Investigating the Structure of the Stokes Wave of Greatest Height in a Domain of Infinite Depth

Spencer Locke

Supervisor: Dr. Dmitry Pelinovsky

Submitted to the Department of Mathematics and Statistics at McMaster University for MATH 4P06

April 2024

Abstract

Stokes waves are periodic travelling wave solutions of the Euler equations in an inviscid and incompressible fluid with irrotational flow. These waves travel at a given speed and amplitude. It has been shown that there exists an admissible wave speed corresponding to a Stokes wave with greatest height. The Stokes waves are smooth for all amplitudes less than this greatest amplitude, while the wave becomes peaked when it reaches its greatest height. We work in a fluid domain of infinite depth, and analyze the structure of the singularity at the peak. We will begin with an investigation of the smooth waves, deriving the equations of motion and their linearization, and obtaining the Babenko equation for water waves. We continue by performing an analysis of the small amplitude expansions for the surface profile and speed of the smooth waves, as well as of the eigenvalues in the even subspace for the linearized Babenko operator. We then transition to the limiting wave, showing that the Babenko equation reproduces known results about the singularity and its second-order correction for the limiting Stokes wave.

Table of Contents

Acknowledgements

I would like to thank my supervisor, Dr. Dmitry Pelinovsky, for his guidance throughout the entirety of this project. His unwavering dedication to the success of the thesis has been inspiring. Not only did he provide a plethora of helpful resources and opportunities for understanding the material, but he was also always available to offer assistance when I was stuck. I am very thankful for all the instruction he has provided, as well as his patience as I proposed my ideas and questions.

I would also like to thank all my friends and family for their constant support, in addition to all of our many entertaining mathematical discussions.

1 Introduction

Consider an inviscid, incompressible fluid with irrotational flow and a velocity v, given by a velocity potential $\Phi(x, y, t)$ so that $v = \nabla\Phi$. The incompressibility of the fluid is given by the condition that $\nabla \cdot v = 0$ and hence this gives us the following requirement for our velocity potential:

$$\Delta\Phi = 0 \tag{1}$$

which we assume to hold in the 2π-periodic domain $\Omega = \left\{(x, y) \in \mathbb{R}^2 : -\pi \leq x \leq \pi, -\infty < y \leq \eta(x, t)\right\}$, where $\eta(x, t)$ is the surface profile.

We are interested in obtaining solutions to the Euler equations, which are such that Φ satisfies (1) and where the surface profile $\eta(x, t)$ satisfies the kinematic boundary condition

$$\frac{\partial \eta}{\partial t} = \left(-\frac{\partial \eta}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}\right)\bigg|_{y=\eta(x,t)} \tag{2}$$

as well as the dynamic boundary condition

$$\left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2\right)\bigg|_{y=\eta(x,t)} + g\eta = 0, \tag{3}$$

taking $\Phi \to 0$ as $y \to -\infty$ (as seen in [4], [6]).

The travelling wave profiles $\eta(x - ct)$ (with c the wave speed) which satisfy these equations are known as *Stokes Waves,* and correspond to symmetric profiles, each of which rises and falls once per period within the 2π-periodic domain [13]. As described by [4], the Stokes waves solutions η generally correspond to smooth profiles which are even in x with a zero mean condition. These smooth solutions correspond to waves of varying amplitudes, which can be constructed with a small amplitude expansion.

However, it was also shown by [15] that one cannot continue to increment the amplitude of such waves indefinitely and, in fact, there exists a maximal height for the Stokes waves. The Stokes wave solution corresponding to this maximal height is sometimes referred to as the *limiting Stokes wave*, where the height H refers to the distance between the crest and the trough of the wave over its spatial period. The limiting Stokes wave is different from the other waves of varying amplitudes in that it has the greatest nonlinearity [6], which contributes to the formation of a peak singularity of the surface profile over one spatial period. In fact, it was conjectured by Stokes that this limiting wave forms an angle equal to exactly $2\pi/3$ radians at the crest, thus allowing for an explicit computation of the size of the jump discontinuity in the derivative of the surface at that point. It's notable that the singularity appears only in the waves of greatest height – all Stokes waves with heights smaller than this greatest height will be smooth [6].

As we will show in this thesis, a special relation for the Stokes wave moving with the speed c is given by the Babenko equation for travelling waves in an infinite depth:

$$\left(c^2 K - 1\right)\eta = \frac{1}{2}K\eta^2 + \eta K\eta$$

where $K := -H\partial_u$ is called Babenko's operator, with H the Hilbert transform.

1.1 Mathematics Behind the Conformal Transformation

In order to better investigate the Stokes waves, it is convenient to apply a time-dependent conformal transformation $x + iy = z(u + iv)$ with holomorphic $z : \mathbb{C} \to \mathbb{C}$ which reduces the free boundary conditions to flat surface conditions. In particular, we employed the transformation as given by [6], mapping the strip of the complex plane $w = u + iv$ given by

$$\{u + iv \in \mathbb{C} : -\pi \leq u \leq \pi, -\infty < v \leq 0\}$$

into the original strip of interest, represented in the complex plane $z = x + iy$ by

$$\{x + iy \in \mathbb{C} : -\pi \leq x \leq \pi, -\infty < y \leq \eta(x,t)\},$$

where the map is such that the upper line segment $u + 0i$ maps to the free surface $x + i\eta(x,t)$. The details of this conformal transformation can be found in [6], and in particular it is shown that the initial boundary value problem for the Euler equations in a fluid of infinite depth can be recast into these conformal coordinates. For notational consistency, let us denote $x(u,t), \eta(u,t), \xi(u,t)$ as the quantities $x, y, \Phi(x, y = \eta(x,t), t)$ (respectively) after applying a conformal transformation and restricting to the horizontal line $w = u$. In other words, we obtain $x(u,t), \eta(u,t), \xi(u,t)$ as the result of a conformal transformation on $x, \eta(x,t), \psi(x,t) = \Phi(x,y,t)|_{y=\eta(x,t)}$, respectively. Note that we use an abuse of notation for the surface profile η, using the same symbol after the transformation. Using this notation, we can transform the equations, first observing that since the Laplace equation is invariant under a conformal transformation, then our incompressibility condition remains to be $\Delta\Phi(u,v,t) = 0$. Moreover, it is shown [6] that the kinematic boundary condition (2) reduces to

$$\eta_t x_u - x_t \eta_u + H\xi_u = 0 \tag{4}$$

while the dynamic boundary condition (3) reduces to

$$\xi_t \eta_u - \xi_u \eta_t + g\eta\eta_u = -H(\xi_t x_u - \xi_u x_t + g\eta x_u) \tag{5}$$

where the operator H is the Hilbert transform, defined by

$$H(f)(u) = \frac{1}{\pi} pv \int_{-\infty}^{\infty} \frac{f(u')}{u' - u} du' := \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{f(u')}{u' - u} du', \tag{6}$$

and g is the acceleration due to gravity. For a bounded, periodic function f with period 2π (that is continuous almost everywhere), then it can be shown [6] that this Hilbert transform has the Fourier representation

$$H(f)(u) = \sum_{k=-\infty}^{\infty} i\mathrm{sgn}(k) f_k e^{iku} \tag{7}$$

where, for all $k \in \mathbb{Z}$ with $-\infty < k < \infty$ we see that f_k is the k-th Fourier mode of the function f.

Analyticity of $x + iy = z(u + iv)$ implies that

$$x - u - x_0 = -H\eta$$

where x_0 is the zeroth Fourier harmonic of $x - u$, so that we have

$$x_u = 1 - H\eta_u \text{ and } x_t = -H\eta_t. \tag{8}$$

Substituting (8) into the kinematic boundary condition (4) and the dynamic boundary condition (5) yields the following closed equations of motion:

$$\eta_t(1 - H\eta_u) + \eta_u H\eta_t + H\xi_u = 0 \tag{9}$$

$$\xi_t \eta_u - \xi_u \eta_t + g\eta\eta_u = -H(\xi_t(1 - H\eta_u) + \xi_u H\eta_t + g\eta(1 - H\eta_u)) \tag{10}$$

1.2 Main Results of the Thesis

In this thesis, we focus on investigating how the Babenko equation for travelling waves in a fluid domain with infinite depth may be used as a tool for understanding and interpreting results relating to the Stokes waves solutions of the Euler equations. We begin with an investigation of the governing equations, deriving the equations of motion and their linearization, while demonstrating their relation to the Babenko equation for water waves. We will continue by analyzing the smooth waves and performing an analysis of the small amplitude expansions for the surface profile and speed of the smooth waves, as well as for the eigenvalues in the even subspace for the linearized Babenko operator. We demonstrate in both cases that the coefficients obtained match what is obtained in the literature. We will then transition to the limiting wave, analyzing the structure of the peaked singularity. In particular, we will show that the Babenko equation reproduces known results about the powers on the first two terms in a Stokes wave expansion for the limiting wave.

2 Deriving the Equations of Motion

2.1 The Travelling Wave Solutions

Given that the Stokes waves correspond to the travelling wave solutions of the equations of motion, we can obtain these solutions after having applied the conformal transformation. In order to do so, we observe that we are seeking solutions of the form:

$$\xi\left(u,t\right)=\xi\left(u-ct\right)\ \text{and}\ \eta\left(u,t\right)=\eta\left(u-ct\right)$$

where c is the velocity of the Stokes wave. Thus, to obtain the travelling waves, we simply compute the derivatives of these terms using the chain rule, and we can insert them back into our equations of motion. Inserting the travelling wave form into (9) we see that

$$\left(\eta_t-c\eta_u\right)\left(1-H\eta_u\right)+\eta_u H\left(\eta_t-c\eta_u\right)+H\xi_u=0$$

Simplifying this equation we obtain

$$\eta_t-\eta_t H\eta_u-c\eta_u+c\eta_u H\eta_u+\eta_u H\eta_t-c\eta_u H\eta_u+H\xi_u=0$$

and hence

$$\eta_t-\eta_t H\eta_u-c\eta_u+\eta_u H\eta_t+H\xi_u=0 \tag{11}$$

which gives us the travelling wave formulation of our kinematic boundary condition. Moreover, we may perform a similar analysis upon the dynamic boundary condition. Inserting the travelling wave form into (10) we see that

$$\left(\xi_t-c\xi_u\right)\eta_u-\xi_u\left(\eta_t-c\eta_u\right)+g\eta\eta_u=-H\left(\left(\xi_t-c\xi_u\right)\left(1-H\eta_u\right)+\xi_u H\left(\eta_t-c\eta_u\right)+g\eta\left(1-H\eta_u\right)\right)$$

Performing simple algebraic expansions on both sides gives that

$$\xi_t\eta_u-c\xi_u\eta_u-\xi_u\eta_t+c\xi_u\eta_u+g\eta\eta_u=-H\left(\xi_t-\xi_t H\eta_u-c\xi_u+c\xi_u H\eta_u+\xi_u H\eta_t-c\xi_u H\eta_u+g\eta-g\eta H\eta_u\right)$$

and so using linearity of the Hilbert transform, we obtain that:

$$\begin{aligned}\xi_t\eta_u-c\xi_u\eta_u-\xi_u\eta_t+c\xi_u\eta_u+g\eta\eta_u= &-H\xi_t+H\left(\xi_t H\eta_u\right)+cH\xi_u-cH\left(\xi_u H\eta_u\right)-H\left(\xi_u H\eta_t\right)\\ &+cH\left(\xi_u H\eta_u\right)-gH\eta+gH\left(\eta H\eta_u\right)\end{aligned}$$

and finally cancelling out the common terms, we obtain that

$$\xi_t\eta_u-\xi_u\eta_t+g\eta\eta_u+H\xi_t-H\left(\xi_t H\eta_u\right)-cH\xi_u+H\left(\xi_u H\eta_t\right)+gH\eta-gH\left(\eta H\eta_u\right)=0 \tag{12}$$

Therefore, the equations of motions for the travelling wave solutions are given by:

$$\begin{aligned}&\eta_t-\eta_t H\eta_u-c\eta_u+\eta_u H\eta_t+H\xi_u=0\\ \text{and}\ &\xi_t\eta_u-\xi_u\eta_t+g\eta\eta_u+H\xi_t-H\left(\xi_t H\eta_u\right)-cH\xi_u+H\left(\xi_u H\eta_t\right)+gH\eta-gH\left(\eta H\eta_u\right)=0\end{aligned}$$

and so the Stokes waves correspond to a solution of this system of equations.

2.2 The Babenko Equation

Note that an important case of these travelling wave solutions corresponds to when we switch into the frame of reference of the travelling wave. To do so, we may apply the transformation $u-ct\rightarrow u$ so that our travelling wave solutions become time-independent. In this case, the kinematic condition (11) reduces down to:

$$H\xi_u=c\eta_u \tag{13}$$

and further the dynamic condition (12) will become:

$$g\eta\eta_u - cH\xi_u + gH\eta - gH\left(\eta H\eta_u\right) = 0 \tag{14}$$

Of course, we can combine both equations (13) and (14) to obtain the following closed form expression describing the Stokes waves:

$$g\eta\eta_u - c^2\eta_u + gH\eta - gH\left(\eta H\eta_u\right) = 0.$$

Without loss of generality, if we take $g = 1$ then we may rewrite this equation as

$$\eta\eta_u - c^2\eta_u + H\eta - H\left(\eta H\eta_u\right) = 0. \tag{15}$$

Now, recall that $H^2 = -Id$ in the space of zero-mean solutions. To ensure zero mean under the Hilbert transform, note that the zero-mean constraint after applying a conformal transformation becomes

$$\oint \eta\left(1 - H\partial_u\eta\right) du = 0. \tag{16}$$

This is because, when applying H to both sides of (15), and using $H^2 = -Id$ and $H0 = 0$, then we obtain

$$H\left(\eta\eta_u\right) - c^2 H\eta_u - \eta + \eta H\eta_u = 0. \tag{17}$$

However, the mean-value of this equation is $\oint \eta\left(1 - H\partial_u\eta\right) du$ and so we see that the constraint (16) ensures we have zero-mean under the Hilbert transform, which allows for (17) to be justified. Note that we can rewrite equation (17) as

$$\frac{1}{2}H\left(\frac{\partial}{\partial u}\eta^2\right) - c^2 H\left(\frac{\partial}{\partial u}\eta\right) - \eta + \eta H\left(\frac{\partial}{\partial u}\eta\right) = 0.$$

It is this equation that motivates the introduction of a new self-adjoint operator,

$$K = -H\frac{\partial}{\partial u},$$

which is referred to as the Babenko operator. Reducing our equation with this operator, we obtain

$$-\frac{1}{2}K\eta^2 + c^2K\eta - \eta - \eta K\eta = 0, \tag{18}$$

which is known as *Babenko's equation for Travelling Waves*, as this equation relating to the Euler equations after conformal transformation was originally derived by Babenko in his work [11]. This closed form Babenko equation gives rise to the steady-state travelling wave solutions and will soon be shown to be a key equation in our investigation into the structure of the peaked travelling wave solutions.

2.3 Linearization of The Equations of Motion and the Linearized Equations of Stability

In investigating the structure of the Stokes wave solutions, a 'brute force' linearization of the equations of motion may be performed, and the resulting linearized quantities may be used to further understand the smooth waves. To do this, we split the solutions into the sum of a steady state and time-dependent perturbation:

$$\eta\left(u,t\right) = \eta^0\left(u\right) + \eta^1\left(u,t\right)$$
$$\xi\left(u,t\right) = \xi^0\left(u\right) + \xi^1\left(u,t\right)$$

Of course, given that η^0 and ξ^0 are the steady-state solutions, then, as we showed, they must satisfy (13) so that

$$H\xi_u^0 = c\eta_u^0 \tag{19}$$

and further they must also satisfy (14)

$$g\eta^0\eta_u^0 - cH\xi_u^0 + gH\eta^0 - gH\left(\eta^0 H\eta_u^0\right) = 0. \tag{20}$$

Thus, linearizing the first (kinematic) equation of motion about the perturbative term, we may insert our perturbative equations into (11) to obtain that

$$\eta_t^1 - \eta_t^1 H\left(\eta_u^0 + \eta_u^1\right) - c\left(\eta_u^0 + \eta_u^1\right) + \left(\eta_u^0 + \eta_u^1\right) H\eta_t^1 + H\left(\xi_u^0 + \xi_u^1\right) = 0,$$

which reduces down to

$$\eta_t^1 - \eta_t^1 H\eta_u^0 - \eta_t^1 H\eta_u^1 - c\eta_u^0 - c\eta_u^1 + \eta_u^0 H\eta_t^1 + \eta_u^1 H\eta_t^1 + H\xi_u^0 + H\xi_u^1 = 0.$$

Dropping the terms that are nonlinear in the time–dependent term, we see that:

$$\eta_t^1 - \eta_t^1 H\eta_u^0 - c\eta_u^0 - c\eta_u^1 + \eta_u^0 H\eta_t^1 + H\xi_u^0 + H\xi_u^1 = 0$$

and thus inserting (19) into this equation then we have

$$\eta_t^1 - \eta_t^1 H\eta_u^0 - c\eta_u^0 - c\eta_u^1 + \eta_u^0 H\eta_t^1 + c\eta_u^0 + H\xi_u^1 = 0$$

which can be rewritten as

$$\eta_t^1\left(1 - H\eta_u^0\right) - c\eta_u^1 + \eta_u^0 H\eta_t^1 + H\xi_u^1 = 0$$

which is the linearized (travelling wave) kinematic boundary condition. Note that it is more insightful to write this equation in terms of the Babenko operator $K = -\partial_u H$, and so we see the linearized kinematic boundary condition reads equivalently as

$$\eta_t^1\left(1 + K\eta^0\right) - c\eta_u^1 + \eta_u^0 H\eta_t^1 - K\xi^1 = 0$$

For the second (dynamic) equation of motion, we again linearize about the perturbative term. Inserting the decomposition into (12), we obtain that

$$\begin{aligned}\xi_t^1\left(\eta_u^0 + \eta_u^1\right) - \left(\xi_u^0 + \xi_u^1\right)\eta_t^1 + \left(\eta^0 + \eta^1\right)\left(\eta_u^0 + \eta_u^1\right) + H\xi_t^1 - H\left(\xi_t^1 H\left(\eta_u^0 + \eta_u^1\right)\right)\\ - cH\left(\xi_u^0 + \xi_u^1\right) + H\left(\left(\xi_u^0 + \xi_u^1\right)H\eta_t^1\right) + H\left(\eta^0 + \eta^1\right) - H\left(\left(\eta^0 + \eta^1\right)H\left(\eta_u^0 + \eta_u^1\right)\right) \quad &= 0\end{aligned}$$

Expanding this equation out and dropping the terms that are nonlinear in the time-dependent terms, as well as utilizing (14), then the linearized dynamic boundary condition will become:

$$\begin{aligned}\xi_t^1\eta_u^0 - \xi_u^0\eta_t^1 + \left(\eta^0\eta_u^1 + \eta_u^0\eta^1\right) + H\left(\xi_t^1\left(1 - H\eta_u^0\right)\right)\\ - cH\xi_u^1 + H\left(\xi_u^0 H\eta_t^1\right) + H\left(\eta^1\left(1 - H\eta_u^0\right)\right) - H\left(\eta^0 H\eta_u^1\right) \quad &= 0\end{aligned}$$

which can be written in terms of the Babenko operator as:

$$\begin{aligned}\xi_t^1\eta_u^0 - \xi_u^0\eta_t^1 + \left(\eta^0\eta_u^1 + \eta_u^0\eta^1\right) + H\left(\xi_t^1\left(1 + K\eta^0\right)\right)\\ + cK\xi^1 + H\left(\xi_u^0 H\eta_t^1\right) + H\left(\eta^1\left(1 + K\eta^0\right)\right) + H\left(\eta^0 K\eta^1\right) \quad &= 0\end{aligned}$$

When linearizing the Babenko equation for travelling wave solutions in an infinite depth, we define the linearized Babenko operator, given by:

$$\mathcal{L} \coloneqq c^2K - 1 - K\left(\eta\cdot\right) - \left(K\eta\right)\cdot - \eta\left(K\cdot\right) \tag{21}$$

Now, taking the substitution

$$\phi^1 \coloneqq \xi^1 + cH\eta^1$$

so that $\phi_u^1 = \xi_u^1 - cK\eta^1$ and $\phi_t^1 = \xi_t^1 + cH\eta_t^1$, then it can be shown that one can rewrite the linearized equations of motion above in the form as given by [13]:

$$\begin{aligned}K\phi^1 &= \eta_t^1\left(1 + K\eta^0\right) + \eta_u^0 H\eta_t^1,\\ L\eta^1 &= \phi_t^1\left(1 + K\eta^0\right) - H\left(\phi_t^1\eta_u^0\right) - 2cH\eta_t^1,\end{aligned}$$

and so in matrix form we can write this as:

$$\begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} \phi^1 \\ \eta^1 \end{bmatrix} = \begin{bmatrix} 0 & 1 + \left(K\eta^0\right) \cdot + \eta_u^0 H \cdot \\ 1 + \left(K\eta^0\right) \cdot - H\left(\eta_u^0 \cdot\right) & -2cH \cdot \end{bmatrix} \begin{bmatrix} \phi_t^1 \\ \eta_t^1 \end{bmatrix}.$$

Note that we may take the operator $M := 1 + \left(K\eta^0\right) \cdot + \eta_u^0 H \cdot$ whose adjoint is given by $M^* = 1 + \left(K\eta^0\right) \cdot - H\left(\eta_u^0 \cdot\right)$ and hence we may rewrite the equations of motion as

$$\begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} \phi^1 \\ \eta^1 \end{bmatrix} = \begin{bmatrix} 0 & M \\ M^* & -2cH \cdot \end{bmatrix} \begin{bmatrix} \phi_t^1 \\ \eta_t^1 \end{bmatrix}.$$

Since we're interested in the eigenvalues, we are looking at solutions where $\phi_t^1 = \lambda\phi^1$ and $\eta_t^1 = \lambda\eta^1$ and so the system of equations becomes

$$\begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} \phi^1 \\ \eta^1 \end{bmatrix} = \lambda \begin{bmatrix} 0 & M \\ M^* & -2cH \cdot \end{bmatrix} \begin{bmatrix} \phi^1 \\ \eta^1 \end{bmatrix},$$

which gives the *linearized equations of stability:*

$$K\phi^1 = \lambda M\eta^1 \text{ and } L\eta^1 = \lambda M^*\phi^1 - 2c\lambda H\eta^1. \tag{22}$$

Note that, as seen in [13], then these equations of stability can reduce to a single quadratic eigenvalue problem. This follows the first linearized equation of stability shows that $\phi^1 = K^{-1}\lambda M\eta^1 = \lambda K^{-1}M\eta^1$, where the invertibility of $K = -\partial_u H$ is because H is invertible. Thus, the second linearized equation of stability gives the desired closed form quadratic representation:

$$L\eta^1 - \lambda^2 M^* K^{-1} M\eta^1 + 2c\lambda H\eta^1 = 0.$$

Obtaining the spectrum from these linearized operators will determine the stability of the Stokes waves.

3 Investigating the Smooth Stokes Waves

3.1 Small Amplitude Expansions of Smooth Stokes Waves

Here we explore smooth Stokes waves in the limit of small amplitudes. The small amplitude expansions converge fast enough to determine the leading order behaviour of the solutions [6]. Therefore, to understand such expansions for the small amplitude waves, we will use their smoothness to add the extra condition of infinite differentiability to the list of properties of Stokes waves that we had detailed in the introduction. We can then expand our solutions in a power series in terms of a small perturbation parameter ε, which represents the amplitude. In particular, we may expand both the surface profile $\eta(u)$ and wave speed c in the following small-amplitude expansion, which is in terms of the small perturbation ε:

$$\eta(u, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \eta_i(u) \text{ and } c(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i c_i = \sum_{i=0}^{\infty} \varepsilon^{2i} c_{2i}, \tag{23}$$

where η_i and c_i are the values of η and c up to order $\mathcal{O}\left(\varepsilon^i\right)$. Note that c is expanded in even powers, for we already know that c must be even in ε [4]. Moreover, the surface η must be even in u given its symmetric profile.

To begin, we are interested in obtaining the coefficients in the expansions (23). To do this, we will use the Babenko equation for travelling waves. Writing out the Babenko equation (18) more explicitly in terms of the Hilbert transform, then we see that

$$-c^2 H\eta_u - \eta + H\left(\eta\eta_u\right) + \eta H\eta_u = 0 \tag{24}$$

and so we will simply expand equation (24) to $\mathcal{O}\left(\varepsilon^i\right)$ in order to determine the values to a given order i. Note that we seek single-lobe solutions, which correspond to waves which have one crest and one trough on its periodic domain, and we saw that the Stokes waves have this property [13].

Performing the routine computations allows for us to deduce the coefficients explicitly:

Theorem 3.1. The smooth, even, single-lobe Stokes waves η and the speed c are expanded as a power series as in (23), and the coefficients to third order are given by:

- To $\mathcal{O}(1)$ we have $\eta_0 = 0$.
- To $\mathcal{O}(\varepsilon)$ we have that $\eta_1 = \cos(u)$ and $c_0 = 1$.
- To $\mathcal{O}(\varepsilon^2)$ we have that $\eta_2 = \cos(2u) - \frac{1}{2}$.
- To $\mathcal{O}(\varepsilon^3)$ we have that $\eta_3 = \frac{3}{2}\cos(3u)$ and and $c_2 = \frac{1}{2}$.

Proof. We know that the surface profile η is expanded as $\eta(u,\varepsilon) = \eta_0(u) + \eta_1(u)\varepsilon + \eta_2(u)\varepsilon^2 + \eta_3(u)\varepsilon^3 + \mathcal{O}(\varepsilon^4)$ and $c(\varepsilon) = c_0 + c_2\varepsilon^2 + \mathcal{O}(\varepsilon^4)$. We wish to determine the coefficients in each of these expansions. Now, in order to determine such terms, we can simply insert our expansion into the Babenko equation, matching the terms of equal order. Note that, first, for the $\mathcal{O}(1)$ case then since the mean-value of the Stokes wave is taken as zero, it follows that η_0 must be constant and set to zero. Thus, $\eta_0 = 0$.

If η' denotes η_u, then we obtain:

$$\begin{aligned}
\eta_u &= \eta_1'\varepsilon + \eta_2'\varepsilon^2 + \eta_3'\varepsilon^3 + \mathcal{O}(\varepsilon^4)\\
H\eta_u &= H\eta_1'\varepsilon + H\eta_2'\varepsilon^2 + H\eta_3'\varepsilon^3 + \mathcal{O}(\varepsilon^4)\\
\eta H\eta_u &= \eta_1 H\eta_1'\varepsilon^2 + (\eta_1 H\eta_2' + \eta_2 H\eta_1')\varepsilon^3 + \mathcal{O}(\varepsilon^4)\\
\eta\eta_u &= \eta_1\eta_1'\varepsilon^2 + (\eta_1\eta_2' + \eta_2\eta_1')\varepsilon^3 + \mathcal{O}(\varepsilon^4)\\
H\eta\eta_u &= H\eta_1\eta_1'\varepsilon^2 + (H\eta_1\eta_2' + H\eta_2\eta_1')\varepsilon^3 + \mathcal{O}(\varepsilon^4)
\end{aligned}$$

and since the square of the wave speed is $c^2 = c_0^2 + 2c_0c_2\varepsilon^2 + \mathcal{O}(\varepsilon^4)$, then we denoting $\tilde{c}_2 := 2c_0c_2$ we have that $c^2 = c_0^2 + \tilde{c}_2\varepsilon^2 + \mathcal{O}(\varepsilon^4)$. As a result, it follows that

$$c^2 H\eta_u = c_0^2 H\eta_1'\varepsilon + c_0^2 H\eta_2'\varepsilon^2 + \left(c_0^2 H\eta_3' + \tilde{c}_2 H\eta_1'\right)\varepsilon^3 + \mathcal{O}(\varepsilon^4),$$

and hence the Babenko equation can be written in the small amplitude form as

$$\begin{aligned}
&c_0^2 H\eta_1'\varepsilon - c_0^2 H\eta_2'\varepsilon^2 - \left(c_0^2 H\eta_3' + \tilde{c}_2 H\eta_1'\right)\varepsilon^3 - \eta_1\varepsilon - \eta_2\varepsilon^2 - \eta_3\varepsilon^3\\
&\quad + H\eta_1\eta_1'\varepsilon^2 + (H\eta_1\eta_2' + H\eta_2\eta_1')\varepsilon^3 + \eta_1 H\eta_1'\varepsilon^2 + (\eta_1 H\eta_2' + \eta_2 H\eta_1')\varepsilon^3 + \mathcal{O}(\varepsilon^4) = 0
\end{aligned}$$

which is the same as writing

$$\begin{aligned}
&\left(c_0^2 H\eta_1' - \eta_1\right)\varepsilon + \left(H\eta_1\eta_1' + \eta_1 H\eta_1' - c_0^2 H\eta_2' - \eta_2\right)\varepsilon^2 + \left(-c_0^2 H\eta_3' - \tilde{c}_2 H\eta_1' - \eta_3\right.\\
&\qquad\left. + H\eta_1\eta_2' + H\eta_2\eta_1' + \eta_1 H\eta_2' + \eta_2 H\eta_1'\right)\varepsilon^3 + \mathcal{O}(\varepsilon^4) = 0
\end{aligned} \tag{25}$$

Now, using (25), we see that, that to $\mathcal{O}(\varepsilon)$, then the Babenko equation reads as

$$-c_0^2 H\eta_1' - \eta_1 = 0.$$

Note that since we require even solutions, then $\eta_1(u) = \cos(u)$ will be the even solution which solves this. The reason this is the unique solution is because the period is 2π and we are considering a single-lobe wave with only one maximum and minimum on the period. Hence, this is the solution and it will allow us to determine c_0^2 very explicitly. In particular, observe that $-c_0^2 H\eta_1' - \eta_1 = c_0^2 H\sin(u) - \cos(u) = c_0^2\cos(u) - \cos(u) = 0$ and hence we must take $c_0^2 = 1$ to obtain this unique even solution. Thus, $\eta_1(u) = \cos(u)$ and $c_0^2 = 1$.

Now, note that with $c_0^2 = 1$, then η_1 satisfies the homogeneous part, observing that $-H\partial_u - 1 = K - 1$. We have shown that $\cos(u)$ is in the null-space of $K - 1$, and hence it follows by Fredholm theory [9] that all further corrections η_2 and η_3 must be orthogonal to $\cos(u)$ in order to have a unique definition of ε.

Next, using (25), then we see that, to $\mathcal{O}(\varepsilon^2)$ then the Babenko equation reads as

$$-H\eta_2' - \eta_2 + H\eta_1\eta_1' + \eta_1 H\eta_1' = 0.$$

Of course, we can simplify this with our finding that $\eta_1(u) = \cos(u)$ thus observing that

$$\begin{aligned} H\eta_1\eta_1' + \eta_1 H\eta_1' &= -H\left(\cos(u)\sin(u)\right) - \cos(u)H\left(\sin(u)\right) \\ &= -\frac{1}{2}H\left(\sin(2u)\right) - \cos^2(u) \\ &= -\frac{1}{2}\cos(2u) - \frac{1}{2} - \frac{1}{2}\cos(2u) \\ &= -\cos(2u) - \frac{1}{2}, \end{aligned}$$

hence showing the Babenko equation to $\mathcal{O}\left(\varepsilon^2\right)$ becomes

$$-H\eta_2' - \eta_2 - \cos(2u) - \frac{1}{2} = 0.$$

Note that the choice $\eta_2(u) = \cos\left(2u\right) - \frac{1}{2}$ solves this, for the fact that

$$\begin{aligned} -H\eta_2' - \eta_2 - \cos(2u) - \frac{1}{2} &= 2H\sin\left(2u\right) - \cos\left(2u\right) + \frac{1}{2} - \cos(2u) - \frac{1}{2} \\ &= 2\cos\left(2u\right) - \cos\left(2u\right) + \frac{1}{2} - \cos(2u) - \frac{1}{2} \\ &= 0 \end{aligned}$$

and it is the unique choice that will allow for η_2 to be orthogonal to η_1. Hence, $\eta_2(u) = \cos\left(2u\right) - \frac{1}{2}$. Finally, going to $\mathcal{O}\left(\varepsilon^3\right)$ then we again see that, using (25), the Babenko equation reads as

$$-H\eta_3' - \tilde{c}_2 H\eta_1' - \eta_3 + H\eta_1\eta_2' + H\eta_2\eta_1' + \eta_1 H\eta_2' + \eta_2 H\eta_1' = 0$$

Now, from before we thus have that $\eta_1'(u) = -\sin(u)$ and $\eta_2'(u) = -2\sin(2u)$ and hence inserting this into our $\mathcal{O}\left(\varepsilon^3\right)$ Babenko equation, we obtain

$$\begin{aligned} -H\eta_3' + \tilde{c}_2 H\sin(u) - \eta_3 - H\left(2\cos(u)\sin(2u)\right) - H\left(\left(\cos\left(2u\right) - \frac{1}{2}\right)\sin(u)\right) \\ -2\cos(u)H\sin(2u) - \left(\cos\left(2u\right) - \frac{1}{2}\right)H\sin(u) = 0 \end{aligned}$$

Recall from elementary trigonometry that $2\cos(u)\sin(2u) = \sin(u) + \sin(3u)$ and therefore $H\left(2\cos(u)\sin(2u)\right) = H(\sin(u)) + H(\sin(3u)) = \cos(u) + \cos(3u)$. Further, $\sin(u)\cos(2u) = \frac{1}{2}(-\sin(u) + \sin(3u))$ and thus we have that

$$\begin{aligned} H\left(\left(\cos\left(2u\right) - \frac{1}{2}\right)\sin(u)\right) &= H(\sin(u)\cos(2u)) - \frac{1}{2}H(\sin(u)) \\ &= \frac{1}{2}(-\cos(u) + \cos(3u)) - \frac{1}{2}\cos(u) \\ &= -\cos(u) + \frac{1}{2}\cos(3u). \end{aligned}$$

Thus, to $\mathcal{O}\left(\varepsilon^3\right)$ then Babenko's equation is

$$-H\eta_3' + \tilde{c}_2\cos(u) - \eta_3 - \cos(u) - \cos(3u) + \cos(u) - \frac{1}{2}\cos(3u) - 2\cos(u)\cos(2u) - \cos(u)\cos(2u) + \frac{1}{2}\cos(u) = 0,$$

which reduces down to

$$-H\eta_3' + \tilde{c}_2\cos(u) - \eta_3 - \cos(3u) - \frac{1}{2}\cos(3u) - 3\cos(u)\cos(2u) + \frac{1}{2}\cos(u) = 0.$$

Of course, we can write that $3\cos(u)\cos(2u) = \frac{3}{2}\cos(u) + \frac{3}{2}\cos(3u)$ and thus

$$-H\eta_3' + \tilde{c}_2\cos(u) - \eta_3 - \cos(3u) - \frac{1}{2}\cos(3u) - \frac{3}{2}\cos(u) - \frac{3}{2}\cos(3u) + \frac{1}{2}\cos(u) = 0,$$

so that

$$-H\eta_3' + \tilde{c}_2\cos(u) - \eta_3 - 3\cos(3u) - \cos(u) = 0$$

and hence

$$-H\eta_3' - \eta_3 = 3\cos(3u) + (1 - \tilde{c}_2)\cos(u)$$

Note that the choice $\eta_3(u) = \frac{3}{2}\cos(3u)$ along with $\tilde{c}_2 = 1$ solves this. Now, we require that this as these are the unique choices such that the right-hand side is orthogonal to $\cos(u)$. Hence, by definition of $\tilde{c}_2$ then we have that $c_2 = \frac{1}{2}$, as desired. □

In Figure 1, we plot the approximation of the smooth Stokes wave $\eta(u, \varepsilon)$ in the limit of small amplitudes, with the values as determined in Theorem 3.1, for a small value of ε.

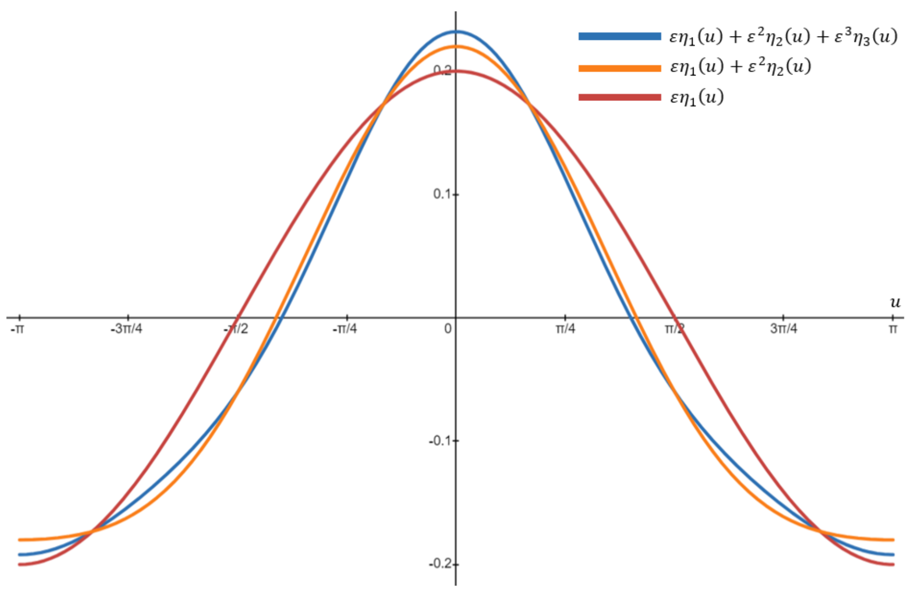

Figure 1: Plot of the approximation to the smooth Stokes wave $\eta(u, \varepsilon)$ for $\varepsilon = 0.2$, showing each successive approximation up to third order. The coefficients η_1, η_2, and η_3 in the expansion are given as in Theorem 3.1. Each approximation is shown in a different colour, as indicated by the legend.

Remark 3.2. We thus find that

$$\eta(u, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \eta_i(u) = \varepsilon\cos(u) + \varepsilon^2\left(\cos(2u) - \frac{1}{2}\right) + \frac{3}{2}\varepsilon^3\cos(3u) + h.o.t. \text{ and } c(\varepsilon) = 1 + \frac{1}{2}\varepsilon^2 + h.o.t.$$

so that, in particular, $c^2(\varepsilon) = 1 + \varepsilon^2 + h.o.t.$. This gives us the leading order behaviour of both the surface profile η and travelling wave speed c in the limit of small amplitude Stokes waves. Note that such small-amplitude computations have been previously performed in the literature, and one finds that these results produced with the Babenko equation match what has been computed [5]. Observe moreover from Figure 1 that as more terms are added to the expansion of the small amplitude wave, the expansion itself seems to converge towards a smooth wave.

In a similar fashion, one can also perform a small amplitude expansion of the even eigenvalues of the linearized Babenko operator (21), and such computations are demonstrated in the following theorem:

Theorem 3.3. Consider the eigenvalues of $\mathcal{L}v = \mu v$ for the linearized Babenko operator (21) corresponding to the even subspace. Take these quantities in the limit of small amplitudes, so that we may expand both in terms of the pertubation parameter ε as $v(u,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i v_i(u)$ and $\mu(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^{2i}\mu_{2i}$. This results in the coefficients up to the second order being given by:

- To $\mathcal{O}(1)$ then we have $v_0(u) = \cos(u)$ and $\mu_0 = 0$.
- To $\mathcal{O}(\varepsilon)$ then we have that $v_1(u) = 2\cos(2u) - 1$.
- To $\mathcal{O}\left(\varepsilon^2\right)$ then we have that $v_2(u) = \frac{9}{2}\cos(3u)$ and $\mu_2 = -2$.

Proof. We are considering the even eigenvalues for the linearized Babenko operator, which satisfy $\mathcal{L}v = \mu v$, in the limit of small amplitudes. Since we're considering the small amplitude limit, then we take the expansions

$$v \coloneqq v(u,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i v_i(u) \text{ and } \mu(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^{2i}\mu_{2i} \tag{26}$$

and thus we have the expansions $v(u,\varepsilon) = v_0(u) + v_1(u)\varepsilon + v_2(u)\varepsilon^2 + v_3(u)\varepsilon^3 + \mathcal{O}\left(\varepsilon^4\right)$ and $\mu(\varepsilon) = \mu_0 + \mu_2\varepsilon^2 + \mathcal{O}\left(\varepsilon^4\right)$, where μ is in even powers as we consider the even subspace. Now, recall that we can write equation (21) as

$$\mathcal{L} \coloneqq c^2 K - 1 - K(\eta\cdot) - (K\eta)\cdot - \eta(K\cdot)$$

and thus using the small amplitude expansion $\eta = \eta_0 + \eta_1\varepsilon + \eta_2\varepsilon^2 + \eta_3\varepsilon^3 + \mathcal{O}\left(\varepsilon^4\right)$ and $c = 1 + \varepsilon^2 + \mathcal{O}\left(\varepsilon^4\right)$ from Theorem 3.1 then we see that we may expand this linearized operator. In particular, expanding each of the individual terms in the operator, then

$$\begin{aligned}
c^2 K &= K + \varepsilon^2 K + \mathcal{O}\left(\varepsilon^4\right) \\
K(\eta\cdot) &= K(\eta_1\cdot)\varepsilon + K(\eta_2\cdot)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) \\
(K\eta)\cdot &= \varepsilon(K\eta_1)\cdot + \varepsilon^2(K\eta_2)\cdot + \mathcal{O}\left(\varepsilon^3\right) \\
\eta(K\cdot) &= \eta_1\varepsilon(K\cdot) + \eta_2\varepsilon^2(K\cdot) + \mathcal{O}\left(\varepsilon^3\right)
\end{aligned}$$

and hence we see that (21) becomes

$$\begin{aligned}
\mathcal{L} = {} & (K-1) + \varepsilon\left(-K(\eta_1\cdot) - (K\eta_1)\cdot - \eta_1(K\cdot)\right) \\
& + \varepsilon^2\left(K - K(\eta_2\cdot) - (K\eta_2)\cdot - \eta_2(K\cdot)\right) + \mathcal{O}\left(\varepsilon^3\right).
\end{aligned}$$

Inserting (26) into this equation, then we see that

$$\begin{aligned}
\mathcal{L}v = {} & (K-1)v_0 + \varepsilon\left(-K(\eta_1 v_0) - (K\eta_1)v_0 - \eta_1(Kv_0)\right) \\
& + \varepsilon^2\left(Kv_0 - K(\eta_2 v_0) - (K\eta_2)v_0 - \eta_2(Kv_0)\right) \\
& + \varepsilon(K-1)v_1 + \varepsilon^2\left(-K(\eta_1 v_1) - (K\eta_1)v_1 - \eta_1(Kv_1)\right) + \varepsilon^2(K-1)v_2 + \mathcal{O}\left(\varepsilon^3\right) \\
= {} & (K-1)v_0 + \varepsilon\left(-K(\eta_1 v_0) - (K\eta_1)v_0 - \eta_1(Kv_0) + (K-1)v_1\right) \\
& + \varepsilon^2\left(-K(\eta_1 v_1) - (K\eta_1)v_1 - \eta_1(Kv_1) + Kv_0 - K(\eta_2 v_0) - (K\eta_2)v_0 - \eta_2(Kv_0) + (K-1)v_2\right) + \mathcal{O}\left(\varepsilon^3\right)
\end{aligned} \tag{27}$$

Of course, since we are looking for the eigenvalues, then we will align the terms up to a given order in the equation $\mathcal{L}v = \mu v$, where we may use (26) to expand the right-hand side as

$$\begin{aligned}
\mu v = {} & \left(\mu_0 + \mu_2\varepsilon^2 + \mathcal{O}\left(\varepsilon^4\right)\right)\left(v_0 + v_1\varepsilon + v_2\varepsilon^2 + v_3\varepsilon^3 + \mathcal{O}\left(\varepsilon^4\right)\right) \\
= {} & \mu_0 v_0 + \mu_0 v_1\varepsilon + \left(\mu_0 v_0 + \mu_2 v_0\right)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right)
\end{aligned} \tag{28}$$

and thus we see that to $\mathcal{O}(1)$ then $\mathcal{L}v = \mu v$ becomes $(K-1)\,v_0 = \mu_0 v_0$. Since we are taking eigenvalues from the even subspace, we may set $\mu_0 = 0$ and thus we have that $(K-1)\,v_0 = 0$. Now, note that the choice $v_0 = \cos(u)$ solves this, as then

$$\begin{aligned}(K-1)\,v_0 = & -(\partial_u H + 1)\cos(u) = -\partial_u H\cos(u) - \cos(u) \\ = & \ \partial_u \sin(u) - \cos(u) = \cos(u) - \cos(u) = 0\end{aligned}$$

as desired.

$$(K-1)\,v_0 = -(\partial_u H + 1)\cos(u)$$

Thus, $v_0(u) = \cos(u)$ and $\mu_0 = 0$ solves this. Since we seek solutions with a period of 2π and are considering single-lobe waves with only one maximum and one minimum on the period, then this is the unique choice for v_0. Similar to the proof of Theorem 3.1, we observe that we may define the homogeneous part, writing that $-H\partial_u - 1 = K - 1$. We have shown that $\cos(u)$ is in the null-space of $K-1$, and hence it follows by Fredholm theory [9] that all further corrections v_1 and v_2 must be orthogonal to $\cos(u)$ in order to have a unique definition of ε.

Next, to $\mathcal{O}(\varepsilon)$ then equating (27) and (28) we see that $\mathcal{L}v = \mu v$ becomes $-K(\eta_1 v_0) - (K\eta_1)\,v_0 - \eta_1(Kv_0) + (K-1)\,v_1 = \mu_0 v_1 = 0$. Inserting the values derived from Theorem 3.1, as well as $v_0(u) = \cos(u)$ then we see that

$$(K-1)\,v_1 - K\left(\cos^2(u)\right) - (K\cos(u))\cos(u) - \cos(u)\,(K\cos(u)) = 0.$$

However, since $\cos^2(u) = \frac{1}{2} + \frac{\cos(2u)}{2}$ then it follows that

$$\begin{aligned}0 = & \ (K-1)\,v_1 + H\partial_u\left(\frac{1}{2} + \frac{\cos(2u)}{2}\right) + (H\partial_u\cos(u))\cos(u) + \cos(u)\,(H\partial_u\cos(u)) \\ = & \ (K-1)\,v_1 - H\sin(2u) - (H\sin(u))\cos(u) - \cos(u)\,(H\sin(u)) \\ = & \ (K-1)\,v_1 - \cos(2u) - 2\cos^2(u) = (K-1)\,v_1 - \cos(2u) - 2\left(\frac{1}{2} + \frac{\cos(2u)}{2}\right) \\ = & \ (K-1)\,v_1 - 2\cos(2u) - 1\end{aligned}$$

and thus $(K-1)\,v_1 = 2\cos(2u) + 1$. Thus, choosing $v_1(u) = 2\cos(2u) - 1$ we see that

$$\begin{aligned}(K-1)\,v_1 = & \ Kv_1 - v_1 = -H\partial_u\,(2\cos(2u) - 1) - 2\cos(2u) + 1 \\ = & \ 4H\sin(2u) - 2\cos(2u) + 1 = 4\cos(2u) - 2\cos(2u) + 1 \\ = & \ 2\cos(2u) + 1\end{aligned}$$

as desired. Thus, $v_1(u) = 2\cos(2u) - 1$ solves the homogeneous part, and it is the unique choice that will allow for v_1 to be orthogonal to v_0.

Finally, to $\mathcal{O}(\varepsilon^2)$, then equating (27) and (28) we see that $\mathcal{L}v = \mu v$ becomes

$$-K(\eta_1 v_1) - (K\eta_1)\,v_1 - \eta_1(Kv_1) + Kv_0 - K(\eta_2 v_0) - (K\eta_2)\,v_0 - \eta_2(Kv_0) + (K-1)\,v_2 = \mu_0 v_0 + \mu_2 v_0 = \mu_2 v_0$$

Inserting the values derived from Theorem 3.1, as well as the previous values of v_0 and v_1, then we see that this says

$$\begin{aligned}0 = & -K(\cos(u)\,(2\cos(2u) - 1)) - (K\cos(u))\,(2\cos(2u) - 1) - \cos(u)\,(K\,(2\cos(2u) - 1)) \\ & + K\cos(u) - K\left(\left(\cos(2u) - \frac{1}{2}\right)\cos(u)\right) - \left(K\left(\cos(2u) - \frac{1}{2}\right)\right)\cos(u) - \left(\cos(2u) - \frac{1}{2}\right)K\cos(u) \\ & + (K-1)\,v_2 - \mu_2\cos(u) \\ = & -K(2\cos(u)\cos(2u) - \cos(u)) - \cos(u)\,(2\cos(2u) - 1) - 4\cos(u)\cos(2u) + \cos(u) \\ & - K\left(\cos(2u)\cos(u) - \frac{\cos(u)}{2}\right) - 2\cos(2u)\cos(u) - \cos(2u)\cos(u) + \frac{\cos(u)}{2} \\ & + (K-1)\,v_2 - \mu_2\cos(u)\end{aligned}$$

Now, using the fact that $2\cos(u)\cos(2u)-\cos(u)=\cos(3u)$ then we see that $K\left(2\cos(u)\cos(2u)-\cos(u)\right)=K(\cos(3u))=3\cos(3u)$ and further $K\left(\cos\left(2u\right)\cos(u)-\frac{\cos(u)}{2}\right)=\frac{1}{2}K(2\cos(u)\cos(2u)-\cos(u))=\frac{3\cos(3u)}{2}$ and so simplifying this equation shows that

$$\begin{aligned}
0 = & -3\cos(3u)-\cos(u)\left(2\cos(2u)-1\right)-4\cos(u)\cos(2u)+\cos(u)-\frac{3\cos(3u)}{2}-2\cos(2u)\cos(u) \\
& -\cos(2u)\cos(u)+\frac{\cos(u)}{2}+\left(K-1\right)v_2-\mu_2\cos(u) \\
= & -3\cos(3u)-2\cos(u)\cos(2u)+\cos(u)-4\cos(u)\cos(2u)+\cos(u)-\frac{3\cos(3u)}{2}-2\cos(2u)\cos(u) \\
& -\cos(2u)\cos(u)+\frac{\cos(u)}{2}+\left(K-1\right)v_2-\mu_2\cos(u) \\
= & -\frac{9}{2}\cos(3u)-9\cos(u)\cos(2u)+\frac{5\cos(u)}{2}+\left(K-1\right)v_2-\mu_2\cos(u)
\end{aligned}$$

Now, since $9\cos(u)\cos(2u)=\frac{9}{2}\cos(3u)+\frac{9}{2}\cos(u)$ then we see that

$$\begin{aligned}
0 = & -\frac{9}{2}\cos(3u)-\frac{9}{2}\cos(3u)-\frac{9}{2}\cos(u)+\frac{5\cos(u)}{2}+\left(K-1\right)v_2-\mu_2\cos(u) \\
= & -9\cos(3u)-2\cos(u)+\left(K-1\right)v_2-\mu_2\cos(u) \\
= & -9\cos(3u)+\left(K-1\right)v_2-\left(\mu_2+2\right)\cos(u)
\end{aligned}$$

and we see that $v_2=\frac{9}{2}\cos(3u)$ and $\mu_2=-2$ solve this, for the fact that:

$$\begin{aligned}
-9\cos(3u)+\left(K-1\right)v_2-\left(\mu_2+2\right)\cos(u) = & -9\cos(3u)+\frac{9}{2}K\cos(3u)-\frac{9}{2}\cos(3u) \\
= & -9\cos(3u)+\frac{9}{2}3\cos(3u)-\frac{9}{2}\cos(3u) \\
= & -9\cos(3u)+9\cos(3u)=0.
\end{aligned}$$

Now, we require that $\mu_2=-2$ as this is needed to ensure that these are the unique choices such that the right-hand side is orthogonal to $\cos(u)$. Thus, $v_2(u)=\frac{9}{2}\cos(3u)$ and $\mu_2=-2$. ☐

Remark 3.4. We thus find that when considering a splitting into the even subspace of eigenvalues of the linearized Babenko operator given by $\mathcal{L}v=\mu v$ in the limit of small amplitudes, then we have that

$$v\left(u,\varepsilon\right)=\cos(u)+\left(2\cos(2u)-1\right)\varepsilon+\left(\frac{9}{2}\cos(3u)\right)\varepsilon^2+\mathcal{O}\left(\varepsilon^3\right)\ \text{ and }\ \mu\left(\varepsilon\right)=-2\varepsilon^2+\mathcal{O}\left(\varepsilon^4\right).$$

Note that such small-amplitude computations have been previously performed in the literature, and one finds that these results produced with the Babenko equation match what has been computed [4].

3.2 Stability Spectrum of Small Amplitude Stokes Waves

In the preceding section, we have considered smooth Stokes waves in the limit of small amplitudes, specifically analyzing the coefficients in the expansions in Theorem (3.1) as well as the coefficients in the expansion of the even eigenvalues of the linearized Babenko operator in Theorem (3.3). However, we may also employ similar techniques to understand the spectrum of the Stokes wave in this small amplitude limit, specifically by expanding the eigenvalue λ in the linearized system of stability (22). In particular, considering the expansion $\lambda=\varepsilon\lambda_1+\varepsilon^2\lambda_2+\mathcal{O}\left(\varepsilon^3\right)$, where the mean-value has been removed, it was shown in [4] that these eigenvalues trace out a figure-8 curve centered at the origin, when the imaginary part is plotted against the real part. To see this, each term λ_i in the expansion of λ was parametrized in terms of a parameter $\mu=\mu_1\varepsilon$ with $\mu_1\in\mathcal{O}(1)$ and using the same equations of linear stability (22), it was shown in [4] that the coefficients up to the second order in the expansion of λ are:

- To $\mathcal{O}(\varepsilon)$ then we have that $\lambda_1 = i\frac{\mu_1}{2}$.
- To $\mathcal{O}(\varepsilon^2)$ then we have that $\lambda_2 = \pm\frac{1}{8}\mu_1\sqrt{8-\mu_1^2}$.

Note that the derivation of equation (22) in [4] is slightly different than the method we outlined in section 2.3, as they use the Ablowitz-Fokas-Musslimani (AFM) formulation of the water waves. Moreover, note that, as was done in [4], when we plot $\mathrm{Im}(\lambda)$ against $\mathrm{Re}(\lambda)$ up to $\mathcal{O}(\varepsilon^2)$, then the 'figure-8' pattern is developed. Observe this done in Figure 2:

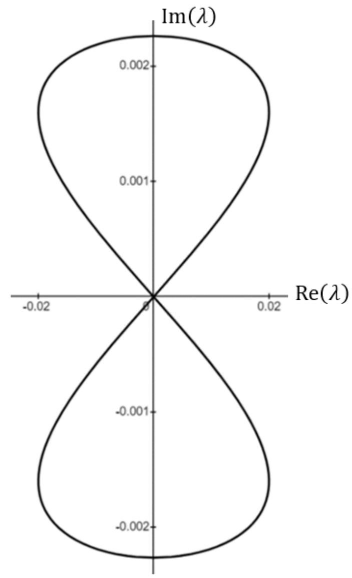

Figure 2: Plot of the equation $\varepsilon\lambda_1 + \varepsilon^2\lambda_2 = \pm\frac{1}{8}\mu_1\sqrt{8-\mu_1^2}\varepsilon + i\frac{\mu_1}{2}\varepsilon^2$ with $\varepsilon = 0.04$ in the complex plane.

This indicates the existence of the so-called 'Benjamin-Feir instability' for these small-amplitude waves, since it shows that four of the eigenvalues collide together at the origin before separating, with each eigenvalue occupying a distinct quadrant [4]. As a result, we see that we have been able to explore the stability and structure of the small-amplitude Stokes waves using these linearized equations of stability, and the Babenko equation allowed us to explore these equations.

4 The Structure of the Peaked Stokes Wave

We are now interested in understanding various properties of the Stokes wave of greatest height, which is also known as the peaked, or limiting, Stokes wave.

4.1 Understanding the Peaked Stokes Wave and the Stokes Expansion

Recall that the Stokes wave is an irrotational periodic gravity wave flowing with a given speed c, which is symmetric about its crest, and which corresponds to the travelling wave solutions of the Euler equations [17]. The height – or amplitude – of the wave, which is defined as being the distance from the crest to the trough of the wave when measured vertically, is often used as a parameterization of the Stokes wave. As we've investigated in Section 3, many Stokes waves are smooth and can have varying amplitudes. However, in 1880 it was conjectured by Stokes [17] that there should exist a wave of greatest height. Moreover, he concluded that if such a limiting wave exists, then there must be a sharp corner about the crest, which forms an angle of $2\pi/3$ radians [17]. It was not until 1978 when the global existence of the peaked Stokes wave was demonstrated by Toland [18] and the full Stokes conjecture, including a proof of the corner angle at the crest, was eventually proved by Plotnikov in 1982 [15] as well as by Amick, Fraenkel, and Toland [1]. Given the existence of the sharp corner at the peak, the limiting Stokes wave is continuous but not differentiable at its peak, and thus is not a smooth wave. Notably, however, we see that only the Stokes wave of greatest height is not smooth [3]. Note that it has not been shown that the limiting wave is unique, and hence the uniqueness of the peaked wave as a limit of smooth near limiting waves is not yet fully understood [16]. Moreover, it is not necessarily known if there is a unique speed c at which the limiting wave occurs, though it was shown in [19] that the horizontal fluid speed is equal to the wave speed c only at the limiting wave, and hence the horizontal speed is never to equal to the wave speed c for smooth waves.

There has been a great deal of work investigating the structure of the peaked Stokes wave. For instance, Constantin [3] provides a description of the paths of particles in the limiting Stokes wave for a domain of finite depth, demonstrating that the paths they follow are 'looping curves', and that no particle will remain stationary at the peak. Lyons [13] extended the results from Constantin's finite depth analysis to show that particles in the limiting Stokes wave will also never remain in a closed loop when considering a domain of infinite depth. It is noteworthy that both [3] and [13] use a different conformal transformation than was considered in Section 1.2, where instead they utilized a hodograph transform induced from the introduction of a stream function.

In this section, we will be interested in understanding the surface profile of the peaked Stokes wave via an expansion in the powers of its amplitude. This type of expansion is known as the Stokes expansion [14]. In such an expansion, the higher order terms become important to be able to describe the behaviour away from the peak, while the lower order terms describe the behaviour closer to the peak. As Lushnikov [12] notes, if we assume that the singularity of the extreme wave touches the fluid surface at $w = 0$ in the conformal coordinates, then the expansion of $z(w) = x(w) + i\eta(w)$, which is the profile after conformal transformation, given by

$$z(w) = i\frac{c^2}{2} - i\left(\frac{3c}{2}\right)^{2/3}(iw)^{2/3} + h.o.t. \tag{29}$$

will ensure the corner angle of $2\pi/3$ radians at the peak is conserved. Hence, with $w = u + iv$ the conformal coordinates, then we may reduce the expansion (29) to the form

$$x\left(u\right) = u + \frac{3^{7/6}}{2^{5/3}}c^{2/3}\left|u\right|^{2/3} + h.o.t.$$

$$\eta\left(u\right) = \frac{c^2}{2} - \frac{3^{2/3}}{2^{5/3}}c^{2/3}\left|u\right|^{2/3} + h.o.t.$$

This motivates the form of the Stokes expansion, as we know that, in order to preserve the $2\pi/3$ angle at the peak, we should have a $2/3$ singularity in the derivative of our surface profile, after conformal transformation. Nonetheless, this exact expansion and its corresponding coefficients have not been shown to be unique [14] and thus for our purposes we will assume a general expansion of the surface profile to be given by

$$\eta(u) = \frac{c^2}{2} - A|u|^{2/3} + o(|u|^{2/3}) \text{ as } |u| \to 0 \tag{30}$$

for some admissible values of $A > 0$. Note that, while we know the first power in the expansion must be $2/3$ to preserve the $2\pi/3$ angle, there is also information regarding the second-order correction term in the expansion. In particular, Grant [14] showed interest in understanding this correction term, and in fact, assuming the expansion $\eta(u) = \frac{c^2}{2} + A|u|^{2/3} + B|u|^{\mu} + o(|u|^{\mu})$ with $\mu > 2/3$ as $|u| \to 0$, then he showed that μ must be an irrational number which satisfies

$$\tan\left(\frac{\pi\mu}{2}\right) = \frac{-\left(4 + 3\mu\right)}{3^{3/2}\mu}. \tag{31}$$

The first solution to this equation, and thus the first possible root, is $\mu \approx 1.468$.

Our goal is to the use the Babenko equation (18) to better understand the properties of the expansion (30).

4.2 Results for the Peaked Stokes Wave

In this section, we will be interested in exploring the expansion of the surface profile given by equation (30) and its interaction with the Babenko equation. In particular, since we saw the Babenko equation was derived for waves of smooth amplitudes, it is questionable if it holds for the limiting Stokes wave. Our main results relate to providing evidence that the Babenko equation holds for the limiting waves, by demonstrating that it agrees with previous analytical results surrounding the Stokes wave expansion.

We begin with the following proposition, which provides a very useful reformulation of the Babenko equation, demonstrating that we can remove the constant term from the Babenko equation and obtain an equivalent equation which simplifies our computations:

Proposition 4.1. Given η a solution to the Babenko equation, then we will have that $\tilde{\eta}$ solves

$$\tilde{\eta} = \frac{c^2}{2} + \frac{1}{2}K\tilde{\eta}^2 + \tilde{\eta}K\tilde{\eta} \tag{32}$$

where

$$\tilde{\eta} := \frac{c^2}{2} - \eta$$

Proof. Taking η a solution to the Babenko equation, let us define $\tilde{\eta} := \frac{c^2}{2} - \eta$ and hence $\eta = \frac{c^2}{2} - \tilde{\eta}$. Inserting this into the left-hand side of the Babenko equation (18) gives that

$$\left(c^2K - 1\right)\eta = \left(c^2K - 1\right)\left(\frac{c^2}{2} - \tilde{\eta}\right) = c^2K\left(\frac{c^2}{2}\right) - c^2K\tilde{\eta} - \frac{c^2}{2} + \tilde{\eta} = -c^2K\tilde{\eta} - \frac{c^2}{2} + \tilde{\eta}$$

Moreover, the right-hand side of the Babenko equation becomes

$$\begin{aligned}\frac{1}{2}K\eta^2 + \eta K\eta &= \frac{1}{2}K\left(\frac{c^4}{4} - c^2\tilde{\eta} + \tilde{\eta}^2\right) + \left(\frac{c^2}{2} - \tilde{\eta}\right)K\left(\frac{c^2}{2} - \tilde{\eta}\right)\\ &= -\frac{1}{2}c^2K\tilde{\eta} + \frac{1}{2}K\tilde{\eta}^2 - \frac{c^2}{2}K\tilde{\eta} + \tilde{\eta}K\tilde{\eta}\\ &= -c^2K\tilde{\eta} + \frac{1}{2}K\tilde{\eta}^2 + \tilde{\eta}K\tilde{\eta}\end{aligned}$$

and thus equation (18) is equivalent to saying that $-c^2K\tilde{\eta} - \frac{c^2}{2} + \tilde{\eta} = -c^2K\tilde{\eta} + \frac{1}{2}K\tilde{\eta}^2 + \tilde{\eta}K\tilde{\eta}$, so that $\tilde{\eta} = \frac{c^2}{2} + \frac{1}{2}K\tilde{\eta}^2 + \tilde{\eta}K\tilde{\eta}$ is reproduced. □

With this useful reformulation of the Babenko equation, we may now continue by verifying that the expansion of the surface profile (30) satisfies the Babenko equation, for some admissible value of c. This is demonstrated in the following proposition, where it is shown that, when taking the general Stokes expansion of form (30), then the Babenko equation does not derive a contradiction. To do this, we have computed the Hilbert transform at the leading-order approximation as $|u| \to 0$, and the computations required for this are demonstrated in Proposition 7.3 and Corollary 7.4.

Proposition 4.2. In assuming the surface profile for the Stokes wave of greatest height is given by $\eta(u) = \frac{c^2}{2} - A|u|^{2/3} + o(|u|^{2/3})$, and hence where $\tilde{\eta} := A|u|^{2/3} + o(|u|^{2/3})$, as $|u| \to 0$ for some admissible values of $A > 0$, the Babenko equation at the limiting height (32) will lead to no contradictions.

Proof. The proof follows immediately from our prior computations for the Hilbert transform. In particular, observe that

$$K\tilde{\eta} = -\frac{\partial}{\partial u}H\tilde{\eta} = -A\frac{\partial}{\partial u}H|u|^{2/3} + o\left(|u|^{-1/3}\right) = -A\frac{\partial}{\partial u}\sqrt{3}|u|^{2/3}\text{sgn}(u) + o\left(|u|^{-1/3}\right) = -A\frac{2}{\sqrt{3}}|u|^{-1/3} + o\left(|u|^{-1/3}\right)$$

while we also have that

$$\tilde{\eta}K\tilde{\eta} = \left(A\,|u|^{2/3} + o(|u|^{2/3})\right)\left(-A\frac{2}{\sqrt{3}}|u|^{-1/3} + o\left(|u|^{-1/3}\right)\right) = -A^2\frac{2}{\sqrt{3}}|u|^{1/3} + o\left(|u|^{2/3}\right)$$

and finally one can compute that

$$\begin{aligned} K\tilde{\eta}^2 = -\frac{\partial}{\partial u}H\tilde{\eta}^2 &= \; -A^2\frac{\partial}{\partial u}H\,|u|^{4/3} + o\left(|u|^{1/3}\right) \\ &= \sqrt{3}A^2\frac{\partial}{\partial u}\left(|u|^{4/3}\mathrm{sgn}(u)\right) + o\left(|u|^{1/3}\right) \\ &= \frac{4A^2}{\sqrt{3}}|u|^{1/3} + o\left(|u|^{1/3}\right) \end{aligned}$$

and thus observe that the right-hand side of the Babenko equation reads as:

$$\begin{aligned} \frac{c^2}{2} + \frac{1}{2}K\tilde{\eta}^2 + \tilde{\eta}K\tilde{\eta} &= \frac{c^2}{2} + \frac{2A^2}{\sqrt{3}}|u|^{1/3} - A^2\frac{2}{\sqrt{3}}|u|^{1/3} + \mathcal{O}\left(|u|^{2/3}\right) \\ &= \frac{c^2}{2} + \mathcal{O}\left(|u|^{2/3}\right) \end{aligned}$$

Thus, there is no contradiction arising from the Babenko equation when assuming that $\eta = \frac{c^2}{2} - A|u|^{2/3} + o\left(|u|^{2/3}\right)$, which shows that this expansion is possible. □

Now, it follows from Proposition 4.2 that we have thus shown that the Stokes expansion (30), which assumes a leading order singularity of 2/3, is in agreement with the Babenko equation. However, this does not necessarily imply that the surface profile must take this form. While well-known results show that the singularity in the derivative must be of order 2/3, these results were not derived with the Babenko equation. Moreover, Grant showed that, in fact, the second-order correction term in the expansion of the surface profile must itself be a continuously differentiable term [14]. We wish to obtain these results definitively by utilizing the Babenko equation.

Note, however, that the method we have used in the previous theorems to compute the Hilbert transform apply only when taking the Hilbert transform of a 2π-periodic function $f(u) = |u|^{n/3}$ for n a natural number. In order to make conclusive comments about the expansion of the peaked wave, we need a stronger and more effective method for computing the Hilbert transforms of functions of the form $f(u) = |u|^{\beta}$ for more general values of β. To do this, observe that [10] shows that, for any $\nu \in (0,1)$ and $u \in \mathbb{R}$ then

$$\int_{-\infty}^{\infty} \frac{|x|^{\nu-1}}{x-u}dx = -\pi\cot\left(\frac{\nu\pi}{2}\right)|u|^{\nu-1}\mathrm{sgn}(u) \tag{33}$$

Equation (33) gives us a very explicit formula for the analogous formulation of the Hilbert transform on the infinite line $\mathbb{R}$, from which we can easily derive our desired Hilbert transform of 2π-periodic functions. In particular, this well-known formula gives rise to the following result, which extends Proposition 7.3 to allow us to very explicitly understand the leading order term in the evaluation of the Hilbert transform of a function $|u|^{\beta}$:

Lemma 4.3. Given $\beta \in (-1,0)$ and the 2π-periodic function $f(u) = |u|^{\beta}$ on $[-\pi,\pi]$, which is extended periodically to all of $\mathbb{R}$, then we have that

$$H\left(f\right)(u) = -\cot\left(\frac{(\beta+1)\,\pi}{2}\right)|u|^{\beta}\mathrm{sgn}(u) + F_{\beta}(u) \tag{34}$$

for some $F_{\beta} \in C^{\omega}(D)$ for any compact subset D of $(-\pi,\pi)$.

Proof. Let us first note that, by Theorem 7.1, since $\beta \in (-1,0) \subset (-1,\infty)$ then we have that

$$H(f)(u) = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(x)}{x-u}dx + G_{\beta}(u) = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{|x|^{\beta}}{x-u}dx + G_{\beta}(u)$$

where $G_\beta \in C^\omega(D)$. Now, defining

$$M_\beta(u) := -\frac{1}{\pi}\left(\int_{-\infty}^{-\pi} \frac{|x|^\beta}{x-u}dx + \int_{\pi}^{\infty} \frac{|x|^\beta}{x-u}dx\right)$$

then observe that we may write

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{|x|^\beta}{x-u}dx &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|x|^\beta}{x-u}dx - \frac{1}{\pi}\left(\int_{-\infty}^{-\pi} \frac{|x|^\beta}{x-u}dx + \int_{\pi}^{\infty} \frac{|x|^\beta}{x-u}dx\right) \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{|x|^{(\beta+1)-1}}{x-u}dx + M_\beta(u) \\
&= -\cot\left(\frac{(\beta+1)\,\pi}{2}\right)|u|^\beta \mathrm{sgn}(u) + M_\beta(u)
\end{aligned}$$

where the third equality follows since $\beta \in (-1,0)$ so that $\beta+1 \in (0,1)$ and so we have employed equation 33. Now, observe that since $u \in [-\pi,\pi]$, then the function

$$\frac{|x|^\beta}{x-u}$$

is analytic on the intervals $(-\infty,-\pi)$ and (π,∞) and hence it follows easily that $M_\beta \in C^\omega(D)$. Therefore, taking

$$F_\beta(u) := G_\beta(u) + M_\beta(u)$$

then we have that F_β is a sum of analytic functions and thus $F_\beta \in C^\omega(D)$. Combining the equations above we thus obtain that

$$\begin{aligned}
H(f)(u) &= -\cot\left(\frac{(\beta+1)\,\pi}{2}\right)|u|^\beta \mathrm{sgn}(u) + M_\beta(u) + G_\beta(u) \\
&= -\cot\left(\frac{(\beta+1)\,\pi}{2}\right)|u|^\beta \mathrm{sgn}(u) + F_\beta(u)
\end{aligned}$$

which proves the claim. ☐

From Lemma 4.3, we may deduce the general form for the Hilbert transform of 2π-periodic functions of the form $|u|^\beta \mathrm{sgn}(u)$, as is demonstrated in the following corollary:

Corollary 4.4. For $g(u) = |u|^\beta \mathrm{sgn}(u)$ with $\beta \in (-1,0)$, which is defined on $(-\pi,\pi)$ and extended periodically to all of $\mathbb{R}$, then we have that

$$H\,(g) = \tan\left(\frac{(\beta+1)\,\pi}{2}\right)|u|^\beta + G_\beta(u)$$

for some $G_\beta \in C^\omega(D)$ for any compact subset D of $(-\pi,\pi)$.

Proof. To demonstrate this fact, let us simply take the Hilbert transform of (34) to observe that

$$H^2(f) = -\cot\left(\frac{(\beta+1)\,\pi}{2}\right)H\left(|u|^\beta \mathrm{sgn}(u)\right) + H\left(F_\beta(u)\right)$$

where we have used the fact that the Hilbert transform is linear. Since f is assumed to have zero mean, then $H^2(f) = -f$ and further we have denoted $g(u) = |u|^\beta \mathrm{sgn}(u)$, therefore we can rewrite this as saying

$$-f = -\cot\left(\frac{(\beta+1)\,\pi}{2}\right)H\,(g) + H\left(F_\beta(u)\right)$$

and hence

$$\cot\left(\frac{(\beta+1)\,\pi}{2}\right)H(g) = f + H\left(F_\beta(u)\right)$$

and multiplying both sides by $\tan\left(\frac{(\beta+1)\pi}{2}\right)$ results in

$$H(g) = \tan\left(\frac{(\beta+1)\,\pi}{2}\right) f + \tan\left(\frac{(\beta+1)\,\pi}{2}\right) H\left(F_\beta(u)\right)$$

Using the fact that $f(u) = |u|^\beta$, along with the notation $G_\beta(u) \coloneqq \tan\left(\frac{(\beta+1)\pi}{2}\right) H\left(F_\beta(u)\right) \in C^\omega(D)$, then we have that

$$H(g) = \tan\left(\frac{(\beta+1)\,\pi}{2}\right) |u|^\beta + G_\beta(u)$$

as desired. □

Utilizing Lemma 4.3 and Corollary 4.4, we may now use the Babenko equation to demonstrate Stokes' main result that the leading order singularity for the surface profile of the peaked Stokes wave gives us a 2/3 power law, which will demonstrate that the Babenko equation requires the first term in our Stokes expansion (30) to have power 2/3:

Theorem 4.5. If Equation (32) admits a solution such that $\tilde{\eta} = A|u|^\beta + o(|u|^\beta)$ as $|u| \to 0$, where $\beta \in (0, 1)$ and an admissible value of $A > 0$, then it must be that $\beta = 2/3$.

Proof. We may split the proof into three cases to be considered. First, suppose that $\beta \in (0, 1/2)$. Then it will follow that $\beta - 1 \in (-1, 0)$ and so from the preceding formulas we may write

$$\begin{aligned} K|u|^\beta &= -\frac{\partial}{\partial u} H|u|^\beta = -\beta H|u|^{\beta-1}\mathrm{sgn}(u) \\ &= -\beta \tan\left(\frac{\pi\beta}{2}\right) |u|^{\beta-1} + h.o.t. \end{aligned}$$

Moreover, observe that $2\beta \in (0, 1)$ and thus we have, again by the preceding formulas, that

$$\begin{aligned} K|u|^{2\beta} &= -\frac{\partial}{\partial u} H|u|^{2\beta} = -2\beta H|u|^{2\beta-1}\mathrm{sgn}(u) \\ &= -2\beta \tan(\pi\beta)\, |u|^{2\beta-1} + h.o.t. \end{aligned}$$

Therefore, we see that the Babenko equation reads as

$$\begin{aligned} A|u|^\beta = \tilde{\eta} &= \frac{c^2}{2} + \frac{1}{2}K\tilde{\eta}^2 + \tilde{\eta}K\tilde{\eta} \\ &= \frac{c^2}{2} + \frac{1}{2}KA^2|u|^{2\beta} + A|u|^\beta KA|u|^\beta \\ &= \frac{c^2}{2} - A^2\beta\tan(\pi\beta)\,|u|^{2\beta-1} - A^2\beta\tan\left(\frac{\pi\beta}{2}\right)|u|^{\beta-1} + h.o.t. \\ &= \frac{c^2}{2} - A^2\beta\left(\tan(\pi\beta) + \tan\left(\frac{\pi\beta}{2}\right)\right)|u|^{2\beta-1} + h.o.t \end{aligned}$$

Now, in order for this equation to hold, it must be that the coefficient on the $|u|^{2\beta-1}$ term is precisely zero, for the fact that there is no other instance of such a power of $|u|$ in the equation. Therefore, we require that

$$\tan(\pi\beta) + \tan\left(\frac{\pi\beta}{2}\right) = 0$$

Of course, $\tan(\pi\beta) = \frac{\sin(\pi\beta)}{\cos(\pi\beta)}$. By the double angle formulas for sine and cosine, then $\sin(\pi\beta) = \sin\left(2\frac{\pi\beta}{2}\right) =$

$2\sin\left(\frac{\pi\beta}{2}\right)\cos\left(\frac{\pi\beta}{2}\right)$ and $\cos(\pi\beta) = \cos\left(2\frac{\pi\beta}{2}\right) = \cos^2\left(\frac{\pi\beta}{2}\right) - \sin^2\left(\frac{\pi\beta}{2}\right)$ so that

$$\begin{aligned}\tan(\pi\beta) + \tan\left(\frac{\pi\beta}{2}\right) &= \frac{\sin(\pi\beta)}{\cos(\pi\beta)} + \frac{\sin\left(\frac{\pi\beta}{2}\right)}{\cos\left(\frac{\pi\beta}{2}\right)} \\ &= \frac{2\sin\left(\frac{\pi\beta}{2}\right)\cos\left(\frac{\pi\beta}{2}\right)}{\cos^2\left(\frac{\pi\beta}{2}\right) - \sin^2\left(\frac{\pi\beta}{2}\right)} + \frac{\sin\left(\frac{\pi\beta}{2}\right)}{\cos\left(\frac{\pi\beta}{2}\right)} = 0\end{aligned}$$

so multiplying by $\cos\left(\frac{\pi\beta}{2}\right)\left(\cos^2\left(\frac{\pi\beta}{2}\right) - \sin^2\left(\frac{\pi\beta}{2}\right)\right)$ on both sides yields

$$2\sin\left(\frac{\pi\beta}{2}\right)\cos^2\left(\frac{\pi\beta}{2}\right) + \sin\left(\frac{\pi\beta}{2}\right)\cos^2\left(\frac{\pi\beta}{2}\right) - \sin^3\left(\frac{\pi\beta}{2}\right) = 0$$

which is to say that

$$3\cos^2\left(\frac{\pi\beta}{2}\right) = \sin^2\left(\frac{\pi\beta}{2}\right) = 1 - \cos^2\left(\frac{\pi\beta}{2}\right)$$

and thus $4\cos^2\left(\frac{\pi\beta}{2}\right) = 1$ so that $\cos\left(\frac{\pi\beta}{2}\right) = \frac{1}{2}$. This immediately implies that $\frac{\pi\beta}{2} = \frac{\pi}{3}$ and therefore $\beta = 2/3$. Therefore, $\beta \in (0, 1/2)$ is impossible.

Now, in the case where $\beta = 1/2$, then we see that $\tilde{\eta} = |u|^{1/2}$ and so we can compute that

$$K\tilde{\eta} = AK|u|^{1/2} = -\frac{A}{2}\tan\left(\frac{\pi}{4}\right)|u|^{1/2} = -\frac{A}{2}|u|^{1/2}$$

Moreover, we see that, using Proposition 7.5 then

$$\begin{aligned}K\tilde{\eta}^2 = K|u| &= -A^2\frac{\partial}{\partial u}H|u| \\ &= -A^2\left(\frac{\partial}{\partial u}\frac{u}{\pi}\left((\ln(|\pi - u|) - \ln(|\pi + u|)) - 2\ln(|u|)\right) + \frac{\partial}{\partial u}G_1(u)\right) \\ &= A^2\left(\frac{1}{\pi}\left(\ln(|\pi + u|) - \ln(|\pi - u|)\right) + \frac{2}{\pi}\ln(|u|) + \frac{u}{\pi(\pi - u)} + \frac{u}{\pi(\pi + u)} + \frac{2}{\pi}\right) + h.o.t\end{aligned}$$

and thus the Babenko equation becomes

$$\begin{aligned}A|u|^{1/2} &= \frac{c^2}{2} + \frac{A}{2}K|u| + A|u|^{1/2}KA|u|^{1/2} \\ &= \frac{c^2}{2} + \frac{A^2}{2}\left(\frac{1}{\pi}\left(\ln(|\pi + u|) - \ln(|\pi - u|)\right) + \frac{2}{\pi}\ln(|u|) + \frac{u}{\pi(\pi - u)} + \frac{u}{\pi(\pi + u)} + \frac{2}{\pi} - |u|\right) + h.o.t\end{aligned}$$

However, this cannot hold, given that there is a logarithmic singularity which appears on the right-hand side that does not occur on the left-hand side. Therefore, the case $\beta = 1/2$ cannot occur.

Finally, let us suppose that $\beta \in (1/2, 1)$. In this case, it will still follow that $K|u|^{\beta} = -\beta\tan\left(\frac{\pi\beta}{2}\right)|u|^{\beta-1}$. However, the computation for $K|u|^{2\beta}$ is slightly more involved, given that $2\beta \in (1, 2)$. In order to use the preceding formulas, we can simply compute the derivative of this quantity and then integrate the answer to obtain the desired term. In particular, observe that

$$\begin{aligned}\frac{\partial}{\partial u}K|u|^{2\beta} = K\frac{\partial}{\partial u}|u|^{2\beta} &= 2\beta K|u|^{2\beta-1}\mathrm{sgn}(u) \\ &= -2\beta(2\beta - 1)H|u|^{2\beta-2} \\ &= 2\beta(2\beta - 1)\cot\left(\frac{(2\beta - 1)\pi}{2}\right)|u|^{2\beta-2}\mathrm{sgn}(u) + h.o.t.\end{aligned}$$

Now, integrating this equation we obtain that

$$K|u|^{2\beta} = 2\beta \cot\left(\frac{(2\beta-1)\,\pi}{2}\right)|u|^{2\beta-1} + h.o.t.$$

and hence we may again write our Babenko equation in the following way:

$$\begin{aligned} A|u|^{\beta} &= \frac{c^2}{2} + \frac{1}{2}K|u|^{2\beta} + |u|^{\beta}K|u|^{\beta} \\ &= \frac{c^2}{2} + A\beta\cot\left(\frac{(2\beta-1)\,\pi}{2}\right)|u|^{2\beta-1} - A\beta\tan\left(\frac{\pi\beta}{2}\right)|u|^{2\beta-1} + h.o.t. \\ &= \frac{c^2}{2} + A\beta\left(\cot\left(\frac{(2\beta-1)\,\pi}{2}\right) - \tan\left(\frac{\pi\beta}{2}\right)\right)|u|^{2\beta-1} + h.o.t \end{aligned}$$

and so just as before in order for this equation to hold, it must be that the coefficient on the $|u|^{2\beta-1}$ term is precisely zero, which means that

$$\cot\left(\frac{(2\beta-1)\,\pi}{2}\right) - \tan\left(\frac{\pi\beta}{2}\right) = 0$$

Of course, $\cot\left(\frac{(2\beta-1)\pi}{2}\right) = \cot\left(\beta\pi - \frac{\pi}{2}\right) = -\tan(\pi\beta)$ so we require that $-\tan(\pi\beta) - \tan\left(\frac{\pi\beta}{2}\right) = 0$ and thus $\tan(\pi\beta) + \tan\left(\frac{\pi\beta}{2}\right) = 0$, which we saw gives precisely $\beta = 2/3$.

$$\tan(\pi\beta) + \tan\left(\frac{\pi\beta}{2}\right) = 0$$

Hence, we conclude that it is only possible to have $\beta = 2/3$, as desired. □

Therefore, we see that the requirement that the first term in the Stokes wave expansion of the limiting wave be 2/3 in order to preserve the $2\pi/3$ angle at the peak is reproduced and enforced by the Babenko equation. Note that in Theorem 4.5, we took $\beta \in (0,1)$ as we know there must exist a singularity in the derivative, so β must be restricted to this interval. Additionally, we may perform a similar computation to determine the behaviour of the second-order correction term in such an expansion (30), as is shown in the following theorem:

Theorem 4.6. If Equation (32) admits a solution such that $\tilde{\eta} = A|u|^{2/3} + B|u|^{\mu} + o(|u|^{\mu})$ as $|u| \to 0$, where the power of the first term is 2/3 by Theorem 4.5 and $\mu > 2/3$, then it must be that $\mu \geq \mu^*$ where μ^* is the second root of the equation

$$(\mu + 2/3)\cot\left(\frac{(\mu - 1/3)\pi}{2}\right) - \mu\tan\left(\frac{\mu\pi}{2}\right) - \frac{2}{\sqrt{3}} = 0. \tag{35}$$

Proof. Just as in the proof of Theorem 4.5, we may split the proof into three cases to be considered. However, we will need to adjust the cases depending upon which interval μ is assumed to be residing. We will, nonetheless, follow the same argument, matching powers in the Babenko equation. Now, since we are taking $\tilde{\eta} = A|u|^{2/3} + B|u|^\mu + o(|u|^\mu)$ then Equation (32) reads as

$$\begin{aligned}
A|u|^{2/3} + B|u|^\mu + o(|u|^\mu) = \tilde{\eta} &= \frac{c^2}{2} + \frac{1}{2}K\tilde{\eta}^2 + \tilde{\eta}K\tilde{\eta} \\
&= \frac{c^2}{2} + \frac{1}{2}K\left(A^2|u|^{4/3} + 2AB|u|^{\mu+2/3} + B^2|u|^{2\mu} + o(|u|^{2\mu})\right) \\
&\quad + \left(A|u|^{2/3} + B|u|^\mu + o(|u|^\mu)\right) K \left(A|u|^{2/3} + B|u|^\mu + o(|u|^\mu)\right) \\
&= \frac{c^2}{2} + \frac{A^2}{2}K|u|^{4/3} + ABK|u|^{\mu+2/3} + \frac{B^2}{2}K|u|^{2\mu} + o(|u|^{2\mu-1}) \\
&\quad + \left(A|u|^{2/3} + B|u|^\mu + o(|u|^\mu)\right)\left(AK|u|^{2/3} + BK|u|^\mu + o(|u|^{\mu-1})\right) \\
&= \frac{c^2}{2} + \frac{A^2}{2}K|u|^{4/3} + ABK|u|^{\mu+2/3} + \frac{B^2}{2}K|u|^{2\mu} + A^2|u|^{2/3}K|u|^{2/3} + AB|u|^{2/3}K|u|^\mu \\
&\quad + AB|u|^\mu K|u|^{2/3} + B^2|u|^\mu K|u|^\mu + o(|u|^{\mu-1/3}) \\
&= \frac{c^2}{2} + \frac{A^2}{2}K|u|^{4/3} + ABK|u|^{\mu+2/3} + A^2|u|^{2/3}K|u|^{2/3} + AB|u|^{2/3}K|u|^\mu \\
&\quad + \mathcal{O}(|u|^{2\mu-1}) + o(|u|^{\mu-1/3})
\end{aligned}$$

where we see the last line follows since $K|u|^{2\mu}$ and $|u|^\mu K|u|^\mu$ are $\mathcal{O}(|u|^{2\mu-1})$. However, note that we are assuming $\mu > 2/3$ and hence this implies $2\mu - 1 > \mu - 1/3$, so that $\mathcal{O}(|u|^{2\mu-1}) + o(|u|^{\mu-1/3}) = o(|u|^{\mu-1/3})$. Therefore, the Babenko equation becomes

$$A|u|^{2/3} + B|u|^\mu + o(|u|^\mu) = \frac{c^2}{2} + \frac{A^2}{2}K|u|^{4/3} + ABK|u|^{\mu+2/3} + A^2|u|^{2/3}K|u|^{2/3} + AB|u|^{2/3}K|u|^\mu + o(|u|^{\mu-1/3})$$

Now, the evaluation of these leading order terms under the Babenko operator generally depends on the intervals for which μ is contained. Of course, there are a few terms that we can begin by immediately evaluating. Let us observe that our proceeding formulas imply that

$$\begin{aligned}
K|u|^{2/3} &= -\frac{\partial}{\partial u}H|u|^{2/3} = -\frac{\partial}{\partial u}\sqrt{3}|u|^{2/3}\mathrm{sgn}(u) + h.o.t. \\
&= -\frac{2}{\sqrt{3}}|u|^{-1/3} + h.o.t.
\end{aligned}$$

Moreover, let us see that

$$\begin{aligned}
K|u|^{4/3} &= -\frac{\partial}{\partial u}H|u|^{4/3} = \frac{\partial}{\partial u}\sqrt{3}|u|^{4/3}\mathrm{sgn}(u) + h.o.t. \\
&= \frac{4}{\sqrt{3}}|u|^{1/3} + h.o.t.
\end{aligned}$$

and thus the Babenko equation now reads as

$$\begin{aligned}
A|u|^{2/3} + B|u|^\mu + o(|u|^\mu) &= \frac{c^2}{2} + \frac{2A^2}{\sqrt{3}}|u|^{1/3} + ABK|u|^{\mu+2/3} - A^2\frac{2}{\sqrt{3}}|u|^{1/3} + AB|u|^{2/3}K|u|^\mu \\
&\quad - AB|u|^\mu\frac{2}{\sqrt{3}}|u|^{-1/3} + o(|u|^{\mu-1/3}) \\
&= \frac{c^2}{2} + AB\left(-\frac{2}{\sqrt{3}}|u|^{\mu-1/3} + K|u|^{\mu+2/3} + |u|^{2/3}K|u|^\mu\right) + h.o.t.
\end{aligned}$$

and so in general one needs to be able to compute the quantities $K|u|^\mu$ and $K|u|^{\mu+2/3}$ for a given value of μ in order to reduce this equation. Of course, just as before, this will involve a case analysis on the value of μ, differentiating the terms sufficiently so that Lemma 4.3 may be invoked. However, we may streamline the computations by computing $K|u|^\gamma$ for γ in a few intervals:

- If $\gamma \in (0,1)$, then we see that $\gamma - 1 \in (-1,0)$ so we may apply Lemma 4.3. Hence

$$K|u|^\gamma = -H\frac{\partial}{\partial u}|u|^\gamma = -\gamma H\left(|u|^{\gamma-1}\mathrm{sgn}(u)\right) = -\gamma\tan\left(\frac{\gamma\pi}{2}\right)|u|^{\gamma-1} + h.o.t.$$

- If $\gamma \in (1,2)$ then we see that $\gamma - 2 \in (-1,0)$ so we may apply Lemma 4.3. Hence

$$\frac{\partial}{\partial u}K|u|^\gamma = -H\frac{\partial^2}{\partial^2 u}|u|^\gamma = -\gamma(\gamma-1)H\left(|u|^{\gamma-2}\right) = \gamma(\gamma-1)\cot\left(\frac{(\gamma-1)\pi}{2}\right)|u|^{\gamma-2} + h.o.t.$$

  and so integrating this equation shows that

$$K|u|^\gamma = \gamma\cot\left(\frac{(\gamma-1)\pi}{2}\right)|u|^{\gamma-1} + h.o.t.$$

  as desired.

- If $\gamma \in (2,3)$ then we see that $\gamma - 3 \in (-1,0)$ so we may apply Lemma 4.3. Hence

$$\frac{\partial^2}{\partial^2 u}K|u|^\gamma = -H\frac{\partial^3}{\partial^3 u}|u|^\gamma = -\gamma(\gamma-1)(\gamma-2)H\left(|u|^{\gamma-3}\mathrm{sgn}(u)\right) = -\gamma(\gamma-1)(\gamma-2)\tan\left(\frac{(\gamma-2)\pi}{2}\right)|u|^{\gamma-3} + h.o.t.$$

  and so integrating this equation shows that

$$\frac{\partial}{\partial u}K|u|^\gamma = -\gamma(\gamma-1)\tan\left(\frac{(\gamma-2)\pi}{2}\right)|u|^{\gamma-2} + h.o.t.$$

  Moreover, integrating again yields

$$K|u|^\gamma = -\gamma\tan\left(\frac{(\gamma-2)\pi}{2}\right)|u|^{\gamma-1} + h.o.t.$$

Now, that we have performed these computations, we may utilize the formulas to reduce our Babenko equation.

First, suppose that $\mu \in \left(\frac{2}{3},1\right)$. Then in particular, since $\mu \in (0,1)$ we may use our previous formula to observe that

$$K|u|^\mu = -\mu\tan\left(\frac{\mu\pi}{2}\right)|u|^{\mu-1} + h.o.t.$$

Furthermore, it will follow that $\mu + 2/3 \in \left(\frac{4}{3},\frac{5}{3}\right) \subset (1,2)$ and hence $\mu + 2/3 \in (1,2)$ so by our previous calculations,

$$K|u|^{\mu+2/3} = (\mu+2/3)\cot\left(\frac{(\mu-1/3)\pi}{2}\right)|u|^{\mu-1/3} + h.o.t.$$

Therefore, in this case the Babenko equation reads as

$$A|u|^{2/3} + B|u|^\mu + o(|u|^\mu) = \frac{c^2}{2} + AB\left(-\frac{2}{\sqrt{3}} + (\mu+2/3)\cot\left(\frac{(\mu-1/3)\pi}{2}\right) - \mu\tan\left(\frac{\mu\pi}{2}\right)\right)|u|^{\mu-1/3} + h.o.t$$

Now, note that we have $\mu - 1/3 \neq 2/3$ since $\mu \in (2/3,1)$ and $\mu - 1/3 < \mu$. Therefore, there is no $|u|^{\mu-1/3}$ term that appears on the left-hand side of the Babenko equation above. In order to have the proper balance in the equation, we will thus require that the $|u|^{\mu-1/3}$ term cancels on the right-hand side. That is, we need to have that

$$AB\left(-\frac{2}{\sqrt{3}} + (\mu+2/3)\cot\left(\frac{(\mu-1/3)\pi}{2}\right) - \mu\tan\left(\frac{\mu\pi}{2}\right)\right) = 0$$

and thus

$$(\mu+2/3)\cot\left(\frac{(\mu-1/3)\pi}{2}\right) - \mu\tan\left(\frac{\mu\pi}{2}\right) - \frac{2}{\sqrt{3}} = 0 \tag{36}$$

Now, let us determine the roots of this equation. Consider Figure 3, which plots this equation as a function of μ:

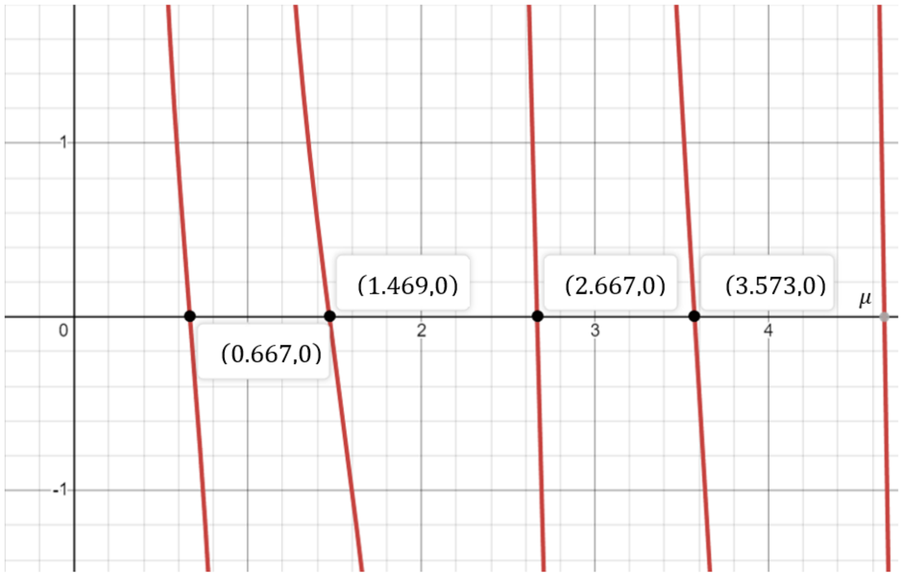

Figure 3: Plot of the equation $(\mu + 2/3)\cot\left(\frac{(\mu-1/3)\pi}{2}\right) - \mu\tan\left(\frac{\mu\pi}{2}\right) - \frac{2}{\sqrt{3}} = 0$ for values of μ. The roots have been computed numerically and are displayed.

Observe that the first possible root is $\mu = 2/3$. Since we took $\mu > 2/3$, then this root is inadmissible. Since the next root to the equation (36) is $\mu \approx 1.469 > 1$ then this contradicts our assumption that $\mu \in \left(\frac{2}{3}, 1\right)$, which shows this case is not possible. Now, suppose that we have $\mu = 1$. Then we see that

$$\begin{aligned}
K|u|^{\mu} &= -\frac{\partial}{\partial u} H|u| \\
&= -\frac{\partial}{\partial u}\frac{u}{\pi}\left((\ln(|\pi-u|) - \ln(|\pi+u|)) - 2\ln(|u|)\right) + \frac{\partial}{\partial u}G_1(u) \\
&= \frac{1}{\pi}\left(\ln(|\pi+u|) - \ln(|\pi-u|)\right) + \frac{2}{\pi}\ln(|u|) + \frac{u}{\pi(\pi-u)} + \frac{u}{\pi(\pi+u)} + 2 + h.o.t
\end{aligned}$$

Moreover, $\mu + 2/3 = 5/3 \in (1, 2)$ and therefore we see that

$$K|u|^{\mu+2/3} = K|u|^{5/3} = \frac{5}{3}\cot\left(\frac{\pi}{3}\right)|u|^{2/3} + h.o.t. = \frac{5}{3\sqrt{3}}|u|^{2/3} + h.o.t.$$

Therefore, in this case the Babenko equation reads as

$$\begin{aligned}
A|u|^{2/3} + B|u| + h.o.t. = \frac{c^2}{2} + AB\Bigg(&-\frac{2}{\sqrt{3}} + \frac{5}{3\sqrt{3}} - \frac{1}{\pi}\left(\ln(|\pi-u|) - \ln(|\pi+u|)\right) + \frac{2}{\pi}\ln(|u|) + \frac{u}{\pi(\pi-u)} \\
&+ \frac{u}{\pi(\pi+u)} + 2\Bigg)|u|^{2/3} + h.o.t.
\end{aligned}$$

Thus, we see that there are logarithmic terms on the right-hand side of this equation, however, such logarithmic singularities do not appear on the left-hand side of the equation. This contradicts the statement of the Babenko equation, and thus we have that $\mu \neq 1$.

Next, suppose that $\mu \in \left(1, \frac{4}{3}\right)$. Then in particular, since $\mu \in (1, 2)$ we may use our previous calculations to observe that

$$K|u|^{\mu} = \mu\cot\left(\frac{(\mu-1)\pi}{2}\right)|u|^{\mu-1} + h.o.t.$$

Furthermore, it will follow that $\mu + 2/3 \in \left(\frac{5}{3}, 2\right) \subset (1,2)$ and hence $\mu + 2/3 \in (1,2)$ so it will follow by the previous formula that

$$K|u|^{\mu+2/3} = (\mu + 2/3)\cot\left(\frac{(\mu-1/3)\pi}{2}\right)|u|^{\mu-1/3} + h.o.t.$$

Therefore, in this case the Babenko equation reads as

$$|u|^{2/3} + |u|^{\mu} + o(|u|^{\mu}) = \frac{c^2}{2} + AB\left(-\frac{2}{\sqrt{3}} + (\mu+2/3)\cot\left(\frac{(\mu-1/3)\pi}{2}\right) + \mu\cot\left(\frac{(\mu-1)\pi}{2}\right)\right)|u|^{\mu-1/3} + h.o.t$$

Now, note that we have $\mu - 1/3 \neq 2/3$ since $\mu \in (1, 4/3)$ and $\mu - 1/3 < \mu$. Therefore, there is no $|u|^{\mu-1/3}$ term that appears on the left-hand side of the Babenko equation above. In order to have the proper balance in the equation, we will thus require that the $|u|^{\mu-1/3}$ term cancels on the right-hand side. That is, we need to have that

$$AB\left(-\frac{2}{\sqrt{3}} + (\mu+2/3)\cot\left(\frac{(\mu-1/3)\pi}{2}\right) + \mu\cot\left(\frac{(\mu-1)\pi}{2}\right)\right) = 0$$

and thus

$$(\mu+2/3)\cot\left(\frac{(\mu-1/3)\pi}{2}\right) + \mu\cot\left(\frac{(\mu-1)\pi}{2}\right) - \frac{2}{\sqrt{3}} = 0 \tag{37}$$

Of course, since $\cot\left(\frac{(\mu-1)\pi}{2}\right) = \cot\left(\frac{\mu\pi}{2} - \frac{\pi}{2}\right) = -\tan\left(\frac{\mu\pi}{2}\right)$ then we have that (37) reduces immediately to (36). Since Figure (3) shows that the next root is $\mu \approx 1.469 > \frac{4}{3}$, then this contradicts the assumption that $\mu \in \left(1, \frac{4}{3}\right)$, showing this cannot happen.

Now, suppose that we have $\mu = 4/3$. As we have computed in the Appendix in Corollary 7.4, then $K|u|^{\mu} = K|u|^{4/3} = \frac{4}{\sqrt{3}}|u|^{1/3}$. Moreover, $\mu + 2/3 = 4/3 + 2/3 = 2$ and we so we thus see it follows from Proposition 7.5 that

$$K|u|^{\mu+2/3} = K|u|^2 = \frac{2u}{\pi}\left(\ln(|\pi+u|) - \ln(|\pi-u|)\right) + \frac{u^2}{\pi(\pi-u)} + \frac{u^2}{\pi(\pi+u)} + h.o.t$$

Therefore, in this case the Babenko equation reads as

$$|u|^{2/3} + |u| = \frac{c^2}{2} + AB\left(-\frac{2}{\sqrt{3}}|u| + \frac{2u}{\pi}\left(\ln(|\pi+u|) - \ln(|\pi-u|)\right) + \frac{u^2}{\pi(\pi-u)} + \frac{u^2}{\pi(\pi+u)} + \frac{4}{\sqrt{3}}|u|^{1/3}\right)$$

Thus, we see that there are logarithmic terms on the right-hand side of this equation which do not cancel, however, such logarithmic singularities do not appear on the left-hand side of the equation. This contradicts the statement of the Babenko equation, and thus we have that $\mu \neq 4/3$.

Finally, suppose that $\mu \in \left(\frac{4}{3}, 2\right)$. Then in particular, since $\mu \in (1,2)$ we may use our previous formula to observe that

$$K|u|^{\mu} = \mu\cot\left(\frac{(\mu-1)\pi}{2}\right)|u|^{\mu-1} + h.o.t.$$

Furthermore, it will follow that $\mu + 2/3 \in \left(2, \frac{8}{3}\right) \subset (2,3)$ and hence $\mu + 2/3 \in (2,3)$ so it will follow by the previous formula that

$$K|u|^{\mu+2/3} = -(\mu+2/3)\tan\left(\frac{(\mu-4/3)\pi}{2}\right)|u|^{\mu-1/3} + h.o.t.$$

Therefore, in this case the Babenko equation reads as

$$|u|^{2/3} + |u|^{\mu} + o(|u|^{\mu}) = \frac{c^2}{2} + AB\left(-\frac{2}{\sqrt{3}} - (\mu+2/3)\tan\left(\frac{(\mu-4/3)\pi}{2}\right) + \mu\cot\left(\frac{(\mu-1)\pi}{2}\right)\right)|u|^{\mu-1/3} + h.o.t.$$

Note again that we have $\mu - 1/3 \neq 2/3$ since $\mu \neq 1$ and $\mu - 1/3 < \mu$. Therefore, there is no $|u|^{\mu-1/3}$ term that appears on the left-hand side of the Babenko equation above. In order to have the proper balance in the equation, we will thus require that the $|u|^{\mu-1/3}$ term cancels on the right-hand side. That is, we need to have that

$$AB\left(-\frac{2}{\sqrt{3}}-(\mu+2/3)\tan\left(\frac{(\mu-4/3)\pi}{2}\right)+\mu\cot\left(\frac{(\mu-1)\pi}{2}\right)\right)=0$$

and thus

$$-(\mu+2/3)\tan\left(\frac{(\mu-4/3)\pi}{2}\right)+\mu\cot\left(\frac{(\mu-1)\pi}{2}\right)-\frac{2}{\sqrt{3}}=0$$

Of course, since $\cot\left(\frac{(\mu-1)\pi}{2}\right)=\cot\left(\frac{\mu\pi}{2}-\frac{\pi}{2}\right)=-\tan\left(\frac{\mu\pi}{2}\right)$ then this equation can be rewritten as

$$-(\mu+2/3)\tan\left(\frac{(\mu-4/3)\pi}{2}\right)-\mu\tan\left(\frac{\mu\pi}{2}\right)-\frac{2}{\sqrt{3}}=0$$

Further, given that $\tan\left(\frac{(\mu-4/3)\pi}{2}\right)=\tan\left(\frac{\mu\pi}{2}-\frac{\pi/3}{2}-\frac{\pi}{2}\right)=-\cot\left(\frac{(\mu-1/3)\pi}{2}\right)$ then we see that this equation reduces again to

$$(\mu+2/3)\cot\left(\frac{(\mu-1/3)\pi}{2}\right)-\mu\tan\left(\frac{\mu\pi}{2}\right)-\frac{2}{\sqrt{3}}=0$$

and this is identical to (36). Since Figure (3) shows there does, in fact, exist a root in this interval $\mu^*\approx 1.469$, then we see that this is the first admissible root (and it is the second root of this equation). It is therefore possible that μ takes this value. Hence, we have that $\mu\geq\mu^*$ where μ^* is the second solution to the equation

$$(\mu+2/3)\cot\left(\frac{(\mu-1/3)\pi}{2}\right)-\mu\tan\left(\frac{\mu\pi}{2}\right)-\frac{2}{\sqrt{3}}=0$$

In particular, we have shown that it must be that $\mu>1$. □

Remark 4.7. Theorem 4.6 demonstrates that the power μ of the second-order correction term in the Stokes wave expansion for the peaked wave satisfies Equation (35), and in fact it must be greater than or equal to μ^*, the second root of this equation. In the proof of this Theorem, we demonstrated numerically that $\mu^*\approx 1.469$ and this matches the lower bound that Grant obtained for the second-order correction [14]. Moreover, Grant showed that the power of the second-order correction term is an irrational number that satisfies the equation (31). However, one may verify that any solution to (31) must also satisfy the equation (36) derived using the Babenko equation. Hence, this shows that the Babenko equation is able to recover all the possible powers for the second-order correction term in the Stokes wave expansion for the peaked wave. Moreover, it is able to verify that the only contribution to the singularity in the first derivative in the Stokes expansion (30) is due to the first term in the expansion.

Now that we have utilized the Babenko equation to verify existing results surrounding the nature of the leading order singularity in the expansion of the surface profile of the peaked Stokes wave, as well as the second-order correction term, it is natural to wonder whether the Babenko equation provides information surrounding the coefficients in the Stokes expansion. While there exists numerical results as to what these coefficients should be, it is not a straightforward result analytically, and thus the current literature suggests that there are no analytical results which describe exactly which coefficients are admissible [12].

Remark 4.8. Using the Babenko equation for the Stokes wave expansion, we also do not have information about the admissible values for $c>0$ and $A>0$ in the asymptotic expansion $\eta(u)=\frac{c^2}{2}-A|u|^{2/3}+\mathcal{O}\left(|u|^{\mu^*}\right)$ as $|u|\to 0$, where μ^* is the second root in the equation (35). This is because these coefficients are defined for the Babenko equation (18) by the constants in the expansions

$$K\tilde{\eta}=-\frac{2A}{\sqrt{3}}|u|^{-1/3}+c_1+o(1)$$

$$\text{and } K\tilde{\eta}^2=\frac{4A^2}{\sqrt{3}}|u|^{1/3}+c_2+o(1)$$

where c_1, c_2 are defined on the entire profile $\tilde{\eta}$. Nevertheless, we can see that the expansion (18) is consistent if $c_1 = 1$ and $c_2 = -c^2$. To show this, take $\tilde{\eta}(u) = A|u|^{2/3} + o\left(|u|^{\mu^*}\right)$ as $|u| \to 0$, and so we may thus directly compute

$$\begin{aligned} K\tilde{\eta} =& -A\frac{\partial}{\partial u}H|u|^{2/3} + o(1) \\ =& -\frac{2A}{\sqrt{3}}|u|^{-1/3} + c_0 + o(1) \end{aligned}$$

where we've used the fact that $K|u|^{2/3} = -A\frac{\partial}{\partial u}\sqrt{3}|u|^{2/3}\text{sgn}(u) + h.o.t. = -\frac{2A}{\sqrt{3}}|u|^{-1/3} + h.o.t.$ and we've taken c_0 constant that is determined from the higher order terms in the evaluation of $K|u|^{2/3}$ as well as $Ko(|u|)$. Moreover, we see that $\tilde{\eta}^2 = A^2|u|^{4/3} + o\left(|u|^{2/3}\right)$ and so we thus see that

$$\begin{aligned} K\tilde{\eta}^2 &= A^2K|u|^{4/3} + o\left(|u|^{2/3}\right) \\ &= A^2\frac{\partial}{\partial u}\sqrt{3}|u|^{4/3}\text{sgn}(u) + o\left(|u|^{2/3}\right) \\ &= A^2\frac{4}{\sqrt{3}}|u|^{1/3} + \tilde{c}_0 + o\left(|u|^{2/3}\right) \end{aligned}$$

where again we've used the fact that $K|u|^{4/3} = A^2\frac{\partial}{\partial u}\sqrt{3}|u|^{4/3}\text{sgn}(u) + h.o.t. = A^2\frac{4}{\sqrt{3}}|u|^{1/3} + h.o.t.$ and we've taken $\tilde{c}_0$ to be a constant that is determined from the higher order terms in the evaluation of $K|u|^{4/3}$ as well as $Ko\left(|u|^{2/3}\right)$. Thus, inserting these quantities into the Babenko equation (32), then we observe that

$$A|u|^{2/3} + \mathcal{O}\left(|u|^{\mu^*}\right) = \frac{c^2}{2} + \frac{1}{2}\left(\frac{4A^2}{\sqrt{3}}|u|^{1/3} + \tilde{c}_0 + o\left(|u|^{2/3}\right)\right) + \left(A|u|^{2/3} + \mathcal{O}\left(|u|^{\mu^*}\right)\right)\left(-\frac{2A}{\sqrt{3}}|u|^{-1/3} + c_0 + o(1)\right)$$

and hence

$$\begin{aligned} A|u|^{2/3} + \mathcal{O}\left(|u|^{\mu^*}\right) &= \frac{c^2}{2} + \frac{\tilde{c}_0}{2} + A^2\frac{2}{\sqrt{3}}|u|^{1/3} - \frac{2A^2}{\sqrt{3}}|u|^{1/3} + c_0A|u|^{2/3} + o\left(|u|^{2/3}\right) \\ &= \frac{c^2}{2} + \frac{\tilde{c}_0}{2} + c_0A|u|^{2/3} + o\left(|u|^{2/3}\right) \end{aligned}$$

and thus since the left-hand side only contains one term involving $|u|^{2/3}$ then we must take $Ac_0 = A$ and thus $c_0 = 1$. Moreover, the constant terms must also cancel to give $c^2 = -\tilde{c}_0$. Thus, in performing this expansion, we have obtained that

$$\begin{aligned} \tilde{\eta} &= A|u|^{2/3} + \mathcal{O}\left(|u|^{\mu^*}\right) \\ K\tilde{\eta} &= -\frac{2A}{\sqrt{3}}|u|^{-1/3} + 1 + o(1) \\ K\tilde{\eta}^2 &= A^2\frac{4}{\sqrt{3}}|u|^{1/3} - c^2 + o\left(|u|^{2/3}\right) \end{aligned} \tag{38}$$

and hence we observe that the constant A has not been uniquely identified after inserting the expansion $\tilde{\eta}(u) = A|u|^{2/3} + \mathcal{O}\left(|u|^{\mu^*}\right)$ into the Babenko equation. In fact, we obtain the nonlocal problem (38) which demonstrates that the constant A must depend itself on the other terms in $\mathcal{O}\left(|u|^{\mu^*}\right)$ in the expansion. Hence, the Babenko equation has confirmed that, in general, we do not have information relating to the coefficients in the peaked Stokes wave expansion.

5 Conclusion

Throughout this thesis, we have reviewed the literature, noting that the non-limiting Stokes waves in a fluid of infinite depth are smooth, with properties that may be analyzed through small amplitude expansions. We also reviewed the fact that there exists a wave of greatest amplitude, which has a singularity at its peak. We have shown that, from the governing equations of motion when applied to smooth Stokes waves, we may derive the Babenko equation (18) for travelling wave solutions of the Euler equations. With the Babenko equation, we analyzed smooth Stokes waves, determining the coefficients in the small amplitude expansion of the surface profile and wave speed, as well as for the small amplitude expansion of the eigenvalues for the linearized Babenko operator existing in the even subspace. We were able to demonstrate that the Babenko equation reproduces the known results about these quantities in the small amplitude limit. Moreover, we were able to verify that the Babenko equation yields no contradictions to the current literature when applied to the limiting Stokes waves, thus lending credence to its ability to gain new information for the wave of greatest height. Finally, we were able to use the Babenko equation to explicitly determine the first two powers in the Stokes expansion of the surface profile near the singularity. Altogether, we are able to see that the Babenko equation reproduces known results for both the limiting and non-limiting Stokes waves, thus providing evidence that the Babenko equation holds for the peaked wave. Hence, the results of this thesis allow for us to conclude that the Babenko equation is a powerful tool that provides another avenue to investigate the Stokes waves solutions to the Euler equations.

There are several opportunities for future work to extend the results of this thesis. To begin, one can investigate the properties surrounding the coefficients in the Stokes wave expansion at the limiting wave. As demonstrated in Remark 4.8, we do not have information relating to the coefficients in the expansion, and so one could investigate properties relating to the magnitude and possible uniqueness of such coefficients. Moreover, while it is known that the surface profile may be expanded asymptotically in the Stokes wave expansion (30), another opportunity for future research is the investigation of the rate of convergence of this expansion, which could include utilizing tools from the current literature as well as the Babenko equation. Additionally, an analysis of the spectrum of the operators in the linearized equations of stability (22) can be performed, in order to understand the stability of the limiting Stokes wave. Finally, an extension of the results to fluids at a finite depth may also be pursued. One might derive an analogous Babenko equation in the shallow water limit, and a Babenko operator K_h for a fluid at a depth h. Corresponding asymptotic expansions of the surface profile may be examined, with similar analyses of the structure of the wave as well as an analysis of the resulting stability.

6 Appendix A: Common Properties of Hilbert Transform

In this Appendix, we will prove some basic properties of the Hilbert transform which are employed throughout the proofs of the various claims in this thesis. To begin, we analyze the action of the Hilbert transform on the sine and cosine functions.

Proposition 6.1. The Hilbert transform H is such that, for all $m \in \mathbb{Z}$ then

$$H(\sin(mx)) = \cos(mx) \text{ and } H(\cos(mx)) = -\sin(mx)$$

Proof. First, to determine $H(\sin(mx))$, we'll need to compute the Fourier modes of $f(x) := \sin(mx)$. These are given by the functions f_k (for $-\infty < k < \infty$) where

$$f(x) = \sum_{k=-\infty}^{\infty} f_k e^{ikx\frac{2\pi}{\lambda}} = \sum_{k=-\infty}^{\infty} f_k e^{ikmx}$$

since $f(x)$ is a $\lambda = 2\pi/m$ periodic function. Now, since we can write that

$$\sin(mx) = \frac{e^{imx} - e^{-imx}}{2i}$$

then it follows that

$$f_k = \begin{cases} \frac{1}{2i} & k = 1 \\ -\frac{1}{2i} & k = -1 \\ 0 & \text{else} \end{cases}$$

and therefore we have that equation (7) shows that

$$
\begin{aligned}
H(f(x)) = & \sum_{k=-\infty}^{\infty} i\text{sgn}\,(k)\, f_k e^{ikx\frac{2\pi}{\lambda}} = \sum_{k=-\infty}^{\infty} i\text{sgn}\,(k)\, f_k e^{ikmx} \\
&= i\text{sgn}\,(-1)\, f_{-1} e^{-imx} + i\text{sgn}\,(1)\, f_1 e^{imx} \\
&= -i\left(\frac{-1}{2i}\right) e^{-imx} + i\left(\frac{1}{2i}\right) e^{imx} \\
&= \frac{e^{imx} + e^{-imx}}{2} = \cos(mx)
\end{aligned}
$$

and hence we see that $H(\sin(mx)) = \cos(mx)$. Moreover, to determine $H(\cos(mx))$, we perform a similar analysis, first computing the Fourier modes of $g(x) := \cos(mx)$. These are given by the functions g_k (for $-\infty < k < \infty$) for which

$$
g(x) = \sum_{k=-\infty}^{\infty} g_k e^{ikx\frac{2\pi}{\lambda}} = \sum_{k=-\infty}^{\infty} g_k e^{ikmx}
$$

since $g(x)$ is a $\lambda = 2\pi/m$ periodic function. Now, since we can write that

$$
\cos(mx) = \frac{e^{imx} + e^{-imx}}{2}
$$

then it follows that

$$
g_k = \begin{cases} \frac{1}{2} & k = 1 \\ \frac{1}{2} & k = -1 \\ 0 & \text{else} \end{cases}
$$

and therefore we have that equation (7) shows that

$$
\begin{aligned}
H(g(x)) = & \sum_{k=-\infty}^{\infty} i\text{sgn}\,(k)\, g_k e^{ikx\frac{2\pi}{\lambda}} = \sum_{k=-\infty}^{\infty} i\text{sgn}\,(k)\, g_k e^{ikmx} \\
&= i\text{sgn}\,(-1)\, g_{-1} e^{-imx} + i\text{sgn}\,(1)\, g_1 e^{imx} \\
&= -i\left(\frac{1}{2}\right) e^{-imx} + i\left(\frac{1}{2}\right) e^{imx} \\
&= \frac{ie^{imx} - ie^{-imx}}{2} = \frac{e^{-imx} - e^{imx}}{2i} = -\sin(mx)
\end{aligned}
$$

and hence we see that $H(\cos(mx)) = -\sin(mx)$, as desired. □

Moreover, it can be useful to observe the following property about Hilbert transforms, effectively allowing us to nearly reverse the Hilbert transform of a function by applying it twice:

Proposition 6.2. For a periodic function f with period λ, we have that $H^2 f = -(f - f_0)$, where f_0 is the zeroth Fourier coefficient of f.

Proof. Consider such a periodic function f. As per equation (7), then we know that we can write

$$
Hf = \sum_{k=-\infty}^{\infty} i\text{sgn}\,(k)\, f_k e^{ikx\frac{2\pi}{\lambda}} := \sum_{k=-\infty}^{\infty} (Hf)_k\, e^{ikx\frac{2\pi}{\lambda}}
$$

which shows that $(Hf)_k = i\text{sgn}(k) f_k$ is the kth Fourier mode of Hf. Now, applying H once more results in

$$\begin{aligned}
H^2 f &= \sum_{k=-\infty}^{\infty} i\text{sgn}(k) (Hf)_k e^{ikx\frac{2\pi}{\lambda}} = \sum_{k=-\infty}^{\infty} i\text{sgn}(k)\, i\text{sgn}(k) f_k e^{ikx\frac{2\pi}{\lambda}} \\
&= -\sum_{k=-\infty}^{\infty} \text{sgn}(k)^2 f_k e^{ikx\frac{2\pi}{\lambda}} \\
&= -\sum_{k=-\infty}^{-1} f_k e^{ikx\frac{2\pi}{\lambda}} - \sum_{k=1}^{\infty} f_k e^{ikx\frac{2\pi}{\lambda}} \\
&= -\sum_{k=-\infty}^{-1} f_k e^{ikx\frac{2\pi}{\lambda}} - f_0 - \sum_{k=1}^{\infty} f_k e^{ikx\frac{2\pi}{\lambda}} + f_0 \\
&= -\sum_{k=-\infty}^{\infty} f_k e^{ikx\frac{2\pi}{\lambda}} + f_0 = -(f - f_0)
\end{aligned}$$

and hence we see that $H^2 f = -(f - f_0)$ as desired. □

Remark 6.3. Note that it follows from Proposition (6.2) that when the zeroth fourier coefficient f_0 of a periodic function f is zero, then $H^2 f = -f$. In particular, this result will thus hold if we take f to be a periodic function with zero mean on its interval of periodicity.

7 Appendix B: Properties of Hilbert Transforms for the Stokes Expansion

In this Appendix, we will prove properties relating to the Hilbert transform which are relevant for our proofs in Section 4. These properties are useful for being able to perform computations when applying the Babenko equation (18) to the Stokes wave expansion (30), which we are taking as an expansion in powers of the wave amplitude. The first property we will prove relates to the general form of the Hilbert transform when acting upon one of the terms in this expansion:

Theorem 7.1. Suppose that $f(u) = |u|^\beta$, where $\beta \in (-1, \infty)$, is defined on $[-\pi, \pi]$, and is extended to $\mathbb{R}$ with a period of 2π. Then it follows that we may compute the Hilbert transform of f as:

$$H(f)(u) = \frac{1}{\pi} \fint_{-\pi}^{\pi} \frac{|x|^\beta}{x - u} dx + F(u)$$

where $F \in C^\omega(D)$ is some analytic function, for any compact subset D of $(-\pi, \pi)$. In particular, for instance, $F \in C^\omega\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$, since $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is a compact subset of $(-\pi, \pi)$.

Proof. Recall that, by definition, as stated in [6], since f is a 2π-periodic function, then we may write the Hilbert transform of f as

$$H(f)(u) = \frac{1}{2\pi} \fint_{-\pi}^{\pi} \frac{|x|^\beta}{\tan\left(\frac{x-u}{2}\right)} dx.$$

Further, reducing the definition of the Hilbert transform on the real line given by equation (6), we obtain the expansion

$$H(f)(u) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \fint_{-\pi}^{\pi} \frac{|x|^\beta}{x - u + 2\pi n} dx = \frac{1}{\pi} \fint_{-\pi}^{\pi} \frac{|x|^\beta}{x - u} dx + \frac{1}{\pi} \sum_{n \in \mathbb{Z}\setminus\{0\}} \fint_{-\pi}^{\pi} \frac{|x|^\beta}{x - u + 2\pi n} dx.$$

Now, letting $F(u) := \frac{1}{\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}} \fint_{-\pi}^{\pi} \frac{|x|^\beta}{x-u+2\pi n}dx$ then it follows that

$$H(f)(u) = \frac{1}{\pi}\fint_{-\pi}^{\pi} \frac{|x|^\beta}{x-u}dx + F(u).$$

Taking any D which is a compact subset of $(-\pi,\pi)$, then we need only show that $F \in C^\omega(D)$. From above, we see that

$$\frac{1}{\pi}\fint_{-\pi}^{\pi} \frac{|x|^\beta}{x-u}dx + F(u) = \frac{1}{2\pi}\fint_{-\pi}^{\pi} \frac{|x|^\beta}{\tan\left(\frac{x-u}{2}\right)}dx$$

and hence

$$\begin{aligned} F(u) &= \frac{1}{2\pi}\fint_{-\pi}^{\pi} \frac{|x|^\beta}{\tan\left(\frac{x-u}{2}\right)}dx - \frac{1}{\pi}\fint_{-\pi}^{\pi} \frac{|x|^\beta}{x-u}dx \\ &= \frac{1}{\pi}\fint_{-\pi}^{\pi} |x|^\beta \left(\frac{1}{2\tan\left(\frac{x-u}{2}\right)} - \frac{1}{x-u}\right)dx \end{aligned}$$

Now, taking the function k given by

$$k(x) := \frac{1}{2\tan\left(\frac{x}{2}\right)} - \frac{1}{x}$$

then we have that

$$F(u) = \frac{1}{\pi}\fint_{-\pi}^{\pi} |x|^\beta k(x-u)dx.$$

Moreover, note that k is analytic everywhere except where there exists singularities, and given that it may be extended analytically to include 0, then the first set of singularities exists at $x = \pm 2\pi$, so that $k \in C^\omega((-2\pi, 2\pi))$. So, we may expand k in a Taylor series about the point $x = 0$:

$$k(x) = \sum_{n=0}^{\infty} \frac{1}{n!}k^{(n)}(0)x^n$$

where the radius of convergence of this Taylor series expansion is $R := 2\pi$. Now, we thus have that

$$\begin{aligned} F(u) &= \frac{1}{\pi}\fint_{-\pi}^{\pi} \sum_{n=0}^{\infty} |x|^\beta \frac{1}{n!}k^{(n)}(0)(x-u)^n dx \\ &= \frac{1}{\pi}\sum_{n=0}^{\infty}\fint_{-\pi}^{\pi} |x|^\beta \frac{1}{n!}k^{(n)}(0)(x-u)^n dx \end{aligned}$$

where we may exchange the integral and the series since the series converges uniformly within its interval of convergence. Now, letting M_R denote the maximum value of $|k(x)|$ on the interval of convergence, then it follows from Cauchy's integral formula [2] that

$$\frac{\left|k^{(n)}(0)\right|}{n!} \leq \frac{M_R}{R^n}$$

and so we see that

$$\begin{aligned} |F(u)| &\leq \sum_{n=0}^{\infty} \frac{1}{\pi n!}\left|k^{(n)}(0)\right| \left|\fint_{-\pi}^{\pi} |x|^\beta (x-u)^n dx\right| \leq \sum_{n=0}^{\infty} \frac{M_R}{\pi R^n}\left|\fint_{-\pi}^{\pi} |x|^\beta (x-u)^n dx\right| \\ &\leq \sum_{n=0}^{\infty} \frac{M_R}{\pi R^n}\fint_{-\pi}^{\pi} |x|^\beta |x-u|^n dx. \end{aligned}$$

Of course, taking $x \in [-\pi,\pi]$, then we have that $|x| \leq \pi$. Moreover, because D is a compact subset of $(-\pi,\pi)$, then there exists some constant $K \in \mathbb{R}$ for which $|u| \leq K < \pi$ for all $u \in D$. Thus, combining these facts with the triangle inequality we obtain that

$$|x-u|^n \leq (|x|+|u|)^n \leq (\pi+K)^n.$$

Hence, we obtain that

$$|F(u)| \leq \sum_{n=0}^{\infty} \frac{M_R(\pi+K)^n}{\pi R^n} \fint_{-\pi}^{\pi} |x|^\beta dx.$$

Taking

$$C_\beta := \fint_{-\pi}^{\pi} |x|^\beta dx$$

then observe that for $\beta \in (-1, \infty)$ we have that $C_\beta < \infty$ and it is thus a finite constant independent of index of summation n. Therefore, we see, using $R = 2\pi$, that

$$|F(u)| \leq \frac{M_R C_\beta}{\pi} \sum_{n=0}^{\infty} \left(\frac{\pi+K}{2\pi}\right)^n < \infty$$

where $\sum_{n=0}^{\infty} \left(\frac{\pi+K}{2\pi}\right)^n < \infty$, because $K < \pi$ so that $\pi + K < 2\pi$ and therefore $\frac{\pi+K}{2\pi} < 1$ and hence this is a convergent geometric series. In particular, given that $u \in D$ was arbitrary, then $|F(u)|$ is bounded above by such a convergent geometric series, which shows that F is majorized by this series, thus implying, by Majorant's Lemma [9] that F is real analytic for all $u \in D$. Thus, we have that $F \in C^\omega(D)$, as was desired. ☐

Remark 7.2. The previous theorem allows for us to identify and isolate the singular behaviour of the Hilbert transform acting on a 2π-periodic function of the form $f(u) = |u|^\beta$, where $\beta \in (-1, \infty)$. In particular, in order to understand the singular behaviour of $H(f)$ for such a function f, then the previous theorem shows that any singularity will come from the integral $\fint_{-\pi}^{\pi} \frac{|x|^\beta}{x-u} dx$ contribution. Also note that, while the remainder term is not guaranteed to be analytic over all of $\mathbb{R}$, it is analytic on every compact subset of $(-\pi, \pi)$. Hence, since our Stokes expansion for the surface profile $\eta(u)$ given by (30) holds in the limit that $u \to 0$, then we are working arbitrarily close to the peak. Thus, Theorem (7.1) tells us that, upon applying the Hilbert transform to a given term $|u|^\beta$ in the Stokes expansion, the singularity will not arise due to the remainder terms, but rather due to the integral term $\fint_{-\pi}^{\pi} \frac{|x|^\beta}{x-u} dx$.

It is useful to understand the Hilbert transform of functions of the form $f(u) = |u|^{\beta/3}$ for $\beta = 1, 2$, and so in the following proposition we will investigate an explicit computation of the leading order terms after applying the Hilbert transform:

Proposition 7.3. The Hilbert transform acts on the functions $|u|^{1/3}$ and $|u|^{2/3}$ as follows:

$$H\left(|u|^{1/3}\right) = \frac{1}{\sqrt{3}} |u|^{1/3} \mathrm{sgn}(u) + F_{1/3}(u)$$

$$H\left(|u|^{2/3}\right) = \sqrt{3} |u|^{2/3} \mathrm{sgn}(u) + F_{2/3}(u)$$

for $F_{1/3} = \mathcal{O}(|u|)$ and $F_{2/3} = \mathcal{O}(|u|)$.

Proof. Let us first utilize the notation

$$G_1(x) := \frac{1}{2} \ln\left(\frac{(x+1)^2}{x^2 - x + 1}\right) \text{ and } G_2(x) := G_1(-x) = \frac{1}{2} \ln\left(\frac{(1-x)^2}{x^2 + x + 1}\right) \tag{39}$$

and

$$K_1(x) := \sqrt{3} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) \text{ and } K_2(x) := K_1(-x) = -\sqrt{3} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) \tag{40}$$

Note that

$$\begin{aligned} G_1\left(\pi^{1/3} u^{-1/3}\right) &= \frac{1}{2} \ln\left(\frac{(\pi^{1/3} u^{-1/3} + 1)^2}{\pi^{2/3} u^{-2/3} - \pi^{1/3} u^{-1/3} + 1}\right) = \frac{1}{2} \ln\left(\frac{(\pi^{1/3} + u^{1/3})^2}{\pi^{2/3} - \pi^{1/3} u^{1/3} + u^{2/3}}\right) \\ &= \ln\left(\pi^{1/3} + u^{1/3}\right) - \frac{1}{2} \ln\left(\pi^{2/3} - \pi^{1/3} u^{1/3} + u^{2/3}\right) \end{aligned}$$

Now, observe that $\pi^{2/3} - \pi^{1/3}u^{1/3} + u^{2/3} = \frac{\pi+u}{\pi^{1/3}+u^{1/3}}$ and therefore we see that

$$\begin{aligned}\ln\left(\pi^{2/3} - \pi^{1/3}u^{1/3} + u^{2/3}\right) &= \ln\left(\frac{\pi+u}{\pi^{1/3}+u^{1/3}}\right) \\ &= \ln(\pi+u) - \ln\left(\pi^{1/3}+u^{1/3}\right)\end{aligned}$$

so that

$$G_1\left(\pi^{1/3}u^{-1/3}\right) = \frac{3}{2}\ln\left(\pi^{1/3}+u^{1/3}\right) - \frac{1}{2}\ln(\pi+u)$$

Note as well that $G_1(0) = \frac{1}{2}\ln\left(\frac{(0+1)^2}{0+1}\right) = \frac{1}{2}\ln(1) = 0$ so that $G_1(0) = 0$. We also see that

$$\begin{aligned}G_2\left(\pi^{1/3}u^{-1/3}\right) &= \frac{1}{2}\ln\left(\frac{(1-\pi^{1/3}u^{-1/3})^2}{\pi^{2/3}u^{-2/3}+\pi^{1/3}u^{-1/3}+1}\right) = \frac{1}{2}\ln\left(\frac{(\pi^{1/3}-u^{1/3})^2}{\pi^{2/3}+\pi^{1/3}u^{1/3}+u^{2/3}}\right) \\ &= \ln\left(\pi^{1/3}-u^{1/3}\right) - \frac{1}{2}\ln\left(\pi^{2/3}+\pi^{1/3}u^{1/3}+u^{2/3}\right)\end{aligned}$$

Now, observe that $\pi^{2/3} + \pi^{1/3}u^{1/3} + u^{2/3} = \frac{\pi-u}{\pi^{1/3}-u^{1/3}}$ and therefore we see that

$$\begin{aligned}\ln\left(\pi^{2/3} + \pi^{1/3}u^{1/3} + u^{2/3}\right) &= \ln\left(\frac{\pi-u}{\pi^{1/3}-u^{1/3}}\right) \\ &= \ln(\pi-u) - \ln\left(\pi^{1/3}-u^{1/3}\right)\end{aligned}$$

and hence

$$G_2\left(\pi^{1/3}u^{-1/3}\right) = \frac{3}{2}\ln\left(\pi^{1/3}-u^{1/3}\right) - \frac{1}{2}\ln(\pi-u)$$

Moreover, we see that $G_2(0) = \frac{1}{2}\ln\left(\frac{(1-0)^2}{0+1}\right) = \frac{1}{2}\ln(1) = 0$ so that $G_2(0) = 0$. Let us further compute limiting values for K_1 and K_2. Observe that

$$K_1\left(\pi^{1/3}u^{-1/3}\right) = \sqrt{3}\arctan\left(\frac{2\pi^{1/3}u^{-1/3}-1}{\sqrt{3}}\right) = \sqrt{3}\arctan\left(\frac{2\pi^{1/3}-u^{1/3}}{\sqrt{3}u^{1/3}}\right)$$

as well as $K_1(0) = \sqrt{3}\arctan\left(\frac{2\cdot 0-1}{\sqrt{3}}\right) = \sqrt{3}\arctan\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\sqrt{3}\pi}{6} = \frac{-\pi}{2\sqrt{3}}$ so $K_1(0) = \frac{-\pi}{2\sqrt{3}}$. Moreover, we also have that

$$K_2\left(\pi^{1/3}u^{-1/3}\right) = -\sqrt{3}\arctan\left(\frac{2\pi^{1/3}u^{-1/3}+1}{\sqrt{3}}\right) = -\sqrt{3}\arctan\left(\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\right)$$

as well as $K_2(0) = -\sqrt{3}\arctan\left(\frac{2\cdot 0+1}{\sqrt{3}}\right) = -\sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right) = -\frac{\sqrt{3}\pi}{6} = -\frac{\pi}{2\sqrt{3}}$ so $K_2(0) = -\frac{\pi}{2\sqrt{3}}$.

In order to perform these computations, let us observe from [10] that Equation 2.126 (1) shows that

$$\int \frac{x^3}{1+x^3}dx = x - \frac{1}{3}\left(G_1(x) + K_1(x)\right) \tag{41}$$

and thus taking $x \to -x$ allows us to see that

$$\int \frac{x^3}{1-x^3}dx = -x - \frac{1}{3}\left(G_2(x) + K_2(x)\right) \tag{42}$$

Moreover, we also see from [10] that Equation 2.126 (2) shows that

$$\int \frac{x^4}{1+x^3}dx = \frac{x^2}{2} + \frac{1}{3}\left(G_1(x) - K_1(x)\right) \tag{43}$$

and so, again, taking $x \to -x$ shows that

$$\int \frac{x^4}{1-x^3}dx = -\frac{x^2}{2} - \frac{1}{3}\left(G_2(x) + K_2(x)\right) \tag{44}$$

Now, using these four explicit integrals, we may evaluate both of these Hilbert transforms.

Let us first verify that $H\left(|u|^{1/3}\right) = \frac{1}{\sqrt{3}}|u|^{1/3}\text{sgn}(u) + F_{1/3}(u)$. Now, by Theorem 7.1, we know that

$$H\left(|u|^{1/3}\right) = \frac{1}{\pi} \fint_{-\pi}^{\pi} \frac{|x|^{1/3}}{x-u}dx + \tilde{F}_{1/3}(u) \tag{45}$$

for some $\tilde{F}_{1/3} \in C^{\omega}(D_1)$ where D_1 is a compact subset of $(-\pi, \pi)$. Now, let us begin by taking $u > 0$. We will evaluate the first integral in this summation. Note that

$$\int_{-\pi}^{\pi} \frac{|x|^{1/3}}{x-u}dx = \int_{-\pi}^{0} \frac{|x|^{1/3}}{x-u}dx + \int_{0}^{\pi} \frac{|x|^{1/3}}{x-u}dx = \int_{-\pi}^{0} \frac{(-x)^{1/3}}{x-u}dx + \int_{0}^{\pi} \frac{x^{1/3}}{x-u}dx$$

To handle the first integral in this sum, simply perform the change of variables $v = (-x)^{1/3} u^{-1/3}$ and hence we see that

$$\begin{aligned}
\int_{-\pi}^{0} \frac{(-x)^{1/3}}{x-u}dx = \frac{1}{u}\int_{-\pi}^{0} \frac{(-x)^{1/3}}{\frac{x}{u}-1}dx = \frac{1}{u}\int_{\pi^{1/3}u^{-1/3}}^{0} \frac{\left(vu^{1/3}\right)\left(-3v^2u\right)}{-v^3-1}dv = \frac{1}{u}\int_{\pi^{1/3}u^{-1/3}}^{0} \frac{3u^{4/3}v^3}{v^3+1}dv \\
&= -3u^{1/3}\int_{0}^{\pi^{1/3}u^{-1/3}} \frac{v^3}{v^3+1}dv \\
&= -3u^{1/3}\left[v - \frac{1}{3}\left(G_1(v) + K_1(v)\right)\right]_{v=0}^{v=\pi^{1/3}u^{-1/3}} \\
&= -3u^{1/3}\left(\pi^{1/3}u^{-1/3} - \frac{1}{3}\left(G_1(\pi^{1/3}u^{-1/3}) + K_1(\pi^{1/3}u^{-1/3})\right) + \frac{1}{3}\left(G_1(0) + K_1(0)\right)\right) \\
&= -3u^{1/3}\Bigg(\pi^{1/3}u^{-1/3} - \frac{1}{3}\Bigg(\frac{3}{2}\ln\left(\pi^{1/3} + u^{1/3}\right) - \frac{1}{2}\ln\left(\pi + u\right) + \sqrt{3}\arctan\left(\frac{2\pi^{1/3} - u^{1/3}}{\sqrt{3}u^{1/3}}\right) \\
&\qquad + \frac{\pi}{2\sqrt{3}}\Bigg)\Bigg) \\
&= -3\pi^{1/3} + u^{1/3}\left(\frac{3}{2}\ln\left(\pi^{1/3} + u^{1/3}\right) - \frac{1}{2}\ln\left(\pi + u\right) + \sqrt{3}\arctan\left(\frac{2\pi^{1/3} - u^{1/3}}{\sqrt{3}u^{1/3}}\right) + \frac{\pi}{2\sqrt{3}}\right) \\
&= -3\pi^{1/3} + \frac{3}{2}u^{1/3}\ln\left(\pi^{1/3} + u^{1/3}\right) - \frac{1}{2}u^{1/3}\ln\left(\pi + u\right) + \sqrt{3}u^{1/3}\arctan\left(\frac{2\pi^{1/3} - u^{1/3}}{\sqrt{3}u^{1/3}}\right) + \frac{\pi u^{1/3}}{2\sqrt{3}}
\end{aligned}$$

Likewise, we may perform the change of variables $v = x^{1/3}u^{-1/3}$ for the second integral in the sum and observe that

$$\int_0^{\pi} \frac{x^{1/3}}{x-u}dx = \frac{1}{u}\int_0^{\pi} \frac{x^{1/3}}{\frac{x}{u}-1}dx = \frac{1}{u}\int_0^{\pi^{1/3}u^{-1/3}} \frac{\left(vu^{1/3}\right)\left(3v^2u\right)}{v^3-1}dv = \frac{1}{u}\int_0^{\pi^{1/3}u^{-1/3}} \frac{3u^{4/3}v^3}{v^3-1}dv$$

$$= -3u^{1/3}\int_0^{\pi^{1/3}u^{-1/3}} \frac{v^3}{1-v^3}dv$$

$$= -3u^{1/3}\left[-v-\frac{1}{3}\left(G_2(v)+K_2(v)\right)\right]_{v=0}^{v=\pi^{1/3}u^{-1/3}}$$

$$= 3u^{1/3}\left(\pi^{1/3}u^{-1/3}+\frac{1}{3}\left(G_2(\pi^{1/3}u^{-1/3})+K_2(\pi^{1/3}u^{-1/3})\right)-\frac{1}{3}\left(G_2(0)+K_2(0)\right)\right)$$

$$= 3u^{1/3}\left(\pi^{1/3}u^{-1/3}+\frac{1}{3}\left(\frac{3}{2}\ln\left(\pi^{1/3}-u^{1/3}\right)-\frac{1}{2}\ln\left(\pi-u\right)-\sqrt{3}\arctan\left(\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\right)\right.\right.$$
$$\left.\left.+\frac{\pi}{2\sqrt{3}}\right)\right)$$

$$= 3\pi^{1/3}+u^{1/3}\left(\frac{3}{2}\ln\left(\pi^{1/3}-u^{1/3}\right)-\frac{1}{2}\ln\left(\pi-u\right)-\sqrt{3}\arctan\left(\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\right)+\frac{\pi}{2\sqrt{3}}\right)$$

$$= 3\pi^{1/3}+\frac{3}{2}u^{1/3}\ln\left(\pi^{1/3}-u^{1/3}\right)-\frac{1}{2}u^{1/3}\ln\left(\pi-u\right)-\sqrt{3}u^{1/3}\arctan\left(\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\right)+\frac{\pi u^{1/3}}{2\sqrt{3}}$$

Now, combining the above equations, we see that

$$\int_{-\pi}^{\pi} \frac{|x|^{1/3}}{x-u}dx = \frac{\pi}{\sqrt{3}}u^{1/3}+\frac{3}{2}u^{1/3}\ln\left(\pi^{2/3}-u^{2/3}\right)-\frac{1}{2}u^{1/3}\ln\left(\pi^2-u^2\right)$$
$$+\sqrt{3}u^{1/3}\left(\arctan\left(\frac{2\pi^{1/3}-u^{1/3}}{\sqrt{3}u^{1/3}}\right)-\arctan\left(\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\right)\right)$$
$$:= \frac{\pi}{\sqrt{3}}u^{1/3}+u^{1/3}\left(\frac{3}{2}\ln\left(1-\frac{u^{2/3}}{\pi^{2/3}}\right)-\frac{1}{2}\ln\left(1-\frac{u^2}{\pi^2}\right)\right)+\sqrt{3}u^{1/3}M_1(u)$$
$$:= \frac{\pi}{\sqrt{3}}u^{1/3}+u^{1/3}L(u)+\sqrt{3}u^{1/3}M_1(u) \tag{46}$$

Now, let us first recognize that it follows from Equation 1.511 in [10] that we may write, for all $-1 < x \le 1$,

$$\ln(1-x) = \sum_{k=1}^{\infty}(-1)^{2k+1}\frac{x^k}{k} = -\sum_{k=1}^{\infty}\frac{x^k}{k}$$

and hence since $u \le \pi$, it follows that

$$\ln\left(1-\frac{u^{2/3}}{\pi^{2/3}}\right) = \sum_{k=1}^{\infty}\frac{-u^{2k/3}}{k\pi^{2k/3}} \text{ and } \ln\left(1-\frac{u^2}{\pi^2}\right) = \sum_{k=1}^{\infty}\frac{-u^{2k}}{k\pi^{2k}}$$

and therefore

$$L(u) = \frac{3}{2}\ln\left(1-\frac{u^{2/3}}{\pi^{2/3}}\right)-\frac{1}{2}\ln\left(1-\frac{u^2}{\pi^2}\right) = \frac{3}{2}\sum_{k=1}^{\infty}\frac{-u^{2k/3}}{k\pi^{2k/3}}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{-u^{2k}}{k\pi^{2k}}$$
$$= \sum_{k=1}^{\infty}\left(\frac{u^{2k}}{2k\pi^{2k}}-\frac{3u^{2k/3}}{2k\pi^{2k/3}}\right) = \mathcal{O}(u^{2/3})$$

and thus $u^{1/3}L(u) = \mathcal{O}(u)$. Moreover, since $\frac{2\pi^{1/3}-u^{1/3}}{\sqrt{3}u^{1/3}} \cdot \frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}} = \frac{4\pi^{2/3}-u^{2/3}}{3u^{2/3}} \ge 1 > -1$ then it follows by 1.625 (8) in [10] that we have

$$\arctan\left(\frac{2\pi^{1/3}-u^{1/3}}{\sqrt{3}u^{1/3}}\right)-\arctan\left(\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\right) = \arctan\left(\frac{\frac{-2u^{1/3}}{\sqrt{3}u^{1/3}}}{1+\frac{4\pi^{2/3}-u^{2/3}}{3u^{2/3}}}\right) = \arctan\left(\sqrt{3}\frac{-\frac{u^{2/3}}{2\pi^{2/3}}}{1+\frac{u^{2/3}}{2\pi^{2/3}}}\right)$$

and since

$$\frac{-x}{1+x} = \sum_{k=1}^{\infty} (-1)^k x^k$$

then

$$\sqrt{3}\frac{\frac{u^{2/3}}{2\pi^{2/3}}}{1+\frac{u^{2/3}}{2\pi^{2/3}}} = \sqrt{3}\sum_{k=1}^{\infty}(-1)^k \frac{u^{2k/3}}{(2\pi^{2/3})^{2k/3}}$$

and hence

$$M_1(u) = \arctan\left(\sqrt{3}\sum_{k=1}^{\infty}(-1)^k \frac{u^{2k/3}}{(2\pi^{2/3})^{2k/3}}\right) := \arctan\left(y(u)\right)$$

where $y(u) = \mathcal{O}(u^{2/3})$. Since we can expand arctan as $\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = \mathcal{O}(u)$ then it follows that $M_1(u) = \mathcal{O}(u^{2/3})$ and thus $\sqrt{3}u^{1/3}M_1(u) = \mathcal{O}(u)$. Therefore, defining $Q_{1/3}(u) := u^{1/3}L(u) + \sqrt{3}u^{1/3}M_1(u) + \tilde{F}_{1/3}(u)$, we have that $Q_{1/3} = \mathcal{O}(u)$ and so inserting (46) into (45), then we see that

$$H\left(|u|^{1/3}\right) = \frac{1}{\sqrt{3}}u^{1/3} + \frac{1}{\pi}Q_{1/3}(u)$$

Now, if we take $F_{1/3}(u) := -\frac{1}{\pi}\mathrm{sgn}(-u)Q_{1/3}(u)$ then we see that $H\left(|u|^{1/3}\right) = \frac{1}{\sqrt{3}}u^{1/3} + F_{1/3}(u)$ where $F_{1/3} = \mathcal{O}(u)$ for $u > 0$. Having shown this for $u > 0$, consider the case where $u < 0$. Then Theorem 7.1 shows that we may write

$$-H\left(|u|^{1/3}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{|x|^{1/3}}{x-(-u)}dx - \tilde{F}_{1/3}(u)$$

for $\tilde{F}_{1/3}$ as defined previously. However, then $-u > 0$ and so

$$\int_{-\pi}^{\pi} \frac{|x|^{1/3}}{x-(-u)}dx = \frac{\pi}{\sqrt{3}}(-u)^{1/3} + (-u)^{1/3}L(-u) - \sqrt{3}(-u)^{1/3}M_1(-u)$$

showing that $H\left(|u|^{1/3}\right) = \frac{-1}{\sqrt{3}}(-u)^{1/3} - \frac{1}{\pi}Q_{1/3}(-u)$. Since $F_{1/3}(-u) = -\frac{1}{\pi}\mathrm{sgn}(-u)Q_{1/3}(-u) = -Q_{1/3}(-u)$ as $u < 0$, then we can write that $H\left(|u|^{1/3}\right) = \frac{-1}{\sqrt{3}}(-u)^{1/3} + F_{1/3}(-u)$, where $F_{1/3} = \mathcal{O}(-u)$. Hence, to cover both cases, we may write that

$$H\left(|u|^{2/3}\right) = \frac{1}{\sqrt{3}}|u|^{1/3}\mathrm{sgn}(u) + F_{1/3}(|u|)$$

where $F_{1/3} = \mathcal{O}(|u|)$, as desired.

Next, let us first verify that $H\left(|u|^{2/3}\right) = \sqrt{3}|u|^{2/3}\mathrm{sgn}(u) + F_{2/3}(u)$. Just as before, Theorem 7.1 shows us that

$$H\left(|u|^{2/3}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{|x|^{2/3}}{x-u}dx + \tilde{F}_{2/3}(u)$$

for some $\tilde{F}_{2/3} \in C^{\omega}(D_2)$ where D_2 is a compact subset of $(-\pi, \pi)$. We may take $u > 0$ to begin. We evaluate the integral in this summation as follows:

$$\int_{-\pi}^{\pi} \frac{|x|^{2/3}}{x-u}dx = \int_{-\pi}^{0} \frac{|x|^{2/3}}{x-u}dx + \int_{0}^{\pi} \frac{|x|^{2/3}}{x-u}dx = \int_{-\pi}^{0} \frac{(-x)^{2/3}}{x-u}dx + \int_{0}^{\pi} \frac{x^{2/3}}{x-u}dx$$

To handle the first integral in this sum, we again perform the change of variables $v = (-x)^{1/3}u^{-1/3}$ and hence we see that

$$\begin{aligned}
\int_{-\pi}^{0} \frac{(-x)^{2/3}}{x-u} dx = \frac{1}{u}\int_{-\pi}^{0} \frac{(-x)^{2/3}}{\frac{x}{u}-1} dx = \frac{1}{u}\int_{\pi^{1/3}u^{-1/3}}^{0} \frac{\left(v^2u^{2/3}\right)\left(-3v^2u\right)}{-v^3-1} dv = \frac{1}{u}\int_{\pi^{1/3}u^{-1/3}}^{0} \frac{3u^{5/3}v^4}{v^3+1} dv \\
&= -3u^{2/3}\int_{0}^{\pi^{1/3}u^{-1/3}} \frac{v^4}{v^3+1} dv \\
&= -3u^{2/3}\left[\frac{v^2}{2} + \frac{1}{3}\left(G_1(v) - K_1(v)\right)\right]_{v=0}^{v=\pi^{1/3}u^{-1/3}} \\
&= -3u^{2/3}\left(\frac{\pi^{2/3}u^{-2/3}}{2} + \frac{1}{3}\left(G_1(\pi^{1/3}u^{-1/3}) - K_1(\pi^{1/3}u^{-1/3})\right) + \frac{1}{3}\left(G_1(0) - K_1(0)\right)\right) \\
&= -3u^{2/3}\left(\frac{\pi^{2/3}u^{-2/3}}{2} + \frac{1}{3}\left(\frac{3}{2}\ln\left(\pi^{1/3}+u^{1/3}\right) - \frac{1}{2}\ln(\pi+u) - \sqrt{3}\arctan\left(\frac{2\pi^{1/3}-u^{1/3}}{\sqrt{3}u^{1/3}}\right)\right.\right. \\
&\qquad \left.\left. - \frac{\pi}{2\sqrt{3}}\right)\right) \\
&= -\frac{3\pi^{2/3}}{2} - u^{2/3}\left(\frac{3}{2}\ln\left(\pi^{1/3}+u^{1/3}\right) - \frac{1}{2}\ln(\pi+u) - \sqrt{3}\arctan\left(\frac{2\pi^{1/3}-u^{1/3}}{\sqrt{3}u^{1/3}}\right) - \frac{\pi}{2\sqrt{3}}\right) \\
&= -\frac{3\pi^{2/3}}{2} - \frac{3}{2}u^{2/3}\ln\left(\pi^{1/3}+u^{1/3}\right) + \frac{1}{2}u^{2/3}\ln(\pi+u) + \sqrt{3}u^{2/3}\arctan\left(\frac{2\pi^{1/3}-u^{1/3}}{\sqrt{3}u^{1/3}}\right) + \frac{\pi u^{2/3}}{2\sqrt{3}}
\end{aligned}$$

Likewise, we may perform the change of variables $v = x^{1/3}u^{-1/3}$ for the second integral in the sum and observe that

$$\begin{aligned}
\int_{0}^{\pi} \frac{x^{2/3}}{x-u} dx = \frac{1}{u}\int_{0}^{\pi} \frac{x^{2/3}}{\frac{x}{u}-1} dx = \frac{1}{u}\int_{0}^{\pi^{1/3}u^{-1/3}} \frac{\left(v^2u^{2/3}\right)\left(3v^2u\right)}{v^3-1} dv = \frac{1}{u}\int_{0}^{\pi^{1/3}u^{-1/3}} \frac{3u^{5/3}v^4}{v^3-1} dv \\
&= -3u^{2/3}\int_{0}^{\pi^{1/3}u^{-1/3}} \frac{v^4}{1-v^3} dv \\
&= -3u^{2/3}\left[-\frac{x^2}{2} - \frac{1}{3}\left(G_2(x) + K_2(x)\right)\right]_{v=0}^{v=\pi^{1/3}u^{-1/3}} \\
&= 3u^{2/3}\left(\frac{\pi^{2/3}u^{-2/3}}{2} + \frac{1}{3}\left(G_2(\pi^{1/3}u^{-1/3}) + K_2(\pi^{1/3}u^{-1/3})\right) + \frac{1}{3}\left(G_2(0) - K_2(0)\right)\right) \\
&= 3u^{2/3}\left(\frac{\pi^{2/3}u^{-2/3}}{2} + \frac{1}{3}\left(\frac{3}{2}\ln\left(\pi^{1/3}-u^{1/3}\right) - \frac{1}{2}\ln(\pi-u) + \sqrt{3}\arctan\left(\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\right)\right.\right. \\
&\qquad \left.\left. - \frac{\pi}{2\sqrt{3}}\right)\right) \\
&= \frac{3\pi^{2/3}}{2} + u^{2/3}\left(\frac{3}{2}\ln\left(\pi^{1/3}-u^{1/3}\right) - \frac{1}{2}\ln(\pi-u) + \sqrt{3}\arctan\left(\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\right) - \frac{\pi}{2\sqrt{3}}\right) \\
&= \frac{3\pi^{2/3}}{2} + \frac{3}{2}u^{2/3}\ln\left(\pi^{1/3}-u^{1/3}\right) - \frac{1}{2}u^{2/3}\ln(\pi-u) + \sqrt{3}u^{2/3}\arctan\left(\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\right) - \frac{\pi u^{2/3}}{2\sqrt{3}}
\end{aligned}$$

Again combining the above equations, we see that

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{|x|^{2/3}}{x-u}dx &= \frac{3}{2}u^{2/3}\left(\ln\left(\pi^{1/3}-u^{1/3}\right)-\ln\left(\pi^{1/3}+u^{1/3}\right)\right)+\frac{1}{2}u^{2/3}\left(\ln\left(\pi+u\right)-\ln\left(\pi-u\right)\right)\\
&\quad+\sqrt{3}u^{2/3}\left(\arctan\left(\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\right)+\arctan\left(\frac{2\pi^{1/3}-u^{1/3}}{\sqrt{3}u^{1/3}}\right)\right)\\
&= u^{2/3}\left(\frac{3}{2}\ln\left(1-\frac{u^{2/3}}{\pi^{2/3}}\right)-\frac{1}{2}\ln\left(1-\frac{u^2}{\pi^2}\right)\right)+\sqrt{3}u^{2/3}\Bigg(\arctan\left(\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\right)\\
&\quad+\arctan\left(\frac{2\pi^{1/3}-u^{1/3}}{\sqrt{3}u^{1/3}}\right)\Bigg)\\
&= u^{2/3}L(u)+\sqrt{3}u^{2/3}\left(\arctan\left(\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\right)+\arctan\left(\frac{2\pi^{1/3}-u^{1/3}}{\sqrt{3}u^{1/3}}\right)\right)\end{aligned}$$

where $L(u)$ is as was defined previously, and $M_2(u) := \arctan\left(\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\right)+\arctan\left(\frac{2\pi^{1/3}-u^{1/3}}{\sqrt{3}u^{1/3}}\right)$. Note that since $\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\cdot\frac{2\pi^{1/3}-u^{1/3}}{\sqrt{3}u^{1/3}} = \frac{4\pi^{2/3}-u^{2/3}}{3u^{2/3}} \geq 1$ then it follows by 1.625 (8) in [10] that we have

$$\arctan\left(\frac{2\pi^{1/3}+u^{1/3}}{\sqrt{3}u^{1/3}}\right)+\arctan\left(\frac{2\pi^{1/3}-u^{1/3}}{\sqrt{3}u^{1/3}}\right) = \pi+\arctan\left(\frac{\frac{4\pi^{1/3}}{\sqrt{3}u^{1/3}}}{1-\frac{4\pi^{2/3}-u^{2/3}}{3u^{2/3}}}\right) = \pi+\arctan\left(\sqrt{3}\frac{\frac{u^{1/3}}{\pi^{1/3}}}{\frac{u^{2/3}}{\pi^{2/3}}-1}\right)$$

Defining $M_2(u) := \arctan\left(\sqrt{3}\frac{\frac{u^{1/3}}{\pi^{1/3}}}{\frac{u^{2/3}}{\pi^{2/3}}-1}\right)$ then we see that

$$\int_{-\pi}^{\pi}\frac{|x|^{2/3}}{x-u}dx = \sqrt{3}\pi u^{2/3}+u^{2/3}L(u)+\sqrt{3}u^{2/3}M_2(u). \tag{47}$$

Now, since

$$\frac{x}{x^2-1} = -\sum_{k=1}^{\infty}x^{2k-1}$$

then

$$\sqrt{3}\frac{\frac{u^{1/3}}{\pi^{1/3}}}{\frac{u^{2/3}}{\pi^{2/3}}-1} = -\sqrt{3}\sum_{k=1}^{\infty}\frac{u^{2/3k-1/3}}{(\pi^{1/3})^{2k-1}}$$

and hence

$$M_2(u) = \arctan\left(-\sqrt{3}\sum_{k=1}^{\infty}\frac{u^{2/3k-1/3}}{(\pi^{1/3})^{2k-1}}\right) := \arctan\left(z(u)\right)$$

where $z(u) = \mathcal{O}(u^{1/3})$. As we saw before, $\arctan(z(u))$ with $z(u) = \mathcal{O}(u^{1/3})$ will also be order $u^{1/3}$ so that $M_2(u) = \mathcal{O}(u^{1/3})$ and hence $\sqrt{3}u^{2/3}M_2(u) = \mathcal{O}(u)$. We also showed before that $u^{1/3}L(u) = \mathcal{O}(u)$ and hence we have $u^{2/3}L(u) = \mathcal{O}(u^{4/3})$. Therefore, defining $Q_{2/3}(u) := u^{2/3}L(u)+\sqrt{3}u^{2/3}M_2(u)+\tilde{F}_{2/3}(u)$, we have that $Q_{2/3} = \mathcal{O}(u)$ and so inserting (47) into (45) then we see that

$$H\left(|u|^{2/3}\right) = \sqrt{3}u^{2/3}+\frac{1}{\pi}Q_{2/3}(u)$$

Now, if we take $F_{2/3}(u) := -\frac{1}{\pi}\text{sgn}(-u)Q_{2/3}(u)$ then we see that $H\left(|u|^{2/3}\right) = \sqrt{3}u^{2/3}+F_{2/3}(u)$ where $F_{2/3} = \mathcal{O}(u)$ when $u > 0$. Having shown this for $u > 0$, consider the case where $u < 0$. Then Theorem 7.1 shows that we may write

$$-H\left(|u|^{2/3}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{|x|^{2/3}}{x-(-u)}dx - \tilde{F}_{2/3}(u)$$

for $\tilde{F}_{2/3}$ as defined previously. However, then $-u > 0$ and so

$$\int_{-\pi}^{\pi} \frac{|x|^{2/3}}{x-(-u)} dx = \sqrt{3}\pi(-u)^{2/3} + (-u)^{2/3}L(-u) + \sqrt{3}(-u)^{2/3}M_2(-u)$$

showing that $H\left(|u|^{1/3}\right) = -\sqrt{3}(-u)^{2/3} - \frac{1}{\pi}Q_{2/3}(-u)$. Since $F_{2/3}(-u) = -\frac{1}{\pi}\text{sgn}(-u)Q_{2/3}(-u) = -\frac{1}{\pi}Q_{2/3}(-u)$ as $u < 0$, then we can write that $H\left(|u|^{2/3}\right) = -\sqrt{3}(-u)^{1/3} + F_{2/3}(-u)$, where $F_{2/3} = \mathcal{O}(-u)$. Hence, to cover both cases, we may write that

$$H\left(|u|^{2/3}\right) = \sqrt{3}|u|^{2/3}\text{sgn}(u) + F_{2/3}(|u|)$$

where $F_{2/3} = \mathcal{O}(|u|)$, as desired. □

In the following corollary, we will use our results from Proposition (7.3) in order to obtain the Hilbert transform of functions of the form $f(u) = |u|^{\beta/3}$ for $\beta = 4, 5$:

Corollary 7.4. The Hilbert transform acts on the functions $|u|^{4/3}$ and $|u|^{5/3}$ with the following evaluation:

$$H\left(|u|^{4/3}\right) = -\sqrt{3}|u|^{4/3}\text{sgn}(u) + F_{4/3}(u)$$

$$H\left(|u|^{5/3}\right) = -\frac{1}{\sqrt{3}}|u|^{5/3}\text{sgn}(u) + F_{5/3}(u)$$

for $F_{4/3} = \mathcal{O}(|u|^2)$ and $F_{5/3} = \mathcal{O}(|u|^2)$.

Proof. Observe that

$$\frac{\partial}{\partial u}|u|^{4/3} = \frac{4}{3}|u|^{1/3}\text{sgn}(u)$$

and hence it follows that

$$H\left(\frac{\partial}{\partial u}|u|^{4/3}\right) = H\left(\frac{4}{3}|u|^{1/3}\text{sgn}(u)\right) \tag{48}$$

from which linearity of the Hilbert transform, along with the fact that it commutes with the derivative operator, implies that $\frac{\partial}{\partial u}H\left(|u|^{4/3}\right) = \frac{4}{3}H\left(|u|^{1/3}\text{sgn}(u)\right)$. Now, Proposition 7.3 above has shown us that

$$H\left(|u|^{1/3}\right) = \frac{1}{\sqrt{3}}|u|^{1/3}\text{sgn}(u) + F_{1/3}(u)$$

where $F_{1/3}(u) = \mathcal{O}(|u|)$. Thus it follows that

$$H^2\left(|u|^{1/3}\right) = \frac{1}{\sqrt{3}}H\left(|u|^{1/3}\text{sgn}(u)\right) + H\left(F_{1/3}\right)$$

and hence since $H^2 = -Id$ for functions with zero-mean, then we have that $-|u|^{1/3} = \frac{1}{\sqrt{3}}H\left(|u|^{1/3}\text{sgn}(u)\right) + H\left(F_{1/3}\right)$ and thus

$$H\left(|u|^{1/3}\text{sgn}(u)\right) = -\sqrt{3}|u|^{1/3} - H\left(F_{1/3}\right)$$

and therefore using 48 we have that

$$\frac{\partial}{\partial u}H\left(|u|^{4/3}\right) = -\frac{4\sqrt{3}}{3}|u|^{1/3} - \frac{4}{3}H\left(F_{1/3}\right).$$

Integrating both sides from 0 to u gives us that

$$\int_0^u \frac{\partial}{\partial x}H\left(|x|^{4/3}\right)dx = -\frac{4\sqrt{3}}{3}\int_0^u |x|^{1/3}dx - \frac{4}{3}\int_0^u H\left(F_{1/3}(x)\right)dx,$$

and hence by the Fundamental Theorem of Calculus we see that

$$H\left(|u|^{4/3}\right) = -\sqrt{3}|u|^{4/3}\text{sgn}(u) - \frac{4}{3}\int_0^u H\left(F_{1/3}(x)\right)dx$$

Now, designating $F_{4/3}(u) := -\frac{4}{3}\int_0^u H\left(F_{1/3}(x)\right)dx$, then we observe that

$$H\left(|u|^{4/3}\right) = -\sqrt{3}|u|^{4/3}\text{sgn}(u) + F_{4/3}(u).$$

Of course, we know that, for any $H(F_{1/3}(u)) = \mathcal{O}(|u|)$ and thus taking the indefinite integral $\int H\left(F_{1/3}(x)\right)dx$ will raise the powers of each term in the expansion by one. Evaluating this expansion at zero gives zero and therefore

$$F_{4/3}(u) = -\frac{4}{3}\int_0^u H\left(F_{1/3}(x)\right)dx = \mathcal{O}(|u|^2)$$

as desired.

For the second identity, we may follow a very similar procedure. Since we know that $\frac{\partial}{\partial u}|u|^{5/3} = \frac{5}{3}|u|^{2/3}\text{sgn}(u)$ then we have that $H\left(\frac{\partial}{\partial u}|u|^{5/3}\right) = H\left(\frac{5}{3}|u|^{2/3}\text{sgn}(u)\right)$. Linearity of the Hilbert transform, along with the fact that it commutes with the derivative operator, implies that

$$\frac{\partial}{\partial u}H\left(|u|^{5/3}\right) = \frac{5}{3}H\left(|u|^{2/3}\text{sgn}(u)\right). \tag{49}$$

Now, Theorem 7.1 above has shown us that

$$H\left(|u|^{2/3}\right) = \sqrt{3}|u|^{2/3}\text{sgn}(u) + F_{2/3}(u)$$

where $F_{2/3}(u) = \mathcal{O}(|u|)$. Thus it follows that

$$H^2\left(|u|^{2/3}\right) = \sqrt{3}H\left(|u|^{2/3}\text{sgn}(u)\right) + H\left(F_{2/3}\right)$$

and hence since $H^2 = -Id$ for functions with zero-mean, then we have that $-|u|^{2/3} = \sqrt{3}H\left(|u|^{2/3}\text{sgn}(u)\right) + H\left(F_{2/3}\right)$ and thus

$$H\left(|u|^{2/3}\text{sgn}(u)\right) = -\frac{1}{\sqrt{3}}|u|^{2/3} - H\left(F_{2/3}\right)$$

and so using equation 49 then we have that

$$\frac{\partial}{\partial u}H\left(|u|^{5/3}\right) = -\frac{5}{3\sqrt{3}}|u|^{2/3} - \frac{5}{3}H\left(F_{2/3}\right)$$

Integrating both sides from 0 to u gives us that

$$\int_0^u \frac{\partial}{\partial x}H\left(|x|^{5/3}\right)dx = -\frac{5}{3\sqrt{3}}\int_0^u |x|^{2/3}dx - \frac{5}{3}\int_0^u H\left(F_{2/3}(x)\right)dx$$

and hence by the Fundamental Theorem of Calculus we see that

$$H\left(|u|^{5/3}\right) = -\frac{1}{\sqrt{3}}|u|^{5/3}\text{sgn}(u) - \frac{5}{3}\int_0^u H\left(F_{2/3}(x)\right)dx$$

Now, designating $F_{5/3}(u) := -\frac{5}{3}\int_0^u H\left(F_{2/3}(x)\right)dx$, then we observe that

$$H\left(|u|^{5/3}\right) = -\frac{1}{\sqrt{3}}|u|^{5/3}\text{sgn}(u) + F_{5/3}(u)$$

Of course, we know that, for any $H(F_{2/3}(u)) = \mathcal{O}(|u|)$ and thus taking the indefinite integral $\int H\left(F_{2/3}(x)\right) dx$ will raise the powers of each term in the expansion by one. Evaluating this expansion at zero gives zero and therefore

$$F_{5/3}(u) = -\frac{5}{3}\int_0^u H\left(F_{2/3}(x)\right) dx = \mathcal{O}(|u|^2)$$

as desired. Therefore, we have the integrated formulas $H\left(|u|^{4/3}\right) = -\sqrt{3}|u|^{4/3}\text{sgn}(u) + F_{4/3}(u)$ and $H\left(|u|^{5/3}\right) = -\frac{1}{\sqrt{3}}|u|^{5/3}\text{sgn}(u) + F_{5/3}(u)$ as desired. □

Now we will show in general that for $\beta \in \mathbb{N}$ then Hilbert transform maps $|u|^\beta$ to functions with a leading order logarithmic singularity:

Proposition 7.5. For any $n \in \mathbb{N}$ with $n \geq 1$, then we observe that

$$H(|u|^n) = \frac{u^n}{\pi}\left((\ln(|\pi - u|) - \ln(|\pi + u|)) + ((-1)^n - 1)\ln(|u|)\right) + G_n(u)$$

for some $G_n \in C^\omega(D)$, where D is any compact subset of $(-\pi, \pi)$.

Proof. Now, by Theorem 7.1, we know that

$$H\left(|u|^n\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{|x|^n}{x-u} dx + F_n(u) \tag{50}$$

for some $F_n \in C^\omega(D)$, with D any compact subset of $(-\pi, \pi)$. Therefore, let us simply evaluate the first integral in this summation. We observe that

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{|x|^n}{x-u} dx &= \int_{-\pi}^{0} \frac{(-x)^n}{x-u} dx + \int_0^{\pi} \frac{x^n}{x-u} dx \\ &= (-1)^n \int_{-\pi}^{0} \frac{x^n}{x-u} dx + \int_0^{\pi} \frac{x^n}{x-u} dx\end{aligned}$$

Now, from Equation 2.111 (3) in [10], we see that

$$\begin{aligned}\int \frac{x^n}{x-u} dx &= \sum_{k=0}^{n-1} (-1)^k \frac{(-u)^k x^{n-k}}{(n-k)} + (-1)^n(-u)^n \ln(|x-u|) \\ &= \sum_{k=0}^{n-1} (-1)^{2k} \frac{u^k x^{n-k}}{(n-k)} + (-1)^{2n} u^n \ln(|x-u|) \\ &= \sum_{k=0}^{n-1} \frac{u^k x^{n-k}}{(n-k)} + u^n \ln(|x-u|)\end{aligned}$$

and hence it follows that

$$\begin{aligned}\int_0^{\pi} \frac{x^n}{x-u} dx &= \left[\sum_{k=0}^{n-1} \frac{u^k x^{n-k}}{(n-k)} + u^n \ln(|x-u|)\right]_{x=0}^{x=\pi} \\ &= \sum_{k=0}^{n-1} \frac{u^k \pi^{n-k}}{(n-k)} + u^n \ln(|\pi - u|) - u^n \ln(|u|)\end{aligned}$$

Moreover, we have that

$$\int_{-\pi}^{0} \frac{x^n}{x-u} dx = \left[\sum_{k=0}^{n-1} \frac{u^k x^{n-k}}{(n-k)} + u^n \ln(|x-u|) \right]_{x=-\pi}^{x=0}$$
$$= u^n \ln(|u|) - \sum_{k=0}^{n-1} \frac{u^k (-\pi)^{n-k}}{(n-k)} + u^n \ln(|-\pi-u|)$$
$$= u^n \ln(|u|) - \sum_{k=0}^{n-1} (-1)^{n-k} \frac{u^k \pi^{n-k}}{(n-k)} + u^n \ln(|\pi+u|)$$
$$= u^n \ln(|u|) - \sum_{k=0}^{n-1} (-1)^{n+k} \frac{u^k \pi^{n-k}}{(n-k)} + u^n \ln(|\pi+u|)$$

so that

$$\int_{-\pi}^{\pi} \frac{|x|^n}{x-u} dx = (-1)^n \int_{-\pi}^{0} \frac{x^n}{x-u} dx + \int_{0}^{\pi} \frac{x^n}{x-u} dx$$
$$= (-1)^n u^n \ln(|u|) - \sum_{k=0}^{n-1} (-1)^{2n+k} \frac{u^k \pi^{n-k}}{(n-k)} - u^n \ln(|\pi+u|)$$
$$+ \sum_{k=0}^{n-1} \frac{u^k \pi^{n-k}}{(n-k)} + u^n \ln(|\pi-u|) - u^n \ln(|u|)$$
$$= u^n \left(\ln(|\pi-u|) - \ln(|\pi+u|) \right) + \left((-1)^n - 1 \right) u^n \ln(|u|)$$
$$+ \sum_{k=0}^{n-1} \left(1 - (-1)^k \right) \frac{u^k \pi^{n-k}}{(n-k)} \tag{51}$$

Now, because we are taking $u \neq 0$ then we have that $\ln(|\pi-u|) - \ln(|\pi+u|) \neq 0$ and so this integral will always have a logarithmic singularity. Moreover, we may take

$$G_n(u) = \frac{1}{\pi} \sum_{k=0}^{n-1} \left(1 - (-1)^k \right) \frac{u^k \pi^{n-k}}{(n-k)} + \frac{1}{\pi} F_n(u)$$

and observe that it is clearly the case that $G_n \in C^\omega(D)$. Thus, we see that when inserting equation 51 into equation 50 then we have

$$H(|u|^n) = \frac{u^n}{\pi} \left(\left(\ln(|\pi-u|) - \ln(|\pi+u|) \right) + \left((-1)^n - 1 \right) \ln(|u|) \right) + G_n(u)$$

for $n \geq 1$. □

Note that it will be useful to have the leading order logarithmic singularities in the Hilbert transform of $|u|$ and $|u|^2$ computed explicitly. Thus, observe that in Proposition 7.5, taking $n = 1$ shows that

$$H(|u|) = \frac{u}{\pi} \left(\left(\ln(|\pi-u|) - \ln(|\pi+u|) \right) - 2\ln(|u|) \right) + G_1(u)$$

Moreover, taking $n = 2$ allows for us to see that

$$H(|u|^2) = \frac{u^2}{\pi} \left(\ln(|\pi-u|) - \ln(|\pi+u|) \right) + G_2(u)$$

References

[1] Amick, C.J., L. E. Fraenkel, and Toland, J.F., On the Stokes conjecture for the wave of extreme form, *Acta Math.*, **148**, 1982, 193–214.

[2] Brown, J. W., & Churchill, R. V., Complex variables and applications, *McGraw-Hill Education*, 9th ed., 2018.

[3] Constantin, A., Particle trajectories in extreme Stokes waves, *IMA Journal of Applied Mathematics*, **77**(3), 2012, 293–307.

[4] Creedon, R. P., Deconinck, B., A high-order asymptotic analysis of the Benjamin-Feir instability spectrum in arbitrary depth, *Journal of Fluid Mechanics*, **956**, 2023, A29.

[5] Creedon, R. P., Deconinck, B. & Trichtchenko, O., High-frequency instabilities of Stokes waves, *Journal of Fluid Mechanics*, **937**, 2022, A24.

[6] Dyachenko, S. A., Lushnikov, P. M. and Korotkevich, A. O., Branch cuts of Stokes wave on deep water, Part I: Numerical solution and Pade approximation, *Stud. Appl. Math.*, **137**, 2016, 419–472.

[7] Dyachenko, S. A., Semenova, A., Quasiperiodic perturbations of Stokes waves: Secondary bifurcations and stability, *Journal of Computational Physics*, **492**, 2023, 112411.

[8] Dyachenko, A.I., Zakharov, V.E., High-Jacobian approximation in the free surface dynamics of an ideal fluid, *Physica D: Nonlinear Phenomena*, **98**, 1996, 652–664.

[9] Evans, L. C., Partial Differential Equations, *AMS*, 2nd ed., 2010.

[10] Gradshteyn, I. S., & Ryzhik, I. M., Table of integrals, series, and products, *Academic Press*, 8th ed., 2015.

[11] K. I. Babenko, Some remarks on the theory of surface waves of finite amplitude, *Sov. Math. Dokl*, **35**, 1987, 599–603.

[12] Lushnikov, P., Structure and location of branch point singularities for Stokes waves on deep water. *Journal of Fluid Mechanics*, **800**, 2016, 557–594.

[13] Lyons, T., Particle trajectories in extreme Stokes waves over infinite depth, *Discrete and Continuous Dynamical Systems*, **34**(8), 2014, 3095–3107.

[14] M. A. Grant, The singularity at the crest of a finite amplitude progressive Stokes wave, *Journal of Fluid Mechanics*, **59**(2), 1973, 257–262.

[15] Plotnikov, P.I., A Proof of the Stokes Conjecture in the Theory of Surface Waves, *Dinamika Splosh. Sredy*, **57**, 1982, 41–76.

[16] Plotnikov, P.I., & Toland, J. F. T., Convexity of Stokes waves of extreme form, *Arch. Ration. Mech. Anal.*, **171**, 2004, 349–416.

[17] Stokes, G. G., On the theory of oscillatory waves, *Math. Phys. Pap. **1***, 1880, 197–229.

[18] Toland, J. F., On the existence of a wave of greatest height and Stokes's conjecture, *Proc. R. Soc. Lond. A*, **363**, 1978, 469–485.

[19] Toland, J. F., Stokes waves, *Topol. Methods Nonlinear Anal.*, **7**, 1996, 1–48.