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Interaction of lumps with a line soliton for the DSII equation

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Abstract

Exact solutions of the focussing Davey–Stewartson II equation are presented, which describe the interaction of N-lumps with a line soliton. These solutions are constructed by analysing the inverse spectral problem of the associated Lax pair. The dynamical properties of these solutions are also discussed. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Gravity–capillary surface waves are described by the Davey–Stewartson (DS) system [1] which, in the shallow water limit, is integrable by the inverse spectral method [2,19]. The integrable cases are known as the DSI and DSII equations. The DSI equation possesses dromion solutions [3], while the DSII equation possesses lump solutions [4,5].

In this paper, we study the interaction of lumps with a line soliton for the focussing DSII equation. This equation can be written in the form

$$\mathrm{i}u_t + u_{zz} + u_{\bar{z}\bar{z}} + 4(g+\bar{g})u = 0, \qquad 2g_{\bar{z}} - (|u|^2)_z = 0, \tag{1.1}$$

where $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$, and $u(z, \bar{z}, t)$, $g(z, \bar{z}, t)$ are complex valued functions. Eq. (1.1) is the compatibility condition of the two-dimensional Dirac problem

$$\varphi_{1\bar{z}} = -u\varphi_2, \qquad \varphi_{2z} = \bar{u}\varphi_1, \tag{1.2}$$

coupled with the time-evolution problem

$$\mathrm{i}\varphi_{1t} + \varphi_{1zz} + u\varphi_{2\bar{z}} - u_{\bar{z}}\varphi_2 + 4g\varphi_1 = 0, \qquad -\mathrm{i}\varphi_{2t} + \varphi_{2\bar{z}\bar{z}} + \bar{u}_z\varphi_1 - \bar{u}\varphi_{1z} + 4\bar{g}\varphi_2 = 0. \tag{1.3}$$

In Eqs. (1.1)–(1.3), the bar denotes complex conjugation.

The DSII equation (1.1) was solved formally in [6]. Lump solutions for the focussing DSII equation decaying like $\mathrm{O}(|z|^{-1})$ and $\mathrm{O}(|z|^{-2})$ as $|z| \to \infty$, were formally incorporated in the inverse spectral formalism in [5] and in [7],

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respectively. It was rigorously proven in [8] that the defocusing DSII equation does not have lump solutions. Rigorous results for the focussing DSII equation were obtained in [9]; however, these results are based on a "small-norm assumption" (see Section 2) which excludes lump solutions. It was shown in [10,11] that a single lump is unstable under a small variation of initial data. A similar result for multilump solutions was obtained in [12].

Several classes of exact solutions of the DSII equation have been constructed, using a variety of "direct methods". These include the method of Liouville–Laplace reductions [13], gauge transformations [14], Darboux transformations [15] and the $\bar{\partial}$ dressing method [16]. However, the spectral meaning and therefore the genericity of these solutions remains open.

An interesting exact solution of the focussing DSII equation was obtained in [17] by using the canonical Zakharov–Shabat dressing method. This solution describes the interaction of a lump and a line soliton. The lump–line soliton solution is given by

$$u = \frac{\bar{w}(t)[1 + \bar{\lambda}(\bar{z} - \bar{\zeta}(t))]}{|z - \zeta(t)|^2 + |w(t)|^2}, \tag{1.4}$$

$$g = \frac{2\lambda(\bar{z} - \bar{\zeta})|w(t)|^2 + \lambda^2|z - \zeta(t)|^2|w(t)|^2 - (\bar{z} - \bar{\zeta}(t))^2}{2(|z - \zeta(t)|^2 + |w(t)|^2)^2}, \tag{1.5}$$

where

$$\zeta(t) = 2k_0 t + z_0, \qquad w(t) = c_0\,\mathrm{e}^{-(\lambda + \mathrm{i}k_0)(z - \zeta(t)) - \mathrm{i}\bar{k}_0(\bar{z} - \bar{\zeta}(t)) - \mathrm{i}(\lambda^2 + k_0^2 + \bar{k}_0^2)t}, \tag{1.6}$$

and λ, k_0, z_0, and c_0 are four arbitrary complex parameters. The lump solution follows from this formula when $\lambda = 0$.

If λ is either real or imaginary, the fields $|u|^2(z, \bar{z})$ and $g(z, \bar{z})$ do not evolve in time t. For example, the field $|u|^2(x, y)$ for $\lambda = \mathrm{i}$ is plotted in Fig. 1, where $z = x + \mathrm{i}y$. Clearly, the field consists of a lump solution located at

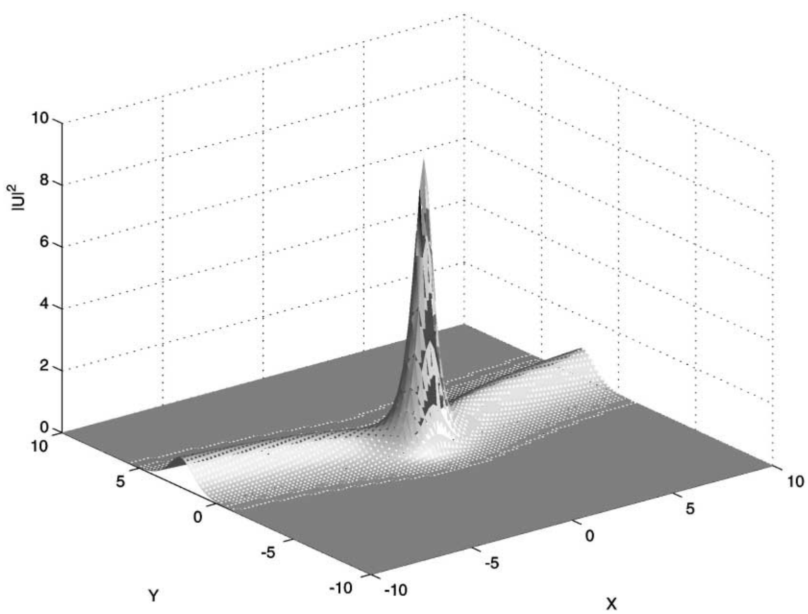

Fig. 1. The exact lump–line soliton (1.4) for $\lambda = \mathrm{i}$.

the background of a line soliton. The nonlocalized direction of the line soliton is given by the curve

$$\{(x, y) \in \mathcal{R}^2 : \theta_r \equiv x\,\mathrm{Re}(\lambda) - y\,\mathrm{Im}(\lambda) + \ln\sqrt{x^2 + y^2} = \text{const.}\}. \tag{1.7}$$

This direction is defined by the complex parameter λ, while the logarithmic curvature of the line soliton is induced by its interaction with the lump. In the asymptotic limit $|z| \to \infty$, Eqs. (1.4) and (1.5) can be reduced to the form

$$u_\infty = \frac{\bar{c}_0 \bar{\lambda}\,\mathrm{e}^{-\bar{\theta}}}{(1 + |c_0|^2\,\mathrm{e}^{-\theta-\bar{\theta}})}, \tag{1.8}$$

$$g_\infty = \frac{\lambda^2 |c_0|^2\,\mathrm{e}^{-\theta-\bar{\theta}}}{2(1 + |c_0|^2\,\mathrm{e}^{-\theta-\bar{\theta}})^2}, \tag{1.9}$$

where $\theta = (\lambda + \mathrm{i}k_0)(z - \zeta) + \mathrm{i}\bar{k}_0(\bar{z} - \bar{\zeta}) + \mathrm{i}(\lambda^2 + k_0^2 + \bar{k}_0^2)t$. The fields $u_\infty(z, \bar{z}, t)$ and $g_\infty(z, \bar{z}, t)$ are exact solutions of the DSII equation (1.1) and corresponds to a line soliton along the direction given by (1.7).

For a general complex value of λ, $|u|^2(z, \bar{z}, t)$ and $g(z, \bar{z}, t)$ evolve in time. In this case, the two-dimensional localized field (the DSII lump) decays, the curvature of the front of the one-dimensional soliton disappears and the line soliton moves with an asymptotically constant propagation speed. Thus, the exact solution (1.4) and (1.5) displays structural sensitivity with respect to variations of the parameter λ. When this parameter slightly deviates from real or imaginary axes, the lump–line soliton is destroyed and the localized field decays. This observation indicates that lump–line solitons are unstable with respect to variations of the initial data. The dynamical instability of these solutions is reviewed in [17].

In this paper, we describe the spectral theory associated to the lump–line solution (1.4) and (1.5). We also generalize the exact solution to describe the interaction of N lumps and a line soliton.

2. Review of the spectral theory for small initial data

The Dirac system (1.2) has the following fundamental matrix solution:

$$\boldsymbol{\varphi} = [\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k})\,\mathrm{e}^{\mathrm{i}kz - \mathrm{i}k^2 t}, \boldsymbol{\chi}(z, \bar{z}, t, k, \bar{k})\,\mathrm{e}^{-\mathrm{i}\bar{k}\bar{z} + \mathrm{i}\bar{k}^2 t}], \tag{2.1}$$

where k is a spectral parameter, and $\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k})$, $\boldsymbol{\chi}(z, \bar{z}, t, k, \bar{k})$ satisfy the system

$$\mu_{1\bar{z}} = -u\mu_2, \qquad \mu_{2z} = -\mathrm{i}k\mu_2 + \bar{u}\mu_1, \tag{2.2}$$

$$\chi_{1\bar{z}} = \mathrm{i}\bar{k}\chi_1 - u\chi_2, \qquad \chi_{2z} = \bar{u}\chi_1. \tag{2.3}$$

It follows from Eqs. (2.2) and (2.3) that $\boldsymbol{\mu}$ and $\boldsymbol{\chi}$ are related by the symmetry constraint

$$\boldsymbol{\chi}(z, \bar{z}, t, k, \bar{k}) = \boldsymbol{\sigma}\bar{\boldsymbol{\mu}}(z, \bar{z}, t, k, \bar{k}), \quad \boldsymbol{\sigma} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

Therefore, it is sufficient to consider only the eigenfunction $\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k})$ which satisfies the Dirac system (2.2), the time-evolution problem

$$\begin{aligned} &\mathrm{i}\mu_{1t} + \mu_{1zz} + 2\mathrm{i}k\mu_{1z} + u\mu_{2\bar{z}} - u_{\bar{z}}\mu_2 + 4g\mu_1 = 0, \\ &-\mathrm{i}\mu_{2t} + \mu_{2\bar{z}\bar{z}} - \mathrm{i}k\mu_{2z} + \bar{u}_z\mu_1 - \bar{u}\mu_{1z} + 4\bar{g}\mu_2 = 0, \end{aligned} \tag{2.5}$$

and the boundary condition

$$\lim_{|z|\to\infty} \boldsymbol{\mu}(z,\bar{z},t,k,\bar{k}) = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.6}$$

In what follows, we review the general methodology of how the Lax pair (2.2) and (2.5) are used to establish the solutions of the initial value problem of the DSII equation for sufficiently small initial data [9]. The starting point is to assume that there exists a solution $u(z,\bar{z},t)$ with sufficient smoothness and decay. Under this assumption, a formula is obtained which expresses $u(z,\bar{z},t)$ in terms of the initial data. One then postulates this formula and proves that it solves the Cauchy problem without the a priori assumption of existence.

Proposition 2.1. *Let $u(z,\bar{z},t)$ satisfy the focussing DSII equation in the complex z-plane for all $t>0$, with initial data*

$$u(z,\bar{z},0) = u_0(z,\bar{z}) \in S(\mathcal{R}^2), \tag{2.7}$$

where $S(\mathcal{R}^2)$ denotes the space of Schwartz functions. Assume that there exists a solution which has sufficient smoothness and decay, and its L^1 and L^∞ norms are sufficiently small. Then this solution can be represented in the form

$$\bar{u}(z,\bar{z},t) = -\frac{1}{2\pi}\iint \mathrm{d}k\wedge\mathrm{d}\bar{k}\, T_0(k,\bar{k})\bar{\mu}_1(z,\bar{z},t,k,\bar{k})\,\mathrm{e}^{-\mathrm{i}(kz+\bar{k}\bar{z})-\mathrm{i}(k^2+\bar{k}^2)t}, \tag{2.8}$$

where the functions $T_0(k,\bar{k})$ and $\mu_1(z,\bar{z},t,k,\bar{k})$ are defined as follows. The function $\mu_1(z,\bar{z},t,k,\bar{k})$ is the first component of the vector solution of the system of the linear integral equations

$$\boldsymbol{\mu}(z,\bar{z},t,k,\bar{k}) = \mathbf{e}_1 + \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}l\wedge\mathrm{d}\bar{l}}{l-k} T_0(l,\bar{l})\boldsymbol{\sigma}\bar{\boldsymbol{\mu}}(z,\bar{z},t,k,\bar{k})\,\mathrm{e}^{-\mathrm{i}(lz+\bar{l}\bar{z})-\mathrm{i}(l^2+\bar{l}^2)t}, \tag{2.9}$$

and $T_0(k,\bar{k})$ is given by

$$T_0(k,\bar{k}) = \frac{1}{2\pi}\iint \mathrm{d}z\wedge\mathrm{d}\bar{z}\,\bar{u}_0(z,\bar{z})\nu_1(z,\bar{z},k,\bar{k})\,\mathrm{e}^{\mathrm{i}(kz+\bar{k}\bar{z})}. \tag{2.10}$$

Here the eigenfunction $\nu_1(z,\bar{z},k,\bar{k})$ is the first component of the vector solution of the system of the linear integral equations

$$\nu_1(z,\bar{z},k,\bar{k}) = 1 - \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}}{\zeta-z}u_0(\zeta,\bar{\zeta})\nu_2(\zeta,\bar{\zeta},k,\bar{k}), \tag{2.11}$$

$$\nu_2(z,\bar{z},k,\bar{k}) = \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}}{\bar{\zeta}-\bar{z}}\bar{u}_0(\zeta,\bar{\zeta})\nu_1(\zeta,\bar{\zeta},k,\bar{k})\,\mathrm{e}^{-\mathrm{i}k(z-\zeta)-\mathrm{i}\bar{k}(\bar{z}-\bar{\zeta})}. \tag{2.12}$$

Proof. Details are given in [9]. The proof involves the following steps. The solution of Eq. (2.2) supplemented with the boundary conditions (2.6) is given by the system of the linear integral equations (2.11) and (2.12) with $u_0(\zeta,\bar{\zeta})$ replaced by $u(\zeta,\bar{\zeta},t)$. Using this equation as well as the symmetry condition (2.4), it can be shown that

$$\frac{\partial\boldsymbol{\mu}}{\partial\bar{k}} = T(k,\bar{k},t)\boldsymbol{\sigma}\bar{\boldsymbol{\mu}}(z,\bar{z},t,k,\bar{k})\,\mathrm{e}^{-\mathrm{i}(kz+\bar{k}\bar{z})}, \tag{2.13}$$

where $T(k,\bar{k},t)$ is the spectral data

$$T(k,\bar{k},t) = \frac{1}{2\pi}\iint \mathrm{d}z\wedge\mathrm{d}\bar{z}\,\bar{u}(z,\bar{z},t)\mu_1(z,\bar{z},t,k,\bar{k})\,\mathrm{e}^{\mathrm{i}(kz+\bar{k}\bar{z})}.$$

The time-evolution problem (2.5) implies that $T(k,\bar{k},t) = T_0(k,\bar{k})\exp(-\mathrm{i}(k^2+\bar{k}^2)t)$, where $T_0(k,\bar{k})$ is given by (2.10).

The above formulas can be used to prove the existence of a solution under the assumption that the L^1 and L^∞ norms of $u_0(z,\bar{z})$ are small. □

Theorem 2.2 (Fokas and Sung [9]). *Let $u_0(z,\bar{z}) \in S(\mathcal{R}^2)$. Assume that the L^1 and L^∞ norms of $u_0(z,\bar{z})$ and of its Fourier transform $\hat{u}_0(k,\bar{k})$ satisfy*

$$\|u_0\|_\infty \|u_0\|_1 < \frac{\pi}{2}\frac{\|\hat{u}_0\|_\infty \|\hat{u}_0\|_1}{(1-\tau)^2} < \frac{\pi}{2}, \tag{2.14}$$

where

$$\tau = \frac{1}{\sqrt{2\pi^3}}\sqrt{\|\hat{u}\|_1\|u\|_1}.$$

Then, the Cauchy problem (2.7) *for the focussing DSII equation has a unique solution. This solution can be obtained through Eqs.* (2.8)–(2.12).

Proof. Details can be found in [9]. Given $u_0(z,\bar{z})$, define $\boldsymbol{\nu}(z,\bar{z},k,\bar{k})$ by Eqs. (2.11) and (2.12). This system has a unique solution provided that $\|u_0\|_\infty\|u_0\|_1 < \pi/2$. Given $u_0(z,\bar{z})$ and $\nu_1(z,\bar{z},k,\bar{k})$, define $T_0(k,\bar{k})$ by Eq. (2.10). The norms of $T_0(k,\bar{k})$ can be bounded in terms of the norms of $u_0(z,\bar{z})$,

$$\|T_0\|_\infty \le \frac{\|\hat{u}_0\|_\infty}{1-\tau}, \qquad \|T_0\|_1 \le \frac{\|\hat{u}_0\|_1}{1-\tau}.$$

Given $T_0(k,\bar{k})$, define $\boldsymbol{\mu}(z,\bar{z},t,k,\bar{k})$ by Eq. (2.9). If $\|T_0\|_\infty\|T_0\|_1 < \pi/2$, this equation also has a unique solution. Then, from Eq. (2.8) define $u(z,\bar{z})$ and prove directly that $u(z,\bar{z},0) = u_0(z,\bar{z})$ and that $u(z,\bar{z},t)$ satisfies Eq. (1.1). □

3. The lump solutions

If the values of $u_0(z,\bar{z})$ are not too small, the system (2.11) and (2.12) can have homogeneous solutions $\mathbf{\Phi}_j(z,\bar{z})$ associated with eigenvalues $k = k_j$. These solutions give rise to lumps. When such solutions exist, the formalism presented earlier must be modified, see [5,7].

Let $\boldsymbol{\mu}(z,\bar{z},t,k,\bar{k})$ satisfy the system (2.11) and (2.12) with $u_0(z,\bar{z})$ replaced by $u(z,\bar{z},t)$. Assume that this equation has a homogeneous solution $\mathbf{\Phi}_j(z,\bar{z},t)$ corresponding to $k = k_j$. This implies that $\boldsymbol{\mu}(z,\bar{z},t,k,\bar{k})$ is singular at $k = k_j$. Assuming that this singularity is a simple pole, it follows that

$$\boldsymbol{\mu}(z,\bar{z},t,k,\bar{k}) = \mathbf{e}_1 + \tilde{\boldsymbol{\mu}}(z,\bar{z},t,k,\bar{k}) + \frac{\mathrm{i}\mathbf{\Phi}_j(z,\bar{z},t)}{k-k_j}, \tag{3.1}$$

where $\tilde{\boldsymbol{\mu}}(z,\bar{z},t,k,\bar{k})$ is bounded at $k = k_j$ and $\mathbf{\Phi}_j(z,\bar{z},t)$ is normalized by the boundary condition

$$\lim_{|z|\to\infty} z\mathbf{\Phi}_j(z,\bar{z},t) = \mathbf{e}_1. \tag{3.2}$$

Using the symmetry relation (2.4), it follows that:

$$\mathbf{\Phi}'_j(z,\bar{z},t) = \boldsymbol{\sigma}\bar{\mathbf{\Phi}}_j(z,\bar{z},t)\,\mathrm{e}^{-\mathrm{i}(k_j z+\bar{k}_j\bar{z})} \tag{3.3}$$

is also a homogeneous solution associated with $k = k_j$. It was shown in [9] that the function

$$\boldsymbol{\mu}_j(z, \bar{z}, t) = \lim_{k \to k_j} \left(\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k}) - \frac{\mathrm{i}\boldsymbol{\Phi}_j(z, \bar{z}, t)}{k - k_j} \right)$$

satisfies an equation, whose homogeneous part is identical to the equation satisfied by $\boldsymbol{\Phi}_j(z, \bar{z}, t)$, and which has a particular solution given by $z\boldsymbol{\Phi}_j(z, \bar{z}, t)$. As a result, the following limiting relation is valid:

$$\mathbf{e}_1 + \tilde{\boldsymbol{\mu}}(z, \bar{z}, t, k, \bar{k}) = (z + \alpha_j(t))\boldsymbol{\Phi}_j(z, \bar{z}, t) + \beta_j(t)\boldsymbol{\sigma}\bar{\boldsymbol{\Phi}}_j(z, \bar{z}, t)\,\mathrm{e}^{-\mathrm{i}(k_j z + \bar{k}_j \bar{z})}, \tag{3.4}$$

where $\alpha_j(t)$ and $\beta_j(t)$ are arbitrary coefficients. If $\tilde{\boldsymbol{\mu}}(z, \bar{z}, t, k, \bar{k}) = 0$, the system of Eq. (3.4) and its complex conjugate reduces to two algebraic equations for $\boldsymbol{\Phi}_j(z, \bar{z}, t)$ and $\bar{\boldsymbol{\Phi}}_j(z, \bar{z}, t)$. Their solution is

$$\boldsymbol{\Phi}_j(z, \bar{z}, t) = \frac{1}{|z + \alpha_j(t)|^2 + |\beta_j(t)|^2} \begin{pmatrix} \bar{z} + \bar{\alpha}_j(t) \\ -\beta_j(t)\,\mathrm{e}^{-\mathrm{i}(k_j z + \bar{k}_j \bar{z})} \end{pmatrix}. \tag{3.5}$$

This equation together with the relation, $\bar{u}(z, \bar{z}, t) = -\Phi_{j2}(z, \bar{z})$, where the subscript denotes the second component of the vector $\boldsymbol{\Phi}(z, \bar{z}, t)$, yields the lump soliton

$$\bar{u}(z, \bar{z}, t) = \frac{\beta_j(t)\,\mathrm{e}^{-\mathrm{i}(k_j z + \bar{k}_j \bar{z})}}{|z + \alpha_j(t)|^2 + |\beta_j(t)|^2}. \tag{3.6}$$

The time-evolution problem (2.5) implies

$$\alpha_j(t) = -2k_j t - z_0, \qquad \beta_j(t) = c_0\,\mathrm{e}^{-\mathrm{i}(k_j \alpha_j(t) + \bar{k}_j \bar{\alpha}_j(t)) - \mathrm{i}(k_j^2 + \bar{k}_j^2)t}, \tag{3.7}$$

where z_0 and c_0 are constants. The lump solution (3.6) coincides with Eq. (1.4) for $\lambda = 0$.

The above analysis can be generalized to the N-lump solution. The eigenfunction $\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k})$, solving Eqs. (2.2), (2.5) and (2.6), has the following meromorphic representation [7]

$$\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k}) = \mathbf{e}_1 + \sum_{j=1}^{N} \frac{\mathrm{i}\boldsymbol{\Phi}_j(z, \bar{z}, t)}{k - k_j}, \tag{3.8}$$

where the bound states $\boldsymbol{\Phi}_j(z, \bar{z}, t)$ are solutions of the algebraic system

$$\mathbf{e}_1 + \sum_{l \neq j} \frac{\mathrm{i}\boldsymbol{\Phi}_l(z, \bar{z}, t)}{k_j - k_l} = (z + \alpha_j(t))\boldsymbol{\Phi}_j(z, \bar{z}, t) + \beta_j(t)\boldsymbol{\sigma}\bar{\boldsymbol{\Phi}}_j(z, \bar{z}, t)\,\mathrm{e}^{-\mathrm{i}(k_j z + \bar{k}_j \bar{z})}. \tag{3.9}$$

Then, the N-lump exact solutions of the DSII equation (1.1) are given by the relation

$$\bar{u}(z, \bar{z}, t) = -\sum_{j=1}^{N} \Phi_{j2}(z, \bar{z}, t). \tag{3.10}$$

The linear system (3.9) and its complex conjugate is solvable for $\Phi_{j1}(z, \bar{z}, t)$ and $\bar{\Phi}_{j2}(z, \bar{z}, t)$ since the determinant is positive definite

$$\det = \begin{bmatrix} M & N \\ -\bar{N} & \bar{M} \end{bmatrix}, \tag{3.11}$$

where the matrix elements M_{ij} and N_{ij} are given by

$$M_{ij} = (z + \alpha_j(t))\delta_{ij} - \frac{\mathrm{i}}{k_i - k_j}(1 - \delta_{ij}), \tag{3.12}$$

$$N_{ij} = -\beta_j(t)\,\mathrm{e}^{-\mathrm{i}(k_j z + \bar{k}_j \bar{z})}\delta_{ij}, \tag{3.13}$$

and δ_{ij} is the Kroneker's symbol.

4. The lump–line soliton solutions

A simple calculation shows that if $\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k})$ satisfies Eq. (2.2), then the vector

$$\boldsymbol{\sigma}\bar{\boldsymbol{\mu}}(z, \bar{z}, t, k, \bar{k})\,\mathrm{e}^{-\mathrm{i}(kz + \bar{k}\bar{z}) - \lambda z} \tag{4.1}$$

satisfies the same equation with k replaced by $k - \mathrm{i}\lambda$. Thus, if the bound state $\boldsymbol{\Phi}_j(z, \bar{z}, t)$ satisfies Eqs. (2.2) and (2.5) for $k = k_j$, the bound state $\boldsymbol{\Phi}'_j(z, \bar{z}, t)$ defined by

$$\boldsymbol{\Phi}'_j(z, \bar{z}, t) = \boldsymbol{\sigma}\bar{\boldsymbol{\Phi}}_j(z, \bar{z}, t)\,\mathrm{e}^{-\mathrm{i}(k_j z + \bar{k}_j \bar{z}) - \lambda z}, \tag{4.2}$$

also solves the system (2.2) and (2.5) for $k = k_j - \mathrm{i}\lambda$. Denote $\boldsymbol{\mu}_j(z, \bar{z}, t)$ as

$$\boldsymbol{\mu}_j(z, \bar{z}, t) = \lim_{k \to k_j}\left(\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k}) - \frac{\mathrm{i}\boldsymbol{\Phi}_j(z, \bar{z}, t)}{k - k_j}\right). \tag{4.3}$$

If the eigenfunction $\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k})$ satisfies Eq. (2.2), then $\boldsymbol{\mu}_j(z, \bar{z}, t)$ satisfies the system

$$\mu_{j1\bar{z}} = -u\mu_{j2}, \qquad \mu_{j2z} + \mathrm{i}k_j\mu_{j2} - \Phi_{j2} = \bar{u}\mu_{j1}.$$

A particular solution of this system is $\boldsymbol{\mu}_j = z\boldsymbol{\Phi}_j(z, \bar{z}, t)$. Thus,

$$\lim_{k \to k_j}\left(\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k}) - \frac{\mathrm{i}\boldsymbol{\Phi}_j(z, \bar{z})}{k - k_j}\right) = (z + \alpha_j(t))\boldsymbol{\Phi}_j(z, \bar{z}, t). \tag{4.4}$$

If the equation for $\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k})$ has homogeneous solutions at $k = k_j$ and $k = k_j - \mathrm{i}\lambda$, the eigenfunction $\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k})$ is singular at these points. As before, assuming that these singularities are simple poles, we postulate the following expansion:

$$\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k}) = \mathbf{e}_1 + \tilde{\boldsymbol{\mu}}(z, \bar{z}, t, k, \bar{k}) + \frac{\mathrm{i}\boldsymbol{\Phi}_j(z, \bar{z}, t)}{k - k_j} - \frac{\mathrm{i}\lambda c_j(t)\boldsymbol{\sigma}\bar{\boldsymbol{\Phi}}_j(z, \bar{z}, t)\,\mathrm{e}^{-\mathrm{i}(k_j z + \bar{k}_j \bar{z}) - \lambda z}}{k - (k_j - \mathrm{i}\lambda)}, \tag{4.5}$$

where $c_j(t)$ is arbitrary coefficient and $\tilde{\boldsymbol{\mu}}(z, \bar{z}, t, k, \bar{k})$ is bounded at $k = k_j$ and $k = k_j - \mathrm{i}\lambda$. This equation together with Eq. (4.4) yields the system

$$\mathbf{e}_1 + \tilde{\boldsymbol{\mu}}(z, \bar{z}, t, k, \bar{k}) - \mathrm{i}c_j(t)\boldsymbol{\sigma}\bar{\boldsymbol{\Phi}}_j(z, \bar{z}, t)\,\mathrm{e}^{-\mathrm{i}(k_j z + \bar{k}_j \bar{z}) - \lambda z} = (z + \alpha_j(t))\boldsymbol{\Phi}_j(z, \bar{z}, t). \tag{4.6}$$

If $\tilde{\boldsymbol{\mu}}(z, \bar{z}, t, k, \bar{k}) = 0$, this equation and its complex conjugate are two algebraic equations for $\boldsymbol{\Phi}_j(z, \bar{z}, t)$ and $\bar{\boldsymbol{\Phi}}_j(z, \bar{z}, t)$. Their solutions yield

$$\boldsymbol{\Phi}_j(z, \bar{z}, t) = \frac{1}{|z - \zeta(t)|^2 + |w(t)|^2}\begin{pmatrix} \bar{z} - \bar{\zeta}(t) \\ -w(t) \end{pmatrix}, \tag{4.7}$$

where $\zeta(t) = -\alpha_j(t)$ and $w(t) = c_j(t)\,\mathrm{e}^{-\mathrm{i}(k_j z+\bar{k}_j\bar{z})-\lambda z}$. The time-dependence of $\zeta(t)$ and $w(t)$ follows from the time-evolution problem (2.5) in the form (1.6). Finally, the exact solution $u(z, \bar{z}, t)$ given by Eq. (1.4) follows from the relation:

$$u = -(\Phi_{j2}(z, \bar{z}, t) - \lambda_j c_j(t)\bar{\Phi}_{j1}(z, \bar{z}, t)\,\mathrm{e}^{-\mathrm{i}(k_j z+\bar{k}_j\bar{z})}). \tag{4.8}$$

The generalization of the lump–line soliton solution to the N-lump–line soliton solutions is straightforward

$$\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k}) = \mathbf{e}_1 + \sum_{j=1}^{N} \frac{\mathrm{i}\boldsymbol{\Phi}_j(z, \bar{z}, t)}{k - k_j} - \sum_{j=1}^{N} \frac{\mathrm{i}\lambda c_j(t)\boldsymbol{\sigma}\bar{\boldsymbol{\Phi}}_j(z, \bar{z}, t)\,\mathrm{e}^{-\mathrm{i}(k_j z+\bar{k}_j\bar{z})-\lambda z}}{k - (k_j - \mathrm{i}\lambda)}. \tag{4.9}$$

Then, the bound states can be found from the algebraic system for $\boldsymbol{\Phi}_j(z, \bar{z}, t)$ and $\bar{\boldsymbol{\Phi}}_j(z, \bar{z}, t)$,

$$\mathbf{e}_1 + \sum_{l\neq j} \frac{\mathrm{i}\bar{\boldsymbol{\Phi}}_l(z, \bar{z}, t)}{k_j - k_l} - \sum_{l=1}^{N} \frac{\mathrm{i}\lambda c_l(t)\boldsymbol{\sigma}\bar{\boldsymbol{\Phi}}_l(z, \bar{z}, t)\,\mathrm{e}^{-\mathrm{i}(k_l z+\bar{k}_l\bar{z})-\lambda z}}{k_j - (k_l - \mathrm{i}\lambda)} = (z + \alpha_j(t))\boldsymbol{\Phi}_j(z, \bar{z}, t). \tag{4.10}$$

The solution of the DSII equation is related to the bound states by

$$\bar{u}(z, \bar{z}, t) = -\sum_{j=1}^{N} [\Phi_{j2}(z, \bar{z}, t) - \lambda c_j(t)\bar{\Phi}_{j1}(z, \bar{z}, t)\,\mathrm{e}^{-\mathrm{i}(k_j z+\bar{k}_j\bar{z})-\lambda z}]. \tag{4.11}$$

The linear system (4.10) and its complex conjugate is solvable for $\Phi_{j1}(z, \bar{z}, t)$ and $\bar{\Phi}_{j2}(z, \bar{z}, t)$ since the determinant is positive definite. The determinant is given by Eq. (3.11), where M_{ij} are the same as in (3.12) and N_{ij} are given by

$$N_{ij} = -\frac{\mathrm{i}\lambda c_j(t)}{k_i - k_j + \mathrm{i}\lambda}\,\mathrm{e}^{-\mathrm{i}(k_j z+\bar{k}_j\bar{z})-\lambda z}.$$

We conclude with the following remarks:

1. The bound state $\boldsymbol{\Phi}_j(z, \bar{z}, t)$ does not satisfy the boundary condition (3.2). Instead, it satisfies the following asymptotic limit along the direction of the line soliton defined in (1.7):

$$\lim_{|z|\to\infty} z\boldsymbol{\Phi}_j(z, \bar{z}, t) = \frac{1}{1 + |c_0|^2\,\mathrm{e}^{-\theta-\bar{\theta}}} \begin{bmatrix} 1 \\ -c_0\,\mathrm{e}^{-\theta} \end{bmatrix}. \tag{4.12}$$

On the other hand, the bound state $\boldsymbol{\Phi}'_j(z, \bar{z}, t)$ given by Eq. (4.2) is delocalized but bounded along the direction of the line soliton (1.7); it satisfies the boundary condition

$$\lim_{|z|\to\infty} \boldsymbol{\Phi}'_j(z, \bar{z}, t) = \frac{1}{1 + |c_0|^2\,\mathrm{e}^{-\theta-\bar{\theta}}} \begin{bmatrix} \bar{c}_0\,\mathrm{e}^{-\theta-\bar{\theta}} \\ \mathrm{e}^{-\theta} \end{bmatrix}. \tag{4.13}$$

The bound states satisfy the relation

$$\mathbf{e}_1 = (z + z_j)\boldsymbol{\Phi}_{\alpha=0}(z, \bar{z}) + c_j\boldsymbol{\Phi}'_{\alpha=1}(z, \bar{z}). \tag{4.14}$$

2. The eigenfunction $\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k})$ does not satisfy the boundary condition (2.6). Instead, it has the following behaviour:

$$\boldsymbol{\mu}(z, \bar{z}, t, k, \bar{k}) \to \mathbf{e}_1 \quad \text{as } \theta_r \to +\infty, \tag{4.15}$$

$$\boldsymbol{\mu}(z,\bar{z},t,k,\bar{k}) \to \frac{k-k_j}{k-k_j+\mathrm{i}\lambda}\mathbf{e}_1 \quad \text{as } \theta_r \to -\infty. \tag{4.16}$$

By virtue of Eq. (4.16), we can introduce an equivalent (double-pole) representation for a solution of the Dirac system (2.2)

$$\boldsymbol{\mu}'(z,\bar{z},t,k,\bar{k}) = \frac{k-k_j+\mathrm{i}\lambda}{k-k_j}\boldsymbol{\mu}(z,\bar{z},t,k,\bar{k}) = \mathbf{e}_1 + \frac{\mathrm{i}[1+\lambda(z+\alpha_j(t))]\boldsymbol{\Phi}_j(z,\bar{z},t)}{k-k_j} - \frac{\lambda\boldsymbol{\Phi}_j(z,\bar{z},t)}{(k-k_j)^2}, \tag{4.17}$$

where we have used the relation (4.14). We notice that similar boundary conditions appear in one-dimensional spectral problems with line solitons [18].

3. The bound state $\boldsymbol{\Phi}_j(z,\bar{z},t)$ can be generalized into a family of bound states along the line segment between $k=k_j$ and $k=k_j-\mathrm{i}\lambda$. Indeed, the following eigenfunction:

$$\boldsymbol{\Phi}_\alpha(z,\bar{z},t) = \boldsymbol{\Phi}_j(z,\bar{z},t)\,\mathrm{e}^{-\alpha\lambda z}, \tag{4.18}$$

solves Eq. (2.2) for $k=k_j-\mathrm{i}\alpha\lambda$, where real $\alpha \in [0,1]$. Inside the interval $0<\alpha<1$, the bound state $\boldsymbol{\Phi}_\alpha(z,\bar{z},t)$ is localized algebraically in the direction of the line soliton given by (1.7).

5. Conclusion

We have characterized the exact solution of the focussing DSII equation which describes the interaction of N lumps and a line soliton. This solution is given by Eq. (4.11), where $\boldsymbol{\Phi}_j(z,\bar{z},t)$ are solutions of the $2N$ algebraic equations given by Eq. (4.10) and its complex conjugate. The derivation of the above solution is based on the analysis of the inverse problem. The direct problem remains open. This involves deriving the analogue of Eqs. (2.11) and (2.12) for initial data $u(z,\bar{z},0) = u_s(z,\bar{z}) + u_0(z,\bar{z})$, where $u_s(z,\bar{z})$ is the line soliton of the DSII equation at $t=0$ (see Eq. (1.8)) and $u_0(z,\bar{z})$ is a perturbation decaying to zero as $|z| \to \infty$.

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