

Normal form for travelling kinks in discrete Klein–Gordon lattices

Gerard Iooss^a, Dmitry E. Pelinovsky^{b,*}

^a *Institut Universitaire de France, INLN, UMR CNRS-UNSA 6618, 1361 route des Lucioles, 06560 Valbonne, France*

^b *Department of Mathematics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S 4K1, Canada*

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Abstract

We study travelling kinks in the spatial discretizations of the nonlinear Klein–Gordon equation, which include the discrete ϕ^4 lattice and the discrete sine-Gordon lattice. The differential advance-delay equation for travelling kinks is reduced to the normal form, a scalar fourth-order differential equation, near the quadruple zero eigenvalue. We show numerically the non-existence of monotonic kinks (heteroclinic orbits between adjacent equilibrium points) in the fourth-order equation. Making generic assumptions on the reduced fourth-order equation, we prove the persistence of bounded solutions (heteroclinic connections between periodic solutions near adjacent equilibrium points) in the full differential advance-delay equation with the technique of centre manifold reduction. Existence of multiple kinks in the discrete sine-Gordon equation is discussed in connection to recent numerical results of Aigner et al. [A.A. Aigner, A.R. Champneys, V.M. Rothos, A new barrier to the existence of moving kinks in Frenkel–Kontorova lattices, *Physica D* 186 (2003) 148–170] and results of our normal form analysis.

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1. Introduction

Spatial discretizations of the nonlinear partial differential equations represent discrete dynamical systems, which are equivalent to chains of coupled nonlinear oscillators or discrete nonlinear lattices. Motivated by various physical applications and recent advances in the mathematical analysis of discrete lattices, we consider the discrete Klein–Gordon equation in the form:

$$\ddot{u}_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + f(u_{n-1}, u_n, u_{n+1}), \quad (1.1)$$

where $u_n(t) \in \mathbb{R}$, $n \in \mathbb{Z}$, $t \in \mathbb{R}$, h is the lattice step size, and $f(u_{n-1}, u_n, u_{n+1})$ is the nonlinearity function. The discrete lattice (1.1) is a discretization of the continuous Klein–Gordon equation, which emerges in the singular limit $h \rightarrow 0$:

$$u_{tt} = u_{xx} + F(u), \quad (1.2)$$

where $u(x, t) \in \mathbb{R}$, $x \in \mathbb{R}$, and $t \in \mathbb{R}$. In particular, we study two versions of the Klein–Gordon equation (1.2), namely the ϕ^4 model

$$u_{tt} = u_{xx} + u(1 - u^2) \quad (1.3)$$

and the sine-Gordon equation

$$u_{tt} = u_{xx} + \sin(u). \quad (1.4)$$

Both versions of the discrete Klein–Gordon equations, the ϕ^4 model and the sine-Gordon equation, are used for description of a broad range of physical phenomena, such as crystal dislocation, localized excitations in ionic crystals, and thermal denaturation of DNA. A review of the literature can be found in [4].

Throughout our work, we shall assume that the spatial discretization $f(u_{n-1}, u_n, u_{n+1})$ of the nonlinearity function $F(u)$ is symmetric,

$$f(u_{n-1}, u_n, u_{n+1}) = f(u_{n+1}, u_n, u_{n-1}), \quad (1.5)$$

and consistent with the continuous limit,

$$f(u, u, u) = F(u). \quad (1.6)$$

* Corresponding author.

E-mail address: dmpeli@math.mcmaster.ca (D.E. Pelinovsky).

We assume that the zero equilibrium state always exists with $F(0) = 0$ and $F'(0) = 1$. This normalization allows us to represent the nonlinearity function in the form:

$$f(u_{n-1}, u_n, u_{n+1}) = u_n + Q(u_{n-1}, u_n, u_{n+1}), \quad (1.7)$$

where the linear part is uniquely normalized (the parameter h can be chosen so that the linear term of $u_{n-1} + u_{n+1}$ is cancelled) and the nonlinear part is represented by the function $Q(u_{n-1}, u_n, u_{n+1})$. In addition, we assume that (i) $F(u)$ and $Q(u_{n-1}, u_n, u_{n+1})$ are odd such that $F(-u) = -F(u)$ and $Q(-v, -u, -w) = -Q(v, u, w)$ and (ii) a pair of non-zero equilibrium points $u_+ = -u_- \neq 0$ exists, such that

$$F(u_{\pm}) = 0, \quad F'(u_{\pm}) < 0, \quad (1.8)$$

while no other equilibrium points exist in the interval $u \in [u_-, u_+]$. For instance, this assumption is verified for the ϕ^4 model (1.3) with $u_{\pm} = \pm 1$ and for the sine-Gordon equation (1.4) with $u_{\pm} = \pm\pi$.

We address the fundamental question of the existence of travelling wave solutions in the discrete Klein–Gordon lattice (1.1). Since discrete equations have no translational and Lorentz invariance, unlike the continuous Klein–Gordon equation (1.2), the existence of travelling waves, pulsating travelling waves and travelling breathers represents a challenging problem of applied mathematics. A review of the literature can be found in [13].

Our work deals with the travelling kinks between the non-zero equilibrium states u_{\pm} . We are not interested in the travelling breathers and pulsating waves near the zero equilibrium state since the zero state is linearly unstable in the dynamics of the discrete Klein–Gordon lattice (1.1) with $F'(0) = 1 > 0$. Indeed, looking for solutions in the form $u_n(t) = e^{i\kappa hn + \lambda n}$, we derive the dispersion relation for linear waves near the zero equilibrium state:

$$\lambda^2 = 1 - \frac{4}{h^2} \sin^2 \frac{\kappa h}{2}.$$

It follows from the dispersion relation that there exists $\kappa_*(h) > 0$ such that $\lambda^2 > 0$ for $0 \leq \kappa < \kappa_*(h)$. On the other hand, the non-zero equilibrium states u_+ and u_- are neutrally stable in the dynamics of the discrete Klein–Gordon lattice (1.1) with $F'(u_{\pm}) < 0$. We focus hence on bounded heteroclinic orbits which connect the stable non-zero equilibrium states u_- and u_+ in the form:

$$u_n(t) = \phi(z), \quad z = hn - ct, \quad (1.9)$$

where the function $\phi(z)$ solves the differential advance-delay equation:

$$c^2 \phi''(z) = \frac{\phi(z+h) - 2\phi(z) + \phi(z-h)}{h^2} + \phi(z) + Q(\phi(z-h), \phi(z), \phi(z+h)). \quad (1.10)$$

We consider the following class of solutions of the differential advance-delay equation (1.10): (i) $\phi(z)$ is a twice continuously differentiable function of $z \in \mathbb{R}$; (ii) $\phi(z)$ is monotonically

increasing on $z \in \mathbb{R}$ and (iii) $\phi(z)$ satisfies boundary conditions:

$$\lim_{z \rightarrow -\infty} \phi(z) = u_-, \quad \lim_{z \rightarrow +\infty} \phi(z) = u_+. \quad (1.11)$$

It is easy to verify that the travelling kink solutions of the continuous Klein–Gordon equation (1.2) with $F(u)$ satisfying (1.8) yield the form $u = \phi(z)$, $z = x - ct$ for $|c| < 1$. However, the travelling kink can be destroyed in the discrete Klein–Gordon lattice (1.1), which results in violation of one or more conditions on $\phi(z)$. For instance, the bounded twice continuously differentiable solution $\phi(z)$ may develop non-vanishing oscillatory tails around the equilibrium states u_{\pm} [13].

Recent progress on travelling kinks was reported for the discrete ϕ^4 model. Four particular spatial discretizations of the nonlinearity $F(u) = u(1 - u^2)$ were proposed with four independent and alternative methods [3,8,15,22], where the ultimate goal was to construct a family of translation-invariant stationary kinks for $c = 0$, that are given by continuous, monotonically increasing functions $\phi(z)$ on $z \in \mathbb{R}$ with the boundary conditions (1.11). Exceptional discretizations were generalized in [2,6], where multi-parameter families of cubic polynomials $f(u_{n-1}, u_n, u_{n+1})$ were obtained. It was observed in numerical simulations of the discrete ϕ^4 model [22] that the effective translational invariance of stationary kinks implies reduction of radiation diverging from moving kinks. New effects such as self-acceleration were reported for some of the exceptional discretizations in [6]. Nevertheless, from a mathematically strict point of view, we shall ask if exceptional discretizations guarantee existence of true travelling kinks (heteroclinic orbits) at least for small values of c . The question was answered negatively in [18], where numerical analysis of beyond-all-order expansions was developed. It was shown that bifurcations of travelling kink solutions from $h = 0$ to small non-zero h do not generally occur in the discrete ϕ^4 model with small values of c , even if the exceptional discretizations allow these bifurcations for $c = 0$. It was also discovered in [18] that bifurcations of travelling kink solutions may occur for finitely many isolated values of c far from the limit $c = 0$.

The discrete sine-Gordon model was also the subject of recent studies. Numerical computations were used to identify oscillatory tails of small amplitudes on the travelling kink solutions [7,20]. These tails were explained with analysis of centre manifold reductions (carried out without the normal form reductions) [9]. Exceptional discretizations of the nonlinearity $F(u) = \sin(u)$ were suggested by the topological bound method in [21] (see the review in [23]) and by the “inverse” (direct substitution) method in [8]. Simultaneously with the absence of single kinks in the sine-Gordon lattices, multiple kinks (between non-zero equilibrium points of $\pi(\text{mod } 2\pi)$) were discovered with the formal reduction of the discrete lattice (1.1) to the fourth-order ODE problem in [5] and confirmed with numerical analysis of the differential advance-delay equation in [1].

Our work is motivated by the recent advances in studies of differential advance-delay equations from the point of dynamical system methods such as centre manifold reductions

and normal forms (see pioneering work in [11,14]). The same methods were recently applied in [19] to travelling solitary waves in discrete nonlinear Schrödinger lattices near the maximum group velocity of linear waves. It was shown in [19] that the non-existence of travelling solitary waves can be derived already in the truncated (polynomial) normal form. We shall exploit this idea to the class of travelling kinks in discrete Klein–Gordon lattices near the specific speed $c = 1$ and the continuous limit $h = 0$. This particular point corresponds to the quadruple zero eigenvalue on the centre manifold of the dynamical system. A general reversible normal form for the quadruple zero eigenvalue was derived and studied in [10] (see also tutorials in the book [12]).

We note that the discrete Klein–Gordon lattice was considered in [14] but the nonlinearity $F(u)$ was taken to be decreasing near $u = 0$, such that the quadruple zero eigenvalue was not observed in the list of possible bifurcations of travelling wave solutions (see Figure 1 in [14]). On the other hand, the quadruple zero eigenvalue occurred in the discrete Fermi–Pasta–Ulam lattice studied in [11], where the symmetry with respect to the shift transformation was used to reduce the bifurcation problem to a three-dimensional centre manifold. Since the reversible symmetry operator for the Fermi–Pasta–Ulam lattice is minus identity times the reversibility symmetry for the Klein–Gordon lattice, the reversible normal forms for quadruple zero eigenvalue are different in these two problems. We focus here only on the case of the discrete Klein–Gordon lattice.

Our strategy is as follows. We develop a decomposition of solutions of the differential advance-delay equation into two parts: a four-dimensional projection to the subspace associated with the quadruple zero eigenvalue and an infinite-dimensional projection to the hyperbolic part of the problem. By using a suitable scaling, we truncate the resulting system of equations with a scalar fourth-order equation, which is similar to the one formally derived in [5]. This fourth-order equation is referred to as the normal form for travelling kinks. We develop a numerical analysis of the fourth-order equation and show that no monotonic heteroclinic orbits from u_- to u_+ exist both in the discrete ϕ^4 and sine-Gordon lattices. Rigorous persistence analysis of bounded solutions (heteroclinic orbits between periodic solutions near u_- and u_+) is developed with the technique of centre manifold reduction. Our main conclusion is that the differential advance-delay equation (1.10) has no monotonic travelling kinks near the point $c = 1$ and $h = 0$ but it admits families of non-monotonic travelling kinks with oscillatory tails.

It seems surprising that the truncated normal form is independent on the discretizations of the nonlinearity function $f(u_{n-1}, u_n, u_{n+1})$ and the negative result extends to all exceptional discretizations constructed in [2,3,6,8,15,21–23]. Any one-parameter curve of monotonic travelling kinks, which bifurcate in the plane (c, h) from the finite set of isolated points $(c_*, 0)$ (see numerical results in [18]), may only exist far from the point $(1, 0)$.

In addition, we explain bifurcations of multiple kinks from $u_- = -\pi$ to $u_+ = \pi(2n - 1)$, $n > 1$ in the discrete sine-Gordon equation from the point of normal form analysis. These

bifurcations may occur along an infinite set of curves on the plane (c, h) which all intersect the point $(1, 0)$ (see numerical results in [1]). We also derive the truncated normal form from the discretizations of the inverse method reported in [8] and show that it may admit special solutions for monotonic kinks, when the continuity condition (1.6) is violated.

We emphasize that our methods are very different from the computations of beyond-all-order expansions, exploited in the context of difference maps in [24,25] and differential advance-delay equations in [18]. By working near the particular point $c = 1$ and $h = 0$, we avoid beyond-all-order expansions and derive non-existence results from the study of the polynomial normal form. In an asymptotic limit of common validity, bifurcations of heteroclinic orbits in the truncated normal form can be studied with beyond-all-order computations (see recent analysis and review in [26]).

Our paper has the following structure. Section 2 discusses eigenvalues of the linearization problem at the zero equilibrium point and gives a formal derivation of the main result, the scalar fourth-order equation. The rigorous derivation of the scalar fourth-order equation from decompositions of solutions, projection techniques and truncation is described in Section 3. Section 4 contains numerical analysis of the fourth-order equation, where we compute the split function for heteroclinic orbits. Persistence analysis of bounded solutions in the differential advance-delay equation is developed in Section 5. Section 6 concludes the paper with a summary and a formulation of an open problem. A number of important but technical computations are reported in appendices to this paper. Appendix A contains the comparison of the fourth-order equation with the normal form from [10]. Appendix B gives the proof of existence of a centre manifold in the full system, which supplements the analysis in [14]. Appendix C describes computations of the Stokes constant for heteroclinic orbits in the fourth-order differential equation from methods of [26]. Appendix D discusses applications of the fourth-order equation to the inverse method from [8].

2. Resonances in dispersion relations and the scalar normal form

When the differential advance-delay equation (1.10) is linearized near the zero equilibrium point, we look for solutions in the form $\phi(z) = e^{\lambda z}$, where λ belongs to the set of eigenvalues. All eigenvalues of the linearized problems are obtained from roots of the dispersion relation:

$$D(\Lambda; c, h) = 2(\cosh \Lambda - 1) + h^2 - c^2 \Lambda^2 = 0, \quad (2.1)$$

where $\Lambda = \lambda h$ is the eigenvalue in zoomed variable $\zeta = \frac{z}{h}$. We are interested to know how many eigenvalues occur on the imaginary axis and whether the imaginary axis is isolated from the set of complex eigenvalues (that is complex eigenvalues do not accumulate at the imaginary axis). Imaginary eigenvalues $\Lambda = 2iK$ of the dispersion relation (2.1) satisfy the transcendental equation:

$$\sin^2 K = \frac{h^2}{4} + c^2 K^2. \quad (2.2)$$

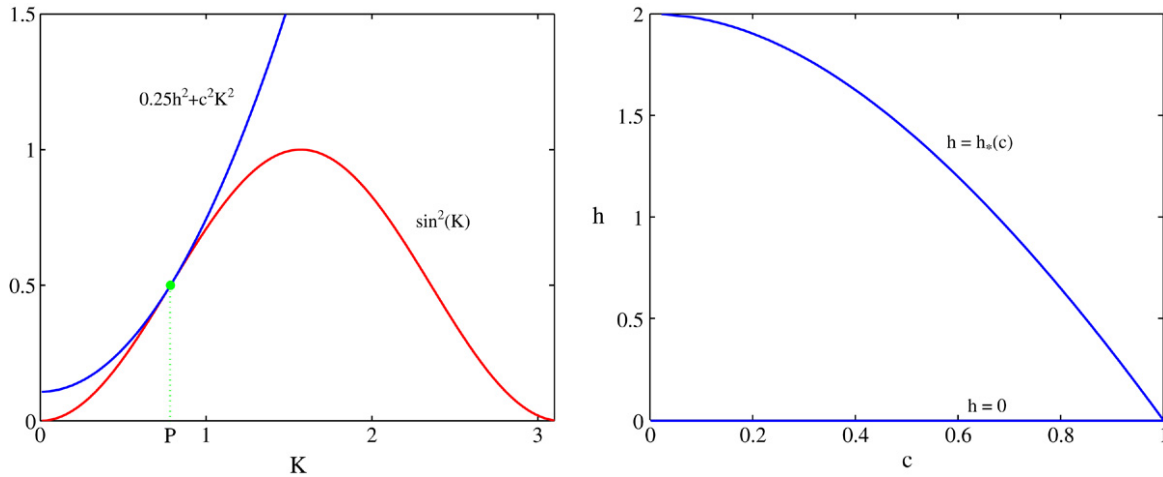


Fig. 1. Left: The graphical solution of the transcendental equation (2.2) for $P = \frac{\pi}{4}$, where (c, h) are given by (2.4). Right: The bifurcation curve for 1:1 resonant Hopf bifurcation $h = h_*(c)$ and double zero resonance bifurcation $h = 0$.

Particular results on roots of the transcendental equation (2.2) are easily deduced from analysis of the function $\sin(z)$ in complex domain $z \in \mathbb{C}$. Due to the obvious symmetry, we shall only consider the non-negative values of c and h .

- When $c = 0$ and $0 < h < 2$, Eq. (2.2) has only simple real roots K , such that all eigenvalues Λ are purely imaginary and simple. All real roots K become double at $h = 0$ and $h = 2$.
- When $h = 0$ and $c \neq 1$, a double zero root of K (a double zero eigenvalue Λ) exists. When $0 < c < 1$, finitely many purely imaginary eigenvalues Λ exist (e.g. only one pair of roots $K = \pm P$ exists for $c_0 < c < 1$, where $c_0 \approx 0.22$), while infinitely many roots are complex and bounded away from the imaginary axis. When $c > 1$, all non-zero eigenvalues Λ are complex and bounded away from the imaginary axis.

In the general case of $h \neq 0$ and $c \neq 0$, the transcendental equation (2.2) is more complicated but it can be analyzed similarly to the dispersion relation considered in [11,14].

Lemma 2.1. *There exists a curve $h = h_*(c)$, which intersects the points $(1, 0)$ and $(0, 2)$ on the plane (c, h) , such that for $0 < c < 1$ and $0 < h < h_*(c)$ finitely many eigenvalues Λ of the dispersion relation (2.1) are located on the imaginary axis and all other eigenvalues in a complex plane are bounded away from the imaginary axis. The curve $h = h_*(c)$ corresponds to the 1:1 resonance Hopf bifurcation, where the set of imaginary eigenvalues includes only one pair of double eigenvalues.*

Remark 2.1. The 1:1 resonant Hopf bifurcation is illustrated on Fig. 1. Fig. 1(left) shows a graphical solution of the transcendental equation (2.2) for $h = h_*(c)$. Fig. 1(right) shows the bifurcation curve $h = h_*(c)$ on the plane (c, h) . The curve $h = 0$ for $0 < c < 1$ corresponds to the double zero eigenvalue in resonance with pairs of purely imaginary eigenvalues.

Proof. Let $\Lambda = p + iq$ and rewrite the dispersion relation in the equivalent form:

$$\begin{aligned} c^2(q^2 - p^2) + h^2 + 2(\cosh p \cos q - 1) &= 0 \\ c^2 pq - \sinh p \sin q &= 0. \end{aligned}$$

It follows from the system with $c \neq 0$ that the imaginary parts of the eigenvalues is bounded by the real parts of the eigenvalues:

$$q^2 \leq p^2 + \frac{4}{c^2} \cosh^2 \frac{p}{2}. \tag{2.3}$$

Therefore, if complex eigenvalues accumulate to the imaginary axis, such that there exists a sequence (p_n, q_n) with $\lim_{n \rightarrow \infty} p_n = 0$ and $\lim_{n \rightarrow \infty} q_n = q_*$, the accumulation point $(0, q_*)$ is bounded. However, since $D(\Lambda; c, h)$ is analytic in Λ , the dispersion relation (2.1) may have finitely many roots of finite multiplicities in a bounded domain of the complex plane. Therefore, the accumulation point $(0, q_*)$ does not exist and complex eigenvalues are bounded away from the imaginary axis. By the same reason, there exist finitely many eigenvalues on the imaginary axis $\Lambda \in i\mathbb{R}$.

Let $\Lambda = 2iK$ and $K = \pm P$ be double roots of Eq. (2.2). The curve $h = h_*(c)$ can be represented in the parameter form

$$c^2 = \frac{\sin P \cos P}{P}, \quad h^2 = 4 \sin P (\sin P - P \cos P). \tag{2.4}$$

A simple graphical analysis of the transcendental equation (2.2) shows that the double roots at $K = \pm P$ are unique for $0 < P < \frac{\pi}{2}$, when the intersection of the parabola and the trigonometric function occurs at the first fundamental period of the \sin^2 function (see Fig. 1(left)). At $P = 0$ (at the point $(c, h) = (1, 0)$), the pair of double imaginary eigenvalues $\Lambda = \pm 2iP$ merge at $\Lambda = 0$ and form a quadruple zero eigenvalue. At $P = \frac{\pi}{2}$ (at the point $(c, h) = (0, 2)$), a sequence of infinitely many double imaginary eigenvalues exists at $\Lambda = i\pi(1 + 2n)$, $n \in \mathbb{Z}$. \square

We will be interested in the reduction of the differential advance–delay equation (1.10) at the particular point $(c, h) = (1, 0)$. Let ϵ be a small parameter that defines a point on the plane (c, h) , which is locally close to the bifurcation point

(1, 0):

$$c^2 = 1 + \epsilon\gamma, \quad h^2 = \epsilon^2\tau. \tag{2.5}$$

Let $\Lambda = \sqrt{\epsilon}\Lambda_1$ be the scaled eigenvalue near $\Lambda = 0$, so that the Taylor series expansion of the dispersion relation (2.1) allows for a non-trivial balance of ϵ -terms. Truncating the Taylor series beyond the leading order $O(\epsilon^2)$, we obtain the bi-quadratic equation for the scaled eigenvalue Λ_1 :

$$\frac{1}{12}\Lambda_1^4 - \gamma\Lambda_1^2 + \tau = 0. \tag{2.6}$$

The 1:1 resonance Hopf bifurcation corresponds to the point, where the roots of the bi-quadratic equation (2.6) are double and purely imaginary. This point occurs when $\tau = 3\gamma^2$, which agrees with the Taylor series expansion of the system (2.4) in powers of P at the leading orders of P^2 and P^4 . As a result, the bifurcation curve $h = h_*(c)$ has the leading order behavior:

$$h_*(c) = \sqrt{3}(1 - c^2) + O(1 - c^2)^2. \tag{2.7}$$

The normal form for travelling kinks, which is the main result of our paper, can be recovered with a formal asymptotic expansion of the nonlinear differential advance-delay equation (1.10). Let $\zeta = \frac{z}{h}$ and rewrite the main Eq. (1.10) in the form:

$$c^2\phi''(\zeta) = \phi(\zeta + 1) - 2\phi(\zeta) + \phi(\zeta - 1) + h^2\phi(\zeta) + h^2Q(\phi(\zeta - 1), \phi(\zeta), \phi(\zeta + 1)). \tag{2.8}$$

Let (c, h) be the perturbed bifurcation point in the form (2.5). Let $\phi(\zeta)$ be a smooth function of a slowly varying variable $\zeta_1 = \sqrt{\epsilon}\zeta$. Expanding $\phi(\zeta_1 \pm \sqrt{\epsilon})$ in the Taylor series in $\sqrt{\epsilon}$ and truncating by terms beyond the leading order ϵ^2 , we obtain the scalar fourth-order differential equation:

$$\frac{1}{12}\phi^{(iv)}(\zeta_1) - \gamma\phi''(\zeta_1) + \tau F(\phi(\zeta_1)) = 0, \tag{2.9}$$

where we have used the representations (1.6) and (1.7). The linearization of the nonlinear ODE (2.9) near the zero equilibrium point recovers the bi-quadratic dispersion relation (2.6). The nonlinear equation (2.9) has the equilibrium points $u_-, 0$, and u_+ , which are inherited from the equilibrium points of the Klein–Gordon equation (1.2). Our main examples will include the ϕ^4 model with $F(u) = u(1 - u^2)$, $u_{\pm} = \pm 1$ and the sine-Gordon equation with $F(u) = \sin(u)$, $u_{\pm} = \pm\pi$.

3. Decompositions, projections and truncation

We shall derive the normal form Eq. (2.9) with rigorous analysis when the solution of the differential advance-delay equation (2.8) is decomposed near a quadruple zero eigenvalue. We adopt notations of [14] (see review in [13]) and rewrite the differential advance-delay equation as an infinite-dimensional evolution problem. We shall work with the scaled (inner) variable $\zeta = \frac{z}{h}$, where the differential advance-delay equation (1.10) takes the form (2.8). Let p be a new independent variable, such that $p \in [-1, 1]$ and define the vector $\mathbf{U} = (U_1(\zeta), U_2(\zeta), U_3(\zeta, p))^T$, such that

$$U_1 = \phi(\zeta), \quad U_2 = \phi'(\zeta), \quad U_3 = \phi(\zeta + p). \tag{3.1}$$

It is clear that $U_3(\zeta, p) = U_1(\zeta + p)$ and $U_3(\zeta, 0) = U_1(\zeta)$. The difference operators are then defined as

$$U_3(\zeta, \pm 1) = \delta^{\pm}U_3(\zeta, p) = \phi(\zeta \pm 1) = \delta_{\pm}U_1(\zeta) = U_1(\zeta \pm 1). \tag{3.2}$$

Let \mathcal{D} and \mathcal{H} be the following Banach spaces for $\mathbf{U} = (U_1, U_2, U_3(p))^T$,

$$\mathcal{D} = \left\{ (U_1, U_2) \in \mathbb{R}^2, U_3 \in C^1([-1, 1], \mathbb{R}), U_3(0) = U_1 \right\}, \tag{3.3}$$

$$\mathcal{H} = \left\{ (U_1, U_2) \in \mathbb{R}^2, U_3 \in C^0([-1, 1], \mathbb{R}) \right\}, \tag{3.4}$$

with the usual supremum norm. We look for a smooth mapping $\zeta \mapsto \mathbf{U}(\zeta)$ in $C^0(\mathbb{R}; \mathcal{D})$, which represents classical solutions of the infinite-dimensional evolution problem:

$$\frac{d}{d\zeta}\mathbf{U} = \mathcal{L}_{c,h}\mathbf{U} + \frac{h^2}{c^2}\mathbf{M}_0(\mathbf{U}), \tag{3.5}$$

where $\mathcal{L}_{c,h}$ and $\mathbf{M}_0(\mathbf{U})$ are found from the differential advance-delay equation (2.8):

$$\mathcal{L}_{c,h} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{h^2 - 2}{c^2} & 0 & \frac{1}{c^2}(\delta^+ + \delta^-) \\ 0 & 0 & \partial_p \end{pmatrix} \tag{3.6}$$

and

$$\mathbf{M}_0(\mathbf{U}) = (0, Q(\delta^-U_3, U_1, \delta^+U_3), 0)^T. \tag{3.7}$$

The linear operator $\mathcal{L}_{c,h}$ maps \mathcal{D} into \mathcal{H} continuously and it has a compact resolvent in \mathcal{H} . The nonlinearity $\mathbf{M}_0(\mathbf{U})$ is analytic in an open neighborhood of $\mathbf{U} = \mathbf{0}$, maps \mathcal{H} into \mathcal{D} continuously, and $\|\mathbf{M}_0(\mathbf{U})\|_{\mathcal{D}} = O(\|\mathbf{U}\|_{\mathcal{H}}^2)$.

The spectrum of $\mathcal{L}_{c,h}$ consists of an infinite set of isolated eigenvalues of finite multiplicities. By virtue of the Laplace transform, eigenvalues of the linear operator $\mathcal{L}_{c,h}$ are found with the solution $\mathbf{U}(\zeta, p) = (1, \Lambda, e^{\Lambda p})^T e^{\Lambda\zeta}$, when the linear problem $\mathbf{U}'(\zeta) = \mathcal{L}_{c,h}\mathbf{U}$ is reduced to the dispersion relation (2.1), that is $D(\Lambda; c, h) = 0$. We are particularly interested in the bifurcation point $c = 1$ and $h = 0$, when

$$D_1(\Lambda) \equiv D(\Lambda; 1, 0) = 2(\cosh \Lambda - 1) - \Lambda^2. \tag{3.8}$$

The transcendental equation $D_1(\Lambda) = 0$ has the quadruple zero root and no other root in the neighborhood of $\Lambda \in i\mathbb{R}$. The four generalized eigenvectors of the Jordan chain for the zero eigenvalue, $\mathcal{L}_{1,0}\mathbf{U}_j = \mathbf{U}_{j-1}$, $j = 1, 2, 3, 4$ with $\mathbf{U}_{-1} = \mathbf{0}$, are found exactly as:

$$\mathbf{U}_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{U}_1 = \begin{pmatrix} 0 \\ 1 \\ p \end{pmatrix}, \quad \mathbf{U}_2 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}p^2 \end{pmatrix}, \tag{3.9}$$

$$\mathbf{U}_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{6}p^3 \end{pmatrix}.$$

The dynamical system (3.5) has the reversibility symmetry \mathcal{S} , such that

$$-\frac{d}{d\zeta}\mathcal{S}\mathbf{U} = \mathcal{L}_{c,h}\mathcal{S}\mathbf{U} + \frac{h^2}{c^2}\mathbf{M}_0(\mathcal{S}\mathbf{U}), \quad (3.10)$$

where

$$\mathcal{S}\mathbf{U} = (U_1(\zeta), -U_2(\zeta), U_3(\zeta, -p))^T \quad (3.11)$$

and the symmetry property (1.5) has been used. Since both the linear operator (3.6) and the nonlinearity (3.7) anti-commute with the reversibility operator (3.11), the standard property of reversible systems holds: if $\mathbf{U}(\zeta)$ is a solution of (3.5) for forward time $\zeta > 0$, then $\mathcal{S}\mathbf{U}(-\zeta)$ is a solution of (3.5) for backward time $-\zeta < 0$ (see [16] for a review). Applying the reversibility operator \mathcal{S} to the eigenvectors (3.9) at the bifurcation of the quadruple zero eigenvalue, we observe that

$$\mathcal{S}\mathbf{U}_j = (-1)^j \mathbf{U}_j, \quad j = 0, 1, 2, 3. \quad (3.12)$$

This bifurcation case fits to the analysis of [10], where the reversible normal form was derived for the quadruple zero eigenvalue. Appendix A shows that the general reversible normal form from [10] is reduced to the normal form Eq. (2.9) by appropriate scaling. However, we notice that this result cannot be applied directly, since it is only valid in a neighborhood of the origin, while in the present case, the solutions we are interested in are of order $O(1)$.

In order to study the bifurcation for a quadruple zero eigenvalue in the reversible system (3.5), we shall define the perturbed point (c, h) near the bifurcation point $(1, 0)$ with an explicit small parameter ε . Contrary to the definition (2.5), it will be easier to work with the parameters (γ, τ) , defined from the relations:

$$\frac{1}{c^2} = 1 - \varepsilon\gamma, \quad \frac{h^2}{c^2} = \varepsilon^2\tau, \quad (3.13)$$

where ε is different from ϵ used in (2.5). The dynamical system (3.5) with the parametrization (3.13) is rewritten in the explicit form:

$$\frac{d}{d\zeta}\mathbf{U} = \mathcal{L}_{1,0}\mathbf{U} + \varepsilon\gamma\mathbf{L}_1(\mathbf{U}) + \varepsilon^2\tau\mathbf{L}_2(\mathbf{U}) + \varepsilon^2\tau\mathbf{M}_0(\mathbf{U}), \quad (3.14)$$

where $\mathcal{L}_{1,0}$ follows from (3.6) with $(c, h) = (1, 0)$, $\mathbf{M}_0(\mathbf{U})$ is the same as (3.7), and the linear terms $\mathbf{L}_{1,2}(\mathbf{U})$ are

$$\begin{aligned} \mathbf{L}_1(\mathbf{U}) &= (0, 2U_1 - (\delta^+ + \delta^-)U_3, 0)^T, \\ \mathbf{L}_2(\mathbf{U}) &= (0, U_1, 0)^T. \end{aligned} \quad (3.15)$$

Bounded solutions of the ill-posed initial-value problem (3.14) on the entire axis $\zeta \in \mathbb{R}$ are the subject of our interest, with the particular emphasis on kink solutions (heteroclinic orbits between non-zero equilibrium points u_- and u_+). These bounded solutions can be constructed with the decomposition of the solution of the infinite-dimensional system (3.14), projection to the finite-dimensional subspace of zero eigenvalue and to the infinite-dimensional subspace of non-zero eigenvalues, and truncation of the resulting system of equations into the fourth-order differential equation (2.9). Since

the original system is reversible, the fourth-order equation must inherit the reversibility property.

The techniques of decompositions, projections and truncation rely on the solution of the resolvent equation $(\Lambda\mathcal{I} - \mathcal{L}_{1,0})\mathbf{U} = \mathbf{F}$, where $\mathbf{U} \in \mathcal{D}$, $\mathbf{F} \in \mathcal{H}$ and $\Lambda \in \mathbb{C}$. When Λ is different from the roots of $D_1(\Lambda)$, the explicit solution of the resolvent equation is obtained in the form:

$$\begin{aligned} U_1 &= -[D_1(\Lambda)]^{-1}\tilde{F}(\Lambda), \\ U_2 &= \Lambda U_1 - F_1, \\ U_3(p) &= U_1 e^{\Lambda p} - \int_0^p F_3(s) e^{\Lambda(p-s)} ds, \end{aligned} \quad (3.16)$$

where

$$\begin{aligned} \tilde{F}(\Lambda) &= \Lambda F_1 + F_2 \\ &\quad - \int_0^1 (F_3(s) e^{\Lambda(1-s)} - F_3(-s) e^{-\Lambda(1-s)}) ds. \end{aligned} \quad (3.17)$$

The quadruple zero eigenvalue appears as the quadruple pole in the solution $\mathbf{U}(p; \Lambda)$ of the resolvent equation near $\Lambda = 0$. Let us decompose the solution of the dynamical system (3.14) into two parts:

$$\mathbf{U} = \mathbf{X} + \mathbf{Y}, \quad (3.18)$$

where \mathbf{X} is a projection to the fourth-dimensional subspace of the quadruple zero eigenvalue,

$$\mathbf{X} = A(\zeta)\mathbf{U}_0 + B(\zeta)\mathbf{U}_1(p) + C(\zeta)\mathbf{U}_2(p) + D(\zeta)\mathbf{U}_3(p) \quad (3.19)$$

and \mathbf{Y} is the projection on the complementary invariant subspace of $\mathcal{L}_{1,0}$. We notice in particular that

$$U_1(\zeta) = A(\zeta) + Y_1(\zeta) \quad (3.20)$$

$$\begin{aligned} U_3(\zeta, p) &= A(\zeta) + pB(\zeta) + \frac{1}{2}p^2C(\zeta) + \frac{1}{6}p^3D(\zeta) \\ &\quad + Y_3(\zeta, p). \end{aligned} \quad (3.21)$$

With the decomposition (3.18) and (3.19), the problem (3.14) is rewritten in the form:

$$\frac{d}{d\zeta}\mathbf{Y} - \mathcal{L}_{1,0}\mathbf{Y} = \mathbf{F}(A, B, C, D, \mathbf{Y}), \quad (3.22)$$

where

$$\begin{aligned} F_1 &= B - A' \\ F_2 &= C - B' - \varepsilon\gamma(C - 2Y_1 + (\delta^+ + \delta^-)Y_3) \\ &\quad + \varepsilon^2\tau(A + Q + Y_1) \\ F_3 &= B - A' + p(C - B') + \frac{1}{2}p^2(D - C') + \frac{1}{6}p^3(-D'), \end{aligned}$$

and

$$\begin{aligned} Q &= Q \left(A - B + \frac{1}{2}C - \frac{1}{6}D + \delta^-Y_3, A + Y_1, \right. \\ &\quad \left. A + B + \frac{1}{2}C + \frac{1}{6}D + \delta^+Y_3 \right). \end{aligned}$$

We notice that p is a dummy variable in (3.22). Since the vector \mathbf{Y} is projected to the subspace of non-zero eigenvalues of $\mathcal{L}_{1,0}$,

the right-hand-side of the “inhomogeneous” problem (3.22) must be orthogonal to eigenvectors of the adjoint operator $\mathcal{L}_{1,0}^*$ associated to the zero eigenvalue. Equivalently, one can consider the solution of the resolvent Eq. (3.17) and remove the pole singularities from the function $[D_1(\lambda)]^{-1} \tilde{F}(\lambda)$ near $\lambda = 0$. See p. 853 in [11] for the projections on the four-dimensional subspace and, in particular, the formula:

$$\mathcal{P}_h \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{-5} \\ \frac{2}{5}p(1 - 5p^2) \end{pmatrix} \equiv \mathbf{F}_0(p), \tag{3.23}$$

where \mathcal{P}_h is a projection operator to the subspace for \mathbf{Y} . This procedure results in the fourth-order differential system for components $(A, B, C, D) \in \mathbb{R}^4$ in ζ :

$$\begin{aligned} A' &= B \\ B' &= C - \frac{2}{5}\varepsilon\gamma(C - 2Y_1 + (\delta^+ + \delta^-)Y_3) \\ &\quad + \frac{2}{5}\varepsilon^2\tau(A + Q + Y_1) \\ C' &= D \\ D' &= 12\varepsilon\gamma(C - 2Y_1 + (\delta^+ + \delta^-)Y_3) \\ &\quad - 12\varepsilon^2\tau(A + Q + Y_1). \end{aligned} \tag{3.24}$$

While the system (3.22) has the reversibility symmetry \mathcal{S} in (3.11), the system (3.24) has the reduced reversibility symmetry,

$$\mathcal{S}_0 \mathbf{A} = (A, -B, C, -D)^T, \tag{3.25}$$

where $\mathbf{A} = (A, B, C, D)^T$. The closed system of Eqs. (3.22) and (3.24) can be rewritten by using the scaling transformation:

$$\begin{aligned} A &= \phi_1(\zeta_1), \quad B = \sqrt{\varepsilon}\phi_2(\zeta_1), \quad C = \varepsilon\phi_3(\zeta_1), \\ D &= \varepsilon^{3/2}\phi_4(\zeta_1), \quad \mathbf{Y} = \varepsilon^{3/2}\boldsymbol{\psi}(\zeta, p), \end{aligned} \tag{3.26}$$

where $\zeta_1 = \sqrt{\varepsilon}\zeta$. The four-dimensional vector $\boldsymbol{\phi} \in \mathbb{R}^4$ satisfies the vector system:

$$\frac{d}{d\zeta_1} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{2}{-5}\varepsilon \\ 0 \\ 12 \end{pmatrix} g, \tag{3.27}$$

where g is a scalar function given by

$$g = \gamma(\phi_3 - 2\sqrt{\varepsilon}\psi_1 + \sqrt{\varepsilon}(\delta^+ + \delta^-)\psi_3) - \tau(\phi_1 + Q + \varepsilon^{3/2}\psi_1)$$

and

$$\begin{aligned} Q &= Q \left(\phi_1 - \sqrt{\varepsilon}\phi_2 + \frac{1}{2}\varepsilon\phi_3 - \frac{1}{6}\varepsilon^{3/2}\phi_4 + \sqrt{\varepsilon^3}\delta^-\psi_3, \right. \\ &\quad \left. \phi_1 + \varepsilon^{3/2}\psi_1, \phi_1 + \sqrt{\varepsilon}\phi_2 + \frac{1}{2}\varepsilon\phi_3 \right. \\ &\quad \left. + \frac{1}{6}\sqrt{\varepsilon^3}\phi_4 + \varepsilon^{3/2}\delta^+\psi_3 \right). \end{aligned}$$

The infinite-dimensional vector $\boldsymbol{\psi} \in \mathcal{D}$ satisfies the “inhomogeneous” problem:

$$\frac{d}{d\zeta} \boldsymbol{\psi} - \mathcal{L}_{1,0}\boldsymbol{\psi} = \sqrt{\varepsilon}g\mathbf{F}_0(p), \tag{3.28}$$

where \mathbf{F}_0 is given in (3.23). The formal truncation of the fourth-order system (3.27) at $\varepsilon = 0$ recovers the scalar fourth-order equation (2.9). The set of equilibrium points of the coupled system (3.27) and (3.28) contains the set of equilibrium points of the fourth-order equation (2.9), since the constraints $\phi_2 = \phi_3 = \phi_4 = 0$ and $\boldsymbol{\psi} = \mathbf{0}$ reduce the coupled system to the algebraic equation $\phi_1 + Q(\phi_1, \phi_1, \phi_1) = 0$, that is $F(\phi_1) = 0$. Our main result is the proof of existence of the centre manifold in the coupled system (3.27) and (3.28).

Theorem 1. Fix $M > 0$ and let $\varepsilon > 0$ be small enough. Then for any given $\boldsymbol{\phi} \in C_b^0(\mathbb{R}, \mathbb{R}^4)$, such that

$$\|\boldsymbol{\phi}\| \leq M, \tag{3.29}$$

there exists a unique solution $\boldsymbol{\psi} \in C_b^0(\mathbb{R}, \mathcal{D})$ of the system (3.28), such that

$$\|\boldsymbol{\psi}\| \leq \sqrt{\varepsilon}K, \tag{3.30}$$

where the constant $K > 0$ is independent of ε . Moreover, when $\boldsymbol{\phi}(\zeta_1)$ is anti-reversible with respect to \mathcal{S}_0 , i.e. $\mathcal{S}_0\boldsymbol{\phi}(-\zeta_1) = -\boldsymbol{\phi}(\zeta_1)$, then $\boldsymbol{\psi}(\zeta)$ is anti-reversible with respect to \mathcal{S} , i.e. $\mathcal{S}\boldsymbol{\psi}(-\zeta) = -\boldsymbol{\psi}(\zeta)$. If the solution $\boldsymbol{\phi}(\zeta_1)$ tends towards a T -periodic orbit as $\zeta_1 \rightarrow \pm\infty$, then the solution $\boldsymbol{\psi}(\zeta)$ tends towards a $T/\sqrt{\varepsilon}$ -periodic orbit of the system (3.28) as $\zeta \rightarrow \pm\infty$.

In this theorem, $C_b^0(\mathbb{R}, \mathcal{E})$ denotes the Banach space (sup norm) of continuous and bounded functions on \mathbb{R} taking values in the Banach space \mathcal{E} .

The proof of existence of the centre manifold for the class of reversible dynamical systems (3.5) is developed in Lemmas 2–4 of [14] with the Green function technique. Applications of the centre manifold reduction in a similar context can be found in [9,11,13]. We note that the case of the quadruple zero eigenvalue was not studied in any of the previous publications (e.g. the pioneering paper [14] dealt with the case of a pair of double and a pair of simple imaginary eigenvalues). We develop an alternative proof of existence of a centre manifold in the coupled system (3.27) and (3.28) with a more explicit and elementary method in Appendix B.

4. Numerical analysis of the truncated normal form

Eigenvalues of the linear part of the fourth-order equation (2.9), that are roots of the bi-quadratic equation (2.6), are shown in Fig. 2. Due to the definition (3.13), it makes sense to consider only the upper half plane of the parameter plane (γ, τ) . The curve $\Gamma_1 = \{\gamma < 0, \tau = 0\}$ corresponds to the double zero eigenvalue that exists in resonance with a pair of purely imaginary eigenvalues. The curve $\Gamma_2 = \{\gamma < 0, \tau = 3\gamma^2\}$ corresponds to the reversible 1:1 resonance Hopf bifurcation $h = h_*(c)$ in the limit (2.7). These two curves confine the

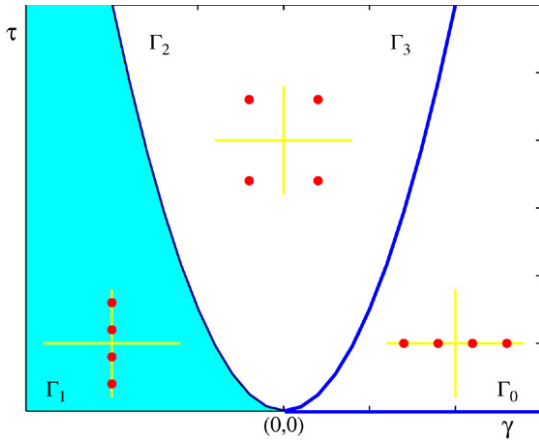


Fig. 2. Eigenvalues of the scalar normal form (2.9) on the parameter plane (γ, τ) . The curves $\Gamma_{0,1,2,3}$ mark various bifurcations of co-dimension one for the zero equilibrium point. The shaded area is a domain of possible bifurcations of a heteroclinic orbit.

parameter domain of our interest, where a heteroclinic orbit between two non-zero equilibrium points u_- and u_+ of the nonlinearity function $F(u)$ may undertake a bifurcation (shown in the shaded area in Fig. 2).

A complete classification of various bifurcations of co-dimension one in the normal form (2.9) can be found in [10]. Homoclinic orbits may bifurcate above the curve Γ_2 and may exist in the domain $D = \{\tau > 3\gamma^2, \gamma < 0\} \cup \{\tau > 0, \gamma \geq 0\}$. Homoclinic orbits correspond to localized solutions to the zero equilibrium state and they are not studied in this paper. We focus hence on bounded heteroclinic orbits which connect the non-zero equilibrium states u_- and u_+ .

Let us rewrite the scalar fourth-order equation (2.9) in the normalized form,

$$\phi^{(iv)} + \sigma\phi'' + F(\phi) = 0, \tag{4.1}$$

where $\phi = \phi(t), t = (12\tau)^{1/4}\zeta_1$, and $\sigma = -\frac{\sqrt{12}\gamma}{\sqrt{\tau}}$. The domain between the curves Γ_2 and Γ_1 corresponds to the semi-infinite interval $\sigma > 2$. We shall consider separately the discrete ϕ^4 model with $F(\phi) = \phi(1 - \phi^2)$ and the sine-Gordon model with $F(\phi) = \sin(\phi)$.

When $F(\phi) = \phi(1 - \phi^2)$, there are two non-zero equilibrium points $u_{\pm} = \pm 1$. Linearization of the scalar equation (4.1) near non-zero equilibrium points gives a pair of real eigenvalues $(\lambda_0, -\lambda_0)$ and a pair of purely imaginary eigenvalues $(i\omega_0, -i\omega_0)$ for any $\sigma \in \mathbb{R}$. Our numerical algorithm is based on the consideration of the solution $\phi_u(t)$ in the unstable manifold, such that

$$\lim_{t \rightarrow -\infty} \phi_u(t) = -1, \quad \lim_{t \rightarrow -\infty} (\phi_u(t) + 1)e^{-\lambda_0 t} = c_0, \tag{4.2}$$

where c_0 is an arbitrary positive constant, which is related to the translation of the solution. Let $t_0(c_0)$ be a zero of the solution $\phi_u(t)$, if it exists. Since the ODE problem (4.1) with odd function $F(-\phi) = -F(\phi)$ has the reflection symmetry $\phi \mapsto -\phi$, we look for a bounded heteroclinic orbit as an odd function of t . Therefore, the distance between solutions in the stable

and unstable manifolds can be measured by the split function $K = \phi_u''(t_0(c_0))$, such that the roots of K give odd heteroclinic orbits. Although the split function K is computed at t_0 , which depends on c_0 , it follows from the translational invariance of the problem (4.1) that K is independent of c_0 . (It may depend on the external parameter σ of the problem (4.1).) The first zero t_0 corresponds to the monotonic heteroclinic orbit when $K = 0$, while subsequent zeros correspond to non-monotonic heteroclinic orbits. We shall here consider monotonic heteroclinic orbits.

The solution $\phi_u(t)$ of the scalar equation (4.1) corresponding to the unstable manifold of the equilibrium point $\phi = -1$, is numerically approximated by the solution of the initial-value problem with the initial data:

$$\begin{pmatrix} \phi(0) \\ \phi'(0) \\ \phi''(0) \\ \phi'''(0) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_0 \begin{pmatrix} 1 \\ \lambda_0 \\ \lambda_0^2 \\ \lambda_0^3 \end{pmatrix}, \tag{4.3}$$

where $\lambda_0 > 0$ and $0 < c_0 \ll 1$. The solution of the initial-value problem is stopped at the first value $t_0 > 0$, where $\phi(t_0) = 0$ and the split function K is approximated as $K = \phi''(t_0)$. Besides the external parameter σ , the numerical approximation of the split function K may depend on the shooting parameter c_0 and the discretization parameter dt of the ODE solver, such that $K = K(\sigma; c_0, dt)$.

The graph of K versus σ for a fixed set of values of c_0 and dt is shown on Fig. 3(left). It is clearly seen that the split function $K(\sigma)$ is non-zero for finite values of σ . It becomes exponentially small in σ^{-1} as $\sigma \rightarrow \infty$, which is the beyond-all-order asymptotic limit. Computations of Stokes constants in this limit give equivalent information to the computation of the split function $K(\sigma)$ (see [24,25]). We apply this technique in Appendix C to show that the Stokes constant for the scalar fourth-order equation (4.1) is non-zero in the limit $\sigma \rightarrow \infty$.

The behavior of the split function K for a fixed value of σ is analyzed on Fig. 4 as the values of dt and c_0 are reduced. When dt is sufficiently reduced, the value of K converges to a constant value monotonically (see Fig. 4(left)). When c_0 changes, the value of K oscillates near a constant level. The amplitude of oscillations depend on the value of dt and it becomes smaller when the value of dt is reduced (see Fig. 4(right)). Oscillations on Fig. 4(right) can be used as the tool to measure the error of numerical approximations for the split function $K(\sigma)$. The maximum relative error is computed from the ratio of the standard deviation to the mean for the values of $K(\sigma)$ when the parameter c_0 varies. The graph of the relative error is plotted on Fig. 3(right) versus parameter σ . The relative error increases for larger values of σ since $K(\sigma)$ becomes exponentially small in the limit $\sigma \rightarrow \infty$.

When $F(\phi) = \sin(\phi)$, there are two non-zero equilibrium points $u_{\pm} = \pm\pi$. Linearization of the scalar equation (4.1) near non-zero equilibrium points still has a pair of real eigenvalues $(\lambda_0, -\lambda_0)$ and a pair of purely imaginary eigenvalues $(i\omega_0, -i\omega_0)$ for any $\sigma \in \mathbb{R}$. Therefore, we adjust the same algorithm to the sine-Gordon model. Fig. 5(left)

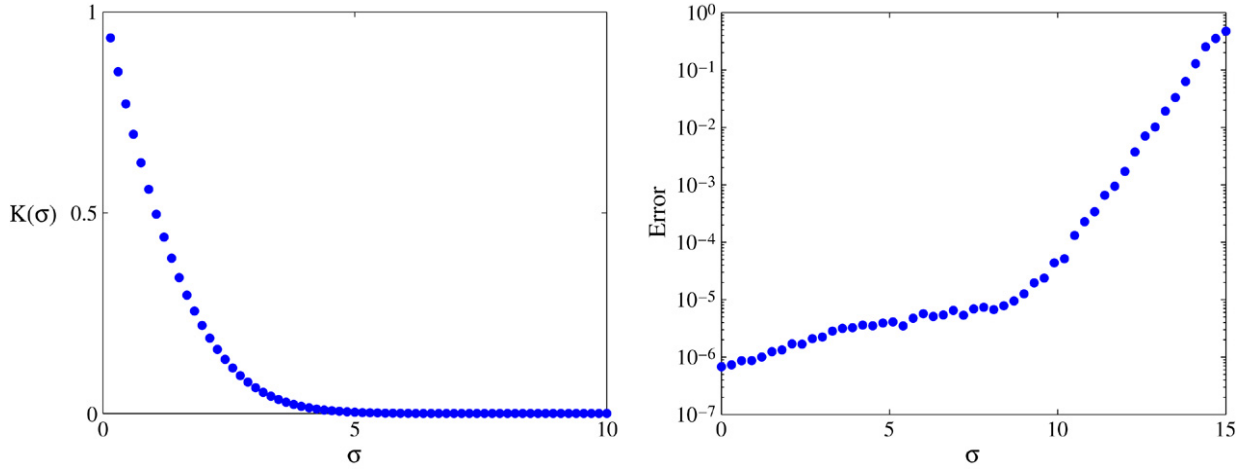


Fig. 3. Left: The graph of the split function $K(\sigma)$ for $c_0 = 0.00001$ and $dt = 0.005$ for kink solutions of the ODE (4.1) with $F(\phi) = \phi(1 - \phi^2)$. Right: Relative error of numerical approximations of $K(\sigma)$ for $dt = 0.005$ under variations of c_0 .

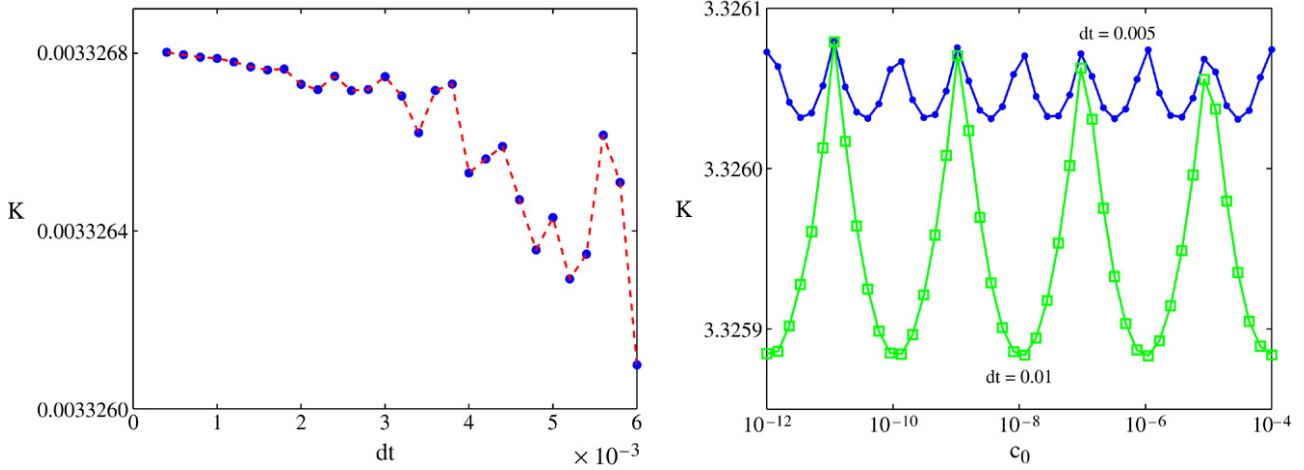


Fig. 4. Left: Convergence of the split function $K(\sigma)$ as $dt \rightarrow 0$ for $\sigma = 5$ and $c_0 = 0.00001$. Right: Oscillations of the split function $K(\sigma)$ as $c_0 \rightarrow 0$ for $\sigma = 5$ and $dt = 0.005$.

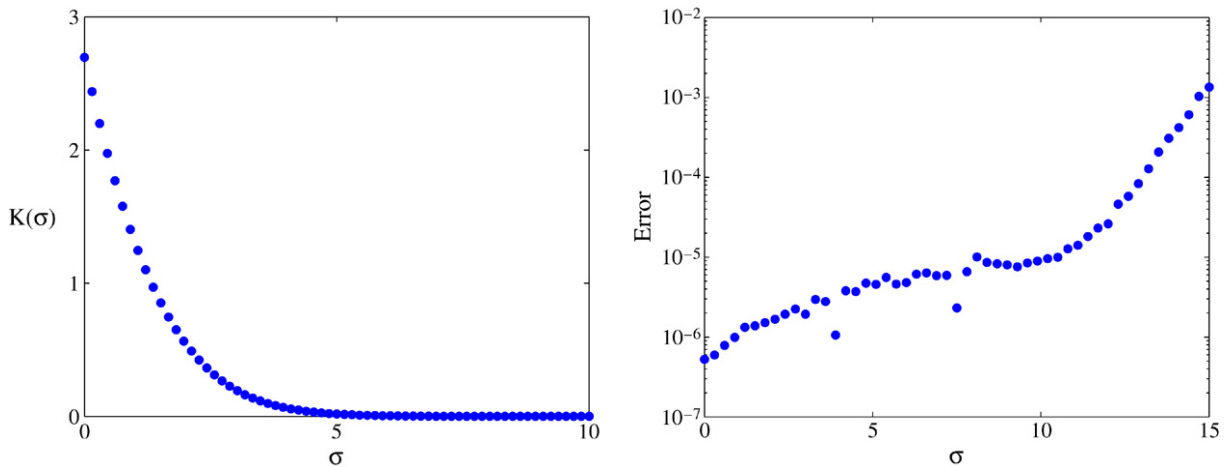


Fig. 5. Left: The graph of the split function $K(\sigma)$ for $c_0 = 0.00001$, $dt = 0.005$, and different values of σ for kink solutions of the ODE (4.1) with $F(\phi) = \sin(\phi)$. Right: Relative error of numerical approximations of $K(\sigma)$ for $dt = 0.005$ under variations of c_0 .

displays the graph of the split function K versus σ for a fixed set of values of c_0 and dt , while Fig. 5(right) shows the maximum relative error of numerical approximations. The split function $K(\sigma)$ displays the same behavior as that for the ϕ^4 model.

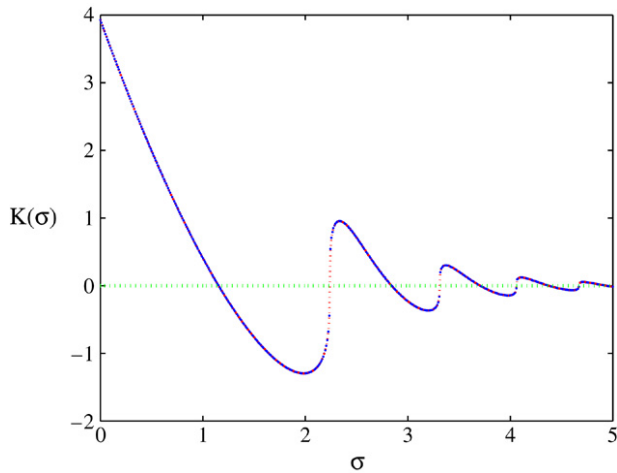


Fig. 6. The graph of the split function $K(\sigma)$ for double kink solutions of the ODE (4.1) with $F(\phi) = \sin(\phi)$.

Since the discrete sine-Gordon lattice has multiple equilibrium states at $\phi = \pi n, n \in \mathbb{Z}$, one can consider multiple monotonic kinks between several consequent equilibrium states [5]. In particular, we shall consider the double kink, which represents a bound state between the heteroclinic orbit from $u_- = -\pi$ to $u_+ = \pi$ and the heteroclinic orbit from $u_+ = \pi$ to $\tilde{u}_+ = 3\pi$.

Double kinks were considered recently in a fourth-order differential model in [5] and in the differential advance-delay equation in [1]. We note that the fourth-order model in [5] is obtained in the continuous limit of the discrete Klein–Gordon lattice (1.1), such that the fourth-order derivative term is small compared to the second-order derivative term. However, numerical results of [5] were derived in the case when both derivatives are comparable. Asymptotic and direct numerical results showed that the families of double kinks intersect the bifurcation point $(c, h) = (1, 0)$, where our normal form (2.9) is applicable. Thus, we can use the normal form Eq. (4.1) to give a rigorous explanation of existence of double kinks in the discrete sine-Gordon lattice.

We modify the numerical algorithm by considering the initial-value problem for the normalized fourth-order equation (4.1) with the initial values (4.3). Let t_0 be defined now from the equation $\phi(t_0) = u_+$, and the split function $K(\sigma)$ be defined by $K = \phi''(t_0)$. When $K = 0$, the bounded solution $\phi_u(t)$ defines a double kink solution with the properties: $\lim_{t \rightarrow -\infty} \phi_u(t) = u_-$, $\lim_{t \rightarrow \infty} \phi_u(t) = \tilde{u}_+$ and $\phi_u(t - t_0) = -\phi_u(t_0 - t) + 2\pi$. The graph of K versus σ on Fig. 6 suggests infinitely many zeros of $K(\sigma)$, which correspond to infinitely many double kink solutions. Although only four such solutions were reported in [5], it was stated that the four solutions are part of a sequence, so the results of [5] are confirmed by the pattern of Fig. 6.

There exists an exact solution of the normal form Eq. (4.1) with $F(\phi) = \sin(\phi)$ for the double kink:

$$\phi(t) = -\pi + 8 \arctan e^{\lambda t},$$

where

$$\lambda = \frac{1}{3^{1/4}}, \quad \sigma = \frac{2}{\sqrt{3}}.$$

The exact solution corresponds to the first zero of $K(\sigma)$ on Fig. 6. The distance between the numerical zero $\sigma \approx 1.1547$ and the exact zero $\sigma = 2/\sqrt{3}$ is computed from the numerical data as 6.3044×10^{-6} , which gives an idea of the numerical error of Fig. 6.

The numerical method of computations of the split function can be tested for other heteroclinic orbits, such as triple monotonic kinks between $(-\pi, \pi), (\pi, 3\pi), (3\pi, 5\pi)$ and triple non-monotonic kinks between $(-\pi, \pi), (\pi, -\pi), (-\pi, \pi)$. It can also be used for construction of homoclinic orbits (see [26]). The accuracy of computations of the split function is only limited to the accuracy of the ODE solver for the scalar fourth-order differential equation (4.1).

5. Persistence analysis of bounded solutions

In this section we consider the existence of a family of heteroclinic connections between periodic solutions of the system (3.5). We prove by a centre manifold argument that such heteroclinic connections of the truncated normal form, if they exist, are inherited by the full dynamical system. The truncation of the system (3.27) takes the form

$$\dot{\phi} = \mathbf{N}(\phi), \tag{5.1}$$

where $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ and the vector field $\mathbf{N} : \mathbb{R}^4 \mapsto \mathbb{R}^4$ is given explicitly in the form

$$\mathbf{N}(\phi) = (\phi_2, \phi_3, \phi_4, 12\gamma\phi_3 - 12\tau F(\phi_1))^T. \tag{5.2}$$

Let us use $t = \zeta_1$ in the notation of this section. The truncated system (5.1) exhibits a number of useful properties:

- P1 Since $F(\phi_1)$ is odd in $\phi_1 \in \mathbb{R}$, the vector field $\mathbf{N}(\phi)$ is odd in $\phi \in \mathbb{R}^4$.
- P2 The system is conservative, such that $\text{div } \mathbf{N} = 0$.
- P3 The system is reversible under the reversibility symmetry \mathcal{S}_0 in (3.25).
- P4 The set of fixed points of $\pm\mathcal{S}_0$ is two-dimensional.
- P5 The system has two symmetric equilibrium points $\mathbf{u}_\pm = (u_\pm, 0, 0, 0)^T$.
- P6 Since $F'(u_\pm) < 0$, the linearized operator $D\mathbf{N}(\mathbf{u}_\pm)$ has two simple eigenvalues on the imaginary axis and two simple real eigenvalues for any $\gamma \in \mathbb{R}$ and $\tau \neq 0$.

We shall add a generic assumption on the existence of a heteroclinic connection between periodic solutions near the non-zero equilibrium points. As it follows from numerical results in Section 4, we cannot assume the existence of a true heteroclinic orbit between the two equilibrium points \mathbf{u}_\pm since the true heteroclinic orbits are not supported by the nonlinear functions of the discrete ϕ^4 and sine-Gordon equations. However, a numerical shooting method implies an existence of a one-parameter family of anti-reversible heteroclinic connections between periodic solutions. Indeed, an initial-value problem for an anti-reversible solution $\phi(t)$ of the truncated system (5.1) (such that $\mathcal{S}_0\phi(-t) = -\phi(t)$) has two parameters, while there is only one constraint on the asymptotic behavior of $\phi(t)$ as $t \rightarrow \infty$. An orthogonality constraint

must be added to remove an eigendirection towards a one-dimensional unstable manifold associated to the real positive eigenvalue λ_0 in the linearization of \mathbf{u}_+ . It is expected that a one-parameter family of solutions of the initial-value problem can be found to satisfy the given constraint. We will make a generic assumption on the existence of the one-parameter family of anti-reversible solutions of the truncated system (5.1) and we prove that such solutions persist for the full system (3.27).

Assumption 1. For a fixed set of parameter (γ, τ) and a fixed nonlinear function $F(\phi_1)$, there exists a smooth family of solutions $\phi_\alpha(t)$, such that

$$\dot{\phi}_\alpha = \mathbf{N}(\phi_\alpha), \tag{5.3}$$

and

$$S_0 \phi_\alpha(-t) = -\phi_\alpha(t). \tag{5.4}$$

When $t \rightarrow \pm\infty$, the family $\phi_\alpha(t)$ tends towards $\mathbf{u}_\alpha^\pm(t)$, where $\mathbf{u}_\alpha^\pm(t)$ are periodic solutions of the system (5.1) near the equilibrium points \mathbf{u}_\pm , such that

$$S_0 \mathbf{u}_\alpha^-(t) = -\mathbf{u}_\alpha^+(t) \tag{5.5}$$

and

$$S_0 \mathbf{u}_\alpha^-(t - \delta_\alpha) = \mathbf{u}_\alpha^-(t - \delta_\alpha). \tag{5.6}$$

We assume that T_α (period at infinity) and δ_α (phase shift) are smooth functions of a parameter α .

Let us choose a particular heteroclinic connection ϕ_0 . We define a linear operator \mathcal{L} associated with the linearized system (5.1) near $\phi_0(t)$:

$$\mathcal{L}(t)\varphi(t) \doteq \dot{\varphi}(t) - DN(\phi_0(t))\varphi(t). \tag{5.7}$$

If the periodic solutions $\mathbf{u}_0^\pm(t)$ have sufficiently small amplitudes, then by the standard perturbation theory for reversible systems, the Floquet spectrum of the linearized system (5.7) at the periodic solutions $\mathbf{u}_0^\pm(t)$ can be deduced from the spectrum at the equilibrium points \mathbf{u}^\pm [12]. Our aim is to study the invertibility of \mathcal{L} in the space of continuous functions on $t \in \mathbb{R}$, which are asymptotically periodic as $t \rightarrow \pm\infty$. In what follows, we prove the invertibility of \mathcal{L} under the additional assumption that the kernel of \mathcal{L} does not contain an anti-reversible decaying solution. If this assumption fails, the Fredholm Alternative Theorem must be used to ensure invertibility of \mathcal{L} under a compatibility condition.

Assumption 2. There is no anti-reversible solution of the homogeneous linearized system that decays to zero as $t \rightarrow \pm\infty$, i.e. such that $\mathcal{L}(t)\varphi(t) = \mathbf{0}$, $S_0\varphi(-t) = -\varphi(t)$, and $\lim_{t \rightarrow \pm\infty} \varphi(t) = \mathbf{0}$.

Then we have the following result about the inverse of operator \mathcal{L} .

Theorem 2. Suppose Assumptions 1 and 2 hold. Let $\mathbf{F} \in C_b^0(\mathbb{R}, \mathbb{R}^4)$ be a function that is reversible with respect to S_0

and tends towards a T_0 -periodic function as $t \rightarrow \pm\infty$. Then, there exists a unique solution $\varphi \in C_b^0(\mathbb{R}, \mathbb{R}^4)$ of the system

$$\mathcal{L}(t)\varphi(t) = \mathbf{F}(t), \tag{5.8}$$

which is anti-reversible with respect to S_0 and tends towards a T_0 -periodic function as $t \rightarrow \pm\infty$. The linear map $\mathbf{F} \mapsto \varphi$ is continuous in $C_b^0(\mathbb{R}, \mathbb{R}^4)$.

Proof. We shall first consider the fundamental matrix for the time-dependent linear system,

$$\dot{\phi} = DN(\phi_0(t))\phi. \tag{5.9}$$

Due to properties of the truncated system (5.1), we have the following relationship,

$$S_0 DN(\phi_0(t))\phi = -DN(\phi_0(-t))S_0\phi. \tag{5.10}$$

By differentiating (5.3) with respect to t , we obtain the first eigenvector of $\mathcal{L}(t)$:

$$\mathcal{L}(t)\dot{\phi}_0(t) = 0. \tag{5.11}$$

By differentiation of (5.3) with respect to α , we can obtain another eigenvector of $\mathcal{L}(t)$. However, since the period depends on α , the second eigenvector is non-periodic at infinity but contains a linearly growing term in t . So, let us define a new “time” τ as

$$t = k_\alpha \tau, \quad k_\alpha = T_\alpha/T_0, \quad k_0 = 1. \tag{5.12}$$

Then $\tilde{\phi}_\alpha(\tau) = \phi_\alpha(t)$ has the period T_0 as $\tau \rightarrow \pm\infty$ and satisfies

$$\frac{d}{d\tau} \tilde{\phi}_\alpha = k_\alpha \mathbf{N}(\tilde{\phi}_\alpha). \tag{5.13}$$

Differentiating this identity with respect to α and denoting $\hat{\phi}_0 = \partial_\alpha \tilde{\phi}_\alpha|_{\alpha=0}$ we obtain the equation,

$$\frac{d}{d\tau} \hat{\phi}_0 = DN(\phi_0)\hat{\phi}_0 + k_1 \mathbf{N}(\phi_0), \tag{5.14}$$

where

$$k_1 = \partial_\alpha k|_{\alpha=0}. \tag{5.15}$$

This explicit computation shows that $\hat{\phi}_0(t)$ is a generalized eigenvector of the operator $\mathcal{L}(t)$,

$$\mathcal{L}(t)\hat{\phi}_0(t) = k_1 \dot{\phi}_0(t), \tag{5.16}$$

which is bounded and T_0 -periodic at infinity as the first eigenvector $\dot{\phi}_0(t)$. Moreover, we note the symmetry properties:

$$S_0 \dot{\phi}_0(-t) = \dot{\phi}_0(t), \tag{5.17}$$

$$S_0 \hat{\phi}_0(-t) = -\hat{\phi}_0(t). \tag{5.18}$$

The two eigenvectors $\dot{\phi}_0(t)$ and $\hat{\phi}_0(t)$ correspond to the “neutral” directions associated with the families of periodic solutions $\mathbf{u}_\alpha^\pm(t)$ near the equilibrium points \mathbf{u}_\pm . Indeed, similarly we denote by $\dot{\mathbf{u}}_0^\pm$ and $\hat{\mathbf{u}}_0^\pm$ the T_0 -periodic solutions of the limiting system with periodic coefficients:

$$\frac{d}{dt} \dot{\mathbf{u}}_0^\pm = DN(\mathbf{u}_0^\pm) \dot{\mathbf{u}}_0^\pm$$

$$\frac{d}{dt} \hat{\mathbf{u}}_0^\pm = DN(\mathbf{u}_0^\pm) \hat{\mathbf{u}}_0^\pm + k_1 \dot{\mathbf{u}}_0^\pm.$$

These two modes are the eigenvector and generalized eigenvector belonging to the double zero Floquet exponent associated with each periodic solution $\mathbf{u}_0^\pm(t)$. Since these periodic solutions are close enough to the equilibrium points \mathbf{u}_\pm , and thanks to perturbation theory, we know that there exist two real Floquet exponents $(\lambda_0^\pm, -\lambda_0^\pm)$ for each solution \mathbf{u}_0^\pm . Let us denote by $\zeta_\pm(t)$ and $\chi_\pm(t)$ the T_0 -periodic eigenvectors (defined up to a factor) such that

$$\begin{aligned}\frac{d}{dt}\zeta_+ + \lambda_0^-\zeta_+ &= DN(\mathbf{u}_0^-)\zeta_+ \\ \frac{d}{dt}\zeta_- - \lambda_0^-\zeta_- &= DN(\mathbf{u}_0^-)\zeta_- \\ \frac{d}{dt}\chi_+ + \lambda_0^+\chi_+ &= DN(\mathbf{u}_0^+)\chi_+ \\ \frac{d}{dt}\chi_- - \lambda_0^+\chi_- &= DN(\mathbf{u}_0^+)\chi_-.\end{aligned}$$

We justify below that $\lambda_0^+ = \lambda_0^-$. Since the function $S_0\zeta_+(-t)$ satisfies

$$\frac{d}{dt}S_0\zeta_+(-t) - \lambda_0 S_0\zeta_+(-t) = DN(S_0\mathbf{u}_0^-(t))S_0\zeta_+(-t) \quad (5.19)$$

and because of (5.6) we can choose ζ_- such that

$$S_0\zeta_+(-t - \delta_0) = \zeta_-(t - \delta_0). \quad (5.20)$$

Now, since we have (5.5) and since $DN(\phi_0(t))$ is even in $\phi_0(t)$, we deduce that

$$DN(\mathbf{u}_0^+(t)) = DN(-S_0(\mathbf{u}_0^-(t))) = DN(\mathbf{u}_0^-(t - 2\delta_0)),$$

which verifies that $\lambda_0^+ = \lambda_0^-$. Finally we can choose the vector functions ζ_\pm and χ_\pm such that

$$\begin{aligned}\chi_+(t + \delta_0) &= \zeta_+(t - \delta_0) \\ \chi_-(t + \delta_0) &= \zeta_-(t - \delta_0) \\ S_0\chi_+(-t + \delta_0) &= \chi_-(t + \delta_0).\end{aligned} \quad (5.21)$$

Since $\zeta_\pm(t)e^{\pm\lambda_0 t}$ and $\chi_\pm(t)e^{\pm\lambda_0 t}$ are solutions of the limiting linear system, we introduce the unique solutions $\phi_{1,2}(t)$ and $\psi_{1,2}(t)$ of the system (5.9) on $t \in \mathbb{R}$, such that

$$\begin{aligned}\phi_1(t) &\sim \zeta_+(t)e^{\lambda_0 t} \quad \text{as } t \rightarrow -\infty \\ \phi_2(t) &\sim \zeta_-(t)e^{-\lambda_0 t} \quad \text{as } t \rightarrow -\infty \\ \psi_2(t) &\sim \chi_+(t)e^{\lambda_0 t} \quad \text{as } t \rightarrow \infty \\ \psi_1(t) &\sim \chi_-(t)e^{-\lambda_0 t} \quad \text{as } t \rightarrow \infty.\end{aligned}$$

Now we have

$$\begin{aligned}\frac{d}{dt}S_0\phi_1(-t) &= DN(S_0\phi_0(-t))S_0\phi_1(-t) \\ &= DN(\phi_0(t))S_0\phi_1(-t)\end{aligned}$$

and

$$S_0\phi_1(-t) \sim S_0\zeta_+(-t)e^{-\lambda_0 t} \quad \text{as } t \rightarrow \infty,$$

where

$$S_0\zeta_+(-t)e^{-\lambda_0 t} = \zeta_-(t - 2\delta_0)e^{-\lambda_0 t} = \chi_-(t)e^{-\lambda_0 t}.$$

Hence

$$S_0\phi_1(-t) = \psi_1(t), \quad (5.22)$$

and, in the same way

$$S_0\phi_2(-t) = \psi_2(t). \quad (5.23)$$

The two pairs of linearly independent solutions (ϕ_1, ϕ_2) and (ψ_1, ψ_2) cannot contain components in $\hat{\phi}_0$ and $\hat{\phi}_0$ thanks to the behavior at infinity, hence there exist constants (a, b, c, d) such that

$$\begin{aligned}\psi_1 &= a\phi_1 + b\phi_2 \\ \psi_2 &= c\phi_1 + d\phi_2.\end{aligned} \quad (5.24)$$

Iterating once the relations (5.22) and (5.23), we find that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If either b or c is non-zero, then $d = -a$ and $a^2 + bc = 1$. If $b = c = 0$, then $a^2 = d^2 = 1$. The choice $a = -d = \pm 1$ is included in the previous solution with $d = -a$ and $a^2 + bc = 1$. The choice $a = d = \pm 1$ is impossible since it would lead to more than two independent vectors in the two-dimensional set of fixed points of the operators $\pm S_0$ at $t = 0$. Thus, we parameterize: $a = -d = \alpha$, $b = \beta$, and $c = \gamma$, where $\alpha^2 + \beta\gamma = 1$ and rewrite the relationship (5.24) in the form:

$$\begin{aligned}\psi_1 &= \alpha\phi_1 + \beta\phi_2 \\ \psi_2 &= \gamma\phi_1 - \alpha\phi_2\end{aligned} \quad (5.25)$$

with the inverse relationship:

$$\begin{aligned}\phi_1 &= \alpha\psi_1 + \beta\psi_2 \\ \phi_2 &= \gamma\psi_1 - \alpha\psi_2.\end{aligned} \quad (5.26)$$

We now wish to solve the linear equation (5.8), where the right-hand-side vector $\mathbf{F}(t)$ is asymptotically T_0 -periodic and reversible with respect to S_0 , i.e. such that

$$S_0\mathbf{F}(-t) = \mathbf{F}(t). \quad (5.27)$$

By construction, we have

$$\begin{aligned}\mathbf{F} &= f_0\dot{\phi}_0 + f_1\hat{\phi}_0 + g_1\phi_1 + g_2\phi_2 \\ \varphi &= \alpha_0\dot{\phi}_0 + \alpha_1\hat{\phi}_0 + \beta_1\phi_1 + \beta_2\phi_2,\end{aligned}$$

where the functions $g_1(t)e^{-\lambda_0|t|}$ and $g_2(t)e^{\lambda_0|t|}$ are bounded as $t \rightarrow -\infty$ and asymptotically T_0 -periodic. Let $\mathcal{M}(t) = [\hat{\phi}_0, \hat{\phi}_0, \phi_1, \phi_2]$, such that the coordinates of \mathbf{F} and φ in the new basis are

$$\tilde{\mathbf{F}}(t) = \mathcal{M}(t)^{-1}\mathbf{F}(t) = (f_0, f_1, g_1, g_2)^T \quad (5.28)$$

$$\tilde{\varphi}(t) = \mathcal{M}(t)^{-1}\varphi(t) = (\alpha_0, \alpha_1, \beta_1, \beta_2)^T. \quad (5.29)$$

The system (5.8) reduces to the simple system

$$\begin{aligned}\frac{d\alpha_0}{dt} &= -k_1\alpha_1 + f_0 \\ \frac{d\alpha_1}{dt} &= f_1 \\ \frac{d\beta_1}{dt} &= g_1\end{aligned}$$

$$\frac{d\beta_2}{dt} = g_2.$$

Since the system (5.1) is conservative (that is $\text{tr}(DN(\phi_0)) = 0$), Liouville’s theorem implies that

$$\det \mathcal{M}(t) = \det \mathcal{M}(0) \neq 0. \tag{5.30}$$

Solving the uncoupled system for $\beta_{1,2}(t)$, we find that

$$\beta_1(t) = \int_0^t g_1(s)ds + \beta_{10}, \quad \beta_2(t) = \int_{-\infty}^t g_2(s)ds,$$

such that the functions $\beta_1(t)e^{-\lambda_0|t|}$ and $\beta_2(t)e^{\lambda_0|t|}$ are bounded as $t \rightarrow -\infty$ and asymptotically T_0 -periodic for any constant β_{10} . The same property holds for $\varphi(t)$ as $t \rightarrow -\infty$. Now, we consider the opposite limit $t \rightarrow +\infty$. Thanks to the relations (5.25), we notice that the functions $(\alpha g_1 + \gamma g_2)(t)e^{-\lambda_0 t}$ and $(\beta g_1 - \alpha g_2)(t)e^{\lambda_0 t}$ are bounded as $t \rightarrow \infty$. Moreover they are asymptotically T_0 -periodic if $\mathbf{F}(t)$ has this property. Now we can write the function $\beta_1\phi_1 + \beta_2\phi_2$ as

$$\begin{aligned} \beta_1\phi_1 + \beta_2\phi_2 &= \psi_1(t) \left(\int_0^t (\alpha g_1(s) + \gamma g_2(s))ds \right. \\ &\quad \left. + \int_{-\infty}^0 \gamma g_2(s)ds + \alpha\beta_{10} \right) + \psi_2(t) \\ &\quad \times \left(\int_0^t (\beta g_1(s) - \alpha g_2(s))ds + \int_{-\infty}^0 -\alpha g_2(s)ds + \beta\beta_{10} \right) \end{aligned}$$

and it is easy to check the boundedness of the function $\beta_1\phi_1 + \beta_2\phi_2$ as $t \rightarrow \infty$, provided that

$$\int_0^\infty (\beta g_1(s) - \alpha g_2(s))ds - \int_{-\infty}^0 \alpha g_2(s)ds + \beta\beta_{10} = 0. \tag{5.31}$$

If $\beta \neq 0$, this equation determines the value of β_{10} . If $\beta = 0$, the value of β_{10} is not determined. If in addition $\alpha = +1$, Eq. (5.31) is satisfied identically since $g_2(t)$ is odd in this case, thanks to the identity (5.34) below. The case $\beta = 0$ and $\alpha = -1$ is excluded by Assumption 2. Thus, the function $\beta_1\phi_1 + \beta_2\phi_2$ is bounded as $t \rightarrow +\infty$ and asymptotically T_0 -periodic. It remains to show that the reversibility constraint (5.27) implies the anti-reversibility constraint for the solution $\varphi(t)$:

$$S_0\varphi(-t) = -\varphi(t). \tag{5.32}$$

By construction, the functions $f_0(t)$ and $f_1(t)$ are even and odd respectively. Since they are defined by a multiplication of an exponentially growing function with an exponentially decaying one of the same rate and with periodic factors, these functions are bounded and asymptotically T_0 -periodic. Solving the uncoupled system for $\alpha_0(t)$ and $\alpha_1(t)$, we have

$$\begin{aligned} \alpha_0(t) &= \int_0^t [f_0(s) - k_1\alpha_1(s)] ds, \\ \alpha_1(t) &= \alpha_1(0) + \int_0^t f_1(s)ds. \end{aligned}$$

Since $f_0(t)$ is even and $f_1(t)$ is odd, it is clear that $\alpha_1(t)$ is even and $\alpha_0(t)$ is odd. Moreover, since $f_1(t)$ is odd, it has asymptotically a zero average at infinity. Therefore, the function

$\alpha_1(t)$ is bounded and asymptotically T_0 -periodic at infinity. Now we can find a unique $\alpha_1(0)$ such that the even function $f_0(s) - k_1\alpha_1(s)$ which is asymptotically T_0 -periodic at infinity, has in addition a zero average at infinity. Then this gives a bounded function $\alpha_0(t)$ which is asymptotically T_0 -periodic at infinity. Now for the two last components of $\mathbf{F}(t)$ we have

$$\begin{aligned} g_1(-t)\psi_1(t) + g_2(-t)\psi_2(t) - g_1(t)\phi_1(t) \\ - g_2(t)\phi_2(t) = 0, \end{aligned} \tag{5.33}$$

which corresponds to the identities

$$\begin{aligned} g_1(t) - \alpha g_1(-t) - \gamma g_2(-t) &= 0, \\ g_2(t) - \beta g_1(-t) + \alpha g_2(-t) &= 0. \end{aligned} \tag{5.34}$$

We need to show that this implies

$$\begin{aligned} \beta_1(t) + \alpha\beta_1(-t) + \gamma\beta_2(-t) &= 0, \\ \beta_2(t) + \beta\beta_1(-t) - \alpha\beta_2(-t) &= 0. \end{aligned}$$

The identities (5.34) lead to

$$\beta g_1(s) - \alpha g_2(s) = g_2(-s) \tag{5.35}$$

hence the constant β_{10} takes the form

$$\beta\beta_{10} = (\alpha - 1) \int_{-\infty}^0 g_2(s)ds. \tag{5.36}$$

We observe that the identity (5.36) is satisfied when $\beta = 0$ and $\alpha = 1$. Now we check that

$$\begin{aligned} \beta_1(t) + \alpha\beta_1(-t) + \gamma\beta_2(-t) &= \int_0^t \{g_1(s) - \alpha g_1(-s)\}ds + \gamma \int_t^\infty g_2(-s)ds \\ &\quad + (1 + \alpha)\beta_{10} \\ &= \gamma \int_0^\infty g_2(-s)ds + (1 + \alpha)\beta_{10} \\ &= (1 + \alpha)\beta_{10} + \gamma \int_{-\infty}^0 g_2(s)ds. \end{aligned}$$

If $\beta = 0$ and $\alpha = 1$, the expression above is zero under a special choice of β_{10} . If $\beta \neq 0$ and $\gamma = (1 - \alpha^2)/\beta$, it is zero because

$$\begin{aligned} (1 + \alpha)\beta_{10} + \gamma \int_{-\infty}^0 g_2(s)ds \\ = \frac{1 + \alpha}{\beta} \left(\beta\beta_{10} + (1 - \alpha) \int_{-\infty}^0 g_2(s)ds \right) = 0. \end{aligned}$$

Similarly,

$$\begin{aligned} \beta_2(t) + \beta\beta_1(-t) - \alpha\beta_2(-t) &= \int_{-\infty}^t g_2(s)ds + \beta\beta_{10} - \alpha \int_{-\infty}^0 g_2(s)ds \\ &\quad + \int_0^t [\alpha g_2(-s) - \beta g_1(-s)] ds \\ &= \beta\beta_{10} + (1 - \alpha) \int_{-\infty}^0 g_2(s)ds = 0. \end{aligned}$$

Hence the statement is proved in all cases except the case $\beta = 0$ and $\alpha = -1$, which is excluded by Assumption 2. \square

Using analysis of the truncated vector system (5.1), we can now prove persistence of bounded solutions of the full system. Since there exists a smooth mapping $\psi(\phi, \varepsilon)$ in the hyperbolic subspace (see Theorem 1), we rewrite the system (3.27) in the form,

$$\dot{\phi} = \mathbf{N}(\phi) + \sqrt{\varepsilon} \mathbf{R}(\phi, \psi(\phi, \varepsilon)), \quad (5.37)$$

where $\psi(\phi, \varepsilon) = O(\sqrt{\varepsilon})$. The perturbation term is non-local, which forces us to use a non-geometric proof for showing the persistence of heteroclinic connections to periodic solutions near equilibrium points \mathbf{u}_{\pm} . Indeed, let us decompose ϕ as

$$\phi = \phi_0 + \varphi \quad (5.38)$$

and define

$$\begin{aligned} \mathbf{R}_1(\phi_0, \varphi) &= \mathbf{N}(\phi_0 + \varphi) - \mathbf{N}(\phi_0) - D\mathbf{N}(\phi_0)\varphi \\ &= O(\|\varphi\|^2), \end{aligned} \quad (5.39)$$

then φ satisfies

$$\mathcal{L}(t)\varphi = \mathbf{R}_1(\phi_0, \varphi) + \sqrt{\varepsilon} \mathbf{R}(\phi_0 + \varphi, \psi(\phi_0 + \varphi, \varepsilon)). \quad (5.40)$$

Using Theorem 2 and the smooth dependence of \mathbf{R} on (ϕ, ψ) , we can then use the implicit function theorem near $\varphi = \mathbf{0}$ and find a unique solution $\varphi \in C_b^0(\mathbb{R}, \mathbb{R}^4)$ for small enough ε . Moreover, this solution $\varphi(t)$ is a T_0 -periodic function at infinity, whose magnitude is bounded by a constant of the order of $O(\sqrt{\varepsilon})$. Using these observations, we assert the following theorem.

Theorem 3. *Let Assumptions 1 and 2 be satisfied and $\phi_0(t)$ be a solution of the truncated system (5.1), which is heteroclinic to small T_0 -periodic solutions $\mathbf{u}_{\pm}(t)$. Then, for small enough ε , there exists a unique solution $\phi(t)$ of the perturbed system (5.37), which is $\sqrt{\varepsilon}$ -close to $\phi_0(t)$ and is heteroclinic to small T_0 -periodic solutions of the system (5.37).*

We note that uniqueness is a remarkable property of the perturbed system (5.37). It is supported here due to the choice that the function $\phi(t)$ has the same asymptotic period at infinity as the function $\phi_0(t)$.

6. Conclusion

We have derived a scalar normal form equation for bifurcations of heteroclinic orbits in the discrete Klein–Gordon equation. The existence of a centre manifold and the persistence of bounded solutions of the normal form equation are proved with rigorous analysis. Bounded solutions may include heteroclinic orbits between periodic perturbations at the equilibrium states and true heteroclinic orbits between equilibrium states. Our numerical results indicate that no true heteroclinic orbits between equilibrium states exist for two important models, the discrete ϕ^4 and sine-Gordon equations. Double and multiple kinks (between several equilibrium states) may exist in the case of the discrete sine-Gordon equation. Existence of double kinks is confirmed in the scalar normal form equation.

We have studied persistence of heteroclinic anti-reversible connections between periodic solutions near the equilibrium

states, but we have not addressed persistence of true heteroclinic orbits between the equilibrium states. It is intuitively clear that non-existence of a true heteroclinic orbit in the truncated normal form must imply the non-existence of such an orbit in the untruncated system at least for sufficiently small ε . Notice that for a one parameter family of reversible vector fields, the existence of a true homoclinic for the truncated system does not imply the persistence of a true heteroclinic for the full system, analogously to the known result for homoclinics proved in [17]. Here, in playing on the two parameters (c, h) the persistence of true homoclinics should be implied for the full system on a certain curve in the parameter plane. Rigorous analysis of such continuations is based on the implicit function argument for a split function, which becomes technically complicated since our full system involves the advance-delay operators, where the resolvent estimates are not simple. This is an open question for further studies.

Appendix A. Truncation of the vector normal form from [10]

The eigenvectors for the quadruple zero eigenvalues satisfy the reversibility symmetry relations (3.12). The general reversible normal form for this case was derived in [10]. We will show here that this normal form can be reduced to the fourth-order equation (2.9) under an appropriate scaling. The general reversible normal form is explicitly written in the vector form (see [10] for details):

$$\begin{aligned} \xi_1' &= \xi_2 + u_3 P_1(u_1, u_2, u_4), \\ \xi_2' &= \xi_3 + q_3 P_1(u_1, u_2, u_4) + u_1 P_2(u_1, u_2, u_4) \\ &\quad + u_2 P_3(u_2, u_4), \\ \xi_3' &= \xi_4 + r_3 P_1(u_1, u_2, u_4) + q_1 P_2(u_1, u_2, u_4) \\ &\quad + q_2 P_3(u_2, u_4) + u_3 P_4(u_1, u_2, u_4), \\ \xi_4' &= s_3 P_1(u_1, u_2, u_4) + r_1 P_2(u_1, u_2, u_4) + r_2 P_3(u_2, u_4) \\ &\quad + q_3 P_4(u_1, u_2, u_4) + P_5(u_1, u_2, u_4), \end{aligned}$$

where $P_{1,2,3,4,5}$ are polynomials in variables:

$$\begin{aligned} u_1 &= \xi_1, \quad u_2 = \xi_2^2 - 2\xi_1\xi_3, \quad u_3 = \xi_2^3 - 3\xi_1\xi_2\xi_3 + 3\xi_1^2\xi_4, \\ u_4 &= 3\xi_2^2\xi_3^2 - 6\xi_2^3\xi_4 - 8\xi_1\xi_3^3 + 18\xi_1\xi_2\xi_3\xi_4 - 9\xi_1^2\xi_4^2, \\ q_1 &= \xi_2, \quad q_2 = -3\xi_1\xi_4 + \xi_2\xi_3, \\ q_3 &= 3\xi_1\xi_2\xi_4 - 4\xi_1\xi_3^2 + \xi_2^2\xi_3, \quad r_1 = \xi_3, \\ r_2 &= -3\xi_2\xi_4 + 2\xi_3^2, \quad r_3 = -3\xi_1\xi_3\xi_4 + 3\xi_2^2\xi_4 - \xi_2\xi_3^2, \end{aligned}$$

and $s_3 = 3\xi_2\xi_3\xi_4 - \frac{4}{3}\xi_3^3 - 3\xi_1\xi_4^2$. If the zero equilibrium point persists in the vector normal form, then $P_5(0, 0, 0) = 0$. The linear part of the vector normal form corresponds to the truncation of $P_{1,2,3,4,5}$ as follows:

$$P_1 = 0, \quad P_2 = \mu_1, \quad P_3 = P_4 = 0, \quad P_5 = \mu_2 u_1,$$

where μ_1 and μ_2 are two parameters of the linear system. The two parameters (μ_1, μ_2) must recover the Taylor series expansion (2.6) of the full dispersion relation (2.1) in scaled variables. Since the dispersion relation from the linear part of the vector normal form is

$$\Lambda^4 - 3\mu_1 \Lambda^2 + \mu_1^2 - \mu_2 = 0,$$

the comparison with the Taylor series expansion (2.6) shows that

$$\mu_1 = 4\varepsilon\gamma + O(\varepsilon^2), \quad \mu_2 = \varepsilon^2(16\gamma^2 - 12\tau) + O(\varepsilon^3). \quad (\text{A.1})$$

The nonlinear part in the dynamical system (3.14) is scaled as ε^2 near the bifurcation point (3.13), such that

$$P_1 = \varepsilon^2 \tilde{P}_1, \quad P_2 = \mu_1 + \varepsilon^2 \tilde{P}_2, \quad P_{3,4} = \varepsilon^2 \tilde{P}_{3,4}, \\ P_4 = \varepsilon^2 \tilde{P}_4, \quad P_5 = \mu_2 u_1 + \varepsilon^2 \tilde{P}_5,$$

where tilded functions \tilde{P}_j depend on the same variables as the original functions P_j . Let $\zeta_1 = \sqrt{\varepsilon}\zeta$ and use the scaling transformation:

$$\xi_1 = \phi_1(\zeta_1), \quad \xi_2 = \sqrt{\varepsilon}\phi_2(\zeta_1), \quad \xi_3 = \varepsilon\phi_3(\zeta_1), \\ \xi_4 = \sqrt{\varepsilon^3}\phi_4(\zeta_1).$$

The scaled version of the vector normal form is written explicitly as follows:

$$\phi'_1 = \phi_2 + \varepsilon^3 \tilde{u}_3 \tilde{P}_1, \\ \phi'_2 = \phi_3 + 4\gamma\phi_1 + \varepsilon\phi_1 \tilde{P}_2 + \varepsilon^2 \tilde{u}_2 \tilde{P}_3 + \varepsilon^3 \tilde{q}_3 \tilde{P}_1, \\ \phi'_3 = \phi_4 + 4\gamma\phi_2 + \varepsilon\phi_2 \tilde{P}_2 + \varepsilon^2 (\tilde{q}_2 \tilde{P}_3 + \tilde{u}_3 \tilde{P}_4) + \varepsilon^3 \tilde{r}_3 \tilde{P}_1, \\ \phi'_4 = 4\gamma\phi_3 + (16\gamma^2 - 12\tau)\phi_1 + \tilde{P}_5 + \varepsilon\phi_3 \tilde{P}_2 \\ + \varepsilon^2 (\tilde{r}_2 \tilde{P}_3 + \tilde{q}_3 \tilde{P}_4) + \varepsilon^3 \tilde{s}_3 \tilde{P}_1,$$

where tilded variables stand for the scaled versions of the original variables. Let $G(\phi_1) = \tilde{P}_5(\phi_1, 0, 0)$. The truncated version of the vector normal form (with $\varepsilon \equiv 0$) is equivalent to the scalar fourth-order equation:

$$\phi^{(iv)} - 12\gamma\phi'' + 12\tau\phi = G(\phi), \quad (\text{A.2})$$

where $\phi \equiv \phi_1$. The comparison with the scalar fourth-order equation (2.9) shows that $G(\phi) = -12\tau(F(\phi) - \phi)$, where $F(\phi)$ is the nonlinearity of the Klein–Gordon equation (1.2). We note that the vector normal form can be simplified near the curves of bifurcations of co-dimension one (see [10]). The truncated normal form (A.2) takes into account the full problem of bifurcation of co-dimension two.

Appendix B. Proof of existence of a centre manifold in the system (3.27) and (3.28)

We shall rewrite the system of coupled equations (3.27) and (3.28) in the form:

$$\dot{\phi} = \mathbf{N}(\phi) + \sqrt{\varepsilon}\mathbf{R}(\phi, \psi), \quad (\text{B.1})$$

$$\dot{\psi} = \mathcal{L}_{1,0}\psi + \sqrt{\varepsilon}\mathbf{F}(\phi, \psi), \quad (\text{B.2})$$

where $\phi \in \mathbb{R}^4$ is a function of $\zeta_1 = \sqrt{\varepsilon}\zeta$ and $\psi \in \mathcal{D}$ is a function of ζ . The vector fields are given by

$$\mathbf{N}(\phi) = (\phi_2, \phi_3, \phi_4, 12\gamma\phi_3 - 12\tau F(\phi_1))^T,$$

$$\mathbf{F}(\phi, \psi) = g\mathbf{F}_0(p)$$

with

$$\mathbf{F}_0 = \begin{pmatrix} 0 \\ \frac{3}{-5} \\ \frac{2}{5}p(1-5p^2) \end{pmatrix}, \quad \mathbf{R}(\phi, \psi) = \begin{pmatrix} 0 \\ -\frac{2}{5}\sqrt{\varepsilon}g \\ 0 \\ \frac{12}{\sqrt{\varepsilon}}\tilde{g} \end{pmatrix},$$

where $\tilde{g} = g - \gamma\phi_3 + \tau F(\phi_1)$ and g is given below (3.27). We shall rewrite g and \tilde{g} in an equivalent form, which is used in our analysis. It follows from identities (3.2), representation (3.18), (3.19) and the scaling (3.26) that

$$\delta^\pm U_3 = \phi_1 \pm \sqrt{\varepsilon}\phi_2 + \frac{1}{2}\varepsilon\phi_3 \pm \frac{1}{6}\varepsilon^{3/2}\phi_4 + \varepsilon^{3/2}\delta^\pm\psi_3,$$

$$\delta_\pm U_1 = \phi_1(\zeta_1 \pm \sqrt{\varepsilon}) + \varepsilon^{3/2}\delta_\pm\psi_1,$$

such that the equality $\delta^\pm U_3 = \delta_\pm U_1$ results in the identity

$$\phi_3 - 2\sqrt{\varepsilon}\psi_1 + \sqrt{\varepsilon}(\delta^+ + \delta^-)\psi_3 = \sqrt{\varepsilon}(\delta_+\psi_1 + \delta_-\psi_1 - 2\psi_1) \\ + \varepsilon^{-1} [\phi_1(\zeta_1 + \sqrt{\varepsilon}) + \phi_1(\zeta_1 - \sqrt{\varepsilon}) - 2\phi_1].$$

As a result, the expression for g below (3.27) can be rewritten as follows:

$$g = \gamma\sqrt{\varepsilon}(\delta_+\psi_1 + \delta_-\psi_1 - 2\psi_1) + \gamma\varepsilon^{-1}(\phi_1(\zeta_1 + \sqrt{\varepsilon}) \\ + \phi_1(\zeta_1 - \sqrt{\varepsilon}) - 2\phi_1) - \tau(\phi_1 + \varepsilon^{3/2}\psi_1 + Q)$$

and

$$Q = Q(\phi_1(\zeta_1 - \sqrt{\varepsilon}) + \varepsilon^{3/2}\delta_-\psi_1, \phi_1 \\ + \varepsilon^{3/2}\psi_1, \phi_1(\zeta_1 + \sqrt{\varepsilon}) + \varepsilon^{3/2}\delta_+\psi_1).$$

Using the system (B.1), we obtain the following representation:

$$\phi_1(\zeta_1 + \sqrt{\varepsilon}) + \phi_1(\zeta_1 - \sqrt{\varepsilon}) - 2\phi_1 \\ = \int_{\zeta_1}^{\zeta_1 + \sqrt{\varepsilon}} \int_{\tau - \sqrt{\varepsilon}}^{\tau} \ddot{\phi}_1(s) ds d\tau \\ = \int_{\zeta_1}^{\zeta_1 + \sqrt{\varepsilon}} \int_{\tau - \sqrt{\varepsilon}}^{\tau} (\phi_3(s) - \frac{2\varepsilon}{5}g(s)) ds d\tau,$$

such that g is rewritten in the implicit form:

$$g = \gamma\sqrt{\varepsilon}(\delta_+\psi_1 + \delta_-\psi_1 - 2\psi_1) - \tau(\phi_1 + \varepsilon^{3/2}\psi_1 + Q) \\ + \gamma\varepsilon^{-1} \int_{\zeta_1}^{\zeta_1 + \sqrt{\varepsilon}} \int_{\tau - \sqrt{\varepsilon}}^{\tau} \left(\phi_3(s) - \frac{2\varepsilon}{5}g(s) \right) ds d\tau. \quad (\text{B.3})$$

In order to obtain an equivalent expression for \tilde{g} , we use the system (B.1) and obtain

$$-\phi_3(\zeta_1) + \varepsilon^{-1} \int_{\zeta_1}^{\zeta_1 + \sqrt{\varepsilon}} \int_{\tau - \sqrt{\varepsilon}}^{\tau} \phi_3(s) ds d\tau \\ = \varepsilon^{-1} \int_{\zeta_1}^{\zeta_1 + \sqrt{\varepsilon}} \int_{\tau - \sqrt{\varepsilon}}^{\tau} \int_{\zeta_1}^s \phi_4(\eta) d\eta ds d\tau.$$

It is also clear that

$$Q(\phi_1(\zeta_1 - \sqrt{\varepsilon}), \phi_1(\zeta_1), \phi_1(\zeta_1 + \sqrt{\varepsilon})) \\ - Q_0(\phi_1(\zeta_1), \phi_1(\zeta_1), \phi_1(\zeta_1)) \\ = \int_{\zeta_1}^{\zeta_1 + \sqrt{\varepsilon}} [\partial_3 Q(\phi_1(\zeta_1 - \sqrt{\varepsilon}), \phi_1(\zeta_1), \phi_1(s))\phi_2(s) \\ - \partial_1 Q(\phi_1(s - \sqrt{\varepsilon}), \phi_1(\zeta_1), \phi_1(\zeta_1))\phi_2(s - \sqrt{\varepsilon})] ds.$$

As a result, the expression for $\tilde{g} = g - \gamma\phi_3 + \tau F(\phi_1)$ is rewritten in an equivalent form,

$$\begin{aligned} \tilde{g} &= \gamma\sqrt{\varepsilon}(\delta_+\psi_1 + \delta_-\psi_1 - 2\psi_1) \\ &\quad - \frac{2}{5}\gamma \int_{\zeta_1}^{\zeta_1+\sqrt{\varepsilon}} \int_{\tau-\sqrt{\varepsilon}}^{\tau} g(s)dsd\tau - \tau(\varepsilon^{3/2}\psi_1 + \tilde{Q}) \\ &\quad + \gamma\varepsilon^{-1} \int_{\zeta_1}^{\zeta_1+\sqrt{\varepsilon}} \int_{\tau-\sqrt{\varepsilon}}^{\tau} \int_{\zeta_1}^s \phi_4(\eta)d\eta dsd\tau, \end{aligned} \tag{B.4}$$

where

$$\begin{aligned} \tilde{Q} &= \int_{\zeta_1}^{\zeta_1+\sqrt{\varepsilon}} [\partial_3 Q(\phi_1(\zeta_1 - \sqrt{\varepsilon}), \phi_1(\zeta_1), \phi_1(s))\phi_2(s) \\ &\quad - \partial_1 Q(\phi_1(s - \sqrt{\varepsilon}), \phi_1(\zeta_1), \phi_1(\zeta_1))\phi_2(s - \sqrt{\varepsilon})] ds \\ &\quad + Q(\phi_1(\zeta_1 - \sqrt{\varepsilon}) + \varepsilon^{3/2}\delta_-\psi_1, \phi_1 + \varepsilon^{3/2}\psi_1, \\ &\quad \phi_1(\zeta_1 + \sqrt{\varepsilon}) + \varepsilon^{3/2}\delta_+\psi_1) - Q(\phi_1(\zeta_1 - \sqrt{\varepsilon}), \\ &\quad \phi_1, \phi_1(\zeta_1 + \sqrt{\varepsilon})). \end{aligned}$$

We observe the fundamental property that if

$$\|\phi\| + \|\psi_1\| \leq M \tag{B.5}$$

then there is $\tilde{M} > 0$ such that

$$\|g\| + \left\| \frac{\tilde{g}}{\sqrt{\varepsilon}} \right\| \leq \tilde{M}. \tag{B.6}$$

In order to prove Theorem 1, we replace the system (B.2) for $\psi(\zeta)$ by an integral formula. However this is not a simple task, since the estimate on the resolvent of $\mathcal{L}_{1,0}$ is not nice on its third component. So, it is necessary here to use the fact that the nonlinear term in the system (B.1) and (B.2) does not depend on $\psi_3(\zeta, p)$, when the system is rewritten in the new formulation (which was precisely the aim of this manipulation). For the study of solutions bounded for $\zeta \in \mathbb{R}$, we use the two first components of the following implicit formula for ψ :

$$\begin{aligned} \psi(\zeta) &= -\sqrt{\varepsilon} \int_{\zeta}^{\infty} e^{(\mathcal{L}_{1,0,+})(\zeta-s)} P_+ \mathbf{F}(\phi(s), \psi(s)) ds \\ &\quad + \sqrt{\varepsilon} \int_{-\infty}^{\zeta} e^{(\mathcal{L}_{1,0,-})(\zeta-s)} P_- \mathbf{F}(\phi(s), \psi(s)) ds, \end{aligned} \tag{B.7}$$

where, by definition

$$\begin{aligned} e^{(\mathcal{L}_{1,0,+})t} P_+ \mathbf{F} &= \frac{1}{2i\pi} \int_{\Gamma_+} e^{A t} (\Lambda - \mathcal{L}_{1,0})^{-1} \mathbf{F} d\Lambda, \quad t < 0, \\ e^{(\mathcal{L}_{1,0,-})t} P_- \mathbf{F} &= \frac{1}{2i\pi} \int_{\Gamma_-} e^{A t} (\Lambda - \mathcal{L}_{1,0})^{-1} \mathbf{F} d\Lambda, \quad t > 0. \end{aligned}$$

The curves Γ_{\pm} are defined by

$$\begin{aligned} \Gamma_+ &= C_+ \cup \overline{C_+} \cup L_+, \\ \Gamma_- &= C_- \cup \overline{C_-} \cup L_-, \end{aligned}$$

where C_{\pm} are curves in the complex plane defined by

$$C_{\pm} = \left\{ \Lambda = \pm x + iy : y = c \cosh(\alpha x), \alpha > 1, c > \sqrt{8}, |x| \geq \xi > 0 \right\},$$

and

$$L_{\pm} = \{ \Lambda = \pm \xi + iy : |y| \leq c \cosh(\alpha \xi) \}. \tag{B.8}$$

The curve Γ_- is oriented with increasing $\text{Im}(\Lambda)$, while the curve Γ_+ is oriented with decreasing $\text{Im}(\Lambda)$. It is clear from the proof of Lemma 2.1 that both curves Γ_{\pm} lie in the resolvent set of $\mathcal{L}_{1,0}$ and that on these curves we have (as on the imaginary axis for $\Lambda = ik, |k|$ large enough),

$$|D_1| \geq c_1 |\Lambda|^2, \tag{B.9}$$

where $D_1(\Lambda) = 2(\cosh \Lambda - 1) - \Lambda^2$. We shall complete the system (B.2) with an explicit formula for $\psi_3(\zeta, p)$. In order to study integrals in the integral equation (B.7), we compute the expression

$$\mathbf{R}(\Lambda) = (\Lambda - \mathcal{L}_{1,0})^{-1} \mathbf{F}_0(p). \tag{B.10}$$

Using solutions of the resolvent Eqs. (3.16) and (3.17) for $\mathbf{F} = \mathbf{F}_0(p)$, we obtain the following expressions:

$$\tilde{F}(\zeta, \Lambda) = -\frac{3}{5} - \frac{4}{5\Lambda^2}(14 + \cosh \Lambda) + \frac{24}{\Lambda^4}(\cosh \Lambda - 1), \tag{B.11}$$

and

$$\begin{aligned} \int_0^p \frac{2}{5} s(1-5s^2) e^{A(p-s)} ds &= -\frac{2p(1-5p^2)}{5\Lambda} + \frac{2(15p^2-1)}{5\Lambda^2} \\ &\quad + \frac{2e^{Ap}}{5\Lambda^2} + \frac{12p}{\Lambda^3} + \frac{12}{\Lambda^4}(1-e^{Ap}). \end{aligned}$$

As a result, the first two components of $\mathbf{R}(\Lambda)$ satisfy the following bounds for $\Lambda \in \Gamma_{\pm}$:

$$\begin{aligned} \left| R_1(\Lambda) - \frac{3}{5\Lambda^2} \right| &\leq \frac{C_1}{|\Lambda|^{4-1/\alpha}} \\ \left| R_2(\Lambda) - \frac{3}{5\Lambda} \right| &\leq \frac{C_2}{|\Lambda|^{3-1/\alpha}} \end{aligned}$$

for some constants $C_{1,2} > 0$. Since the following integrals cancel for $t < 0$ (below we consider integrals on Γ_+ , analogous results hold on Γ_- for $t > 0$),

$$\frac{1}{2i\pi} \int_{\Gamma_+} \frac{e^{At}}{\Lambda} d\Lambda = \frac{1}{2i\pi} \int_{\Gamma_+} \frac{e^{At}}{\Lambda^2} d\Lambda = 0,$$

we obtain for $t < 0$ ($\alpha > 1$):

$$\left| \frac{1}{2i\pi} \int_{\Gamma_+} e^{At} R_j(\Lambda) d\Lambda \right| \leq \tilde{C}'_j \frac{e^{-\xi(\alpha-t)}}{\alpha-t}, \quad j = 1, 2 \tag{B.12}$$

and

$$\left| \frac{d}{dt} \left(\frac{1}{2i\pi} \int_{\Gamma_+} e^{At} R_j(\Lambda) d\Lambda \right) \right| \leq \tilde{C}'_j \frac{e^{-\xi(\alpha-1-t)}}{\alpha-1-t}, \quad j = 1, 2. \tag{B.13}$$

As a consequence, for $\alpha > 1$, the two first components of the integral $\frac{1}{2i\pi} \int_{\Gamma_+} e^{At} \mathbf{R}(\Lambda) d\Lambda$ and $\frac{1}{2i\pi} \int_{\Gamma_-} e^{At} \mathbf{R}(\Lambda) d\Lambda$ are differentiable real functions respectively for $t \leq 0$ and for

$t \geq 0$. We note that no similar estimates can be obtained for the third component of $\mathbf{R}(\lambda)$ for $t < 0$. Now using the identity

$$\Lambda(\Lambda - \mathcal{L}_{1,0})^{-1}\mathbf{F} = (\Lambda - \mathcal{L}_{1,0})^{-1}\mathcal{L}_{1,0}\mathbf{F} + \mathbf{F} \tag{B.14}$$

one can show that a bounded solution $(\psi_1, \psi_2) \in C_b(\mathbb{R}, \mathbb{R}^2)$ of the two first components of the integral equation (B.7) is completed by the following explicit formula for $\psi_3(\zeta, p)$

$$\begin{aligned} \psi_3(\zeta, p) = & \psi_1(\zeta + p) - \frac{\sqrt{\varepsilon}}{10} \int_{\zeta-p}^{\zeta+p} (\zeta + p - \tau) \\ & \times \left(1 - \frac{5}{4}(\zeta + p - \tau)^2 \right) g \left(\frac{\zeta + p + \tau}{2} \right) d\tau, \end{aligned} \tag{B.15}$$

such that $\psi_3(\zeta, 0) = \psi_1(\zeta)$. Therefore, for bounded solutions on $\zeta \in \mathbb{R}$, the system (B.2) with the relation (B.15) reduces to a two-dimensional system. The argument of [27, p. 145] can then be used.

Coming back to the system (B.1) and (B.2), the implicit linear equation (B.3) for g can be solved for small enough ε and the system (B.1) and (B.2) is hence equivalent to the coupled system (3.27) and (3.28). From the above study, the two first components of Eq. (B.7), completed by (B.15), have a meaning for ψ in $C_b^0(\mathbb{R}, \mathcal{D})$, for any known function $\phi \in C_b^0(\mathbb{R}, \mathbb{R}^4)$. Moreover, for ε small enough, and thanks to (B.6), the implicit function theorem applies, and one can find a unique solution of the integral equation (B.7) for $\psi \in C_b^0(\mathbb{R}, \mathcal{D})$, which is bounded by a constant of the order of $O(\sqrt{\varepsilon})$ under the condition that

$$\|\phi\|_{C_b^0(\mathbb{R}, \mathbb{R}^4)} \leq M, \tag{B.16}$$

where M is an arbitrarily fixed number. Moreover, because of the reversibility of the system (B.1), (B.2) and oddness property of the function $Q(v, u, w)$, if $\phi(\zeta_1)$ is anti-reversible with respect to \mathcal{S}_0 , then the unique solution $\psi(\zeta)$ is also anti-reversible with respect to \mathcal{S} .

It remains to prove the last sentence of Theorem 1. When $\phi(\zeta_1)$ is a periodic function, there exists a unique periodic solution $\psi(\zeta)$ of the system (B.2), since the system (B.1) and (B.2) is invariant under a shift of ζ by $T/\sqrt{\varepsilon}$. When $\phi(\zeta_1)$ is an asymptotically periodic function at infinity, the solution $\psi(\zeta)$ is then asymptotic to the corresponding periodic solution of the system (B.2). This follows from the fact that the formula (B.15) for the third component of $\psi(\zeta)$ can be considered in the limit $\zeta \rightarrow \pm\infty$ and from the two-dimensional Eq. (B.7) for the two first components of $\psi(\zeta)$, for which the solution depends continuously on $\phi(\zeta_1)$, as for an ordinary differential equation.

Appendix C. Stokes constant for the fourth-order ODE (4.1)

The Stokes constants are used in rigorous analysis of the non-existence of homoclinic and heteroclinic orbits in singularly perturbed systems of differential and difference equations [25]. If the Stokes constant is non-zero, the heteroclinic orbit of the unperturbed problem does not persist beyond all orders of the perturbation series expansion. We will

develop here computations of the Stokes constant for the fourth-order equation (4.1) in the limit $\sigma \rightarrow \infty$, in the context of the ϕ^4 model. We will show that the Stokes constant is non-zero, which implies that the graph of $K(\sigma)$ on Fig. 3(left) remains non-zero in the limit $\sigma \rightarrow \infty$.

Let us introduce a small parameter ϵ , such that $\epsilon = \sigma^{-1}$, and apply rescaling of t as $x = \sqrt{\epsilon}t$. The fourth-order equation (4.1) with $F = \phi(1 - \phi^2)$ takes the form of the singularly perturbed ODE:

$$\phi'' + \phi - \phi^3 + \epsilon^2 \phi^{(iv)} = 0. \tag{C.1}$$

At $\epsilon = 0$, the ODE (C.1) corresponds to the stationary problem for the continuous ϕ^4 model with the heteroclinic orbit in the form $\phi_0(x) = \tanh(x/\sqrt{2})$. There exists a formal solution to the perturbed ODE (C.1) in the form of the regular perturbation series,

$$\hat{\phi}(x) = \tanh \frac{x}{\sqrt{2}} + \sum_{k \geq 1} \epsilon^{2k} \phi_k(x), \tag{C.2}$$

where an odd function $\phi_k(x)$ is uniquely defined from the solution of the linear inhomogeneous problem:

$$\begin{aligned} \left(-\partial_x^2 + 2 - 3\operatorname{sech}^2 \frac{x}{\sqrt{2}} \right) \phi_k = & H_k(\phi_{k-1}, \phi_{k-2}, \dots, \phi_0), \\ & k \geq 1, \end{aligned} \tag{C.3}$$

where H_k are correction terms which are odd exponentially decaying functions of $x \in \mathbb{R}$. Each term of $\phi_k(x)$ is a real analytic function of $x \in \mathbb{R}$ and is extended meromorphically to the complex plane, with pole singularities at the points $x = \frac{\pi(1+2n)i}{\sqrt{2}}$, $n \in \mathbb{Z}$. The Stokes constants are introduced after the perturbed ODE (C.1) is rescaled near the pole singularities and the formal solution (C.2) is replaced with the inverse power series. Indeed, let us rescale variables of (C.1) by

$$x = \frac{\pi i}{\sqrt{2}} + \epsilon z, \quad \phi(x) = \frac{\psi(z)}{\epsilon}.$$

Then the function $\psi(z)$ satisfies the regularly perturbed ODE:

$$\psi^{(iv)} + \psi'' - \psi^3 + \epsilon^2 \psi = 0. \tag{C.4}$$

We are looking at the formal solution of the ODE (C.4) with the perturbation series

$$\hat{\psi}(z) = \hat{\psi}_0(z) + \sum_{k \geq 1} \epsilon^k \hat{\psi}_k(z), \tag{C.5}$$

where each term $\hat{\psi}_k(z)$ is the inverse power series of z , starting with the zero-order solution:

$$\hat{\psi}_0(z) = \sum_{m \geq 1} \frac{a_m}{z^m}. \tag{C.6}$$

Matching conditions between (C.2), (C.5) and (C.6) imply that $a_1 = \sqrt{2}$ and $a_2 = 0$. The main result of the beyond-all-order perturbation theory is that the formal series (C.2) diverges if the inverse power series (C.6) diverges. Divergence of the formal series implies in its turn that the heteroclinic orbit $\phi_0(x) = \tanh(x/\sqrt{2})$ is destroyed by the singular perturbation of the

ODE (C.1). Divergence of (C.6) is defined by the asymptotical behavior of the coefficients a_m as $m \rightarrow \infty$ (see [26] for details).

In order to show that the inverse power series (C.6) diverges, we find the recurrence relation between coefficients in the set $\{a_m\}_{m=1}^\infty$:

$$m(m+1)(m+2)(m+3)a_m + (m+2)(m+3)a_{m+2} - \sum_{l+k \leq m+3} a_l a_k a_{m+4-l-k} = 0, \tag{C.7}$$

where $m \geq 1$, $a_1 = \sqrt{2}$ and a_2 is arbitrary. Due to the translational invariance, we can always fix $a_2 = 0$. Setting $a_m = (-1)^n (2n)! b_n$ for $m = 2n + 1$, $n \geq 0$ and $a_m = 0$ for $m = 2n$, $n \geq 1$, we reduce the recurrence equation (C.7) to the diagonal form:

$$b_{n+1} = b_n + \sum_{l+k \leq n+1} \frac{(2l)!(2k)!(2n+2-2l-2k)!}{(2n+4)!} \times b_l b_k b_{n+1-l-k}, \quad n \geq 0, \tag{C.8}$$

where $b_0 = \sqrt{2}$. Since

$$1 - \frac{6}{(2n+4)(2n+3)} > 0, \quad n \geq 0,$$

it follows from (C.8) that the sequence $\{b_n\}_{n=0}^\infty$ is increasing, such that $b_{n+1} \geq b_n > b_0 = \sqrt{2}$. Therefore, the sequence $\{b_n\}_{n=0}^\infty$ is bounded from below by a positive constant, so that the Stokes constant in the beyond-all-order theory is strictly positive.

Similar computations can be developed for the fourth-order equation (4.1) with $F = \sin(\phi)$. They are left for a reader's exercise.

Appendix D. The normal form for the inverse method of [8]

It was shown in [8] that the nonlinearity of the discrete Klein–Gordon equation (1.1) can be chosen in such a way that it yields an exact travelling kink solution $u_n(t) = \phi(z)$, $z = n - ct$ for a particular value of velocity $c = s$. In application to the discrete ϕ^4 lattice, one can look for the exact solution in the form $\phi(z) = \tanh(\mu z)$, where μ is an arbitrary parameter, and derive the explicit form of the nonlinearity function $f = f(u_n)$ parameterized by s and μ . (See Eq. (27) and Refs. [13,14] in [8].) Since we are using a particular form of the nonlinearity $f(u_{n-1}, u_n, u_{n+1})$ in the starting Eq. (1.1), we transform the nonlinearity from [8] to the form:

$$f(u_n) = u_n(1 - u_n^2) \frac{1 + \alpha u_n^2}{1 - \beta u_n^2}, \tag{D.1}$$

where

$$\alpha = \frac{s^2 \mu^2 \tanh^2 \mu}{\tanh^2 \mu - s^2 \mu^2}, \quad \beta = \tanh^2 \mu,$$

and h^2 is related to the parameters μ and s by means of the relation:

$$h^2 = 2 \left(\tanh^2 \mu - s^2 \mu^2 \right). \tag{D.2}$$

We note that the nonlinearity (D.1) violates the assumption (1.6) of our paper, since it depends implicitly on the lattice parameter h^2 .

When $s = 0$ ($\alpha = 0$, $\beta = \frac{1}{2}h^2$), the nonlinearity (D.1) represents one of the four exceptional nonlinearities with translationally invariant stationary kinks (see [2]). We consider here the limit $s \rightarrow 1$ from the point of normal form analysis. We will show that existence of the exact travelling kink solutions $\phi(z) = \tanh(\mu z)$ in the differential advance-delay equation (1.10) is preserved within the reduction to the scalar fourth-order equation (2.9). Let

$$s^2 = 1 + \epsilon \gamma_s, \quad c^2 = 1 + \epsilon \gamma, \quad h^2 = \epsilon^2 \tau.$$

The parameter μ is defined from the transcendental equation (D.2) in the form,

$$\mu = \sqrt{\epsilon} \mu_s + O(\epsilon),$$

where μ_s is a real positive root of the bi-quadratic equation:

$$\frac{4}{3} \mu_s^4 + 2 \gamma_s \mu_s^2 + \tau = 0.$$

Two real positive roots μ_s exist in the domain $0 < \tau < \frac{3}{4} \gamma_s^2$ for $\gamma_s < 0$ and no real positive roots exist in the domain $\tau > \frac{3}{4} \gamma_s^2$ for $\gamma_s < 0$ and in the domain $\tau > 0$ for $\gamma_s \geq 0$. The nonlinearity (D.1) is reduced in the limit of small ϵ to the form,

$$F(u) = u(1 - u^2) \left(1 + \frac{2 \mu_s^4}{\tau} u^2 \right) + O(\epsilon),$$

such that the normal form (2.9) becomes now

$$\frac{1}{12} \phi^{(iv)} - \gamma \phi'' + \tau \phi(1 - \phi^2) + 2 \mu_s^4 \phi^3(1 - \phi^2) = 0. \tag{D.3}$$

It is easy to verify that the scalar fourth-order equation (D.3) has the exact heteroclinic orbit $\phi(z) = \tanh(\mu_s z)$ for $\gamma = \gamma_s$ and $\tau > 0$. Since there are two roots for μ_s , two heteroclinic orbits $\phi(z)$ exist. These two orbits disappear at the saddle–node bifurcation at $\tau = \frac{3}{4} \gamma_s^2$. Within the numerical method of our paper, the heteroclinic orbits can be detected as isolated zeros of the split function $K(\sigma)$. We conclude that modifications of assumptions on the nonlinearity of the discrete Klein–Gordon equation (1.1) result in modifications of the scalar fourth-order normal form (2.9), which may hence admit isolated heteroclinic orbits.

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