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PINNING IN THE EXTENDED LUGIATO–LEFEVER EQUATION*

LUKAS BENGEL[†], DMITRY PELINOVSKY[‡], AND WOLFGANG REICHEL[†]

Abstract. We consider a variant of the Lugiato–Lefever equation (LLE), which is a nonlinear Schrödinger equation on a one-dimensional torus with forcing and damping, to which we add a first-order derivative term with a potential $\varepsilon V(x)$. The potential breaks the translation invariance of LLE. Depending on the existence of zeroes of the effective potential V_{eff}, which is a suitably weighted and integrated version of V, we show that stationary solutions from $\varepsilon = 0$ can be continued locally into the range $\varepsilon \neq 0$. Moreover, the extremal points of the ε-continued solutions are located near zeros of V_{eff}. We therefore call this phenomenon *pinning* of stationary solutions. If we assume additionally that the starting stationary solution at $\varepsilon = 0$ is spectrally stable with the simple zero eigenvalue due to translation invariance being the only eigenvalue on the imaginary axis, we can prove asymptotic stability or instability of its ε-continuation depending on the sign of V_{eff}' at the zero of V_{eff} and the sign of ε. The variant of the LLE arises in the description of optical frequency combs in a Kerr nonlinear ring-shaped microresonator which is pumped by two different continuous monochromatic light sources of different frequencies and different powers. Our analytical findings are illustrated by numerical simulations.

Key words. nonlinear Schrödinger equation, bifurcation theory, continuation method

MSC codes. 34C23, 34B15, 35Q55, 34B60

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1. Introduction.

The Lugiato–Lefever equation [22] is the most commonly used model to describe electromagnetic fields inside a resonant cavity that is pumped by a strong continuous laser source. Inside the cavity, the electromagnetic field propagates and suffers losses due to curvature and/or material imperfections. Most importantly, the cavity consists of a Kerr-nonlinear material so that, triggered by modulation instability, the field may experience a nonlinear interaction of the pumped and resonantly enhanced modes of the cavity. Under appropriate driving conditions of the resonant cavity and the laser, a stable Kerr-frequency comb may form in the cavity, which is a spatially localized and spectrally broad waveform.

Since their discovery by the 2005 Noble prize laureate Theodor Hänsch, frequency combs have seen an enormously wide field of applications, e.g., in high capacity optical communications [25], ultrafast optical ranging [38], optical frequency metrology [39], or spectroscopy [32, 35]. The Lugiato–Lefever equation (LLE) is an amplitude equation for the electromagnetic field inside the cavity derived by means of the slowly varying envelope approximation.

In the following we assume that the cavity is a ring-shaped microresonator with normalized perimeter 2π. Using dimensionless quantities and writing $u(x,t) = \sum_{k\in\mathbb{Z}} u_k(t)\mathrm{e}^{\mathrm{i}kx}$ for the slowly varying and 2π-periodic amplitude of the electromagnetic field, the LLE in its original form [22] reads as

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[†]Institute for Analysis, Karlsruhe Institute of Technology (KIT), D-76128 Karlsruhe, Germany (lukas.bengel@kit.edu, Wolfgang.Reichel@kit.edu).
[‡]Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, L8S 4K1, Canada (dmpeli@math.mcmaster.ca).

$$\mathrm{i}\partial_t u = -d\partial_x^2 u + (\zeta - \mathrm{i}\mu)u - |u|^2 u + \mathrm{i}f_0, \qquad (x,t) \in \mathbb{T} \times \mathbb{R}, \tag{1.1}$$

where $\mathbb{T}$ is a circle of length 2π. Here $d > 0$ is the case of anomalous dispersion, whereas $d < 0$ refers to normal dispersion. The laser pump with frequency ω_{p_0} has the power $|f_0|^2$, and the detuning value ζ represents the offset between the pump frequency ω_{p_0} and the closest resonance frequency ω_0 of the resonator. Finally, the value $\mu > 0$ quantifies the damping coefficient.

More recently, novel pumping schemes have been discussed [36], where, instead of one monochromatic laser pump, one uses a dual laser pump with two different frequencies as a source term. Using again dimensionless quantities the resulting equation is given by

$$\mathrm{i}\partial_t u = -d\partial_x^2 u + (\zeta - \mathrm{i}\mu)u - |u|^2 u + \mathrm{i}f_0 + \mathrm{i}f_1 \mathrm{e}^{\mathrm{i}(k_1 x - \nu_1 t)}, \qquad (x,t) \in \mathbb{T} \times \mathbb{R}, \tag{1.2}$$

cf. [13, 14, 36] for a detailed derivation. In contrast to (1.1), there is now a second source term with pump strength f_1, and k_1 stands for the second pumped mode (the first pumped mode is again $k_0 = 0$). This gives rise to two detuning variables $\zeta = \frac{2}{\kappa}(\omega_0 - \omega_{p_0})$, $\zeta_1 = \frac{2}{\kappa}(\omega_{k_1} - \omega_{p_1})$, and they define $\nu_1 = \zeta - \zeta_1 + dk_1^2$. One of the main outcomes of [14] is that the stationary states of (1.2) are far more localized than the stationary states of (1.1), and the best results can be achieved when $f_0 = f_1$ among all power distributions such that $f_0^2 + f_1^2$ is kept constant.

However, there are cases where a power distribution $|f_0| \gg |f_1|$ is more adequate in physical experiments. In this case, it is shown in Appendix A that one can derive from (1.2) the perturbed LLE in the form

$$\mathrm{i}\partial_t u = -d\partial_x^2 u + \mathrm{i}\varepsilon V(x)\partial_x u + (\zeta - \mathrm{i}\mu)u - |u|^2 u + \mathrm{i}f_0, \qquad (x,t) \in \mathbb{T} \times \mathbb{R}, \tag{1.3}$$

where, in the physical context, $V(x) = \nu_1 - 2dk_1^2 \frac{f_1}{f_0}\cos(x)$ and $\varepsilon = 1$. However, if ν_1 and $k_1^2 f_1/f_0$ are small, we will consider (1.3) as the perturbed LLE, with $\varepsilon \in \mathbb{R}$ being small and $V \in C^1([-\pi,\pi],\mathbb{R})$ being a generic periodic potential. Recall that (1.3) is already set in a moving coordinate frame. In its stationary form the equation becomes

$$-du'' + \mathrm{i}\varepsilon V(x)u' + (\zeta - \mathrm{i}\mu)u - |u|^2 u + \mathrm{i}f_0 = 0, \qquad x \in \mathbb{T}. \tag{1.4}$$

The main questions addressed in this paper are the existence and stability of the stationary solution of (1.3). Our main results, which are stated in detail in section 2, can be summarized as follows:

- In Theorem 3 we prove existence of solutions of (1.4) for small ε provided the effective potential V_{eff} changes sign, where V_{eff} is a weighted integrated version of the coefficient function V.
- In Theorems 8 and 9 we prove stability/instability properties of the solution obtained from Theorem 3 with the time evolution of (1.3).
- In section 3 we illustrate the findings of our theorems by numerical simulations. The numerical simulations show that the location of the intensity extremum of the ε-continued solutions does not change significantly for small ε. Therefore, we call this phenomenon *pinning of solutions at zeroes of the effective potential* V_{eff}.

Existence and bifurcation behavior of solutions of (1.1) have been studied quite well, cf. [11, 12, 15, 16, 24, 26, 27, 28, 29], and their stability properties have been investigated in [2, 6, 7, 17, 18, 19, 31, 34, 37]. Analytical and numerical investigations of (1.2) have been reported in [3, 13, 14]. In contrast, we are not aware of any treatment of (1.3). However, a related problem, where instead of $\mathrm{i}\varepsilon V(x)u'$ a term of the form

$\varepsilon V(x)u$ appears in the nonlinear Schrödinger equation, has been quite well studied; cf. [1, 10, 30]. In this case, solutions are pinned near nondegenerate critical points of V_{eff} instead of the zeroes of V_{eff} as in our case. A different kind of pinning at the intensity maximum of an external forcing term occurs in [33].

2. Main results.

In this section we present our main results regarding existence and stability of stationary solutions of (1.3). For $\varepsilon = 0$ there is a plethora of nontrivial (nonconstant) stationary solutions; cf. [12, 24]. We start with such a solution under the assumption of its nondegeneracy according to the following definition.

DEFINITION 1. *A nonconstant solution* $u \in H^2_{\text{per}}([-\pi,\pi],\mathbb{C})$ *of* (1.4) *for* $\varepsilon = 0$ *is called nondegenerate if the kernel of the linearized operator*

$$L_u\varphi := -d\varphi'' + (\zeta - \mathrm{i}\mu - 2|u|^2)\varphi - u^2\bar{\varphi}, \quad \varphi \in H^2_{\text{per}}([-\pi,\pi],\mathbb{C})$$

consists only of $\operatorname{span}\{u'\}$.

Remark 2. Note that $L_u : H^2_{\text{per}}([-\pi,\pi],\mathbb{C}) \to L^2([-\pi,\pi],\mathbb{C})$ is a relatively compact perturbation of the isomorphism $-d\partial_x^2 + \operatorname{sign}(d) : H^2_{\text{per}}([-\pi,\pi],\mathbb{C}) \to L^2([-\pi,\pi],\mathbb{C})$ and hence a Fredholm operator. Notice also that $\operatorname{span}\{u'\}$ always belongs to the kernel of L_u due to translation invariance in x for $\varepsilon = 0$. Nondegeneracy means that except for the obvious candidate u' (and its real multiples), there is no other element in the kernel of L_u.

One can ask whether nonconstant nondegenerate solutions at $\varepsilon = 0$ in Definition 1 may be continued into the regime of $\varepsilon \neq 0$. In order to describe the continuation, we denote such a solution by u_0 and its spatial translations by $u_\sigma(x) := u_0(x - \sigma)$. The nondegeneracy assumption implies that $\ker L_{u_\sigma} = \operatorname{span}\{u'_\sigma\}$. Since the adjoint operator $L^*_{u_0}$ also has a one-dimensional kernel, there exists $\phi^*_0 \in H^2_{\text{per}}([-\pi,\pi],\mathbb{C})$ such that $\ker L^*_{u_0} = \operatorname{span}\{\phi^*_0\}$. Notice that with $\phi^*_\sigma(x) = \phi^*_0(x-\sigma)$ we find $\ker L^*_{u_\sigma} = \operatorname{span}\{\phi^*_\sigma\}$.

Before stating our existence result, let us clarify the assumption on the potential V.

(A1) The potential $V : [-\pi,\pi] \to \mathbb{R}, x \mapsto V(x)$ is a 2π-periodic, continuously differentiable function.

The existence result is given by the following theorem.

THEOREM 3. *Let* $d \in \mathbb{R}\setminus\{0\}, f_0, \zeta \in \mathbb{R}$, $\mu > 0$ *be fixed, and assume that* (A1) *holds. Furthermore, let* $u_0 \in H^2_{\text{per}}([-\pi,\pi],\mathbb{C})$ *be a nonconstant, nondegenerate solution of* (1.4) *for* $\varepsilon = 0$. *If* σ_0 *is a simple zero of the function*

$$\sigma \mapsto V_{\text{eff}}(\sigma) := \operatorname{Re}\int_{-\pi}^{\pi} \mathrm{i}V(x+\sigma)u'_0\bar{\phi}^*_0\,dx, \tag{2.1}$$

then there exists a continuous curve $(-\varepsilon^*, \varepsilon^*) \ni \varepsilon \to u(\varepsilon) \in H^2_{\text{per}}([-\pi,\pi],\mathbb{C})$ *consisting of solutions of* (1.4) *with* $\|u(\varepsilon) - u_0(\cdot - \sigma_0)\|_{H^2} \leq C|\varepsilon|$ *for some constant* $C > 0$.

Remark 4. The value of σ_0 is determined from the existence of a unique solution $v \in H^2_{\text{per}}([-\pi,\pi],\mathbb{C})$ of the linear inhomogeneous equation

$$L_{u_{\sigma_0}} v = -\mathrm{i}V(x)u'_{\sigma_0}$$

with the property that $v \perp_{L^2} u'_{\sigma_0}$. Fredholm's condition shows that σ_0 is a zero of V_{eff} if and only if this equation is uniquely solvable. Simplicity of the zero of V_{eff} yields the result of Theorem 3.

Remark 5. The observation that V_{eff} having a zero is a necessary condition for continuability of solutions in the case where $V(x) \equiv V_0$ is constant occurred in [4], where traveling solitons with speed V_0 were considered.

To investigate the stability of a stationary solution u, we introduce the expansion

$$u(x) + v(x,t) = u_1(x) + \mathrm{i}u_2(x) + v_1(x,t) + \mathrm{i}v_2(x,t)$$

and substitute this into the perturbed LLE (1.3). After neglecting the quadratic and cubic terms in v and separating real and imaginary parts, we obtain the linearized system for $\boldsymbol{v} = (v_1, v_2)$, which reads as

$$\partial_t \boldsymbol{v} = \widetilde{L}_{u,\varepsilon}\boldsymbol{v},$$

and the linearization has the form

$$\widetilde{L}_{u,\varepsilon} = JA_u - I(\mu - \varepsilon V(x)\partial_x) \tag{2.2}$$

with

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$A_u := \begin{pmatrix} -d\partial_x^2 + \zeta - (3u_1^2 + u_2^2) & -2u_1u_2 \\ -2u_1u_2 & -d\partial_x^2 + \zeta - (u_1^2 + 3u_2^2) \end{pmatrix}.$$

In the following, we will often identify functions in $\mathbb{C}$ as vector-valued functions in $\mathbb{R} \times \mathbb{R}$ and use the notation

$$u = u_1 + \mathrm{i}u_2 \in \mathbb{C} \quad \leftrightarrow \quad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2.$$

We denote the spectrum of $\widetilde{L}_{u,\varepsilon}$ in $L^2([-\pi,\pi]) \times L^2([-\pi,\pi])$ by $\sigma(\widetilde{L}_{u,\varepsilon})$ and the resolvent set of $\widetilde{L}_{u,\varepsilon}$ by $\rho(\widetilde{L}_{u,\varepsilon})$.

For our stability results, we require one additional spectral assumption on the nondegenerate solution u_0 regarding the spectrum of $\widetilde{L}_{u_0,0}$.

(A2) The eigenvalue $0 \in \sigma(\widetilde{L}_{u_0,0})$ is algebraically simple, and there exists $\xi > 0$ such that

$$\sigma(\widetilde{L}_{u_0,0}) \subset \{z \in \mathbb{C} : \operatorname{Re} z \le -\xi\} \cup \{0\}.$$

Remark 6. By Fredholm's theory, the assumption of simplicity of the zero eigenvalue of $\widetilde{L}_{u_0,0}$ is equivalent to $\boldsymbol{u}_0' \notin \operatorname{range}\widetilde{L}_{u_0,0} = \operatorname{span}\{J\boldsymbol{\phi}_0^*\}^\perp$. It will be convenient to use the normalization $\langle \boldsymbol{u}_0', J\boldsymbol{\phi}_0^*\rangle_{L^2} = \int_{-\pi}^{\pi} \boldsymbol{u}_0' \cdot J\boldsymbol{\phi}_0^* \, dx = 1$. We also note that

$$\int_{-\pi}^{\pi} \boldsymbol{u}_0' \cdot J\boldsymbol{\phi}_0^* \, dx = \operatorname{Re} \int_{-\pi}^{\pi} \mathrm{i}u_0' \bar{\phi}_0^* \, dx.$$

Before stating the stability results, let us clarify that u_0' and ϕ_0^* are linearly independent if $\mu \neq 0$, and the integrand of V_{eff} is generically nonzero. We also clarify the parity of eigenfunctions in $\ker L_u^*$ and $\ker L_u$ if u_0 is even in x. This is used for many practical computations.

LEMMA 7. *Let $u_0 \in H^2_{per}([-\pi,\pi],\mathbb{C})$ be a nonconstant, nondegenerate solution of (1.4) for $\varepsilon = 0$ and $\mu \neq 0$. Then the following hold:*

(i) *u_0' and ϕ_0^* are linearly independent, and* $\operatorname{Re} \mathrm{i}u_0'\bar{\phi}_0^* \not\equiv 0$*;*

(ii) *if u_0 is even, then ϕ_0^* is odd.*

Proof. Part (i): By using the decomposition (2.2) with $u = u_0$ and $\varepsilon = 0$, the eigenvalue problems $L_{u_0} u_0' = 0$ and $L_{u_0}^* \phi_0^* = 0$ are equivalent to

$$JA_{u_0} \begin{pmatrix} u_{01}' \\ u_{02}' \end{pmatrix} = \mu \begin{pmatrix} u_{01}' \\ u_{02}' \end{pmatrix}, \qquad JA_{u_0} \begin{pmatrix} \phi_{01}^* \\ \phi_{02}^* \end{pmatrix} = -\mu \begin{pmatrix} \phi_{01}^* \\ \phi_{02}^* \end{pmatrix}.$$

But since (u_{01}', u_{02}') and $(\phi_{01}^*, \phi_{02}^*)$ are eigenvectors to the different eigenvalues μ and $-\mu$ of JA_{u_0}, respectively, they are linearly independent. Moreover, the determinant of the matrix with columns $(u_{01}', u_{02}')^T$ and $(\phi_{01}^*, \phi_{02}^*)^T$ coincides with $\mathrm{Re}\, \mathrm{i} u_0' \bar{\phi}_0^*$, and is not identically zero.

Part (ii): By assumption we have that $\ker L_{u_0} = \mathrm{span}\{u_0'\}$ and u_0' is an odd function. Let us define the restriction of L_{u_0} onto the odd functions

$$L_{u_0}^{\#} : H^2_{\mathrm{per,odd}} \to L^2_{\mathrm{per,odd}},\ \varphi \mapsto L_{u_0}\varphi.$$

Then $L_{u_0}^{\#}$ is again a Fredholm operator of index 0 with $\ker L_{u_0}^{\#} = \mathrm{span}\{u_0'\}$. Further we have $(L_{u_0}^{\#})^* = (L_{u_0}^*)^{\#}$, where

$$(L_{u_0}^*)^{\#} : H^2_{\mathrm{per,odd}} \to L^2_{\mathrm{per,odd}},\ \varphi \mapsto L_{u_0}^*\varphi$$

is the restriction of the adjoint operator onto the odd functions. But since $1 = \dim\ker(L_{u_0}^*)^{\#} = \dim\ker L_{u_0}^*$, it follows that $\ker(L_{u_0}^*)^{\#} = \ker L_{u_0}^*$, and hence $\phi_0^* \in H^2_{\mathrm{per,odd}}$ as claimed. ∎

The stability results are given by the following two theorems. A stationary solution u of (1.4) is called spectrally stable if $\mathrm{Re}(\lambda) \le 0$ for all eigenvalues λ of $\widetilde{L}_{u,\varepsilon}$. It is called spectrally unstable if there exists one eigenvalue λ with $\mathrm{Re}(\lambda) > 0$.

THEOREM 8. *Let $d \in \mathbb{R}\setminus\{0\}, f_0, \zeta \in \mathbb{R}, \mu > 0$ be fixed, let $u_0 \in H^2_{\mathrm{per}}([-\pi,\pi],\mathbb{C})$ be as in Theorem* 3, *and assume that* (A1) *and* (A2) *hold. With σ_0 being a simple zero of V_{eff} as in Theorem* 3, *we have*

$$V_{\mathrm{eff}}'(\sigma_0) = \mathrm{Re}\int_{-\pi}^{\pi} \mathrm{i}V'(x+\sigma_0) u_0' \bar{\phi}_0^*\, dx = \langle V'(\cdot+\sigma_0)\boldsymbol{u}_0', J\boldsymbol{\phi}_0^*\rangle_{L^2} \neq 0.$$

Then there exists $\varepsilon_0 \in (0, \varepsilon^*]$ such that, on the solution branch $(-\varepsilon_0, \varepsilon_0) \ni \varepsilon \to u(\varepsilon) \in H^2_{\mathrm{per}}([-\pi,\pi],\mathbb{C})$ of (1.4) *with $u(0) = u_{\sigma_0}$, the solutions $u(\varepsilon)$ are spectrally stable for $V_{\mathrm{eff}}'(\sigma_0)\cdot\varepsilon > 0$ and spectrally unstable for $V_{\mathrm{eff}}'(\sigma_0)\cdot\varepsilon < 0$.*

In the next theorem we will show that spectral stability leads to nonlinear asymptotic stability because $\varepsilon \neq 0$ breaks the translational symmetry. Thus, the zero eigenvalue of the linearization disappears, and the asymptotic orbital stability result from [34] can be adapted and leads to a slightly improved result.

THEOREM 9. *Let $u(\varepsilon) \in H^2_{\mathrm{per}}([-\pi,\pi],\mathbb{C})$ be a spectrally stable stationary solution of* (1.3) *for a small value of $\varepsilon \neq 0$ as in Theorem* 8. *Then $u(\varepsilon)$ is asymptotically stable; i.e., there exist $\eta, \delta, C > 0$ with the following properties. If $\varphi \in C([0,T), H^1_{per}([-\pi,\pi],\mathbb{C}))$ is a solution of* (1.3) *with maximal existence time T and*

$$\|\varphi(\cdot,0) - u(\varepsilon)\|_{H^1} < \delta,$$

then $T = \infty$ and

$$\|\varphi(\cdot,t) - u(\varepsilon)\|_{H^1} \le C\mathrm{e}^{-\eta t}\|\varphi(\cdot,0) - u(\varepsilon)\|_{H^1} \quad \textit{for all } t \ge 0.$$

Remark 10. Due to periodicity of V_{eff} on $\mathbb{T}$, simple zeros of V_{eff} come in pairs. By Theorems 8 and 9, for any sign of ε, there is always one simple zero which provides a solution branch consisting of asymptotically stable solutions. Moreover, at the bifurcation point $\varepsilon = 0$, there is an exchange of stability as in the transcritical bifurcation; i.e., the zero eigenvalue crosses the imaginary axis with nonzero speed.

Remark 11. In [7, 17] the authors constructed spectrally stable stationary solutions u of (1.3) for $\varepsilon = 0$ in the case of anomalous dispersion $d > 0$. These solutions satisfy the spectral condition $\sigma(\widetilde{L}_{u,0}) \subset \{-2\mu\} \cup \{\operatorname{Re} z = -\mu\} \cup \{0\}$ and are therefore nondegenerate starting solutions for which our main results from Theorems 3, 8, and 9 hold. Moreover, in section 3 we provide examples of numerically computed solutions for which we checked (A2) numerically.

Remark 12. If u is a solution of (1.4), then the relation

$$\int_{-\pi}^{\pi} (u'\bar{u} - \bar{u}'u)dx = 0$$

holds. This constraint is satisfied by every even function u. In fact, the only solutions of (1.4) for $\varepsilon = 0$ that we are aware of are even around $x = 0$ (up to a shift).

Remark 13. Using Fourier-series expansions of the potential V and the function $\operatorname{Re}(\mathrm{i}u_0'\bar{\phi}_0^*)$, we can derive the Fourier expansion of the effective potential V_{eff}. Assume that u_0 is a nonconstant, nondegenerate, even solution of (1.4) for $\varepsilon = 0$ and $\mu \neq 0$. Then, according to Lemma 7, the function $f := \operatorname{Re}(\mathrm{i}u_0'\bar{\phi}_0^*)$ is a nonconstant, even, real-valued, periodic function, and we can write $f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k e^{\mathrm{i}kx}$ with Fourier-coefficients satisfying $\hat{f}_k = \hat{f}_{-k} = \overline{\hat{f}_k}$ for all $k \in \mathbb{Z}$. Expanding $V(x) = \sum_{k\in\mathbb{Z}} \hat{V}_k e^{\mathrm{i}kx}$ a straightforward calculation shows

$$V_{\text{eff}}(\sigma) = \operatorname{Re} \int_{-\pi}^{\pi} \mathrm{i}V(x+\sigma)u_0'\bar{\phi}_0^*\, dx = \sum_{k\in\mathbb{Z}} \hat{V}_k \hat{f}_k e^{\mathrm{i}k\sigma}.$$

In particular, since not all $\hat{f}_k$, $|k| \geq 1$, vanish, we can choose V as an average-zero trigonometric polynomial such that V_{eff} is also an average-zero nontrivial trigonometric polynomial, and thus generally has simple zeroes.

Remark 14. In the limit where u_0 is highly localized around 0 (e.g., the limit $d \to 0\pm$) and the potential V is wide, the effective potential V_{eff} is well approximated by the actual potential V. More precisely, we find the asymptotic

$$V_{\text{eff}}(\sigma) = \operatorname{Re} \int_{-\pi}^{\pi} \mathrm{i}V(x+\sigma)u_0'\bar{\phi}_0^*\, dx \approx V(\sigma) \operatorname{Re} \int_{-\pi}^{\pi} \mathrm{i}u_0'\bar{\phi}_0^*\, dx = V(\sigma)$$

provided $\langle \mathrm{i}u_0', \phi_0^* \rangle_{L^2} = 1$. Thus, the asymptotically stable branch bifurcates from a simple zero σ_0 of V with $V'(\sigma_0)\varepsilon > 0$.

Remark 15. The criterion for stability of stationary solutions in Theorem 8 can be written in a more precise form for small μ in the case of solitary waves. This limit is considered in Appendix B.

To summarize, our main results show that nondegenerate solutions of (1.4) for $\varepsilon = 0$ can be extended locally for small $\varepsilon \neq 0$ provided the effective potential V_{eff} has a sign-change. Depending on the derivative of V_{eff} at a simple zero, we determined the stability properties of these solutions. It remains an open problem to give a criterion on V or V_{eff} for the existence/stability of stationary solutions which applies when $|\varepsilon|$ is large.

3. Numerical simulations. In the following we describe numerical simulations of solutions to (1.4). We choose $f_0 = 2$, $\mu = 1$, $V(x) = 0.1 + 0.5\cos(x)$, and $d = \pm 0.1$. All computations are done with help of the MATLAB package `pde2path` (cf. [8, 40]), which has been designed to numerically treat continuation and bifurcation in boundary value problems for systems of PDEs.

We begin with the description of the stationary solutions of the LLE (1.1), which are the same as the solutions of (1.4) for $\varepsilon = 0$. The corresponding results are mainly taken from [12, 24]. There is a curve of trivial, spatially constant solutions, cf. black line in Figure 1, and this is the same curve for anomalous dispersion ($d = 0.1$) and normal dispersion ($d = -0.1$). Next, one finds that there are finitely many bifurcation points on the curve of trivial solutions (blue dots). Depending on the sign of the dispersion parameter d, one can find now the branches of the single solitons on the periodic domain $\mathbb{T}$. In the following descriptions we always follow the path of trivial solutions by starting from negative values of ζ.

For $d = 0.1$ (left panel in Figure 1), along the trivial branch there is a last bifurcation point which gives rise to a single bright soliton branch (red line). This branch has a turning point, at which the solutions change from unstable (dashed) to stable (solid), and after the turning point it tends back toward the trivial branch. Thus, the red line in the left panel of Figure 1 represents two different but almost identical curves, which can be seen in the enlarged inset. We have chosen a solution at the point BP on the stable branch as a starting point for the illustration of Theorems 3 and 8.

In the case where $d = -0.1$ (right panel in Figure 1), along the trivial branch there is a first bifurcation point from which a single dark soliton branch (red line) bifurcates. Near the second turning point of this branch the most localized single solitons live, and we have chosen a stable dark soliton solution at the point BP as a starting point for the illustration of Theorems 3 and 8.

Next we explain the global picture in Figure 2 of the continuation in ε of the chosen points BP from the $\varepsilon = 0$ case in Figure 1. The local picture is covered by Theorem 3. First we note the following symmetry: since $V(x)$ is even around $x = 0$, we find that $(u(x), \varepsilon)$ solves (1.4) if and only if $(u(-x), -\varepsilon)$ satisfies (1.4). Since reflecting u does not affect the L^2-norm, we see for $\varepsilon > 0$ an exact mirror image of the one for $\varepsilon < 0$.

Next we observe that continuation curves in ε appear to be unbounded for $d = 0.1$ (upper left panel of Figure 2) and closed and bounded for $d = -0.1$ (lower left panel

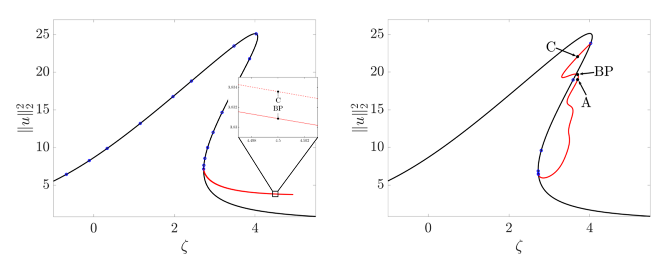

FIG. 1. *Bifurcation diagram for the case $\varepsilon = 0$. Blue dots indicate bifurcation points on the line of trivial solutions (black). The red curve denotes the single soliton solution branch. The point BP is chosen as a starting point for Theorem 3. Further solutions on the same branch for the same value of ζ are denoted by C (left panel) and A, C (right panel). Left panel for $d = 0.1$, right panel for $d = -0.1$.*

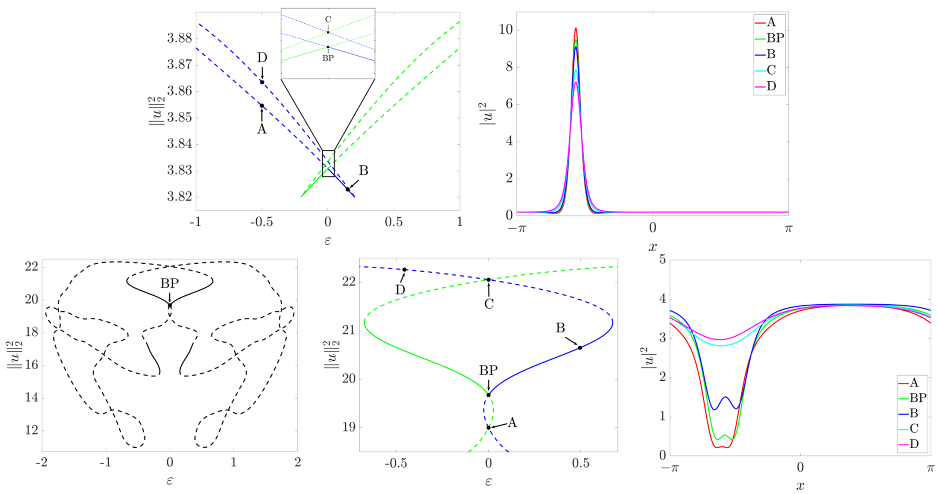

FIG. 2. *Continuation diagrams w.r.t. ε with stability regions (solid = stable; dashed = unstable) and solutions at designated points. The two different zeroes of V_{eff} give rise to two different continuation curves (blue and green). Top panels: $d=0.1$, $\zeta=3.7$. Bottom panels: $d=-0.1$, $\zeta=4.5$ with zoom (middle panel) of the continuation curve near the starting point.*

of Figure 2). In our example the map $\sigma \mapsto V_{\mathrm{eff}}(\sigma) := \mathrm{Re}\int_{-\pi}^{\pi} \mathrm{i}V(x+\sigma)u_0'\bar{\phi}_0^*\,dx$ has two zeroes in the periodic domain $\mathbb{T}$, denoted by σ_0 and σ_1. Since, moreover, u_0 is even and consequently u_0', ϕ_0^* are odd, we see that the effective potential V_{eff} is also even, and hence $\sigma_0 = -\sigma_1$. Thus, continuation in ε works for the starting point $u_0(\cdot - \sigma_0)$ (blue curve) and $u_0(\cdot + \sigma_0)$ (green curve) with $\sigma_0 < 0$. As predicted from Theorem 8, locally on one side of $\varepsilon = 0$ we have stable solutions, and on the other side we have unstable solutions. On the top and bottom right panels of Figure 2 we see the graph of $|u|^2$ for several solutions on the continuation diagram. The top left panel and the bottom middle panel indicate that the ε-continuation curves meet all other nontrivial points (C for $d=0.1$ and A, C for $d=-0.1$) at $\varepsilon=0$ from Figure 1.

In Figure 3 we show the starting solutions $u_0(x-\sigma_0)$ and $u_0(x-\sigma_1)$ together with the potential $V(x)$. Here the zeroes $\sigma_0 < 0 < \sigma_1$ of the effective potential V_{eff} are shown as blue and green dots, and we already observed $\sigma_0 = -\sigma_1$ due to the evenness of both V and V_{eff}. Since u_0 is sufficiently strongly localized, the zeroes of V_{eff} are well approximated by the zeroes of V, and the starting solutions are thus centered near the zeroes of V. Therefore, by applying Remark 14, we see that slope of V at the center of the soliton being positive in the blue bifurcation point indicates that the ε-continuation will be stable for $\varepsilon > 0$ and unstable for $\varepsilon < 0$. The stability behavior is exactly opposite for the green bifurcation point. The stability considerations are valid both for $d=0.1$ and $d=-0.1$.

Finally, let us illustrate the spectral stability properties of the ε-continuations in Figure 4. For $\varepsilon = 0$, in the left panel we see the spectrum of the linearization around u_0, with most of spectrum having real part -1 due to damping $\mu = 1$ and the remaining part of the spectrum in the left half plane together with the zero eigenvalue caused by shift-invariance. Now we consider how the critical eigenvalue behaves when ε varies. We do this for the case where the starting soliton sits at a zero of V_{eff} with positive slope; cf. blue bifurcation point in Figure 3. As predicted, the critical eigenvalue moves into the complex left half plane for $\varepsilon > 0$, rendering the

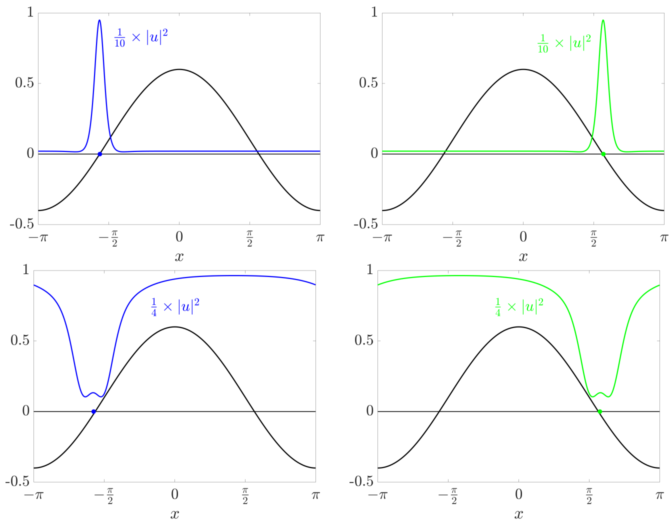

FIG. 3. *Top row: $d = 0.1$, bottom row: $d = -0.1$. Left panels: starting solutions $u_0(x - \sigma_0)$ together with $V(x)$ and negative zero σ_0 of V_{eff} (blue dot). Stability for $\varepsilon > 0$, instability for $\varepsilon < 0$. Right panels: starting solutions $u_0(x + \sigma_0)$ together with $V(x)$ and positive zero $\sigma_1 = -\sigma_0$ of V_{eff} (green dot). Stability for $\varepsilon < 0$, instability for $\varepsilon > 0$.*

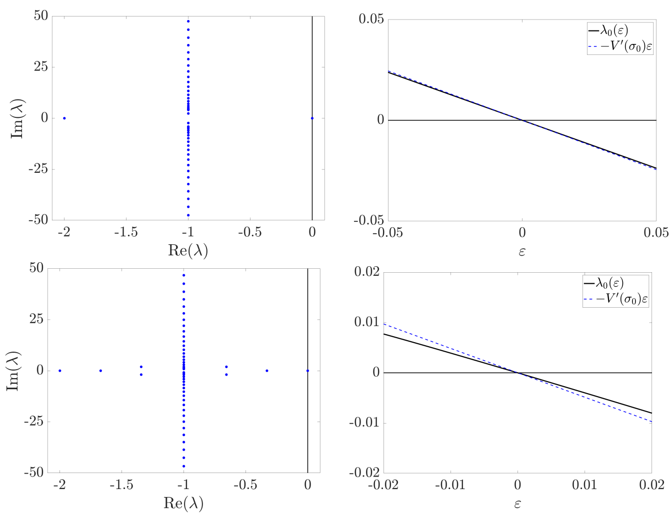

FIG. 4. *Top: $d = 0.1$, bottom: $d = -0.1$. Left: spectrum for $\varepsilon = 0$. Right: critical eigenvalue $\lambda_0(\varepsilon)$ together with $-V'(\sigma_0)\varepsilon$ as functions of ε.*

ε-continuations stable. Since the starting solitons are sufficiently localized, $-V'(\sigma_0)$ predicts well the slope of the critical eigenvalue; cf. Lemma 18 and Remark 14.

4. Proof of the existence result. Theorem 3 will be proved via Lyapunov–Schmidt reduction and the implicit function theorem. Fix the values of d, ζ, μ, and f_0. Let $u_0 \in H^2_{\text{per}}([-\pi, \pi], \mathbb{C})$ be a nondegenerate solution of (1.4) for $\varepsilon = 0$, and recall that, for $\sigma \in \mathbb{R}$, its shifted copy $u_\sigma(x) := u_0(x - \sigma)$ is also a solution of (1.4) for $\varepsilon = 0$.

Proof of Theorem 3. We seek solutions u of (1.4) of the form

$$u = u_\sigma + v, \quad \langle v, u'_\sigma \rangle_{L^2} = 0, \quad v \in H^2_{\text{per}}([-\pi,\pi],\mathbb{C}).$$

Inserting it into (1.4), we obtain the following equation for the correction term v:

$$L_{u_\sigma} v + \mathrm{i}\varepsilon V(u'_\sigma + v') - N(v,\sigma) = 0, \quad \langle v, u'_\sigma \rangle_{L^2} = 0, \tag{4.1}$$

with nonlinearity given by

$$N(v,\sigma) = \bar{u}_\sigma v^2 + 2u_\sigma |v|^2 + |v|^2 v.$$

The nonlinearity is a sum of quadratic and cubic terms in v. Since H^2_{per} is a Banach algebra, it is clear that, for every $R > 0$, there exists $C_R > 0$ such that

$$\|N(v,\sigma)\|_{L^2} \le C_R \|v\|^2_{H^2} \quad \text{for every } \; v \in H^2_{\text{per}}: \; \|v\|_{H^2} \le R. \tag{4.2}$$

Moreover, since $V \in L^\infty$, it follows that

$$\|\mathrm{i}\varepsilon V(u'_\sigma + v')\|_{L^2} \le |\varepsilon| \|V\|_{L^\infty} \|u_\sigma + v\|_{H^2}.$$

Next we solve (4.1) according to the Lyapunov–Schmidt reduction method. Define the orthogonal projections

$$P_\sigma : L^2 \to \operatorname{span}\{u'_\sigma\} \subset L^2, \quad Q_\sigma : L^2 \to \operatorname{span}\{\phi^*_\sigma\}^\perp \subset L^2$$

onto $\ker L_{u_\sigma}$ and $(\ker L^*_{u_\sigma})^\perp = \operatorname{span}\{\phi^*_\sigma\}^\perp = \operatorname{range} L_{u_\sigma}$, respectively. Then (4.1) can be decomposed into a nonsingular and singular equation

$$Q_\sigma \left(L_{u_\sigma}(I - P_\sigma)v + \mathrm{i}\varepsilon V(u'_\sigma + v') - N(v,\sigma) \right) = 0, \tag{4.3}$$

$$\langle \mathrm{i}\varepsilon V u'_\sigma, \phi^*_\sigma \rangle_{L^2} + \langle \mathrm{i}\varepsilon V v' - N(v,\sigma), \phi^*_\sigma \rangle_{L^2} = 0, \tag{4.4}$$

$$\langle v, u'_\sigma \rangle_{L^2} = 0. \tag{4.5}$$

Notice that the linear part $Q_\sigma L_{u_\sigma}(I - P_\sigma)$ in (4.3) is invertible between the σ-dependent subspaces $(\ker L_{u_\sigma})^\perp$ and $\operatorname{range} L_{u_\sigma}$. Therefore, the implicit function theorem cannot be applied directly to solve (4.3). However, (4.3) together with the orthogonality condition (4.5) is equivalent to $F(v,\varepsilon,\sigma) = 0$ with

$$F(v,\varepsilon,\sigma) := Q_\sigma \left(L_{u_\sigma}(I - P_\sigma)v + \mathrm{i}\varepsilon V(u'_\sigma + v') - N(v,\sigma) \right) + \phi^*_\sigma \langle v, u'_\sigma \rangle_{L^2}$$

and $F : H^2_{\text{per}}([-\pi,\pi],\mathbb{C}) \times \mathbb{R} \times \mathbb{R} \to L^2([-\pi,\pi],\mathbb{C})$. Here the added term $\phi^*_\sigma \langle v, u'_\sigma \rangle_{L^2}$ enforces $v \perp u'_\sigma$. For any fixed $\sigma_0 \in \mathbb{R}$, we have $F(0,0,\sigma_0) = 0$. Since

$$D_v F(0,0,\sigma_0)\varphi = L_{u_{\sigma_0}} \varphi + \phi^*_{\sigma_0} \langle \varphi, u'_{\sigma_0} \rangle_{L^2}$$

is an isomorphism from H^2_{per} to L^2, we can apply the implicit function theorem, which gives the existence of a smooth function $v = v(\varepsilon,\sigma)$ solving the problem $F(v(\varepsilon,\sigma),\varepsilon,\sigma) = 0$ for (ε,σ) in a neighborhood of $(0,\sigma_0)$. Moreover from the definition of F, we see that $F(0,0,\sigma) = 0$ so that $v(0,\sigma) = 0$, which implies the bound

$$\|v(\varepsilon,\sigma)\|_{H^2} \le C|\varepsilon| \tag{4.6}$$

for (ε,σ) close to $(0,\sigma_0)$. As a consequence, $\|v'(\varepsilon,\sigma)\|_{L^2} \le C|\varepsilon|$, where $v'(\varepsilon,\sigma)$ denotes the derivative of v with respect to x. Inserting $v(\varepsilon,\sigma)$ into the singular equation (4.4), we end up with the two-dimensional problem

$$f(\varepsilon,\sigma) := \langle \mathrm{i}\varepsilon V u'_\sigma, \phi^*_\sigma \rangle_{L^2} + \langle \mathrm{i}\varepsilon V v'(\varepsilon,\sigma) - N(v(\varepsilon,\sigma),\sigma), \phi^*_\sigma \rangle_{L^2} = 0.$$

To find nontrivial solutions, let us define the C^1-function

$$\tilde{f}(\varepsilon,\sigma) := \begin{cases} \varepsilon^{-1} f(\varepsilon,\sigma), & \varepsilon \neq 0, \\ \partial_\varepsilon f(0,\sigma), & \varepsilon = 0. \end{cases}$$

By (4.2), (4.6), and our assumption on the effective potential V_{eff}, there exists $\sigma_0 \in \mathbb{R}$ such that

$$\tilde{f}(0,\sigma_0) = \langle \mathrm{i} V u'_{\sigma_0}, \phi^*_{\sigma_0} \rangle_{L^2} = \operatorname{Re} \int_{-\pi}^{\pi} \mathrm{i} V(x) u'_{\sigma_0} \bar{\phi}^*_{\sigma_0} dx = V_{\text{eff}}(\sigma_0) = 0$$

and

$$\partial_\sigma \tilde{f}(0,\sigma_0) = \partial_\sigma \langle \mathrm{i} V u'_\sigma, \phi^*_\sigma \rangle_{L^2}|_{\sigma=\sigma_0} = \partial_\sigma \operatorname{Re} \int_{-\pi}^{\pi} \mathrm{i} V(x) u'_\sigma \bar{\phi}^*_\sigma dx \Big|_{\sigma=\sigma_0} = V'_{\text{eff}}(\sigma_0) \neq 0.$$

Hence the implicit function theorem can be applied to the problem $\tilde{f}(\varepsilon,\sigma) = 0$ and yields a curve of unique nontrivial solutions $\sigma = \sigma(\varepsilon)$ to the singular equation $f(\varepsilon,\sigma) = 0$ such that $\sigma(0) = \sigma_0$. Finally we conclude that $u(\varepsilon) = u_0(\cdot - \sigma(\varepsilon)) + v(\varepsilon, \sigma(\varepsilon))$ solves (1.4) for small ε. ∎

5. Proof of the stability result. In this section we will find the condition when the stationary solutions obtained in Theorem 3 as a continuation of a stable solution u_0 of the LLE (1.1) are spectrally stable against co-periodic perturbations in the perturbed LLE (1.3). Moreover, we prove the nonlinear asymptotic stability of stationary spectrally stable solutions.

5.1. Preliminary notes. For our stability analysis we consider (1.3) as a 2-dimensional system by decomposing the function $u = u_1 + \mathrm{i} u_2$ into a real and imaginary part. This leads us to the system of dynamical equations

$$\begin{cases} \partial_t u_1 = -d\partial_x^2 u_2 + \varepsilon V(x)\partial_x u_1 + \zeta u_2 - \mu u_1 - (u_1^2 + u_2^2) u_2 + f_0, \\ \partial_t u_2 = d\partial_x^2 u_1 + \varepsilon V(x)\partial_x u_2 - \zeta u_1 - \mu u_2 + (u_1^2 + u_2^2) u_1, \end{cases} \tag{5.1}$$

equipped with the 2π-periodic boundary condition on $\mathbb{R}$. The spectral problem associated to the nonlinear system (5.1) can be written as

$$\widetilde{L}_{u,\varepsilon} \boldsymbol{v} = \lambda \boldsymbol{v}, \quad \lambda \in \mathbb{C}, \quad \boldsymbol{v} \in H^2_{\text{per}}([-\pi,\pi],\mathbb{C}) \times H^2_{\text{per}}([-\pi,\pi],\mathbb{C}),$$

and the linearized operator $\widetilde{L}_{u,\varepsilon}$ is given by (2.2). Note that the operator A_u in the decomposition (2.2) is self-adjoint on $L^2([-\pi,\pi],\mathbb{C}) \times L^2([-\pi,\pi],\mathbb{C})$, and $\widetilde{L}_{u,\varepsilon}$ is a Fredholm operator of index 0. Moreover we see that if u_0 is a nondegenerate solution of (1.4) for $\varepsilon = 0$, then the following relations for the linearized operators are true:

$$\ker \widetilde{L}_{u_0,0} = \operatorname{span}\{\boldsymbol{u}'_0\}, \quad \ker \widetilde{L}^*_{u_0,0} = \operatorname{span}\{J\boldsymbol{\phi}^*_0\},$$

where the vectors $\boldsymbol{u}'_0 = (u'_{01}, u'_{02})$ and $\boldsymbol{\phi}^*_0 = (\phi^*_{01}, \phi^*_{02})$ are obtained from $u'_0 = u'_{01} + \mathrm{i} u'_{02}$ and $\phi^*_0 = \phi^*_{01} + \mathrm{i}\phi^*_{02}$. We recall that $\langle \boldsymbol{u}'_0, J\boldsymbol{\phi}^*_0 \rangle_{L^2} = 1$ due to normalization; cf. Remark 6.

Finally we observe that since the embedding

$$H^2_{\text{per}}([-\pi,\pi],\mathbb{C}) \times H^2_{\text{per}}([-\pi,\pi],\mathbb{C}) \hookrightarrow L^2([-\pi,\pi],\mathbb{C}) \times L^2([-\pi,\pi],\mathbb{C})$$

is compact, the linearization has compact resolvents, and thus the spectrum of $\widetilde{L}_{u,\varepsilon}$ consists of isolated eigenvalues with finite multiplicity where the only possible accumulation point is at ∞. In the following we will use the spaces

$$H^2_{\text{per}}([-\pi,\pi],\mathbb{C}) =: X, \quad H^1_{\text{per}}([-\pi,\pi],\mathbb{C}) =: Y, \quad L^2([-\pi,\pi],\mathbb{C}) =: Z.$$

The proofs of both Theorem 8 and Theorem 9 rely on the next lemma for the linearized operator $\widetilde{L}_{u(\varepsilon),\varepsilon}$, where $u(\varepsilon)$ lies on the solution branch of Theorem 3, and $|\varepsilon|$ is small. The lemma gives spectral bounds for eigenvalues with large imaginary parts together with a uniform resolvent estimate. The proof is presented in section 5.4.

LEMMA 16. *Denote* $\Lambda_{\lambda^*} := \{\lambda \in \mathbb{C} : \text{Re}(\lambda) \geq 0, |\text{Im}(\lambda)| \geq \lambda^*\}$. *Given* $\varepsilon_1 > 0$ *sufficiently small, there exists* $\lambda^* > 0$ *such that we have the uniform resolvent bound*

$$\sup_{\lambda \in \Lambda_{\lambda^*}} \|(\lambda I - \widetilde{L}_{u(\varepsilon),\varepsilon})^{-1}\|_{L^2 \to L^2} < \infty$$

for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$.

Remark 17. The uniformity of the resolvent estimate on the imaginary axis allows us to sharpen the above result as follows. If we define S as the supremum from Lemma 16 and let $0 < \delta < 1/S$, then the estimate

$$\sup_{\lambda \in \Lambda_{\lambda^*} - \delta} \|(\lambda I - \widetilde{L}_{u(\varepsilon),\varepsilon})^{-1}\|_{L^2 \to L^2} < \infty$$

holds. This follows from taking inverses in the identity

$$((\lambda - \delta)I - \widetilde{L}_{u(\varepsilon),\varepsilon}) = (\lambda I - \widetilde{L}_{u(\varepsilon),\varepsilon})(I - \delta(\lambda I - \widetilde{L}_{u(\varepsilon),\varepsilon})^{-1}).$$

5.2. Proof of Theorem 8. For $\lambda \in \mathbb{C}$ we study the spectral problem

$$\widetilde{L}_{u,\varepsilon}\boldsymbol{v} = \lambda \boldsymbol{v}. \tag{5.2}$$

Since (1.4) has the translational symmetry in the case that $\varepsilon = 0$, we find

$$\widetilde{L}_{u,0}\boldsymbol{u}' = 0.$$

For $\varepsilon \neq 0$, this symmetry is broken, and the zero eigenvalue is expected to move into either the stable or unstable half plane. In our stability analysis, it is therefore important to understand how the critical zero eigenvalue behaves along the bifurcating solution branch given by $(-\varepsilon^*, \varepsilon^*) \ni \varepsilon \mapsto u(\varepsilon) \in X$ with $u(0) = u_{\sigma_0}$, where σ_0 is a simple zero of V_{eff} as in Theorem 3. For the following calculations we will identify the $\mathbb{C}$-valued function $u(\varepsilon) : \mathbb{T} \to \mathbb{C}$ with the $\mathbb{R}^2$ vector-valued function $\boldsymbol{u}(\varepsilon) : \mathbb{T} \to \mathbb{R}^2$ and, understanding that the set of $\mathbb{R}^2$ valued functions is a subset of the set of $\mathbb{C}^2$ valued functions, write this as $\boldsymbol{u}(\varepsilon) \in X \times X$.

We start with the tracking of the simple critical zero eigenvalue and set up the equation for the perturbed eigenvalue $\lambda_0 = \lambda_0(\varepsilon)$ which reads

$$\widetilde{L}_{u(\varepsilon),\varepsilon}\boldsymbol{v}(\varepsilon) = \lambda_0(\varepsilon)\boldsymbol{v}(\varepsilon).$$

After a possible rescaling we find that $\boldsymbol{v}(0) = \boldsymbol{u}'_{\sigma_0}$, and using regular perturbation theory for simple eigenvalues, cf. [21, Proposition I.7.2], the mapping $(-\varepsilon^*, \varepsilon^*) \ni \varepsilon \mapsto \lambda_0(\varepsilon) \in \mathbb{R}$ is continuously differentiable. Our first goal is to derive a formula for $\lambda_0'(0)$. If $\lambda_0'(0) > 0$, this means that the solutions $u(\varepsilon)$ for $\varepsilon > 0$ are spectrally unstable. In contrast, if $\lambda_0'(0) < 0$, the solutions $u(\varepsilon)$ for $\varepsilon > 0$ are spectrally stable.

LEMMA 18. *Let $\varepsilon \mapsto \lambda_0(\varepsilon)$ be the C^1 parametrization of the perturbed zero eigenvalue. Then the following formula holds true:*

$$\lambda_0'(0) = -\int_{-\pi}^{\pi} V'(x)\boldsymbol{u}_{\sigma_0}' \cdot J\boldsymbol{\phi}_{\sigma_0}^* dx.$$

Proof. On the one hand, if we differentiate the equation

$$\widetilde{L}_{u(\varepsilon),\varepsilon}\boldsymbol{v}(\varepsilon) = \lambda_0(\varepsilon)\boldsymbol{v}(\varepsilon)$$

with respect to ε and evaluate at $\varepsilon = 0$, we find

$$\widetilde{L}_{u_{\sigma_0},0}\partial_\varepsilon \boldsymbol{v}(0) - JN_u\boldsymbol{u}_{\sigma_0}' + V(x)\boldsymbol{u}_{\sigma_0}'' = \lambda_0'(0)\boldsymbol{u}_{\sigma_0}', \tag{5.3}$$

where N_u is given by

$$N_u = 2\begin{pmatrix} 3u_{\sigma_0 1}\partial_\varepsilon u_1(0) + u_{\sigma_0 2}\partial_\varepsilon u_2(0) & u_{\sigma_0 1}\partial_\varepsilon u_2(0) + u_{\sigma_0 2}\partial_\varepsilon u_1(0) \\ u_{\sigma_0 1}\partial_\varepsilon u_2(0) + u_{\sigma_0 2}\partial_\varepsilon u_1(0) & u_{\sigma_0 1}\partial_\varepsilon u_1(0) + 3u_{\sigma_0 2}\partial_\varepsilon u_2(0) \end{pmatrix}.$$

On the other hand, if we differentiate (1.4) with respect to ε at $\varepsilon = 0$, then we obtain

$$\widetilde{L}_{u_{\sigma_0},0}\partial_\varepsilon \boldsymbol{u}(0) + V(x)\boldsymbol{u}_{\sigma_0}' = 0.$$

If we differentiate this equation with respect to x, we find

$$\widetilde{L}_{u_{\sigma_0},0}\partial_\varepsilon \boldsymbol{u}'(0) + V(x)\boldsymbol{u}_{\sigma_0}'' + V'(x)\boldsymbol{u}_{\sigma_0}' - JN_u\boldsymbol{u}_{\sigma_0}' = 0. \tag{5.4}$$

Combining (5.3) and (5.4) yields

$$\widetilde{L}_{u_{\sigma_0},0}[\partial_\varepsilon \boldsymbol{v}(0) - \partial_\varepsilon \boldsymbol{u}'(0)] - V'(x)\boldsymbol{u}_{\sigma_0}' = \lambda_0'(0)\boldsymbol{u}_{\sigma_0}',$$

and testing this equation with $J\boldsymbol{\phi}_{\sigma_0}^* \in \ker \widetilde{L}_{u_{\sigma_0},0}^*$, we obtain

$$-\int_{-\pi}^{\pi} V'(x)\boldsymbol{u}_{\sigma_0}' \cdot J\boldsymbol{\phi}_{\sigma_0}^* dx = -\langle V'(x)\boldsymbol{u}_{\sigma_0}', J\boldsymbol{\phi}_{\sigma_0}^*\rangle_{L^2} = \lambda_0'(0)\langle \boldsymbol{u}_{\sigma_0}', J\boldsymbol{\phi}_{\sigma_0}^*\rangle_{L^2} = \lambda_0'(0),$$

which finishes the proof. ∎

By Lemma 18 we can control the critical part of the spectrum close to the origin along the bifurcating solution branch. In fact, using standard perturbation theory, cf. [20], we know that all the eigenvalues of $\widetilde{L}_{u(\varepsilon),\varepsilon}$ depend continuously on the parameter ε. However, this dependence is in general not uniform w.r.t. all eigenvalues, so we have to make sure that no unstable spectrum occurs far from the origin. At this point, it is worth mentioning that we have an a priori bound on the spectrum of the form

$$\exists \lambda_* = \lambda_*(u(\varepsilon),\varepsilon) > 0: \quad \lambda \in \sigma(\widetilde{L}_{u(\varepsilon),\varepsilon}) \implies \mathrm{Re}(\lambda) \le \lambda_*.$$

This bound follows from the Hille–Yosida Theorem since $\widetilde{L}_{u(\varepsilon),\varepsilon}$ generates a C_0-semigroup on $Z \times Z$; cf. Lemma 19 below. It can also be shown directly by testing the eigenvalue problem with the corresponding eigenfunction and integration by parts. As a conclusion, spectral stability holds if we can prove that there exists $\lambda^* > 0$ such that

$$\{\lambda \in \mathbb{C} : 0 \le \mathrm{Re}(\lambda) \le \lambda_*, |\mathrm{Im}(\lambda)| \ge \lambda^*\} \subset \rho(\widetilde{L}_{u(\varepsilon),\varepsilon}).$$

This relation is shown as part of Lemma 16, and it is extended to the left of the origin by the subsequent Remark 17. Since in any rectangle $\{\lambda \in \mathbb{C} : -M \le \mathrm{Re}(\lambda) \le \lambda_*, |\mathrm{Im}(\lambda)| \le \lambda^*\}$ there are only finitely many eigenvalues of $\widetilde{L}_{u(\varepsilon),\varepsilon}$, and they depend (uniformly) continuously on ε, our assumption (A2) on $\widetilde{L}_{u_0,0}$ shows that none of these eigenvalues (except possibly the critical one) can move into the right half plane if $|\varepsilon|$ is small. Therefore, only the movement of the critical eigenvalue determines the spectral stability, and therefore Theorem 8 is true.

5.3. Proof of Theorem 9. In order to prove nonlinear asymptotic stability of stationary solutions of (5.1), it is enough to show exponential stability of the semigroup of the linearization in $Y \times Y$; see, e.g., [5]. For the proof of Theorem 9 we will show the following three steps:

(i) Prove that $\widetilde{L}_{u(\varepsilon),\varepsilon}$ is the generator of a C_0-semigroup on $Z \times Z$.

(ii) Show exponential decay of $(\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t})_{t\geq 0}$ in $Z \times Z$.

(iii) Show exponential decay of $(\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon} t})_{t\geq 0}$ in $Y \times Y$.

For step (i), we establish the generator properties of the linearization in $Z \times Z$.

LEMMA 19. *The operator $\widetilde{L}_{u(\varepsilon),\varepsilon}$ generates a C_0-semigroup on $Z \times Z$.*

Proof. We split the operator into

$$\widetilde{L}_{u(\varepsilon),\varepsilon} = L_1 + L_2 + L_3,$$

where $L_1 : X \times X \to Z \times Z$, $L_2 : Y \times Y \to Z \times Z$, and $L_3 : Z \times Z \to Z \times Z$ are defined by

$$L_1 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} := \begin{pmatrix} -d\varphi_2'' - \mu\varphi_1 \\ d\varphi_1'' - \mu\varphi_2 \end{pmatrix},$$

$$L_2 \boldsymbol{\varphi} := \varepsilon V(x)\boldsymbol{\varphi}' - \frac{|\varepsilon|}{2}\|V'\|_{L^\infty}\boldsymbol{\varphi},$$

and

$$L_3 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} := \begin{pmatrix} \frac{|\varepsilon|}{2}\|V'\|_{L^\infty} - 2u_1u_2 & \zeta - (u_1^2 + 3u_2^2) \\ -\zeta + 3u_1^2 + u_2^2 & \frac{|\varepsilon|}{2}\|V'\|_{L^\infty} + 2u_1u_2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.$$

We will show that

(i) L_1 generates a contraction semigroup.

(ii) L_2 is dissipative and bounded relative to L_1.

(iii) L_3 is a bounded operator on $Z \times Z$.

By using the semigroup theory, this will prove that the sum $L_1 + L_2 + L_3$ is the generator of a C_0-semigroup on $Z \times Z$.

Part (i): It follows that $\mathrm{Re}\langle L_1\boldsymbol{\varphi}, \boldsymbol{\varphi}\rangle_{L^2} = -\mu\|\boldsymbol{\varphi}\|_{L^2}^2 \leq 0$ for every $\boldsymbol{\varphi} \in X \times X$, and $\lambda I - L_1$ is invertible for every $\lambda > 0$, which can be seen using Fourier transform. By the Lumer–Phillips Theorem we find that L_1 generates a contraction semigroup on $Z \times Z$.

Part (ii): We have to show that

$$\forall \boldsymbol{\varphi} \in Y \times Y: \quad \mathrm{Re}\langle L_2\boldsymbol{\varphi}, \boldsymbol{\varphi}\rangle_{L^2} \leq 0$$

and

$$\forall a > 0, \exists b > 0: \quad \|L_2\boldsymbol{\varphi}\|_{L^2} \leq a\|L_1\boldsymbol{\varphi}\|_{L^2} + b\|\boldsymbol{\varphi}\|_{L^2} \quad \forall \boldsymbol{\varphi} \in X \times X.$$

Let $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in Y \times Y$, and observe that integration by parts yields

$$\begin{aligned} \mathrm{Re}\int_{-\pi}^{\pi} \varepsilon V(x)(\varphi_1'\bar{\varphi}_1 + \varphi_2'\bar{\varphi}_2) - \frac{|\varepsilon|}{2}\|V'\|_{L^\infty}|\boldsymbol{\varphi}|^2 dx \\ = \int_{-\pi}^{\pi} -\frac{\varepsilon}{2}V'(x)|\boldsymbol{\varphi}|^2 - \frac{|\varepsilon|}{2}\|V'\|_{L^\infty}|\boldsymbol{\varphi}|^2 dx \leq 0, \end{aligned}$$

which shows that L_2 is dissipative. Further, if $\boldsymbol{\varphi} \in X \times X$, then for every $a > 0$ we have

$$\begin{aligned}
&\|\varepsilon V \boldsymbol{\varphi}' - \frac{|\varepsilon|}{2}\|V'\|_{L^\infty}\boldsymbol{\varphi}\|_{L^2} \\
&\quad \le |\varepsilon|\|V\|_{L^\infty}\|\boldsymbol{\varphi}'\|_{L^2} + \frac{|\varepsilon|}{2}\|V'\|_{L^\infty}\|\boldsymbol{\varphi}\|_{L^2} \\
&\quad \le |\varepsilon|a\|V\|_{L^\infty}\|\boldsymbol{\varphi}''\|_{L^2} + \frac{|\varepsilon|}{4a}\|V\|_{L^\infty}\|\boldsymbol{\varphi}\|_{L^2} + \frac{|\varepsilon|}{2}\|V'\|_{L^\infty}\|\boldsymbol{\varphi}\|_{L^2} \\
&\quad \le \frac{|\varepsilon|a}{|d|}\|V\|_{L^\infty}\|L_1\boldsymbol{\varphi}\|_{L^2} + |\varepsilon|\left(\left(\frac{a\mu}{|d|} + \frac{1}{4a}\right)\|V\|_{L^\infty} + \frac{1}{2}\|V'\|_{L^\infty}\right)\|\boldsymbol{\varphi}\|_{L^2},
\end{aligned}$$

where we used the inequality

$$\forall \boldsymbol{\varphi} \in X \times X, \forall a > 0: \quad \|\boldsymbol{\varphi}'\|_{L^2} \le a\|\boldsymbol{\varphi}''\|_{L^2} + \frac{1}{4a}\|\boldsymbol{\varphi}\|_{L^2}.$$

Hence, by the dissipative perturbation theorem, cf. Chapter III, Theorem 2.7 in [9], for generators, the operator $L_1 + L_2 : X \times X \to Z \times Z$ generates a contraction semigroup.

Part (iii): It follows that L_3 is bounded on $Z \times Z$. Then the bounded perturbation theorem for generators, cf. Chapter III, Theorem 1.3 in [9], yields that $\widetilde{L}_{u(\varepsilon),\varepsilon} = L_1 + L_2 + L_3$ generates a C_0-semigroup on $Z \times Z$ as desired. ∎

Remark 20. Using similar arguments, one can show that $\widetilde{L}_{u(\varepsilon),\varepsilon}$ is the generator of a C_0-semigroup on $Y \times Y$.

For step (ii), we use a characterization of exponential decay of semigroups in Hilbert spaces known as the Gearhart–Greiner–Prüss Theorem; cf. Chapter V, Theorem 1.11 in [9].

THEOREM 21 (Gearhart–Greiner–Prüss Theorem). *Let L be the generator of a C_0-semigroup $(\mathrm{e}^{Lt})_{t\ge0}$ on a complex Hilbert space H. Then $(\mathrm{e}^{Lt})_{t\ge0}$ is exponentially stable in H if and only if*

$$\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \ge 0\} \subset \rho(L) \quad \textit{and} \quad \sup_{\mathrm{Re}\,\lambda\ge0} \|(\lambda I - L)^{-1}\|_{H\to H} < \infty.$$

By the assumption of Theorem 9, spectral stability of the solution $u(\varepsilon)$ is guaranteed, and we are left with the proof of the uniform resolvent estimate on $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \ge 0\}$. Using Lemma 16, we find $\lambda^* \gg 1$ such that $(\lambda I - \widetilde{L}_{u(\varepsilon),\varepsilon})^{-1}$ is uniformly bounded on the set Λ_{λ^*} for sufficiently small ε. Moreover, since $\widetilde{L}_{u(\varepsilon),\varepsilon}$ is the generator of a C_0-semigroup on the state-space $Z \times Z$, the Hille–Yosida Theorem ensures a uniform bound of the resolvent on $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \lambda_*\}$ for some constant $\lambda_* > 0$. From the fact that $\lambda \mapsto (\lambda I - \widetilde{L}_{u(\varepsilon),\varepsilon})^{-1}$ is a meromorphic function with no poles in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \ge 0\}$, the resolvent is uniformly bounded on compact subsets of $\mathbb{C}$ in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \ge 0\}$. Thus, we can conclude that $\widetilde{L}_{u(\varepsilon),\varepsilon}$ satisfies the Gearhart–Greiner–Prüss resolvent bound, and exponential stability in $Z \times Z$ follows.

Finally, for step (iii), we will interpolate the decay estimate between the spaces $Z \times Z$ and $X \times X$. To do so, we have to establish bounds in $X \times X$, which is done in the next lemma. The interpolation argument is then in the spirit of Lemma 5 in [34] and will also lead to decay estimates in the more general interpolation spaces $H^s_{\mathrm{per}} \times H^s_{\mathrm{per}}$ for $s \in [0, 2]$.

LEMMA 22. *For any $s \in [0, 2]$ and sufficiently small ε, the semigroup $(\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t})_{t\ge0}$ has exponential decay in $H^s_{\mathrm{per}}([-\pi, \pi], \mathbb{C}) \times H^s_{\mathrm{per}}([-\pi, \pi], \mathbb{C})$; i.e., there exist $C_s > 0$ such that*

$$\|\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t}\|_{H^s\to H^s} \le C_s \mathrm{e}^{-\eta t} \quad \textit{for } t \ge 0,$$

where $-\eta < 0$ *is the previously established growth bound of the semigroup in* $Z\times Z$.

Proof. We consider only the case $d>0$, since the other case can be shown by rewriting JA_u as $-J(-A_u)$ and using the same arguments as presented below. If $d>0$, the operator $A_{u(\varepsilon)}+\gamma I$ is positive and self-adjoint provided $\gamma>0$ is sufficiently large. Hence, for $z\in\mathbb{C}$, we can define the complex powers by

$$(A_{u(\varepsilon)}+\gamma I)^z \boldsymbol{v} = \int_0^\infty \lambda^z dE_\lambda \boldsymbol{v}, \quad \text{for } \boldsymbol{v}\in \mathrm{dom}(A_u u(\varepsilon)+\gamma I)^z,$$

with domain given by

$$\mathrm{dom}(A_{u(\varepsilon)}+\gamma I)^z = \left\{ \boldsymbol{v}\in Z\times Z : \|(A_{u(\varepsilon)}+\gamma I)^z\boldsymbol{v}\|_{L^2}^2 = \int_0^\infty \lambda^{2\,\mathrm{Re}\, z} d\|E_\lambda \boldsymbol{v}\|_{L^2}^2 < \infty \right\}$$

and where E_λ for $\lambda\in\mathbb{R}$ is the family of self-adjoint spectral projections associated to $A_{u(\varepsilon)}+\gamma I$. Note that for $\theta\in[0,1]$, the relation

$$\mathrm{dom}(A_{u(\varepsilon)}+\gamma I)^\theta = H^{2\theta}_{\mathrm{per}}([-\pi,\pi],\mathbb{C}) \times H^{2\theta}_{\mathrm{per}}([-\pi,\pi],\mathbb{C})$$

is true, cf. [23] Theorem 4.36, and further, for any $r\in\mathbb{R}$, the operator $(A_{u(\varepsilon)}+\gamma I)^{\mathrm{i}r}$ is unitary on $Z\times Z$. If $\theta=0,1$, we will show that there exists $C_\theta>0$ such that

$$\forall r\in\mathbb{R}, \forall t\ge 0, \forall \boldsymbol{v}\in X\times X,: \quad \|(A_{u(\varepsilon)}+\gamma I)^{\theta+\mathrm{i}r}\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t}\boldsymbol{v}\|_{L^2} \le C_\theta \mathrm{e}^{-\eta t}\|\boldsymbol{v}\|_{H^{2\theta}},$$

which implies

$$\forall r\in\mathbb{R}, \forall t\ge 0, \forall\theta\in(0,1), \forall \boldsymbol{v}\in X\times X : \|(A_{u(\varepsilon)}+\gamma I)^{\theta+\mathrm{i}r}\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t}\boldsymbol{v}\|_{L^2} \le C_0^{1-\theta}C_1^\theta \mathrm{e}^{-\eta t}\|\boldsymbol{v}\|_{H^{2\theta}},$$

by complex interpolation; cf. [23] Theorem 2.7. In particular, we see that

$$\|\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t}\|_{H^s\to H^s} \le C_0^{1-s}C_1^s \mathrm{e}^{-\eta t},$$

which is precisely our claim. The estimate for $\theta=0$ has already been shown in the preceding discussion, so it remains for us to check the estimate for $\theta=1$. Let $\boldsymbol{v}\in X\times X$ and observe that

$$\begin{aligned}
\|(A_{u(\varepsilon)}+\gamma I)^{1+\mathrm{i}r}\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t}\boldsymbol{v}\|_{L^2} &= \|(A_{u(\varepsilon)}+\gamma I)\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t}\boldsymbol{v}\|_{L^2}\\
&= \|(\widetilde{L}_{u(\varepsilon),\varepsilon} + J\gamma + I(\mu-\varepsilon V(x)\partial_x))\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t}\boldsymbol{v}\|_{L^2}\\
&\le \|\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t}\widetilde{L}_{u(\varepsilon),\varepsilon}\boldsymbol{v}\|_{L^2} + C\|\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t}\boldsymbol{v}\|_{L^2}\\
&\quad + |\varepsilon|\|V\|_{L^\infty}\|\partial_x \mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t}\boldsymbol{v}\|_{L^2}\\
&\le C\mathrm{e}^{-\eta t}\|\widetilde{L}_{u(\varepsilon),\varepsilon}\boldsymbol{v}\|_{L^2} + C\mathrm{e}^{-\eta t}\|\boldsymbol{v}\|_{L^2}\\
&\quad + |\varepsilon|\|V\|_{L^\infty}\|\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t}\boldsymbol{v}\|_{H^1}\\
&\le C\mathrm{e}^{-\eta t}\|\boldsymbol{v}\|_{H^2} + |\varepsilon| C\|\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t}\boldsymbol{v}\|_{H^2},
\end{aligned}$$

which yields $\|(A_{u(\varepsilon)}+\gamma I)^{1+\mathrm{i}r}\mathrm{e}^{\widetilde{L}_{u(\varepsilon),\varepsilon}t}\boldsymbol{v}\|_{L^2} \le C\mathrm{e}^{-\eta t}\|\boldsymbol{v}\|_{H^2}$ if ε is sufficiently small because of the norm equivalence $\|\boldsymbol{v}\|_{H^2}\sim\|(A_{u(\varepsilon)}+\gamma I)\boldsymbol{v}\|_{L^2}$. ∎

In particular Lemma 22 establishes exponential stability of the linearization in $Y\times Y$; thus we have proved Theorem 9.

5.4. Proof of Lemma 16. The uniform resolvent estimate is proved if we can find a constant $C>0$ independent of $\lambda\in\Lambda_{\lambda^*}$ such that

$$\forall \boldsymbol{\varphi}\in X\times X: \quad \|(\lambda I-\widetilde{L}_{u(\varepsilon),\varepsilon})\boldsymbol{\varphi}\|_{L^2}\geq C\|\boldsymbol{\varphi}\|_{L^2}. \tag{5.5}$$

In order to simplify the situation, let us introduce the rotation on $Z\times Z$ as follows:

$$R\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}:=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

with spatially varying angular $\theta(x)=\frac{\varepsilon}{2d}\int_{-\pi}^{x}[V(y)-\hat{V}_0]dy$, where $\hat{V}_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}V(y)\,dy$ is the mean of the potential V. Since R is an isometry on $Z\times Z$, the resolvent estimate (5.5) is equivalent to

$$\forall \boldsymbol{\varphi}\in X\times X: \quad \|(\lambda I-R\widetilde{L}_{u(\varepsilon),\varepsilon}R^{-1})\boldsymbol{\varphi}\|_{L^2}\geq C\|\boldsymbol{\varphi}\|_{L^2},$$

where we note that $\sigma(\widetilde{L}_{u(\varepsilon),\varepsilon})=\sigma(R\widetilde{L}_{u(\varepsilon),\varepsilon}R^{-1})$. The advantage of considering the operator $R\widetilde{L}_{u(\varepsilon),\varepsilon}R^{-1}$ becomes clear if we calculate

$$R\widetilde{L}_{u(\varepsilon),\varepsilon}R^{-1}=J\tilde{A}_{u(\varepsilon),\varepsilon,V}-I(\mu-\varepsilon\hat{V}_0\partial_x),$$

where the operator $\tilde{A}_{u(\varepsilon),\varepsilon,V}$ is given by

$$\tilde{A}_{u(\varepsilon),\varepsilon,V}:=\begin{pmatrix}-d\partial_x^2+W_1 & W_2+W_4\\ W_2-W_4 & -d\partial_x^2+W_3\end{pmatrix}$$

with potentials

$$\begin{aligned}
W_1&=\zeta+\cos^2\theta U_1+2\cos\theta\sin\theta U_2+\sin^2\theta U_3+d\theta'^2-\varepsilon\theta' V,\\
W_2&=(\cos^2\theta-\sin^2\theta)U_2+\cos\theta\sin\theta(U_3-U_1),\\
W_3&=\zeta+\sin^2\theta U_1-2\cos\theta\sin\theta U_2+\cos^2\theta U_3+d\theta'^2-\varepsilon\theta' V,\\
W_4&=d\theta''
\end{aligned}$$

and functions

$$U_1=-(3u_1^2(\varepsilon)+u_2^2(\varepsilon)),\ U_2=-2u_1(\varepsilon)u_2(\varepsilon),\ U_3=-(u_1^2(\varepsilon)+3u_2^2(\varepsilon)).$$

Clearly, the first-order derivative is now multiplied by a constant instead of a spatially varying potential which will be used in the following calculations. We also note that the functions $W_i\in X$, $i=1,2,3$ depend upon the solution u and the potential V, whereas $W_4\in X$ only depends upon the potential V. For the proof of the resolvent estimate, we use techniques presented in [34], where the authors construct resolvents for the unperturbed LLE (1.1).

We need the following proposition, which is Lemma 4 in [34].

PROPOSITION 23. *Let $d\neq 0$ and $\mu>0$. Then there exists $\lambda^*>0$ depending on d and μ with the property that for all $\omega\geq\lambda^*$ there is at most one $k_0=k_0(\omega,\mu)\in\mathbb{N}$ such that*

$$\omega\geq|d^2k_0^4+\mu^2-\omega^2|.$$

For all other $k\in\mathbb{Z}\setminus\{\pm k_0(\omega,\mu)\}$ we have

$$|d^2k^4+\mu^2-\omega^2|\geq\frac{1}{10}\max\{d^2k^2,\omega\}^{3/2}.$$

Moreover, we find $k_0(\omega,\mu)=\mathcal{O}(\omega^{1/2})$ as $\omega\to\infty$.

Now we can start to construct and bound the resolvent. By the Hille–Yosida Theorem, a uniform resolvent estimate holds whenever $\operatorname{Re}\lambda$ is sufficiently large. It therefore remains for us to consider $\lambda = \delta + \mathrm{i}\omega \in \Lambda_{\lambda^*}$ for some $\lambda^* > 0$ and $\delta \geq 0$ on a compact set. Since δ replaces μ in $\lambda I - \widetilde{L}_{u(\varepsilon),\varepsilon}$ by $\mu + \delta$, and the estimates of Proposition 23 hold for any $\mu > 0$ on a compact set, it suffices for us to prove the uniform estimates for $\delta = 0$. For now, we do not specify the value of λ^*, since this will be done a posteriori. More precisely, we will choose λ^* such that the conditions (5.8), (5.9), (5.10), (5.16), which we derive below, are all satisfied. We can restrict to the case $\omega \geq \lambda^*$, since the proof for $\omega \leq -\lambda^*$ follows from symmetries of the spectral problem under complex conjugation. For $\boldsymbol{v} \in X \times X$ we define

$$(\lambda I - R\widetilde{L}_{u(\varepsilon),\varepsilon}R^{-1})\boldsymbol{v} =: \boldsymbol{\psi} \in Z \times Z \tag{5.6}$$

and show that there exist bounded operators T_1 and T_2 on $Z \times Z$ depending on λ, with norms satisfying $\|T_1\|_{L^2\to L^2} = \mathcal{O}(\omega^{-1/2})$ and $\|T_2\|_{L^2\to L^2} = \mathcal{O}(1)$ as $\omega \to \infty$ such that (5.6) implies

$$(I + T_1)\boldsymbol{v} = T_2\boldsymbol{\psi}. \tag{5.7}$$

If λ^* is sufficiently large such that

$$\forall \omega \geq \lambda^* : \quad \|T_1\|_{L^2\to L^2} \leq \frac{1}{2}, \tag{5.8}$$

we then deduce that $I + T_1$ is a small perturbation of the identity and hence invertible with norm uniformly bounded in λ, which is our claim. Therefore, it remains for us to show (5.7). We introduce the matrix-valued potential

$$W = \begin{pmatrix} W_1 & W_2 + W_4 \\ W_2 - W_4 & W_3 \end{pmatrix}$$

in order to write

$$\lambda I - R\widetilde{L}_{u(\varepsilon),\varepsilon}R^{-1} = \mathrm{i}\omega I - J(-d\partial_x^2 + W) + I(\mu - \varepsilon\hat{V}_0\partial_x).$$

Now, let $A = \lambda I - R\widetilde{L}_{u(\varepsilon),\varepsilon}R^{-1} + JW$, and observe that $A\boldsymbol{v}(x) = \sum_{k\in\mathbb{Z}} A_k\hat{\boldsymbol{v}}_k\mathrm{e}^{\mathrm{i}kx}$ with $\boldsymbol{v}(x) = \sum_{k\in\mathbb{Z}} \hat{\boldsymbol{v}}_k\mathrm{e}^{\mathrm{i}kx}$ and Fourier multiplier

$$A_k = A_k^1 + A_k^2 = \begin{pmatrix} \mathrm{i}\omega + \mu & -dk^2 \\ dk^2 & \mathrm{i}\omega + \mu \end{pmatrix} + \begin{pmatrix} -\mathrm{i}\varepsilon\hat{V}_0 k & 0 \\ 0 & -\mathrm{i}\varepsilon\hat{V}_0 k \end{pmatrix}.$$

The inverse of A_k^1 is given by

$$(A_k^1)^{-1} = \frac{1}{\det(A_k^1)}\begin{pmatrix} \mathrm{i}\omega + \mu & dk^2 \\ -dk^2 & \mathrm{i}\omega + \mu \end{pmatrix},$$

and by Proposition 23 there exists at most one $k_0 = k_0(\omega,\mu) \in \mathbb{N}$ such that

$$\omega \geq |d^2k_0^4 + \mu^2 - \omega^2| \tag{5.9}$$

and

$$|\det(A_k^1)| \geq |d^2k^4 + \mu^2 - \omega^2| \geq \frac{1}{10}\max\{d^2k^2, \omega\}^{3/2} \text{ for all } k \neq \pm k_0 \tag{5.10}$$

provided that λ^* is sufficiently large. Thus A_k^1 is invertible with bound $\|(A_k^1)^{-1}\|_{\mathbb{C}^{2\times 2}} \leq C/\max\{\omega^{1/2}, |k|\}$ for all $k \neq \pm k_0$. Using again Proposition 23, we have the asymptotic $k_0 = k_0(\omega) = \mathcal{O}(\omega^{1/2})$ as $\omega \to \infty$. Consequently, if $|\varepsilon|$ is sufficiently small, then $A_k = A_k^1(I + (A_k^1)^{-1}A_k^2)$, $k \neq \pm k_0$, is also invertible with the bound $\|(A_k)^{-1}\|_{\mathbb{C}^{2\times 2}} = \mathcal{O}(\omega^{-1/2})$ as $\omega \to \infty$. Next, for the above $k_0 \in \mathbb{N}$, we introduce the orthogonal projections $P, Q, Q_1, Q_2 : Z \times Z \to Z \times Z$ as follows:

$$Q_1\boldsymbol{v} = \hat{\boldsymbol{v}}_{k_0}\mathrm{e}^{\mathrm{i}k_0(\cdot)}, \quad Q_2\boldsymbol{v} = \hat{\boldsymbol{v}}_{-k_0}\mathrm{e}^{-\mathrm{i}k_0(\cdot)}$$

and

$$Q = Q_1 + Q_2, \quad P = I - Q.$$

This allows us to decompose (5.6) as follows:

$$PAP\boldsymbol{v} - PJW\boldsymbol{v} = P\boldsymbol{\psi}, \tag{5.11}$$
$$QAQ\boldsymbol{v} - QJW\boldsymbol{v} = Q\boldsymbol{\psi}. \tag{5.12}$$

From the preceding arguments we find

$$\|(PAP)^{-1}\|_{PL^2\to PL^2} = \mathcal{O}(\omega^{-1/2}) \text{ as } \omega \to \infty,$$

which implies that (5.11) is equivalent to

$$P\boldsymbol{v} - (PAP)^{-1}PJW\boldsymbol{v} = (PAP)^{-1}P\boldsymbol{\psi} \tag{5.13}$$

with bound $\|(PAP)^{-1}PJW\|_{L^2\to L^2} = \mathcal{O}(\omega^{-1/2})$ as $\omega \to \infty$.

Next we investigate (5.12), which we decompose a second time to find

$$Q_1AQ_1\boldsymbol{v} - Q_1JWQ_1\boldsymbol{v} - Q_1JWQ_2\boldsymbol{v} - Q_1JWP\boldsymbol{v} = Q_1\boldsymbol{\psi}, \tag{5.14}$$
$$Q_2AQ_2\boldsymbol{v} - Q_2JWQ_1\boldsymbol{v} - Q_2JWQ_2\boldsymbol{v} - Q_2JWP\boldsymbol{v} = Q_2\boldsymbol{\psi}. \tag{5.15}$$

Both equations can be handled similarly, and thus we focus on the first one. Using (5.13), we can write (5.14) as

$$\begin{aligned}&[Q_1AQ_1 - Q_1JWQ_1]\boldsymbol{v} - Q_1JWQ_2\boldsymbol{v} - Q_1JW(PAP)^{-1}PJW\boldsymbol{v}\\ &\quad = Q_1JW(PAP)^{-1}\boldsymbol{\psi} + Q_1\boldsymbol{\psi}.\end{aligned}$$

The operator $B := Q_1AQ_1 - Q_1JWQ_1$ acts like a Fourier multiplier on range Q_1 with matrix

$$B_{k_0} = \begin{pmatrix} \mathrm{i}(\omega - \varepsilon\hat{V}_0k_0) + \mu - (\hat{W}_2)_0 + (\hat{W}_4)_0 & -dk_0^2 - (\hat{W}_3)_0 \\ dk_0^2 + (\hat{W}_1)_0 & \mathrm{i}(\omega - \varepsilon\hat{V}_0k_0) + \mu + (\hat{W}_2)_0 + (\hat{W}_4)_0 \end{pmatrix},$$

and we observe that

$$|\det(B_{k_0})| \geq |\operatorname{Im}\det(B_{k_0})| = 2|\omega - \varepsilon\hat{V}_0k_0||\mu + (\hat{W}_4)_0| \sim \omega$$

since $k_0 = \mathcal{O}(\omega^{1/2})$. This means that we find $\lambda^* \gg 1$ such that

$$\forall \omega \geq \lambda^* : \quad B_{k_0} \text{ is invertible with } \|B_{k_0}^{-1}\|_{\mathbb{C}^{2\times 2}} = \mathcal{O}(1) \text{ as } \omega \to \infty, \tag{5.16}$$

and thus the same holds for the operator B. Inverting B yields

$$Q_1\boldsymbol{v} - B^{-1}[Q_1JWQ_2 + Q_1JW(PAP)^{-1}PJW]\boldsymbol{v} = B^{-1}Q_1JW(PAP)^{-1}P\boldsymbol{\psi} + B^{-1}Q_1\boldsymbol{\psi},$$

and since we have $W_i \in Y$ for $i = 1,2,3,4$ we can exploit decay of the Fourier-coefficients

$$|(\hat{W}_i)_k| \le \frac{C}{\sqrt{1+k^2}} \quad \forall k \in \mathbb{Z}$$

to bound $Q_1 JWQ_2 \boldsymbol{v} = (J\hat{W})_{2k_0} \hat{\boldsymbol{v}}_{-k_0} \mathrm{e}^{\mathrm{i}k_0(\cdot)}$:

$$\|Q_1 JWQ_2\|_{L^2 \to L^2} = \mathcal{O}(k_0(\omega,\mu)^{-1}) = \mathcal{O}(\omega^{-1/2}) \text{ as } \omega \to \infty.$$

Finally from the bounds of the first part, we infer that

$$\|Q_1 JW(PAP)^{-1} PJW\|_{L^2 \to L^2} = \mathcal{O}(\omega^{-1/2}) \text{ as } \omega \to \infty,$$
$$\|Q_1 JW(PAP)^{-1}\|_{L^2 \to L^2} = \mathcal{O}(\omega^{-1/2}) \text{ as } \omega \to \infty,$$

and as a conclusion we arrive at (5.7) which is all we had to prove.

Appendix A. Derivation of the perturbed LLE. The following is a derivation of the perturbed LLE (1.3) from the dual laser pump equation (1.2). We start by taking a solution $u = u(x,t)$ of (1.2). Jumping in a moving coordinate system, we set $\tilde{u}(x,t) = u(k_1 x - \nu_1 t, t)$ and find that $\tilde{u}$ satisfies

$$\mathrm{i}\partial_t \tilde{u} - \mathrm{i}\nu_1 \partial_\xi \tilde{u} = -dk_1^2 \partial_\xi^2 \tilde{u} + (-\mathrm{i}\mu + \zeta)\tilde{u} - |\tilde{u}|^2 \tilde{u} + \mathrm{i} f_0 + \mathrm{i} f_1 \mathrm{e}^{\mathrm{i}\xi}, \tag{A.1}$$

where $\xi := k_1 x - \nu_1 t$. Next, using the approximation $\arctan s \approx s$ for $|s|$ small, we find, for $|f_0| \gg |f_1|$, that

$$f_0 + f_1 \mathrm{e}^{\mathrm{i}\xi} = \sqrt{f_0^2 + 2f_0 f_1 \cos\xi + f_1^2}\, \mathrm{e}^{\mathrm{i} \arctan \frac{f_1 \sin\xi}{f_0 + f_1 \cos\xi}} \approx f_0 \mathrm{e}^{\mathrm{i}\frac{f_1}{f_0}\sin\xi}.$$

Inserting this into (A.1) we find that approximately the following equation holds for $\tilde{u}$:

$$\mathrm{i}\partial_t \tilde{u} - \mathrm{i}\nu_1 \partial_\xi \tilde{u} = -dk_1^2 \partial_\xi^2 \tilde{u} + (-\mathrm{i}\mu + \zeta)\tilde{u} - |\tilde{u}|^2 \tilde{u} + \mathrm{i} f_0 \mathrm{e}^{\mathrm{i}\frac{f_1}{f_0}\sin\xi}. \tag{A.2}$$

This suggests setting $\tilde{u}(\xi,t) = w(\xi,t)\mathrm{e}^{\mathrm{i}\frac{f_1}{f_0}\sin\xi}$ so that w solves

$$\begin{aligned} \mathrm{i}\partial_t w = &-dk_1^2 \partial_\xi^2 w + \left(\mathrm{i}\nu_1 - \mathrm{i}2dk_1^2 \frac{f_1}{f_0}\cos\xi\right)\partial_\xi w \\ &+ \left(-\mathrm{i}\mu + \zeta \underbrace{-\nu_1 \frac{f_1}{f_0}\cos\xi + dk_1^2 \frac{f_1^2}{f_0^2}\cos^2\xi + \mathrm{i}dk_1^2 \frac{f_1}{f_0}\sin\xi}_{=:\alpha(\xi)}\right) w - |w|^2 w + \mathrm{i} f_0. \end{aligned}$$

Using $|f_1| \ll |f_0|$ we see that the term $\alpha(\xi)$ is much smaller than $-\mathrm{i}\mu + \zeta$ for physically relevant (normalized) values of $\mu = \mathcal{O}(1)$ and ζ between $\mathcal{O}(1)$ and $\mathcal{O}(10)$. Neglecting $\alpha(\xi)$, we arrive at

$$\mathrm{i}\partial_t w = -dk_1^2 \partial_\xi^2 w + \mathrm{i}\left(\underbrace{\nu_1 - 2dk_1^2 \frac{f_1}{f_0}\cos\xi}_{=:V(\xi)}\right)\partial_\xi w + (-\mathrm{i}\mu + \zeta)w - |w|^2 w + \mathrm{i} f_0,$$

which is our target equation (1.3) in the case $\varepsilon = 1$ and with d replaced by dk_1^2.

Appendix B. Stability criterion for solitary waves in the limit of small $\boldsymbol{\mu}$. The stability criterion of Theorem 8 becomes more explicit in the limit $\mu \to 0+$ for solitary waves on $\mathbb{R}$ for the focusing case $d>0$. We thus consider the stationary LLE in the form

$$-du'' + (\zeta - \mathrm{i}\mu)u - |u|^2 u + \mathrm{i}\mu f_0 = 0, \qquad x \in \mathbb{R}. \tag{B.1}$$

Here both the pumping term $\mathrm{i}\mu f_0$ and the dissipative term $-\mathrm{i}\mu u$ are small and of equal order in μ. When μ is small, the solution can be expanded asymptotically as

$$u = u^{(0)} + \mu u^{(1)} + \mathcal{O}(\mu^2). \tag{B.2}$$

Here $u^{(0)}$ is the solitary wave of the nonlinear Schrödinger equation which exists if $d>0$ and $u^{(1)}$ is found from the linear inhomogeneous equation

$$(-d\partial_x^2 + \zeta - 2|u^{(0)}|^2)u^{(1)} - (u^{(0)})^2\bar{u}^{(1)} = \mathrm{i}u^{(0)} - \mathrm{i}f_0. \tag{B.3}$$

By using the vector form with $u = u_1 + \mathrm{i}u_2$ and the notation from (2.2), we can rewrite (B.3) in the form $JA_{u^{(0)}}\boldsymbol{u}^{(1)} = \boldsymbol{u}^{(0)} + f_0$. Recall that

$$\ker \widetilde{L}_u = \mathrm{span}\{\boldsymbol{u}'\}, \quad \ker \widetilde{L}_u^* = \mathrm{span}\{J\boldsymbol{\phi}^*\},$$

according to Assumption (A2), which implies that

$$JA_u\boldsymbol{u}' = \mu\boldsymbol{u}', \qquad JA_u\boldsymbol{\phi}^* = -\mu\boldsymbol{\phi}^*.$$

Inserting expansion (B.2) into the operator A_u and using expansions for the eigenfunctions $\boldsymbol{u}'$ and $\boldsymbol{\phi}^*$ in powers of μ up to the order of $\mathcal{O}(\mu)$, one can derive that

$$\begin{aligned}
\boldsymbol{u}' &= (\boldsymbol{u}^{(0)})' + \mu(\boldsymbol{u}^{(1)})' + \mathcal{O}(\mu^2),\\
\boldsymbol{\phi}^* &= C\left[(\boldsymbol{u}^{(0)})' + \mu[(\boldsymbol{u}^{(1)})' + 2\boldsymbol{v}^{(1)}] + \mathcal{O}(\mu^2)\right],
\end{aligned}$$

where $\boldsymbol{v}^{(1)}$ is a solution of the linear inhomogeneous equation $JA_{u^{(0)}}\boldsymbol{v}^{(1)} = -(\boldsymbol{u}^{(0)})'$, and the constant $C = C(\mu) \in \mathbb{C}$ is found from the normalization condition $\langle \boldsymbol{u}', J\boldsymbol{\phi}^*\rangle_{L^2} = 1$. The solution of $JA_{u^{(0)}}\boldsymbol{v}^{(1)} = -(\boldsymbol{u}^{(0)})'$ on the line $\mathbb{R}$ is available explicitly:

$$\boldsymbol{v}^{(1)} = -\frac{1}{2d}xJ\boldsymbol{u}^{(0)},$$

where $\boldsymbol{u}^{(0)}(x) \to 0$ as $|x| \to \infty$ exponentially fast in the case of solitary waves for $d>0$. This allows us to compute the following by using integration by parts:

$$\begin{aligned}
\langle \boldsymbol{u}', J\boldsymbol{\phi}^*\rangle_{L^2} &= C\left[2\mu\int_{\mathbb{R}}[(u_1^{(0)})'v_2^{(1)} - (u_2^{(0)})'v_1^{(1)}]dx + \mathcal{O}(\mu^2)\right]\\
&= C\left[\mu d^{-1}\int_{\mathbb{R}} x[(u_1^{(0)})'u_1^{(0)} + (u_2^{(0)})'u_2^{(0)}]dx + \mathcal{O}(\mu^2)\right]\\
&= C\left[-\frac{\mu}{2d}\|u^{(0)}\|_{L^2}^2 + \mathcal{O}(\mu^2)\right].
\end{aligned}$$

Normalization $\langle \boldsymbol{u}', J\boldsymbol{\phi}^*\rangle_{L^2} = 1$ defines C asymptotically as follows:

$$C = -\frac{2d}{\mu\|u^{(0)}\|_{L^2}^2}\left[1 + \mathcal{O}(\mu)\right].$$

The stability condition of Theorem 8 is expressed in terms of the sign of $V'_{\text{eff}}(\sigma_0)$, where σ_0 is a simple root of V_{eff}. The effective potential can now be written more explicitly as

$$\begin{aligned} V_{\text{eff}}(\sigma_0) &= \langle V(\cdot+\sigma_0)\boldsymbol{u}', J\boldsymbol{\phi}^*\rangle_{L^2} \\ &= C\left[\mu d^{-1}\int_{\mathbb{R}} xV(x+\sigma_0)[(u_1^{(0)})'u_1^{(0)} + (u_2^{(0)})'u_2^{(0)}]dx + \mathcal{O}(\mu^2)\right] \\ &= \frac{1}{\|u^{(0)}\|_{L^2}^2}\int_{\mathbb{R}}[xV'(x+\sigma_0)+V(x+\sigma_0)]|u^{(0)}|^2dx + \mathcal{O}(\mu). \end{aligned}$$

If $V_{\text{eff}}(\sigma_0)=0$, then the solitary wave of the stationary LLE (B.1) with small $\mu\neq 0$ is uniquely continued in the perturbed equation for small ε, and the unique continuation is spectrally stable if $V'_{\text{eff}}(\sigma_0)\cdot\varepsilon > 0$.

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Note Added in Proof. The code used for generating the figures in section 3 is available at https://doi.org/10.5445/IR/1000155820.

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