}}{(I'_{c_\star})^2}, \tag{7.3}$$

where $\lambda'_{c_\star} \neq 0$ by (7.1). System (7.2) has the nonlinear terms $R_j(\eta,\zeta,\boldsymbol{v})$, $j = 2,3$, estimated in Lemma 5.3 for monotonically increasing functions $\eta(t)$, $|\zeta(t)|$ on a local existence interval $0 < t < T_1$. It follows from (3.24) that

$$R_2(0,0,0) = R_3(0,0,0) = 0,$$

so that the point $(\eta,\zeta) = (0,0)$ is a critical point of (7.2) when $\boldsymbol{v} = 0$. This critical point corresponds to the critical traveling wave $\boldsymbol{\phi}_{c_\star}(x)$ itself. The following result establishes a local equivalence between system (7.2) and the truncated system $\ddot\eta = \lambda'_{c_\star}\eta\dot\eta$, thus guaranteeing the instability of the critical point $(\eta,\zeta) = (0,0)$.

Proposition 7.1. *Assume that conditions* (7.1) *are satisfied. Consider the subset of trajectories* $(\eta(t),\zeta(t))$ *of system* (7.2) *that lie inside the* ϵ*-neighborhood* $\mathcal{D}_\epsilon \subset \mathbb{R}^2$ *of the origin and satisfy the condition that both functions* $\eta(t)$ *and* $|\zeta(t)|$ *are monotonically increasing. For sufficiently small* $\epsilon > 0$ *this subset of the trajectories is topologically equivalent to a subset of the trajectories of the truncated normal form,*

$$\dot x = \frac{1}{2}\lambda'_{c_\star} x^2 + E_1, \tag{7.4}$$

where E_1 *is constant.*

Proof. Since $\zeta = \dot\eta - R_2(\eta,\zeta,\boldsymbol{v})$, we can rewrite system (7.2) in the equivalent form

$$\frac{d}{dt}\left(\dot\eta - \frac{1}{2}\lambda'_{c_\star}\eta^2 - R_2(\eta,\zeta,\boldsymbol{v})\right) = R(\eta,\zeta,\boldsymbol{v}), \tag{7.5}$$

where

$$R(\eta,\zeta,\boldsymbol{v}) \equiv R_3(\eta,\zeta,\boldsymbol{v}) - \lambda_c R_2(\eta,\zeta,\boldsymbol{v})) + (\lambda_c - \lambda'_{c_\star}\eta)\zeta.$$

It follows from Lemma 5.3 and (7.3) that there exists a constant $C > 0$ such that $|R| \le C(\zeta^2 + \eta^2|\zeta|)$. The integral form of (7.5) is

$$\dot\eta - \frac{1}{2}\lambda'_{c_\star}\eta^2 - E_1 = \tilde R(t), \tag{7.6}$$

where

$$\tilde{R}(t) \equiv R_2(\eta(t), \zeta(t), \boldsymbol{v}(t)) + \int_0^t R(\eta(t'), \zeta(t'), \boldsymbol{v}(t'))\, dt'$$

and E_1 is the constant of integration. Using Lemma 5.3, the bound $|\zeta| \le \dot{\eta} + C_6\zeta^2$, and integration by parts, we obtain that

$$\int_0^t \zeta^2\, dt' \le \eta|\zeta| + C_6 \int_0^t |\zeta|^3\, dt' \le \eta|\zeta| + C_6\eta|\zeta|^2 + C_6^2 \int_0^t |\zeta|^4\, dt' \le \cdots \le \frac{\eta|\zeta|}{1 - C_6|\zeta|}$$

and

$$\int_0^t \eta^2|\zeta|\, dt' \le \frac{\eta^3}{3} + C_6 \int_0^t \eta^2\zeta^2\, dt' \le \cdots \le \frac{\eta^3}{3(1 - C_6|\zeta|)}.$$

Thus, if $|\zeta|$ is sufficiently small, there exists a constant $\tilde{C} > 0$ such that $|\tilde{R}| \le \tilde{C}(\zeta^2 + |\zeta|\eta + \eta^3)$. The topological equivalence of (7.6) with the above estimate on $|\tilde{R}|$ in the disk $(\eta, \zeta) \in \mathcal{D}_\epsilon$ to the truncated normal form (7.4) with sufficiently small E_1 is proved in [Kuz98, Lemma 3.1]. By definition, two systems are said to be topologically equivalent if there exists a homeomorphism between solutions of these systems. We note that this equivalence holds for a family of trajectories which corresponds to monotonically increasing functions $\eta(t)$, $|\zeta(t)|$ in a subset of the small disk near $(\eta, \zeta) = (0, 0)$. □

Corollary 7.2. *The critical point* $(0, 0)$ *of system* (7.2) *is unstable in the sense that there exists* $\epsilon > 0$ *such that for any* $\delta > 0$ *there are* $(\eta(0), \zeta(0)) \in \mathcal{D}_\delta$ *and* $t_* = t_*(\delta, \epsilon) < \infty$ *such that* $(\eta(t_*), \zeta(t_*)) \notin \mathcal{D}_\epsilon$.

Proof. The normal form equation (7.4) shows that the critical point $x = 0$ is semistable at $E_1 = 0$, such that the trajectory with any $x(0) \ne 0$ of the same sign as $\lambda'_{c_\star}$ escapes the local neighborhood of the point $x = 0$ in a local time $t \in [0, T]$. By Proposition 7.1, local dynamics of (7.4) for $x(t)$ is equivalent to local dynamics of (7.2) for (η, ζ). □

Remark 7.3. The truncated normal form (7.4) is rewritten for $c = c_\star + x$:

$$\dot{c} = \frac{1}{2}\lambda'_{c_\star}(c - c_\star)^2 + E_1. \tag{7.7}$$

The normal form (7.7) corresponds to the standard saddle-node bifurcation. It was derived and studied in [PG96] by using the asymptotic multiscale expansion method. When $E = 0$, the critical point $c = c_\star$ is a degenerate saddle point, which is nonlinearly unstable. Assume for definiteness that $\lambda'_{c_\star} > 0$ (which implies that $\mathcal{N}''_{c_\star} < 0$). Then there are no fixed points for $E_1 > 0$ and two fixed points for $E_1 < 0$ in the normal form equation (7.7). Therefore, there exist initial perturbations (with $E_1 > 0$ and any c_0 or with $E_1 = 0$ and $c_0 > c_\star$) which are arbitrarily close to the traveling wave with $c = c_\star$, but the norm $|c - c_\star|$ exceeds some a priori fixed value at $t = t_* > 0$. Two fixed points exist for $E_1 < 0$:

$$c = c_E^\pm = c_\star \pm \sqrt{\frac{E_1}{\mathcal{N}''_{c_\star}}}\,|I'_{c_\star}|, \tag{7.8}$$

so that $c = c_E^+$ is an unstable saddle point and $c = c_E^-$ is a stable node. The two fixed points correspond to two branches of traveling waves with $\mathcal{N}_c < \mathcal{N}_{\max}$, where

$\mathscr{N}_{\max} = \mathscr{N}(\boldsymbol{\phi}_{c_\star})$. The left branch with $c_E^- < c_\star$ corresponds to $\mathscr{N}'_{c_E^-} > 0$ and the right branch with $c_E^+ > c_\star$ corresponds to $\mathscr{N}'_{c_E^+} < 0$. According to the stability theory for traveling waves [PW92], the left branch is orbitally stable, while the right branch is linearly unstable.

Appendix A. Existence of solitary waves. Let us discuss the existence of standing waves. We assume that f is smooth. Let F denote the primitive of f such that $F(0) = 0$. Thus, by (1.2),

$$F(0) = F'(0) = F''(0) = 0. \tag{A.1}$$

The wave profile $\boldsymbol{\phi}_c$ is to satisfy the equation

$$u'' - cu = f(u), \qquad c > 0.$$

Multiplying the above relation by u' and integrating, and taking into account that we need $\lim_{|x|\to\infty} u(x) = 0$, we get

$$\frac{du(x)}{dx} = \pm\sqrt{cu^2 + 2F(u)}. \tag{A.2}$$

There will be a strictly positive continuous solution exponentially decaying at infinity if there exists $\boldsymbol{\xi}_c > 0$ such that $c\frac{u^2}{2} + F(u) > 0$ for $0 < u < \boldsymbol{\xi}_c$, and also

$$c\frac{\boldsymbol{\xi}_c^2}{2} + F(\boldsymbol{\xi}_c) = 0, \qquad c\boldsymbol{\xi}_c + f(\boldsymbol{\xi}_c) < 0.$$

The last two conditions imply that the map $c \mapsto \boldsymbol{\xi}_c$ is invertible and smooth (as F is). One immediately sees that $\boldsymbol{\phi}_c \in C^\infty(\mathbb{R})$ and, due to the exponential decay at infinity, $\boldsymbol{\phi}_c \in H^\infty(\mathbb{R})$. For each c, the solution $\boldsymbol{\phi}_c$ is unique (up to translations of the origin) and (after a suitable translation of the origin) satisfies the following properties: it is strictly positive, symmetric, and monotonically decreasing (strictly) away from the origin. This result follows from the implicit representation

$$x = \pm\int_{\boldsymbol{\phi}_c}^{\boldsymbol{\xi}_c} \frac{du}{\sqrt{cu^2 + 2F(u)}}. \tag{A.3}$$

See [BL83, section 6] for the exhaustive treatment of this subject.

Lemma A.1. *There exist positive constants C_1, C_2, C_1', and C_2' such that*

$$C_1 e^{-\sqrt{c}|x|} \le |\boldsymbol{\phi}_c(x)| \le C_2 e^{-\sqrt{c}|x|}, \qquad x \in \mathbb{R}, \tag{A.4}$$

$$C_1' e^{-\sqrt{c}|x|} \le |\partial_x \boldsymbol{\phi}_c(x)| \le C_2' e^{-\sqrt{c}|x|}, \qquad |x| \ge 1. \tag{A.5}$$

Proof. Since $\lim_{|x|\to\infty} \boldsymbol{\phi}_c(x) = 0$, there exists $x_1 > 0$ so that $\frac{|F(\boldsymbol{\phi}_c(x))|}{\boldsymbol{\phi}_c^2(x)} < \frac{c}{4}$ for $|x| \ge x_1$. Then, for $x > x_1$, we get from (A.3)

$$x - x_1 = \int_{\boldsymbol{\phi}_c(x)}^{\boldsymbol{\phi}_c(x_1)} \frac{du}{\sqrt{cu^2 + 2F(u)}}.$$

It follows that

$$\begin{aligned} \int_{\boldsymbol{\phi}_c(x)}^{\boldsymbol{\phi}_c(x_1)} \frac{du}{c^{1/2}u} - \int_{\boldsymbol{\phi}_c(x)}^{\boldsymbol{\phi}_c(x_1)} \frac{|F(u)|}{c^{3/2}u^3}\,du &\le x - x_1 \\ \le \int_{\boldsymbol{\phi}_c(x)}^{\boldsymbol{\phi}_c(x_1)} \frac{du}{c^{1/2}u} &+ \int_{\boldsymbol{\phi}_c(x)}^{\boldsymbol{\phi}_c(x_1)} \frac{|F(u)|}{c^{3/2}u^3}\,du. \end{aligned} \tag{A.6}$$

By (A.1), $|F(u)|/u^3$ is bounded for u small, and we conclude from (A.6) that

$$\ln \boldsymbol{\phi}_c(x) - C_3 \le c^{1/2}(x - x_1) \le \ln \boldsymbol{\phi}_c(x) + C_3, \tag{A.7}$$

where $C_3 = c^{-1}\int_0^{\boldsymbol{\phi}_c(x_1)} |F(u)|u^{-3}\,du$. Inequalities (A.7) immediately prove (A.4). Bounds (A.5) immediately follow from (A.2). □

We also need the following result that gives the rate of decay of $\boldsymbol{e}_{2,c} = \partial_c \boldsymbol{\phi}_c$ and $\boldsymbol{e}_{3,c_\star}$ at infinity.

LEMMA A.2. *Let $R \in C^\infty(\mathbb{R})$ satisfy the bound $|R(x)| \le C_1 e^{-\sqrt{c}|x|}$ for $x \ge 0$ for some $c > 0$, $C_1 > 0$. Let $u \in C^\infty(\mathbb{R})$ satisfy*

$$u'' - cu = R, \qquad \lim_{x\to+\infty} u(x) = 0. \tag{A.8}$$

Then there exists $C_2 > 0$ (that depends on c, C_1, and u) such that

$$|u(x)| \le C_2(1 + |x|)e^{-\sqrt{c}|x|}, \qquad x \ge 0. \tag{A.9}$$

Remark A.3. C_2 depends not only on c and C_1 but also on u because the solution to (A.8) is defined up to const $e^{-\sqrt{c}\,x}$.

Proof. First, we notice that if $P \in C^\infty(\mathbb{R})$, $P(x) \ge 0$ for $x \ge 0$, and if $v \in C^\infty(\mathbb{R})$ solves

$$v'' - cv = P(x), \qquad v(0) = 0, \qquad \lim_{x\to+\infty} v(x) = 0, \tag{A.10}$$

then $v(x) \le 0$ for $x \ge 0$. (The existence of a point $x_0 > 0$ where u assumes a positive maximum contradicts the equation in (A.10).)

Now we consider the functions u_- and u_+ that satisfy

$$u_\pm''(x) - cu_\pm = \pm C_1 e^{-\sqrt{c}|x|}, \qquad u_\pm(0) = u(0), \qquad \lim_{x\to+\infty} u_\pm(x) = 0. \tag{A.11}$$

Both $u_\pm$ can be written explicitly; they satisfy (A.9). Since $v = u - u_-$ and $v = u_+ - u$ satisfy (A.10) with $P(x) = C_1 e^{-\sqrt{c}|x|} + R(x)$ and $P(x) = C_1 e^{-\sqrt{c}|x|} - R(x)$, respectively, we conclude that $u_+(x) \le u(x) \le u_-(x)$ for $x \ge 0$, and hence u also satisfies (A.9). □

Appendix B. Fredholm alternative for $\mathcal{H}_c$.

LEMMA B.1 (Fredholm alternative). *Let $R(x) \in \mathscr{S}_{+,m}(\mathbb{R})$, $m \ge 0$ (see Definition* 2.1*). If*

$$\int_{\mathbb{R}} \boldsymbol{e}_{1,c}(x)R(x)\,dx = 0, \tag{B.1}$$

then the equation

$$\mathcal{H}_c u = R \tag{B.2}$$

has a solution $u \in \mathscr{S}_{+,m}(\mathbb{R})$. (This solution is unique if we impose the constraint $\langle \boldsymbol{e}_{1,c}, u\rangle = 0$.) Otherwise, any solution $u(x)$ to (B.2) *such that $\lim_{x\to+\infty} u(x) = 0$ grows exponentially at $-\infty$:*

$$\lim_{x\to-\infty} e^{-\sqrt{c}|x|}u(x) \ne 0.$$

Proof. Let us pick an even function $R_+ \in H^\infty(\mathbb{R})$ so that $R_+(x) = R(x)$ for $x \ge 1$. Since R_+ is even and therefore orthogonal to the kernel of the operator $\mathcal{H}_c$, there is a solution $u_+ \in H^\infty(\mathbb{R})$ to the equation

$$\mathcal{H}_c u_+ = R_+. \tag{B.3}$$

Denote by u the solution to the ordinary differential equation

$$\mathcal{H}_c u \equiv -u'' + (f'(\boldsymbol{\phi}_c) + c)u = R \tag{B.4}$$

such that $u|_{x=1} = u_+|_{x=1}$, $u'|_{x=1} = u'_+|_{x=1}$. Then $u \in C^\infty(\mathbb{R})$ coincides with u_+ for $x \ge 1$ and thus satisfies

$$\lim_{x\to+\infty} u(x) = 0. \tag{B.5}$$

We take the pairing of (B.4) with $\boldsymbol{e}_{1,c}$:

$$\int_x^\infty \boldsymbol{e}_{1,c}(y)\mathcal{H}_c u(y)\,dy = \int_x^\infty \boldsymbol{e}_{1,c}(y)R(y)\,dy \equiv r(x), \qquad x \in \mathbb{R}. \tag{B.6}$$

Since

$$\boldsymbol{e}_{1,c}\mathcal{H}_c u = u\mathcal{H}_c\boldsymbol{e}_{1,c} - \boldsymbol{e}_{1,c}\partial_x^2 u + u\partial_x^2\boldsymbol{e}_{1,c} = -\partial_x(\boldsymbol{e}_{1,c}u') + \partial_x(u\partial_x\boldsymbol{e}_{1,c}),$$

where we took into account that $\mathcal{H}_c\boldsymbol{e}_{1,c} = 0$, we obtain from (B.6) the relation

$$\boldsymbol{e}_{1,c}(x)u'(x) - u(x)\partial_x\boldsymbol{e}_{1,c}(x) = r(x). \tag{B.7}$$

The boundary term at $x = +\infty$ does not contribute into (B.7) due to the limit (B.5). We will use this relation to find the behavior of $u(x)$ as $x \to -\infty$. For $x \le -1$, we divide the relation (B.7) by $\boldsymbol{e}_{1,c}^2$ (we can do this since $\boldsymbol{e}_{1,c}(x) = -\partial_x\boldsymbol{\phi}_c(x) \ne 0$ for $x \ne 0$), getting

$$\partial_x\left(\frac{u(x)}{\boldsymbol{e}_{1,c}(x)}\right) = \frac{r(x)}{\boldsymbol{e}_{1,c}^2(x)}. \tag{B.8}$$

Therefore, for $x \le -1$,

$$u(x) - \boldsymbol{e}_{1,c}(x)\frac{u(-1)}{\boldsymbol{e}_{1,c}(-1)} = \boldsymbol{e}_{1,c}(x)\int_{-1}^x \frac{(r(y) - r_-) + r_-}{\boldsymbol{e}_{1,c}^2(y)}\,dy, \tag{B.9}$$

where $r_- = \lim_{x\to-\infty} r(x)$.

Since $R \in \mathscr{S}_{+,m}(\mathbb{R})$, $|R(x)| \le C(1+|x|)^m$, $m \in \mathbb{Z}$, $m \ge 0$. Using Lemma A.1, we see that

$$|r(x) - r_-| = \left|\int_{-\infty}^x R(y)\boldsymbol{e}_{1,c}(y)\,dy\right| \le \operatorname{const} e^{-\sqrt{c}|x|}(1+|x|)^m, \quad x \le -1. \tag{B.10}$$

At the same time, Lemma A.1 also shows that

$$\int_{-1}^x \frac{dy}{\boldsymbol{e}_{1,c}^2(y)} \ge \operatorname{const} e^{2\sqrt{c}|x|}, \qquad x \le -1. \tag{B.11}$$

Therefore, if $r_- \neq 0$, the right-hand side of (B.9) grows exponentially as $x \to -\infty$. The same is true for $u(x)$, since the second term in the left-hand side of (B.9) decays exponentially when $|x| \to \infty$ by Lemma A.1. If instead $r_- = 0$, Lemma A.1 and the bound (B.10) show that the right-hand side of (B.9) is bounded by $\mathrm{const}(1+|x|)^m$, proving a similar bound for $u(x)$. Using (B.4) to get the bounds on the derivatives $u^{(N)}$, we conclude that $u \in \mathscr{S}_{+,m}(\mathbb{R})$. ◻

Appendix C. Nondegeneracy of $\inf_{s\in\mathbb{R}} \|\boldsymbol{\phi}_c(\cdot) - \boldsymbol{\phi}_{c_\star}(\cdot - s)\|$ at $c_\star$.

LEMMA C.1. *If $\eta_1 > 0$ is sufficiently small, there exist $\mu_1 > 0$ and $\kappa > 0$ so that*

$$\inf_{s\in\mathbb{R}} \|\boldsymbol{\phi}_c(\cdot) - \boldsymbol{\phi}_{c_\star}(\cdot - s)\|_{L^2(\mathbb{R},\min(1,e^{\mu x})\,dx)} \geq \kappa|c - c_\star|, \quad c \in [c_\star, c_\star + \eta_1], \quad \mu \in [0, \mu_1].$$

Proof. Consider the function

$$g_\mu(c,s) = \|\boldsymbol{\phi}_c(\cdot) - \boldsymbol{\phi}_{c_\star}(\cdot - s)\|^2_{L^2(\mathbb{R},\min(1,e^{\mu x})\,dx)}. \tag{C.1}$$

It is a smooth nonnegative function of c and s for $c \in [c_\star, c_\star + \eta_1]$ and $s \in \mathbb{R}$. It also depends smoothly on the parameter $\mu \geq 0$. Zero is its absolute minimum, achieved at the point $(c,s) = (c_\star, 0)$. We also note that the point $(c_\star, 0)$ is nondegenerate when $\mu = 0$:

$$\partial_c^2 g_0(c,s)|_{(c_\star,0)} = 2\|\partial_c \boldsymbol{\phi}_c|_{c=c_\star}\|^2_{L^2} > 0, \qquad \partial_s^2 g_0(c,s)|_{(c_\star,0)} = 2\|\partial_x \boldsymbol{\phi}_{c_\star}\|^2_{L^2} > 0,$$

$$\partial_c\partial_s g_0(c,s)|_{(c_\star,0)} = -2(\partial_c \boldsymbol{\phi}_c|_{c=c_\star}, \partial_x \boldsymbol{\phi}_{c_\star}) = 0.$$

By continuity, the quadratic form $g''_\mu|_{(c_\star,0)}$ is nondegenerate for $0 \leq \mu \leq \mu_1$, with some $\mu_1 > 0$. Therefore, there exist $\kappa > 0$ and an open neighborhood $\Omega \subset \mathbb{R}^2$ of the point $(c_\star, 0)$ such that

$$g_\mu(c,s) \geq \kappa^2((c - c_\star)^2 + s^2), \qquad (c,s) \in \Omega, \qquad 0 \leq \mu \leq \mu_1. \tag{C.2}$$

Moreover, we claim that

$$\Gamma \equiv \inf_{\mu\in(0,\mu_1)} \inf_{(c,s)\in[c_\star,c_\star+\eta_1]\times\mathbb{R})\setminus\Omega} g_\mu(c,s) > 0. \tag{C.3}$$

To prove (C.3), we only need to note that $(c_\star, 0)$ is the only point where $g_\mu(c,s)$ takes the zero value and that $\lim_{|s|\to\infty} g_\mu(c,s) \geq \inf_{c\in[c_\star,c_\star+\eta_1]} \|\boldsymbol{\phi}_c\|^2_{L^2(\mathbb{R},\min(1,e^{\mu_1 x})\,dx)} > 0$.

Now, we assume that $\eta_1 > 0$ is small enough so that $\kappa^2\eta_1^2 < \Gamma$. Then, by (C.2) (valid for $(c,s) \in \Omega$) and (C.3) (valid for $(c,s) \in ([c_\star, c_\star + \eta_1] \times \mathbb{R})\setminus\Omega$), we conclude that

$$\inf_{s\in\mathbb{R}} g_\mu(c,s) \geq \kappa^2(c - c_\star)^2, \qquad c \in [c_\star, c_\star + \eta_1], \quad \mu \in [0, \mu_1]. \tag{C.4}$$

This proves the lemma. ◻

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