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Bounds on the tight-binding approximation for the Gross–Pitaevskii equation with a periodic potential

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ABSTRACT

We justify the validity of the discrete nonlinear Schrödinger equation for the tight-binding approximation in the context of the Gross–Pitaevskii equation with a periodic potential. Both piecewise-constant and smooth potentials are considered in the semi-classical limit. While justification of stationary equations is developed in our previous work (Pelinovsky et al. (2008) [11]), this work deals with time-dependent space-decaying solutions on large but finite time intervals.

1. Introduction

We consider the Gross–Pitaevskii (GP) equation with a periodic potential in the form

$$i\phi_t = -\phi_{xx} + V(x)\phi + \sigma|\phi|^2\phi, \tag{1.1}$$

where a complex-valued function $\phi(x,t)$ on $\mathbb{R}\times\mathbb{R}_+$ decays to zero sufficiently fast as $|x|\to\infty$ for a finite $t\in\mathbb{R}_+$, the potential $V(x)$ is given by a bounded real-valued 2π-periodic function, and the parameter $\sigma=\pm1$ is normalized for convenience. As a particular example, we consider a piecewise-constant potential in the form

$$V(x) = \begin{cases} \varepsilon^{-2}, & x\in(0,a) \ \mathrm{mod}(2\pi), \\ 0, & x\in(a,2\pi) \ \mathrm{mod}(2\pi), \end{cases} \tag{1.2}$$

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for a fixed $0 < a < 2\pi$ and a small $\varepsilon > 0$. The asymptotic limit $\varepsilon \to 0$ represents the so-called *tight-binding approximation*, for which the potential $V(x)$ is a periodic sequence of large walls of a nonzero width and the lowest bands in the purely continuous spectrum of the linear operator $L = -\partial_x^2 + V(x)$ are exponentially narrow with respect to small ε.

The potential V in the form (1.2) is considered in our recent work with MacKay [11], where we show that the limit $\varepsilon \to 0$ corresponds to the limit $L \to \infty$ of the L-periodic potential with two nonequal constant legs, after a rescaling transformation. Another example of $V(x)$ is a squared-sine potential in the form

$$V(x) = \frac{4}{\varepsilon^2} \sin^2\left(\frac{x}{2}\right), \tag{1.3}$$

which is prototypical for analysis of Schrödinger operators with periodic potentials in the semi-classical limit (see books [5,7] for a review of relevant analytical results).

The tight-binding approximation is well known in physics literature (see, e.g., [2,15]), where time-dependent solutions of the GP equation (1.1) are shown to be related to the time-dependent solutions of the discrete nonlinear Schrödinger (DNLS) equation

$$i\dot{\phi}_n = \alpha(\phi_{n+1} + \phi_{n-1}) + \sigma\beta|\phi_n|^2\phi_n, \tag{1.4}$$

where α and β are some numerical coefficients and the sequence $\{\phi_n(t)\}_{n\in\mathbb{Z}}$ represents a small-amplitude solution $\phi(x,t)$ evaluated at a sequence of the potential wells. The precise formulation of the correspondence between solutions of (1.1) and (1.4) is given in our main result, Theorem 1.

We proved in the previous work [11] that stationary localized solutions of the GP equation (1.1) with $V(x)$ in (1.2), which take the form $\phi(x,t) = \Phi(x)e^{-i\omega t}$ with $\Phi \in H^1(\mathbb{R})$ and $\omega \notin \sigma(L)$, are related for small values of ε to stationary localized solutions of the DNLS equation (1.4), which take the form $\phi_n(t) = \Phi_n e^{-i\Omega t}$ with $\vec{\boldsymbol{\Phi}} \in l^1(\mathbb{Z})$ and Ω being related to a rescaled parameter ω. Here and henceworth, we use the standard notations for the Sobolev space $H^1(\mathbb{R})$ of scalar complex-valued functions equipped with the norm

$$\|\phi\|_{H^1(\mathbb{R})} := \left(\int_{\mathbb{R}} \left(\left|\phi'(x)\right|^2 + \left|\phi(x)\right|^2\right) dx\right)^{1/2}$$

and the space $l^1(\mathbb{Z})$ of vectors representing complex-valued sequences equipped with the norm

$$\|\vec{\boldsymbol{\phi}}\|_{l^1(\mathbb{Z})} := \sum_{n\in\mathbb{Z}} |\phi_n|.$$

In this work, we extend our analysis to time-dependent space-decaying solutions of these equations and prove that the formal reduction to the DNLS equation from [2,15] can be justified for small values of ε on large but finite time intervals. Although the piecewise-constant potential (1.2) is our main example, we also show that the main result (Theorem 1) can be extended to the smooth potentials in the semi-classical limit such as the squared-sine potential (1.3).

Because $\|V\|_{L^\infty(\mathbb{R})} \to \infty$ as $\varepsilon \to 0$, it is more convenient to work in the space defined by the quadratic form associated with operator $(I + L)$. We denote this space by $\mathcal{H}^1(\mathbb{R})$ and equip it with the norm

$$\|\phi\|_{\mathcal{H}^1(\mathbb{R})} := \left(\int_{\mathbb{R}} \left(\left|\phi'(x)\right|^2 + V(x)\left|\phi(x)\right|^2 + \left|\phi(x)\right|^2\right) dx\right)^{1/2}.$$

Since $V(x) \geqslant 0$ for all $x \in \mathbb{R}$, it is clear that $\|\phi\|_{H^1(\mathbb{R})} \leqslant \|\phi\|_{\mathcal{H}^1(\mathbb{R})}$.

Our analysis is closely related to the recent works on justifications of nonlinear evolution equations for pulses in space-periodic potentials near edges of spectral bands [4] and in narrow band gaps of one-dimensional [14] and two-dimensional [6] potentials. In the context of lattice equations embedded in partial differential equations with space-periodic coefficients, a similar work was developed by Scheel and Van Vleck in [13] for the nonlinear heat equation with a space-periodic diffusive term. Although the justification of lattice equations for the time-dependent solutions of dissipative (reaction–diffusion) equations can be extended globally for $t \geqslant 0$ with the invariant manifold theory, the justification of the DNLS equation can only be carried out for finite time intervals because the GP equation is a conservative (Hamiltonian) system.

Other works have recently appeared on the subject. Reductions to the DNLS equation on a finite lattice for a finite time interval were discussed by Bambusi and Sacchetti in [3] in the context of the GP equation with an N-well trapping potential. Infinite lattices for smooth potentials in the semi-classical limit were studied by Aftalion and Helffer in [1], where the DNLS equation was justified among other models. Analysis of the coupled N-wave equations for a sequence of modulated pulses in the semi-classical limit of the GP equation with a periodic potential was performed by Giannoulis, Mielke, and Sparber in [8].

Methods of our analysis follow closely to arguments from [9] and rely on the Wannier function decomposition from [11]. The Wannier function decomposition is reviewed in Section 2, where we formulate the main theorem. The proof of the main theorem with analysis of the remainder terms is developed in Section 3. Section 4 is devoted to the extension of our results to smooth potentials in the semi-classical limit.

2. Main result

Let $u_l(x;k)$ be a Bloch function of the operator $L = -\partial_x^2 + V(x)$ for the eigenvalue $\omega_l(k)$, such that $l \in \mathbb{N}$, $k \in \mathbb{T} = [-\frac{1}{2}, \frac{1}{2}]$, $u_l(x+2\pi;k) = u_l(x;k)e^{i2\pi k}$ for all $x \in \mathbb{R}$, and the following orthogonality and normalization conditions are met

$$\int_{\mathbb{R}} \bar{u}_{l'}(x,k') u_l(x,k)\,dx = \delta_{l,l'}\delta(k-k'), \quad \forall l, l' \in \mathbb{N},\ \forall k, k' \in \mathbb{T}, \tag{2.1}$$

where $\delta_{l,l'}$ is the Kronecker symbol and $\delta(k)$ is the Dirac delta function in the sense of distributions. Here and henceforth, we denote the set of positive integers by $\mathbb{N} = \{1,2,3,\ldots\}$. To normalize uniquely the phase factors of the Bloch functions [10], we assume that $u_l(x;-k) = \bar{u}_l(x;k)$ is chosen as a Bloch function for $\omega_l(-k) = \bar{\omega}_l(k) = \omega_l(k)$.

Since the band function $\omega_l(k)$ and the Bloch function $u_l(x;k)$ are 1-periodic with respect to k on $\mathbb{R}$ for any $l \in \mathbb{N}$, we represent them by the Fourier series

$$\omega_l(k) = \sum_{n\in\mathbb{Z}} \hat{\omega}_{l,n} e^{i2\pi nk}, \qquad u_l(x;k) = \sum_{n\in\mathbb{Z}} \hat{u}_{l,n}(x) e^{i2\pi nk}, \quad \forall l \in \mathbb{N},\ \forall k \in \mathbb{R}, \tag{2.2}$$

where the coefficients satisfy the constraints

$$\hat{\omega}_{l,n} = \bar{\hat{\omega}}_{l,-n} = \hat{\omega}_{l,-n}, \qquad \hat{u}_{l,n}(x) = \bar{\hat{u}}_{l,n}(x), \quad \forall n \in \mathbb{Z},\ \forall l \in \mathbb{N},\ \forall x \in \mathbb{R}, \tag{2.3}$$

and

$$\hat{u}_{l,n}(x) = \hat{u}_{l,n-1}(x-2\pi) = \hat{u}_{l,0}(x-2\pi n), \quad \forall n \in \mathbb{Z},\ \forall l \in \mathbb{N},\ \forall x \in \mathbb{R}. \tag{2.4}$$

The real-valued functions $\hat{u}_{l,n}(x)$ are referred to as the Wannier functions.

Now we assume that the potential $V(x)$ depends on a small parameter $\mu > 0$ such that $V(x)$ is continuous in $0 < \mu \ll 1$ and

$$\lim_{\mu\to 0} \|V\|_{L^\infty(\mathbb{R})} = \infty.$$

The parameter μ is exponentially small in ε, which is used for the potentials $V(x)$ in (1.2) and (1.3). We pick a particular l-th spectral band for a fixed $l \in \mathbb{N}$ and assume the following properties on the functions $\omega_l(k)$ and $u_l(x;k)$, which are periodically continued in k with the Fourier series (2.2).

Assumption 1. For a fixed $l \in \mathbb{N}$, there exist $\mu_0 > 0$, μ-independent constants $\omega_0, C_1^{\pm}, C_2, C_s, C > 0$, and a nonzero μ-independent function $\hat{u}_0 \in \mathcal{H}^1(\mathbb{R})$ such that, for any $\mu \in [0,\mu_0)$, the following properties are true:

$$\text{(i)}\quad \text{(band boundness)}\qquad |\hat{\omega}_{l,0}| \leqslant \omega_0, \tag{2.5}$$

$$\text{(ii)}\quad \text{(tight-binding)}\qquad C_1^- \mu \leqslant |\hat{\omega}_{l,1}| \leqslant C_1^+ \mu, \qquad |\hat{\omega}_{l,n}| \leqslant C_2 \mu^2, \quad \forall n \geqslant 2, \tag{2.6}$$

$$\text{(iii)}\quad \text{(band separation)}\qquad \inf_{\forall l' \in \mathbb{N}\backslash\{l\}} \inf_{\forall k \in \mathbb{T}} \left|\omega_{l'}(k) - \hat{\omega}_{l,0}\right| \geqslant C_s, \tag{2.7}$$

$$\text{(iv)}\quad \text{(limiting function)}\qquad \lim_{\mu\downarrow 0} \left\|\hat{u}_{l,0}(x) - \hat{u}_0(x)\right\|_{L^\infty(\mathbb{R})} = 0, \tag{2.8}$$

$$\text{(v)}\quad \text{(compact support)}\qquad \sup_{x\in\mathbb{R}\backslash[0,2\pi]} \left|\hat{u}_{l,0}(x)\right| \leqslant C\mu. \tag{2.9}$$

Remark 1. It is proved in Appendices B and C of [11] that properties of Assumption 1 are satisfied by the piecewise-constant potential (1.2) as $\varepsilon \to 0$ with the correspondence $\mu = \varepsilon e^{-a/\varepsilon}$ and the limiting function

$$\hat{u}_0(x) = \frac{\sqrt{2}}{\sqrt{2\pi - a}} \sin \frac{\pi l(2\pi - x)}{2\pi - a} \mathbf{1}_{[a,2\pi]}, \tag{2.10}$$

for a fixed $l \in \mathbb{N}$, where $\mathbf{1}_S$ is a characteristic function of a set $S \subset \mathbb{R}$.

Fig. 1 illustrates the spectrum of $L = -\partial_x^2 + V(x)$ and the Wannier function $\hat{u}_{1,0}(x)$ for the potential V in (1.2) with $a = \pi$ and $\varepsilon = 0.5$. The left panel shows the first spectral bands of L computed from the trace of the monodromy matrix. The right panel shows the Wannier function $\hat{u}_{1,0}(x)$ computed from the integral representation $\hat{u}_{1,0}(x) = \int_{\mathbb{T}} u_1(x;k)\,dk$ and the finite-difference approximation of the Bloch function $u_1(x;k)$ for $x \in [0,2\pi]$ and $k \in \mathbb{T}$ (see details in [11]). The Wannier functions (solid line) approach the limiting function (2.10) with $l = 1$ (dotted line) as ε gets smaller.

We use the set of Wannier functions $\{\hat{u}_{l,n}\}_{n\in\mathbb{Z}}$ for a fixed $l \in \mathbb{N}$ to represent solutions of the GP equation (1.1) in the form

$$\phi(x,t) = \mu^{1/2}\big(\varphi_0(x,T) + \mu\varphi(x,t)\big)e^{-i\hat{\omega}_{l,0}t}, \tag{2.11}$$

where $T = \mu t$ is the slow time and $\varphi_0(x,T) = \sum_{n\in\mathbb{Z}} \phi_n(T)\hat{u}_{l,n}(x)$ belongs to a subspace of $L^2(\mathbb{R})$ associated with the l-th spectral band of operator $L = -\partial_x^2 + V(x)$. Let us require that the sequence $\{\phi_n(T)\}_{n\in\mathbb{Z}}$ satisfy the DNLS equation (1.4) with

$$\alpha = \frac{\hat{\omega}_{l,1}}{\mu}, \qquad \beta = \|\hat{u}_{l,0}\|^4_{L^4(\mathbb{R})}.$$

Note that the constants α and β are bounded and nonzero for small $\mu > 0$ thanks to assumptions (i), (ii), and (iv) and the bound

$$\|\hat{u}_{l,0}\|_{\mathcal{H}^1(\mathbb{R})} \leqslant (1 + \hat{\omega}_{l,0})^{1/2} \leqslant (1 + \omega_0)^{1/2}. \tag{2.12}$$

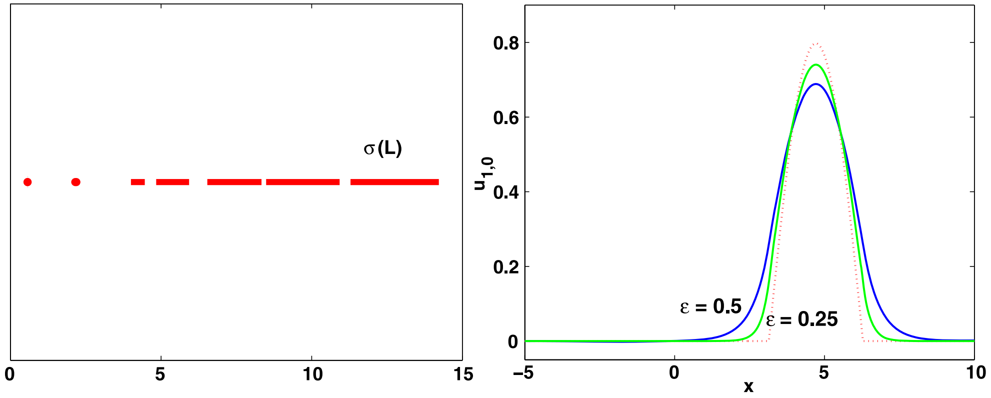

Fig. 1. Left: the band-gap structure of the spectrum $\sigma(L)$ for $\varepsilon = 0.5$. Right: the Wannier functions $\hat{u}_{1,0}(x)$ for $\varepsilon = 0.5$ and $\varepsilon = 0.25$. The dotted line shows the limiting function (2.10).

Since the Wannier functions satisfy the ODE system

$$-\hat{u}''_{l,n}(x) + V(x)\hat{u}_{l,n}(x) = \sum_{n' \in \mathbb{Z}} \hat{\omega}_{l,n-n'}\hat{u}_{l,n'}(x), \quad \forall n \in \mathbb{Z}, \tag{2.13}$$

we obtain an inhomogeneous PDE system for the function $\varphi(x,t)$ in the form

$$i\varphi_t = -\varphi_{xx} + V(x)\varphi - \hat{\omega}_{l,0}\varphi + \frac{1}{\mu}\sum_{n\in\mathbb{Z}}\sum_{m\geqslant 2}\hat{\omega}_{l,m}(\phi_{n+m} + \phi_{n-m})\hat{u}_{l,n}$$
$$+ \sigma\left(|\varphi_0 + \mu\varphi|^2(\varphi_0 + \mu\varphi) - \beta\sum_{n\in\mathbb{Z}}|\phi_n|^2\phi_n\hat{u}_{l,n}\right). \tag{2.14}$$

Our main goal is to establish local well-posedness of the Cauchy problem for the nonlinear evolution equation (2.14) and to control $\|\varphi\|_{\mathcal{H}^1(\mathbb{R})}$ in terms of $\|\vec{\phi}\|_{l^1(\mathbb{Z})}$, where $\vec{\phi}$ is a vector notation for elements of $\{\phi_n\}_{n\in\mathbb{Z}}$ satisfying the DNLS equation (1.4). Global existence of solutions of the DNLS equation (1.4) is also needed to establish our main result, which is the following theorem.

Theorem 1. *Fix $l \in \mathbb{N}$ and use Assumption 1. Let $\vec{\phi}(T) \in C^1(\mathbb{R}, l^1(\mathbb{Z}))$ be a global solution of the DNLS equation (1.4) with initial data $\vec{\phi}(0) = \vec{\phi}_0 \in l^2_p(\mathbb{Z})$ for any $p > \frac{1}{2}$. Let $\phi_0 \in \mathcal{H}^1(\mathbb{R})$ satisfy the bound*

$$\left\| \phi_0 - \mu^{1/2}\sum_{n\in\mathbb{Z}}\phi_n(0)\hat{u}_{l,n}\right\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C_0\mu^{3/2}, \tag{2.15}$$

for some $C_0 > 0$. There exist $\mu_0 > 0$, $T_0 > 0$, and $C > 0$, such that for any $\mu \in (0, \mu_0)$, the GP equation (1.1) with initial data $\phi(0) = \phi_0$ has a solution $\phi(t) \in C([0, T_0/\mu], \mathcal{H}^1(\mathbb{R}))$ satisfying the bound

$$\forall t \in [0, T_0/\mu]: \quad \left\| \phi(\cdot, t) - \mu^{1/2}\left(\sum_{n\in\mathbb{Z}}\phi_n(\mu t)\hat{u}_{l,n}\right)e^{-i\hat{\omega}_{l,0}t}\right\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C\mu^{3/2}. \tag{2.16}$$

Remark 2. Since $\mu = \varepsilon e^{-a/\varepsilon}$ for the potential (1.2), the finite interval $[0, T_0/\mu]$ is exponentially large with respect to parameter ε, similar to the bound obtained in [3].

3. Bounds on the remainder terms

To prove Theorem 1, we shall develop analysis of the nonlinear evolution equation (2.14). Let E_l be the L-invariant closed subspace of $L^2(\mathbb{R})$ associated with the l-th spectral band of $L = -\partial_x^2 + V(x)$. Let Π_l be the orthogonal projection from $L^2(\mathbb{R})$ to $E_l \subset L^2(\mathbb{R})$. We project $|\varphi_0|^2\varphi_0$ onto E_l and its complement in $L^2(\mathbb{R})$, i.e.

$$|\varphi_0|^2\varphi_0 = \Pi_l|\varphi_0|^2\varphi_0 + (I - \Pi_l)|\varphi_0|^2\varphi_0.$$

The following two lemmas allow us to control both terms in the decomposition above.

Lemma 1. *Assume that $E_l \cap E_{l'} = \emptyset$ for a fixed $l \in \mathbb{N}$ and all $l' \neq l$. Then, $\langle \hat{u}_{n,l}, \hat{u}_{n',l}\rangle = \delta_{n,n'}$ for any $n, n' \in \mathbb{Z}$ and there exist constants $\eta_l > 0$ and $C_l > 0$, such that*

$$\left|\hat{u}_{l,n}(x)\right| \leqslant C_l e^{-\eta_l|x-2\pi n|}, \quad \forall n \in \mathbb{Z},\ \forall x \in \mathbb{R}. \tag{3.1}$$

Moreover, if $\vec{\phi} \in l^1(\mathbb{Z})$, $\hat{u}_{l,n} \in \mathcal{H}^1(\mathbb{R})$, and $\phi(x) = \sum_{n\in\mathbb{Z}} \phi_n \hat{u}_{l,n}(x)$ for a fixed $l \in \mathbb{N}$, then $\phi \in \mathcal{H}^1(\mathbb{R})$ and $\langle \phi, \psi\rangle = 0$, $\forall \psi \in \bigcup_{l'\neq l} E_{l'}$.

Proof. The orthogonality and exponential decay of Wannier functions follow from the orthogonality relations (2.1) and complex integration (see [11] for the proof). The assertion that $\langle \phi, \psi\rangle = 0$, $\forall \psi \in \bigcup_{l'\neq l} E_{l'}$ follows from the L^2 spectral theory for the operator $L = -\partial_x^2 + V(x)$ (if $\vec{\phi} \in l^1(\mathbb{Z})$, then $\vec{\phi} \in l^2(\mathbb{Z})$ and $\phi \in E_l \subset L^2(\mathbb{R})$). The assertion that $\phi \in \mathcal{H}^1(\mathbb{R})$ follows from the triangle inequality. □

Remark 3. According to property (iii) of Assumption 1, the l-th spectral band for a fixed $l \in \mathbb{N}$ is disjoint from the rest of the spectrum of L uniformly in $0 < \mu \ll 1$. According to property (i) of Assumption 1 and the bound (2.12), we have $\hat{u}_{l,n} \in \mathcal{H}^1(\mathbb{R})$ for all $n \in \mathbb{N}$ uniformly in $0 < \mu \ll 1$. Therefore, the assumptions of Lemma 1 are satisfied for a sufficiently small $\mu > 0$.

Lemma 2. *There exists a unique solution $\varphi \in \mathcal{H}^1(\mathbb{R})$ of the inhomogeneous equation*

$$\left(-\partial_x^2 + V(x) - \hat{\omega}_{l,0}\right)\varphi = (I - \Pi_l)f, \tag{3.2}$$

for any $f \in L^2(\mathbb{R})$, such that $\langle \varphi, \psi\rangle = 0$, $\forall \psi \in E_l$ and

$$\|\varphi\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C\|f\|_{L^2(\mathbb{R})}, \tag{3.3}$$

for some $C > 0$ uniformly in $0 < \mu \ll 1$.

Proof. By property (iii) of Assumption 1, if $f \in L^2(\mathbb{R})$, then there exists a solution $\varphi \in L^2(\mathbb{R})$ of the inhomogeneous equation (3.2) uniformly in $0 < \mu \ll 1$. By property (i) of Assumption 1, we obtain

$$\left\|\varphi'\right\|^2_{L^2(\mathbb{R})} + \left\|V^{1/2}\varphi\right\|^2_{L^2(\mathbb{R})} \leqslant |\hat{\omega}_{l,0}|\|\varphi\|^2_{L^2(\mathbb{R})} + \left|\langle \varphi, f\rangle\right| \leqslant C\|f\|^2_{L^2(\mathbb{R})}, \tag{3.4}$$

for some $C > 0$ uniformly in $0 < \mu \ll 1$. Therefore, there exists a solution $\varphi \in \mathcal{H}^1(\mathbb{R})$ of the inhomogeneous equation (3.2) satisfying (3.3). Uniqueness of φ such that $\varphi \perp E_l$ follows from the fact that the operator $L - \hat{\omega}_{l,0}$ is invertible in $E_l^\perp = L^2(\mathbb{R})\backslash E_l$. □

We can also use the following elementary result.

Lemma 3. *The space $\mathcal{H}^1(\mathbb{R})$ forms Banach algebra under the pointwise multiplication, such that*

$$\forall u, v \in \mathcal{H}^1(\mathbb{R}): \quad \|uv\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C\|u\|_{\mathcal{H}^1(\mathbb{R})}\|v\|_{\mathcal{H}^1(\mathbb{R})}, \tag{3.5}$$

for some $C > 0$.

Proof. The result follows from the representation $\|u\|^2_{\mathcal{H}^1(\mathbb{R})} = \|u\|^2_{H^1(\mathbb{R})} + \|V^{1/2}u\|^2_{L^2(\mathbb{R})}$ and the Sobolev embedding theorem $\|u\|_{L^\infty(\mathbb{R})} \leqslant C\|u\|_{H^1(\mathbb{R})}$ for some $C > 0$. □

Let us return back to the evolution problem (2.14) and decompose the solution $\varphi(x,t)$ into two parts $\varphi(x,t) = \varphi_1(x,T) + \psi(x,t)$, where φ_1 is a unique solution of the inhomogeneous equation

$$(L - \hat{\omega}_{l,0})\varphi_1 = -\sigma(I - \Pi_l)|\varphi_0|^2\varphi_0, \quad \varphi_1 \in E_l^\perp,$$

thanks to Lemma 2. Using Lemma 1, we can also write explicitly the term

$$\Pi_l|\varphi_0|^2\varphi_0 = \sum_{n\in\mathbb{Z}}\left(\sum_{(n_1,n_2,n_3)\in\mathbb{Z}^3} K_{n,n_1,n_2,n_3}\phi_{n_1}\bar{\phi}_{n_2}\phi_{n_3}\right)\hat{u}_{l,n},$$

where $K_{n,n_1,n_2,n_3} = \langle \hat{u}_{l,n}, \hat{u}_{l,n_1}\hat{u}_{l,n_2}\hat{u}_{l,n_3}\rangle$. Direct substitution shows that ψ satisfies the evolution problem in the abstract form

$$i\psi_t = (L - \hat{\omega}_{l,0})\psi + \mu R(\vec{\boldsymbol{\phi}}) + \mu\sigma N(\vec{\boldsymbol{\phi}}, \psi), \tag{3.6}$$

where

$$\begin{aligned} R(\vec{\boldsymbol{\phi}}) = {} & \frac{1}{\mu^2}\sum_{n\in\mathbb{Z}}\sum_{m\geqslant 2}\hat{\omega}_{l,m}(\phi_{n+m} + \phi_{n-m})\hat{u}_{l,n} \\ & + \frac{\sigma}{\mu}\sum_{n\in\mathbb{Z}}\left(\sum_{(n_1,n_2,n_3)\in\mathbb{Z}^3\setminus\{(n,n,n)\}} K_{n,n_1,n_2,n_3}\phi_{n_1}\bar{\phi}_{n_2}\phi_{n_3}\right)\hat{u}_{l,n} \end{aligned}$$

and

$$\begin{aligned} N(\vec{\boldsymbol{\phi}}, \psi) = {} & -i\sigma\partial_T\varphi_1 + 2|\varphi_0|^2(\varphi_1 + \psi) + \varphi_0^2(\bar{\varphi}_1 + \bar{\psi}) \\ & + \mu\big(2|\varphi_1 + \psi|^2\varphi_0 + (\varphi_1 + \psi)^2\bar{\varphi}_0\big) + \mu^2|\varphi_1 + \psi|^2(\varphi_1 + \psi). \end{aligned}$$

Recall that

$$\varphi_0 = \sum_{n\in\mathbb{Z}}\phi_n\hat{u}_{l,n}, \qquad \varphi_1 = -\sigma(I - \Pi_l)(L - \hat{\omega}_{l,0})^{-1}(I - \Pi_l)|\varphi_0|^2\varphi_0,$$

and $\sigma = \pm 1$. The following lemma gives a bound on the vector field of the nonlinear evolution equation (3.6). Since our results are local in time, it makes sense to consider balls of finite radii in spaces $\mathcal{H}^1(\mathbb{R})$ and $l^1(\mathbb{Z})$, which we denote as $B_\delta(\mathcal{H}^1(\mathbb{R}))$ and $B_{\delta_0}(l^1(\mathbb{Z}))$ respectively.

Lemma 4. *Let $\vec{\phi}$, $\partial_T\vec{\phi} \in B_{\delta_0}(l^1(\mathbb{Z}))$ and $\psi \in B_\delta(\mathcal{H}^1(\mathbb{R}))$ for some $\delta_0, \delta > 0$. There exist $C_R, C_N, K_N > 0$ such that*

$$\|R(\vec{\phi})\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C_R\|\vec{\phi}\|_{l^1(\mathbb{Z})}, \tag{3.7}$$

$$\|N(\vec{\phi},\psi)\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C_N\big(\|\vec{\phi}\|_{l^1(\mathbb{Z})} + \|\psi\|_{\mathcal{H}^1(\mathbb{R})}\big), \tag{3.8}$$

$$\|N(\vec{\phi},\psi) - N(\vec{\phi},\tilde{\psi})\|_{\mathcal{H}^1(\mathbb{R})} \leqslant K_N\|\psi - \tilde{\psi}\|_{\mathcal{H}^1(\mathbb{R})}, \tag{3.9}$$

uniformly in $0 < \mu \ll 1$.

Proof. By the last assertion of Lemma 1 and property (iv) of Assumption 1, if $\vec{\phi} \in l^1(\mathbb{Z})$ and $\varphi_0(x) = \sum_{n\in\mathbb{Z}}\phi_n\hat{u}_{l,n}(x)$ for a fixed $l \in \mathbb{N}$, then $\varphi_0 \in \mathcal{H}^1(\mathbb{R})$ and there exists $C_0 > 0$ uniformly in $0 < \mu \ll 1$ such that

$$\|\varphi_0\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C_0\|\vec{\phi}\|_{l^1(\mathbb{Z})}.$$

By Lemma 3, $\||\varphi_0|^2\varphi_0\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C\|\varphi_0\|^3_{\mathcal{H}^1(\mathbb{R})}$ for some $C > 0$, and, by Lemma 2, there exists $C_1 > 0$ such that

$$\|\varphi_1\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C_1\|\vec{\phi}\|^3_{l^1(\mathbb{Z})}.$$

The vector field $R(\vec{\phi})$ can be represented by $R(\vec{\phi}) = \sum_{n\in\mathbb{Z}} r_n(\vec{\phi})\hat{u}_{l,n}(x)$, where

$$r_n(\vec{\phi}) = \frac{1}{\mu^2}\sum_{m\geqslant 2}\hat{\omega}_{l,m}(\phi_{n+m} + \phi_{n-m}) + \frac{\sigma}{\mu}\sum_{(n_1,n_2,n_3)\in\mathbb{Z}^3\backslash\{(n,n,n)\}} K_{n,n_1,n_2,n_3}\phi_{n_1}\bar{\phi}_{n_2}\phi_{n_3}.$$

Bound (3.7) is proved if $\|\vec{\mathbf{r}}(\vec{\phi})\|_{l^1(\mathbb{Z})} \leqslant C_R\|\vec{\phi}\|_{l^1(\mathbb{Z})}$ for some $C_R > 0$. The first term in $\|\vec{\mathbf{r}}(\vec{\phi})\|_{l^1(\mathbb{Z})}$ is estimated as follows

$$\sum_{n\in\mathbb{Z}}\left|\sum_{n'\in\mathbb{Z}\backslash\{n-1,n,n+1\}}\hat{\omega}_{l,n'-n}\phi_{n'}\right| \leqslant \sum_{n\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}\backslash\{n-1,n,n+1\}}|\hat{\omega}_{l,n'-n}||\phi_{n'}| \leqslant K_1\|\vec{\phi}\|_{l^1(\mathbb{Z})},$$

where

$$K_1 = \sup_{n\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}\backslash\{n-1,n,n+1\}}|\hat{\omega}_{l,n'-n}| = \sum_{n\in\mathbb{Z}\backslash\{-1,0,1\}}|\hat{\omega}_{l,n}|.$$

Since $\omega_l(k)$ is analytically extended along the Riemann surface on $k \in \mathbb{T}$ (by Theorem XIII.95 on p. 301 in [12]), we have $\omega_l \in H^s(\mathbb{T})$ for any $s \geqslant 0$, such that $K_1 < \infty$. By property (ii) of Assumption 1, $\hat{\omega}_{l,n} = \mathcal{O}(\mu^2)$ for all $n \geqslant 2$, so that K_1/μ^2 is uniformly bounded in $0 < \mu \ll 1$.

The second term in $\|\vec{\mathbf{r}}(\vec{\phi})\|_{l^1(\mathbb{Z})}$ is estimated as follows

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\left|\sum_{(n_1,n_2,n_3)\in\mathbb{Z}^3\backslash\{(n,n,n)\}} K_{n,n_1,n_2,n_3}\phi_{n_1}\bar{\phi}_{n_2}\phi_{n_3}\right| &\leqslant \sum_{n\in\mathbb{Z}}\sum_{(n_1,n_2,n_3)\in\mathbb{Z}^3\backslash\{(n,n,n)\}}|K_{n,n_1,n_2,n_3}||\phi_{n_1}||\phi_{n_2}||\phi_{n_3}| \\ &\leqslant K_2\|\vec{\phi}\|^3_{l^1(\mathbb{Z})},\end{aligned}$$

where $K_2 = \sup_{(n_1,n_2,n_3)\in\mathbb{Z}} \sum'_{n\in\mathbb{Z}} |K_{n,n_1,n_2,n_3}|$ and the sum $\sum'$ excludes the term with $n = n_1 = n_2 = n_3$. Using the exponential decay (3.1), we obtain

$$\sum_{n\in\mathbb{Z}} \big|\hat{u}_{l,n}(x)\big| \leqslant C_l \sum_{n\in\mathbb{Z}} e^{-\eta_l|x-2\pi n|} \leqslant A_l,$$

for some $A_l > 0$ uniformly in $x \in \mathbb{R}$. Therefore,

$$\sum_{n\in\mathbb{Z}} |K_{n,n_1,n_2,n_3}| \leqslant A_l \int_{\mathbb{R}} \big|\hat{u}_{l,n_1}(x)\big|\big|\hat{u}_{l,n_2}(x)\big|\big|\hat{u}_{l,n_3}(x)\big|\,dx \leqslant A_l \|\hat{u}_{l,0}\|^3_{\mathcal{H}^1(\mathbb{R})},$$

uniformly in $(n_1, n_2, n_3) \in \mathbb{Z}^3$. By property (v) of Assumption 1, $K_{n,n_1,n_2,n_3} = \mathcal{O}(\mu)$ for all $n_1, n_2, n_3 \in \mathbb{Z}^3 \backslash \{(n,n,n)\}$, so that K_2/μ is uniformly bounded in $0 < \mu \ll 1$. Thus, bound (3.7) is proved.

Bound (3.8) follows from the fact that both $\mathcal{H}^1(\mathbb{R})$ and $l^1(\mathbb{Z})$ form Banach algebras with respect to pointwise multiplication. If $\vec{\boldsymbol{\phi}}, \partial_T\vec{\boldsymbol{\phi}} \in l^1(\mathbb{Z})$, then $\varphi_0, \varphi_1, \partial_T\varphi_1 \in \mathcal{H}^1(\mathbb{R})$ and $N(\vec{\boldsymbol{\phi}}, \psi)$ maps $\psi \in \mathcal{H}^1(\mathbb{R})$ to an element of $\mathcal{H}^1(\mathbb{R})$. Moreover, $N(\vec{\boldsymbol{\phi}}, \psi)$ is uniformly bounded in $0 < \mu \ll 1$. The proof of bound (3.9) follows from the explicit expression for $N(\vec{\boldsymbol{\phi}}, \psi)$. □

We can now prove that the initial-value problem for the nonlinear evolution equation (3.6) is locally well-posed in $\mathcal{H}^1(\mathbb{R})$, whereas that for the DNLS equation (1.4) is globally well-posed in $l^1(\mathbb{Z})$.

Lemma 5. *Let $\vec{\boldsymbol{\phi}}(T) \in C^1(\mathbb{R}, l^1(\mathbb{Z}))$ and $\psi_0 \in \mathcal{H}^1(\mathbb{R})$. Then, there exist a $t_0 > 0$ and a unique solution*

$$\psi(t) \in C\big([0,t_0], \mathcal{H}^1(\mathbb{R})\big) \cap C^1\big([0,t_0], L^2(\mathbb{R})\big)$$

of the time-evolution problem (3.6) *with initial data $\psi(0) = \psi_0$.*

Proof. Since L is a self-adjoint operator in $L^2(\mathbb{R})$ and $\mathcal{H}^1(\mathbb{R})$ is defined by the quadratic form associated with L, the operator $e^{-it(L-\hat{\omega}_{l,0})}$ forms a unitary evolution group satisfying

$$\forall\psi \in \mathcal{H}^1(\mathbb{R}): \quad \big\|e^{-it(L-\hat{\omega}_{l,0})}\psi\big\|_{\mathcal{H}^1(\mathbb{R})} = \|\psi\|_{\mathcal{H}^1(\mathbb{R})},$$

uniformly in $t \in \mathbb{R}$. Using the variation of constant formula, we rewrite the time-evolution problem (3.6) in the integral form

$$\psi(t) = e^{-it(L-\hat{\omega}_{l,0})}\psi_0 + \int_0^t e^{-i(t-s)(L-\hat{\omega}_{l,0})}\big(\mu R\big(\vec{\boldsymbol{\phi}}(\mu s)\big) + \mu\sigma N\big(\vec{\boldsymbol{\phi}}(\mu s), \psi(s)\big)\big)\,ds. \tag{3.10}$$

The vector field of the integral equation (3.10) maps a ball in $\mathcal{H}^1(\mathbb{R})$ to itself for a small $t > 0$, thanks to bounds (3.7)–(3.8). By the Contraction Mapping Principle, there exists a unique fixed point of the integral equation (3.10) in space $\psi(t) \in C([0,t_0], \mathcal{H}^1(\mathbb{R}))$ for sufficiently small $t_0 > 0$ such that

$$\mu K_N t_0 < 1,$$

thanks to bound (3.9). Differentiating the integral equation (3.10) in t and recalling that L provides a uniformly bounded map from $\mathcal{H}^1(\mathbb{R})$ to $L^2(\mathbb{R})$, it follows that the unique fixed point $\psi(t)$ also belongs to $C^1([0,t_0], L^2(\mathbb{R}))$. □

Lemma 6. *Fix $p \geqslant 0$ and let $\vec{\boldsymbol{\phi}}_0 \in l^2_p(\mathbb{Z})$. Then, there exists a unique solution $\vec{\boldsymbol{\phi}}(T) \in C^1(\mathbb{R}, l^2_p(\mathbb{Z}))$ of the DNLS equation* (1.4) *with initial data $\vec{\boldsymbol{\phi}}(0) = \vec{\boldsymbol{\phi}}_0$.*

Proof. To prove local existence, we write the DNLS equation (1.4) as

$$\vec{\boldsymbol{\phi}}(T) = \vec{\boldsymbol{\phi}}_0 - i \int_0^T \left(\alpha \Delta \vec{\boldsymbol{\phi}}(T') + \sigma \beta \boldsymbol{\Gamma}\left(\vec{\boldsymbol{\phi}}(T')\right)\right) dT',$$

where $(\Delta \vec{\boldsymbol{\phi}})_n = \phi_{n+1} + \phi_{n-1}$ and $(\boldsymbol{\Gamma}(\vec{\boldsymbol{\phi}}))_n = |\phi_n|^2 \phi_n$. Using the fact that $l^2_p(\mathbb{Z})$ forms a Banach algebra with respect to pointwise multiplication for any $p \geqslant 0$, we see that the right-hand side of the integral equation defines a Lipschitz continuous map in $C([0, T_0], B_{\delta_0}(l^2_p(\mathbb{Z})))$ for any $T_0 > 0$ and $\delta_0 > 0$. Furthermore, choosing $T_0 > 0$ sufficiently small we can also make the Lipschitz constant strictly smaller than unity. We thus have a contraction in $C([0, T_0], B_{\delta_0}(l^2_p(\mathbb{Z})))$, and a unique fixed point. Since Δ is a bounded operator, the bootstrapping argument shows that $\vec{\boldsymbol{\phi}}(T) \in C^1([0, T_0], B_{\delta_0}(l^2_p(\mathbb{Z})))$.

To extend T_0 to infinity it is sufficient to bound the l^2_p-norm of the solution. Multiplying (1.4) by $\bar{\phi}_n$ and subtracting the complex conjugate equation, we eliminate the nonlinear term in the equation

$$i \frac{d}{dT} |\phi_n|^2 = \alpha (\bar{\phi}_n \Delta \phi_n - \phi_n \Delta \bar{\phi}_n).$$

We write this equation as

$$\left|\phi_n(T)\right|^2 = \left|\phi_n(0)\right|^2 - i\alpha \int_0^T \left[\bar{\phi}_n(T')\left(\phi_{n+1}(T') + \phi_{n-1}(T')\right) - \phi_n(T')\left(\bar{\phi}_{n+1}(T') + \bar{\phi}_{n-1}(T')\right)\right] dT'.$$

By the Cauchy–Schwarz inequality, we obtain

$$\sum_{n \in \mathbb{Z}} \left(1 + n^2\right)^p |\phi_n||\phi_{n+1}| \leqslant C_p \|\vec{\boldsymbol{\phi}}\|^2_{l^2_p},$$

for some $C_p > 0$. Therefore,

$$\left\|\vec{\boldsymbol{\phi}}(T)\right\|^2_{l^2_p} \leqslant \left\|\vec{\boldsymbol{\phi}}(0)\right\|^2_{l^2_p} + 4C_p \int_0^T \left\|\vec{\boldsymbol{\phi}}(T')\right\|^2_{l^2_p} dT'.$$

By Gronwall's inequality, we have

$$\left\|\vec{\boldsymbol{\phi}}(T)\right\|^2_{l^2_p} \leqslant \left\|\vec{\boldsymbol{\phi}}(0)\right\|^2_{l^2_p} e^{4C_p|T|}, \quad \forall T \in \mathbb{R},$$

so that $\boldsymbol{\phi}(T) \in C^1(\mathbb{R}, l^2_p(\mathbb{Z}))$. $\square$

Remark 4. Discrete Sobolev's embedding $l^2_p(\mathbb{Z}) \subset l^1(\mathbb{Z})$ for $p > \frac{1}{2}$ implies that the unique solution of the DNLS equation (1.4) in $C^1(\mathbb{R}, l^2_p(\mathbb{Z}))$ for $p > \frac{1}{2}$ belongs to $\vec{\boldsymbol{\phi}}(T) \in C^1(\mathbb{R}, l^1(\mathbb{Z}))$. Since our approximation result is valid on a finite time interval, we do not have to control $\limsup_{T \to \infty} \|\vec{\boldsymbol{\phi}}(T)\|_{l^1(\mathbb{Z})}$, which may generally diverge.

We can now complete the proof of Theorem 1.

Proof of Theorem 1. Existence of a unique solution $\phi(x,t)$ of the Gross–Pitaevskii equation (1.1) in $C([0,T_0/\mu],\mathcal{H}^1(\mathbb{R}))\cap C^1([0,T_0/\mu],L^2(\mathbb{R}))$ follows from Lemma 5 since $\varphi_0,\varphi_1,\partial_T\varphi_0,\partial_T\varphi_1\in\mathcal{H}^1(\mathbb{R})$ if $\vec{\boldsymbol{\phi}}(T)\in C^1([0,T_0],B_{\delta_0}(l^1(\mathbb{Z})))$ for any $T_0>0$ and $\delta_0>0$. Because of the relation $T=\mu t$ and the constraint in the proof of Lemma 5, we need only local well-posedness of the DNLS equation (1.4) on the time interval $[0,T_0]$ for $T_0=\mu t_0<\frac{1}{K_N}$. In this construction, we need to select $\delta>0$ large enough so that the solution $\psi(t)$ remain in $B_\delta(\mathcal{H}^1(\mathbb{R}))$ on $[0,T_0/\mu]$. Using bounds (3.7)–(3.8) of Lemma 4 and the integral equation (3.10), we obtain

$$\|\psi_\mu(t)\|_{\mathcal{H}^1(\mathbb{R})}\leqslant\|\psi_\mu(0)\|_{\mathcal{H}^1(\mathbb{R})}+\mu\int_0^t\big((C_R+C_N)\|\vec{\boldsymbol{\phi}}(\mu s)\|_{l^1(\mathbb{Z})}+C_N\|\psi_\mu(s)\|_{\mathcal{H}^1(\mathbb{R})}\big)\,ds.$$

By Gronwall's inequality on $[0,T_0/\mu]$, it follows that a local solution $\psi_\mu(t)$ of the integral equation (3.10) satisfies the bound

$$\begin{aligned}\sup_{t\in[0,T_0/\mu]}\|\psi_\mu(t)\|_{\mathcal{H}^1(\mathbb{R})}&\leqslant\Big(\|\psi_\mu(0)\|_{\mathcal{H}^1(\mathbb{R})}+(C_R+C_N)T_0\sup_{T\in[0,T_0]}\|\vec{\boldsymbol{\phi}}(T)\|_{l^1(\mathbb{Z})}\Big)e^{C_NT_0}\\&\leqslant\big(\|\psi_\mu(0)\|_{\mathcal{H}^1(\mathbb{R})}+(C_R+C_N)T_0\delta_0\big)e^{C_NT_0}=:\delta,\end{aligned}$$

where $\|\psi_\mu(0)\|_{\mathcal{H}^1(\mathbb{R})}$ is bounded by the initial bound (2.15). Therefore, for given $\delta_0>0$ and $\delta>0$, one can find a $T_0\in(0,K_N^{-1})$ such that the main bound (2.16) is justified for some constant $C>0$ uniformly in $0<\mu\ll 1$. □

4. Extension for smooth potentials

Our results can be extended to smooth potentials in the semi-classical limit. A prototypical example is the squared-sine potential (1.3) in the limit $\varepsilon\to 0$. Let us review analogues of properties (i)–(v) of Assumption 1 for the potential (1.3). These properties are summarized in [1] based on the previous results [5,7].

Spectral bands of $L=-\partial_x^2+V(x)$ for $\varepsilon>0$ are centered near eigenvalues of $L_0=-\partial_x^2+x^2/\varepsilon^2$, which are located at $\omega_l=\frac{2l-1}{\varepsilon}$ for $l\in\mathbb{N}$. Therefore, properties (i) and (iii) are replaced by

$$|\hat{\omega}_{l,0}|\leqslant\frac{\omega_0}{\varepsilon},\qquad\inf_{\forall l'\in\mathbb{N}\setminus\{l\}}\inf_{\forall k\in\mathbb{T}}|\omega_{l'}(k)-\hat{\omega}_{l,0}|\geqslant\frac{C_s}{\varepsilon},$$

for some $\omega_0,C_s>0$. It also follows from the harmonic approximation of the sine-squared potential $V(x)$ near $x=0$ that if $u_l(x)$ is an L^2-normalized eigenfunction of L_0 for the eigenvalue $\omega_l=\frac{2l-1}{\varepsilon}$, then

$$\|u_l\|^4_{L^4(\mathbb{R})}\leqslant\frac{C_0}{\varepsilon^{1/2}},$$

for some $C_0>0$. This bound replaces property (iv). Properties (ii) and (v) follow from the tunneling effects due to the fact that $V(x)$ consists of an infinite sequence of harmonic potential wells centered at $x=2\pi n$ on $n\in\mathbb{Z}$. The semi-classical analysis shows that property (ii) persists with $\mu=\varepsilon^{-3/2}e^{-S/\varepsilon}$, where

$$S = \int_0^{2\pi} \sqrt{\varepsilon^2 V(x)}\, dx = 8,$$

whereas property (v) is replaced by

$$\left| \int_{\mathbb{R}} \hat{u}_{l,n} \hat{u}_{l,n_1} \hat{u}_{l,n_2} \hat{u}_{l,n_3}\, dx \right| \leqslant \frac{C}{\varepsilon^{1/2}} \mu^{|n_1-n|+|n_2-n|+|n_3-n|+|n_2-n_1|+|n_3-n_1|+|n_3-n_2|},$$

for some $C > 0$ and all $(n, n_1, n_2, n_3) \in \mathbb{Z}^4$. Let us generalize these computations for the particular squared-sine potential (1.3) with a general assumption on smooth 2π-periodic potentials in the semi-classical limit.

Assumption 2. Fix $l \in \mathbb{N}$ and let $\mu = \varepsilon^{-3/2} e^{-S/\varepsilon}$ for some $S > 0$ and $0 < \varepsilon \ll 1$. There exist $\mu_0 > 0$ and μ-independent constants $C_1^{\pm}, C_2, C_s, C > 0$ such that, for any $\mu \in [0, \mu_0)$, the following properties are true:

(i) (tight-binding) $\quad C_1^- \mu \leqslant |\hat{\omega}_{l,1}| \leqslant C_1^+ \mu, \qquad |\hat{\omega}_{l,n}| \leqslant C_2 \mu^2, \quad \forall n \geqslant 2,$

(ii) (band separation) $\quad \inf_{\forall l' \in \mathbb{N}\setminus\{l\}} \inf_{\forall k \in \mathbb{T}} \left| \omega_{l'}(k) - \hat{\omega}_{l,0} \right| \geqslant \frac{C_s}{\varepsilon},$

(iii) (cubic terms)

$$\left| \int_{\mathbb{R}} \hat{u}_{l,n} \hat{u}_{l,n_1} \hat{u}_{l,n_2} \hat{u}_{l,n_3}\, dx \right| \leqslant \frac{C}{\varepsilon^{1/2}} \mu^{|n_1-n|+|n_2-n|+|n_3-n|+|n_2-n_1|+|n_3-n_1|+|n_3-n_2|}.$$

Using a modified expansion,

$$\phi(x,t) = \varepsilon^{1/4} \mu^{1/2} \big(\varphi_0(x,T) + \mu \varphi(x,t) \big) e^{-i\hat{\omega}_{l,0} t},$$

with $T = \mu t$ and $\varphi_0(x,T) = \sum_{n \in \mathbb{Z}} \phi_n(T) \hat{u}_{l,n}(x)$, we now require that the sequence $\{\phi_n(T)\}_{n \in \mathbb{Z}}$ satisfies the DNLS equation (1.4) with

$$\alpha = \frac{\hat{\omega}_{l,1}}{\mu}, \qquad \beta = \varepsilon^{1/2} \|\hat{u}_{l,0}\|^4_{L^4(\mathbb{R})}.$$

By properties (i) and (iii) of Assumption 2, these coefficients are bounded and nonzero in $0 < \varepsilon \ll 1$ (or, equivalently, in $0 < \mu \ll 1$). Now the function $\varphi(x,t)$ satisfies the nonlinear evolution equation

$$\begin{aligned} i\varphi_t = {} & -\varphi_{xx} + V(x)\varphi - \hat{\omega}_{l,0}\varphi + \frac{1}{\mu} \sum_{n \in \mathbb{Z}} \sum_{m \geqslant 2} \hat{\omega}_{l,m} (\phi_{n+m} + \phi_{n-m}) \hat{u}_{l,n} \\ & + \sigma \varepsilon^{1/2} \left(|\varphi_0 + \mu\varphi|^2 (\varphi_0 + \mu\varphi) - \beta \sum_{n \in \mathbb{Z}} |\phi_n|^2 \phi_n \hat{u}_{l,n} \right). \end{aligned} \tag{4.1}$$

All bounds of Lemmas 2 and 4 remain valid due to properties (i), (ii), and (iii) of Assumption 2 and the obvious requirement that $\varepsilon < 1$. As a result, we immediately obtain an extension of Theorem 1, where Assumption 1 is replaced by Assumption 2 and the factor $\mu^{1/2}$ in the left-hand side of bounds (2.15) and (2.16) is replaced by the factor $\varepsilon^{1/4}\mu^{1/2}$.

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