BRIGHT AND DARK BREATHERS ON AN ELLIPTIC WAVE IN THE DEFOCUSING MKDV EQUATION

DMITRY E. PELINOVSKY AND RUDI WEIKARD

ABSTRACT. Breathers on an elliptic wave background consist of nonlinear superpositions of a soliton and a periodic wave, both traveling with different wave speeds and interacting periodically in the space-time. For the defocusing modified Korteweg–de Vries (mKdV) equation, the construction of general breathers has been an open problem since the elliptic wave is related to the elliptic degeneration of the hyperelliptic solutions of genus two. We have found the new representation of eigenfunctions of the Lax operator associated with the elliptic wave, which enables us to solve this open problem and to construct two families of breathers with bright (elevation) and dark (depression) profiles.

1. INTRODUCTION

We consider the integrable model given by the defocusing modified Korteweg–de Vries (mKdV) equation written in the form:

$$u_t - 6u^2 u_x + u_{xxx} = 0, \tag{1.1}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}$ and $u = u(x,t) \in \mathbb{R}$. As is well-known (see reviews in [18, 19]), there exists a three-parameter family of the traveling periodic wave solutions of the defocusing mKdV equation (1.1), which we call *the elliptic wave*. In the symmetric case, it reduces to the snoidal profile, which is related to the genus-one elliptic solutions of integrable equations of the AKNS hierarchy [4]. In the non-symmetric case, it is related to the elliptic degeneration of the genus-two hyperelliptic solutions. Although Riemann theta functions degenerate into Jacobi theta functions in this limit [20], the general theory is too abstract for practical applications of the mKdV equation.

The mKdV equation is one of the fundamental models of the dispersive hydrodynamics, where the elliptic waves and breathers on their backgrounds capture dynamics of dispersive shock waves and soliton gases [1]. *Breathers* are nonlinear superpositions of a soliton and a periodic wave, both traveling with different wave speeds and interacting periodically in the space-time. In the symmetric case, breathers with the dark (depression) profiles were constructed in [37] by using the explicit expression for eigenfunctions of the Lax operator associated with the genus-one elliptic wave from [40]. In the non-symmetric case, breathers with the kink profiles were constructed in [3] for the only case when the spectral parameter is at the origin (which is the center of symmetry of the Lax spectrum). The main purpose of this work is to construct breathers with both bright (elevation) and dark (depression) profiles in the general case of the genus-two elliptic wave and for the arbitrary non-zero values of the spectral parameter.

The mKdV equation (1.1) is related to the Lax system of linear equations written for the eigenfunction $\varphi \in C^2(\mathbb{R}\times\mathbb{R},\mathbb{C}^2)$ and the spectral parameter $\zeta \in \mathbb{C}$:

$$\partial_x \varphi = U(\zeta,u)\varphi, \quad \partial_t \varphi = V(\zeta,u)\varphi, \tag{1.2}$$

where

$$U(\zeta,u) = \left(\begin{array}{cc} i\zeta & u \\ u & -i\zeta \end{array}\right),$$
$$V(\zeta,u) = \left(\begin{array}{cc} 4i\zeta^3 + 2i\zeta u^2 & 4\zeta^2 u - 2i\zeta u_x + 2u^3 - u_{xx} \\ 4\zeta^2 u + 2i\zeta u_x + 2u^3 - u_{xx} & -4i\zeta^3 - 2i\zeta u^2 \end{array}\right).$$

Classical solutions of the mKdV equation (1.1) arise as a compatibility condition of the Lax system (1.2) given by $\partial_x\partial_t\varphi = \partial_t\partial_x\varphi$. Since the spectral problem

$$\left(\begin{array}{cc} -i\partial_x & iu \\ -iu & i\partial_x \end{array}\right)\varphi = \zeta\varphi, \tag{1.3}$$

is self-adjoint in $L^2(\mathbb{R},\mathbb{C}^2)$, the admissible values of the spectral parameter ζ (forming the Lax spectrum) belong to a subset of $\mathbb{R}$. We construct explicit eigenfunctions $\varphi \in C^2(\mathbb{R}\times\mathbb{R},\mathbb{C}^2)$ in the case when $u(x,t) = \phi(x+ct)$ is a general elliptic wave with the (left-propagating) wave speed $c \in \mathbb{R}$ and the smooth periodic profile $\phi(x) : \mathbb{R} \to \mathbb{R}$ and when $\zeta \in \mathbb{R}$ is arbitrary.

1.1. **Main results.** The periodic wave profile ϕ of the traveling wave $u(x,t) = \phi(x+ct)$ with the wave speed $c \in \mathbb{R}$ satisfies the third-order equation

$$\phi''' - 6\phi^2\phi' + c\phi' = 0, \tag{1.4}$$

which can be integrated twice with two real constants of integration b and d as follows:

$$\phi'' - 2\phi^3 + c\phi = b, \tag{1.5}$$
$$(\phi')^2 - \phi^4 + c\phi^2 = 2b\phi + 2d. \tag{1.6}$$

The general periodic solution of the system (1.4), (1.5), and (1.6) has different analytic forms for $b = 0$ (symmetric case) and $b \neq 0$ (non-symmetric case). If $b \neq 0$ (without loss of generality, we consider $b > 0$), the general periodic solution is best parameterized by the real parameters $(\zeta_1,\zeta_2,\zeta_3)$ satisfying $0 < \zeta_3 < \zeta_2 < \zeta_1$ in the form:

$$\begin{cases} b = 4\zeta_1\zeta_2\zeta_3, \\ c = 2(\zeta_1^2+\zeta_2^2+\zeta_3^2), \\ d = \frac{1}{2}(\zeta_1^4+\zeta_2^4+\zeta_3^4) - \zeta_1^2\zeta_2^2 - \zeta_1^2\zeta_3^2 - \zeta_2^2\zeta_3^2. \end{cases} \tag{1.7}$$

The elliptic wave is then represented in the standard form (see, e.g., [19]):

$$\phi(x) = \frac{2(\zeta_1+\zeta_3)(\zeta_2+\zeta_3)}{(\zeta_1+\zeta_3) - (\zeta_1-\zeta_2)\mathrm{sn}^2(\nu x,k)} - \zeta_1 - \zeta_2 - \zeta_3, \tag{1.8}$$

with

$$\nu = \sqrt{\zeta_1^2 - \zeta_3^2}, \quad k = \sqrt{\frac{\zeta_1^2-\zeta_2^2}{\zeta_1^2-\zeta_3^2}}. \tag{1.9}$$

Parameters $\{\zeta_1,\zeta_2,\zeta_3\}$ in (1.7), (1.8), and (1.9) define the end points of the Lax spectrum

$$\sigma_L=(-\infty,-\zeta_1]\cup[-\zeta_2,-\zeta_3]\cup[\zeta_3,\zeta_2]\cup[\zeta_1,\infty), \tag{1.10}$$

with two symmetric bandgaps $(-\zeta_1,-\zeta_2)\cup(\zeta_2,\zeta_1)$ and the central bandgap $(-\zeta_3,\zeta_3)$. If $0<\zeta_3<\zeta_2<\zeta_1$, then the wave profile ϕ is periodic on $\mathbb{R}$ with the period $2\nu^{-1}K(k)$ and on $i\mathbb{R}$ with the period $2\nu^{-1}K'(k)$. In what follows, we drop the dependence of the elliptic functions and integrals on $k\in(0,1)$ if it does not cause a confusion.

The main result of this study is the representation formula for eigenfunctions of the Lax system (1.2) with $u(x,t)=\phi(x+ct)$ given by (1.8) and with arbitrary $\zeta\in\mathbb{R}$. To do so, we use a new representation of the elliptic function ϕ obtained in [3], which is defined by poles $\pm\nu^{-1}(iK'+\alpha)$ and zeros $\pm\nu^{-1}\beta$, where the values of $\alpha\in(0,K)$ and $\beta\in[0,K)\cup i[0,K')$ are uniquely obtained from

$$\operatorname{sn}(\alpha)=\sqrt{\frac{\zeta_1-\zeta_3}{\zeta_1+\zeta_2}},\quad \operatorname{sn}(\beta)=\sqrt{\frac{(\zeta_1+\zeta_3)(\zeta_1-\zeta_2-\zeta_3)}{(\zeta_1-\zeta_2)(\zeta_1+\zeta_2+\zeta_3)}}. \tag{1.11}$$

If $\zeta_1\neq\zeta_2+\zeta_3$, the wave profile ϕ is represented in the factorized form (see [3, Theorem 1]):

$$\phi(x)=(\zeta_1-\zeta_2-\zeta_3)\frac{\Theta^2(\alpha)}{H^2(\beta)}\frac{H(\nu x-\beta)H(\nu x+\beta)}{\Theta(\nu x-\alpha)\Theta(\nu x+\alpha)}, \tag{1.12}$$

where $\beta\in(0,K)$ for $\zeta_1>\zeta_2+\zeta_3$ and $\beta\in i(0,K')$ for $\zeta_1<\zeta_2+\zeta_3$. If $\zeta_1=\zeta_2+\zeta_3$, then $\beta=0$, and the wave profile ϕ can be written in the factorized form:

$$\zeta_1=\zeta_2+\zeta_3:\quad \phi(x)=\frac{2(\zeta_2+\zeta_3)\zeta_3\Theta^2(\alpha)\operatorname{sn}^2(\nu x)\Theta^2(\nu x)}{(\zeta_2+2\zeta_3)\Theta^2(0)\Theta(\nu x-\alpha)\Theta(\nu x+\alpha)}. \tag{1.13}$$

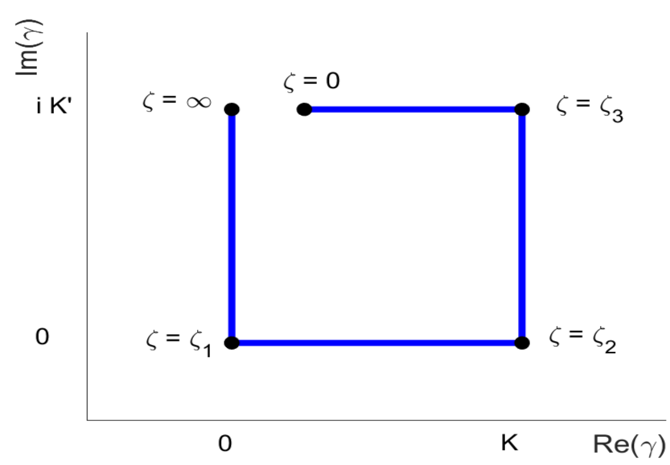

FIGURE 1. The pre-image of the mapping $[0,K]\times[0,iK']\ni\gamma\to\zeta\in[0,\infty)$ on the complex plane when ζ changes from $\zeta=0$ to $\zeta=\infty$.

To introduce the eigenfunctions of the Lax system (1.2), we parameterize the spectral parameter $\zeta \in \mathbb{R}$ by using $\gamma \in [0, K] \times [0, iK']$ from the dispersion relation:

$$\zeta^2 = \zeta_3^2 + (\zeta_1^2 - \zeta_3^2)\,\mathrm{dn}^2(\gamma). \tag{1.14}$$

Figure 1 shows the pre-image of the mapping $[0, K] \times [0, iK'] \ni \gamma \to \zeta \in [0, \infty)$, based on Lemma 1 below. With this parameterization, the following theorem specifies the explicit representation of the eigenfunctions.

Theorem 1. *Let $u(x,t) = \phi(x + ct)$ be defined by (1.8) for $0 < \zeta_3 < \zeta_2 < \zeta_1$ and $\gamma \in [0, K] \times [0, iK']$ be defined by (1.14) for a given $\zeta \in (0, \zeta_3) \cup (\zeta_3, \zeta_2) \cup (\zeta_2, \zeta_1) \cup (\zeta_1, \infty)$. There exist two linearly independent solutions of the Lax system (1.2) in the form:*

$$\varphi(x,t) = \begin{pmatrix} p_1(x+ct) \\ q_1(x+ct) \end{pmatrix} e^{\mu t}, \quad \varphi(x,t) = \begin{pmatrix} p_2(x+ct) \\ q_2(x+ct) \end{pmatrix} e^{-\mu t}, \tag{1.15}$$

where

$$\begin{pmatrix} p_1(x) \\ q_1(x) \end{pmatrix} = \begin{pmatrix} -i\zeta \frac{H(\nu x+\gamma+\alpha)}{\Theta(\nu x+\alpha)} + \zeta_1 \frac{\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)\Theta(2\alpha)} \frac{H(\nu x+\gamma-\alpha)}{\Theta(\nu x-\alpha)} \\ -i\zeta \frac{H(\nu x+\gamma+\alpha)}{\Theta(\nu x+\alpha)} - \zeta_1 \frac{\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)\Theta(2\alpha)} \frac{H(\nu x+\gamma-\alpha)}{\Theta(\nu x-\alpha)} \end{pmatrix} e^{-\nu x Z(\gamma)}, \tag{1.16}$$

$$\begin{pmatrix} p_2(x) \\ q_2(x) \end{pmatrix} = \begin{pmatrix} \zeta_1 \frac{\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)\Theta(2\alpha)} \frac{H(\nu x-\gamma+\alpha)}{\Theta(\nu x+\alpha)} + i\zeta \frac{H(\nu x-\gamma-\alpha)}{\Theta(\nu x-\alpha)} \\ \zeta_1 \frac{\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)\Theta(2\alpha)} \frac{H(\nu x-\gamma+\alpha)}{\Theta(\nu x+\alpha)} - i\zeta \frac{H(\nu x-\gamma-\alpha)}{\Theta(\nu x-\alpha)} \end{pmatrix} e^{\nu x Z(\gamma)}, \tag{1.17}$$

and

$$\mu = -4\nu^3 k^2\,\mathrm{sn}(\gamma)\,\mathrm{cn}(\gamma)\,\mathrm{dn}(\gamma). \tag{1.18}$$

Example 1. *When $\zeta = 0$, we have $\gamma = 2\alpha + iK'$, see Figure 1. By using*

$$H(z + iK') = ie^{\frac{\pi K'}{4K}} e^{-\frac{i\pi z}{2K}} \Theta(z), \quad \Theta(z + iK') = ie^{\frac{\pi K'}{4K}} e^{-\frac{i\pi z}{2K}} H(z), \tag{1.19}$$

expressions (1.16) and (1.17) reduce up to the norming constants to

$$\begin{pmatrix} p_1(x) \\ q_1(x) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{\Theta(\nu x + \alpha)}{\Theta(\nu x - \alpha)} e^{-\nu x \frac{H'(2\alpha)}{H(2\alpha)}} \tag{1.20}$$

and

$$\begin{pmatrix} p_2(x) \\ q_2(x) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{\Theta(\nu x - \alpha)}{\Theta(\nu x + \alpha)} e^{\nu x \frac{H'(2\alpha)}{H(2\alpha)}}, \tag{1.21}$$

which were derived in [3]. Furthermore, by using

$$\mathrm{sn}(z \pm iK') = \frac{1}{k\,\mathrm{sn}(z)}, \quad \mathrm{cn}(z \pm iK') = \frac{\mp i\,\mathrm{dn}(z)}{k\,\mathrm{sn}(z)}, \quad \mathrm{dn}(z \pm iK') = \frac{\mp i\,\mathrm{cn}(z)}{\mathrm{sn}(z)}, \tag{1.22}$$

and

$$\mathrm{sn}(2\alpha) = \frac{\nu}{\zeta_1}, \quad \mathrm{cn}(2\alpha) = \frac{\zeta_3}{\zeta_1}, \quad \mathrm{dn}(2\alpha) = \frac{\zeta_2}{\zeta_1}, \tag{1.23}$$

which follows from

$$\mathrm{sn}(\alpha) = \frac{\sqrt{\zeta_1 - \zeta_3}}{\sqrt{\zeta_1 + \zeta_2}}, \quad \mathrm{cn}(\alpha) = \frac{\sqrt{\zeta_2 + \zeta_3}}{\sqrt{\zeta_1 + \zeta_2}}, \quad \mathrm{dn}(\alpha) = \frac{\sqrt{\zeta_2 + \zeta_3}}{\sqrt{\zeta_1 + \zeta_3}}, \tag{1.24}$$

expression (1.18) yields $\mu = 4\zeta_1\zeta_2\zeta_3$, also in agreement with [3].

Expressions (1.16) and (1.17) of Theorem 1 will now be used to construct breathers on the elliptic wave for eigenvalues ζ chosen in the gaps $(-\zeta_1, -\zeta_2)$, $(-\zeta_3, \zeta_3)$, and (ζ_2, ζ_1) of the Lax spectrum (1.10).

1.2. **Contruction of breathers.** We use the Darboux transformation

$$u = \phi - \frac{4i\zeta pq}{p^2 - q^2}, \tag{1.25}$$

where $p = c_1 p_1 + c_2 p_2$ and $q = c_1 q_1 + c_2 q_2$ for some constant coefficients $c_1, c_2 \in \mathbb{C}$ and for a given choice of the spectral parameter $\zeta \in \mathbb{R}$. Expanding yields

$$u = \phi - \frac{4i\zeta(c_1^2 p_1 q_1 + c_1 c_2(p_1 q_2 + p_2 q_1) + c_2^2 p_2 q_2)}{c_1^2(p_1^2 - q_1^2) + 2c_1 c_2(p_1 p_2 - q_1 q_2) + c_2^2(p_2^2 - q_2^2)}. \tag{1.26}$$

We show in Section 4 below that if $\zeta \in (0, \zeta_3) \cup (\zeta_2, \zeta_1)$, then we can choose scaling factors in (p_1, q_1) and (p_2, q_2) such that

$$p_1^2 - q_1^2, p_2^2 - q_2^2, p_1 q_2 + p_2 q_1 \in i\mathbb{R} \tag{1.27}$$

and

$$p_1 p_2 - q_1 q_2, p_1 q_1, p_2 q_2 \in \mathbb{R}. \tag{1.28}$$

This implies that the solution $u = u(x,t)$ in (1.26) is real if $c_1 c_2 \in i\mathbb{R}$, $c_1^2, c_2^2 \in \mathbb{R}$. The exponential factors in (1.16) and (1.17) are defined differently between $(0, \zeta_3)$ and (ζ_2, ζ_1) by the parameter $\kappa > 0$:

$$\kappa = \begin{cases} \frac{\Theta'(\gamma)}{\Theta(\gamma)}, & \zeta \in (\zeta_2, \zeta_1), \\ \frac{H'(\gamma - iK')}{H(\gamma - iK')}, & \zeta \in (0, \zeta_3). \end{cases} \tag{1.29}$$

It follows from (1.26) that only the factor c_2/c_1 affects the representation of the solutions. To reduce the number of parameters, we introduce the phase translation of the soliton relative to the elliptic wave and define

$$c_1 = e^{-\nu\kappa x_0}, \quad c_2 = -ie^{\nu\kappa x_0}, \tag{1.30}$$

where the negative sign of c_2 was chosen to obtain bounded breather solutions. It follows from (1.15) with (1.16) and (1.17) that the breather propagates along the coordinate $\eta = x + c_s t + x_0$ with the (left-propagating) wave speed c_s given by

$$c_s = c - \frac{\mu}{\nu\kappa}. \tag{1.31}$$

In comparison with the three-parameter family of the elliptic wave with the profile ϕ in (1.8), the breather solution obtained from (1.25), (1.26), and (1.30) has two additional parameters $\zeta \in (0, \zeta_3) \cup (\zeta_2, \zeta_1)$ (which determines uniquely $\gamma \in [0, K] \times [0, iK']$, see Figure 1) and $x_0 \in \mathbb{R}$. Since $\kappa > 0$ in (1.29), it follows from (1.16) and (1.17) that the elliptic wave is shifted across the breather with the phase shift in x given by $2\nu^{-1}(\gamma - \alpha)$.

To illustrate the breather solutions, let us first show the periodic profiles of ϕ in Figure 2 for $(\zeta_1, \zeta_2, \zeta_3) = (2, 1, 0.5)$ (left) and $(\zeta_1, \zeta_2, \zeta_3) = (1.25, 1, 0.5)$ (right). In both cases, the difference between (1.8) and (1.12) is found within the machine precision error. The left panel corresponds to $\beta \in (0, K)$, for which ϕ has zeros on $\mathbb{R}$, and the right panel corresponds to $\beta \in i(0, K')$, for which ϕ has no zeros on $\mathbb{R}$.

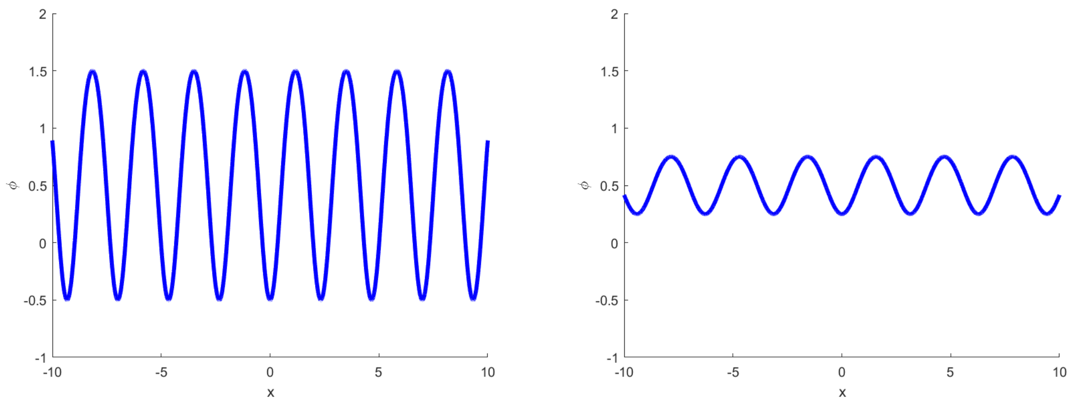

FIGURE 2. The elliptic function ϕ versus x given by either (1.8) or (1.12) for $(\zeta_1, \zeta_2, \zeta_3) = (2, 1, 0.5)$ (left) and $(\zeta_1, \zeta_2, \zeta_3) = (1.25, 1, 0.5)$ (right).

The breather solution $u = u(x, t)$ in (1.26) for the choice of ζ in $(0, \zeta_3)$ is shown for $(\zeta_1, \zeta_2, \zeta_3) = (2, 1, 0.5)$ in Figure 3 and for $(\zeta_1, \zeta_2, \zeta_3) = (1.25, 1, 0.5)$ in Figure 4. In both cases, the left panels show the snapshot versus x at $t = 0$ and the right panels show the solution surface versus (ξ, t), where $\xi = x + ct$. In both cases, we have set $x_0 = 0$ in (1.30) and chosen ζ from (1.14) with $\gamma = 2\alpha + 0.1(K - 2\alpha) + iK'$. For the choice of ζ in $(0, \zeta_3)$, breathers represent the elevation (bright) profile propagating on the traveling elliptic wave to the right in the reference frame $\xi = x + ct$ moving with the wave speed c since $c_s < c$ in (1.31).

In the limit $\gamma \to 2\alpha + iK'$ (when $\zeta \to 0$), the breather profile becomes wide and the solution degenerates to the kink breather constructed in [3, Theorem 2]. There are two kinks by the symmetry: from negative to positive values of the periodic wave and from positive to negative values of the periodic wave (the latter is shown in Figure 4 in [3]). The kinks also moves to the right in the reference frame $\xi = x + ct$ moving with the wave speed c.

The breather solution $u = u(x, t)$ in (1.26) for the choice of ζ in (ζ_2, ζ_1) is shown for $(\zeta_1, \zeta_2, \zeta_3) = (2, 1, 0.5)$ in Figure 5 and for $(\zeta_1, \zeta_2, \zeta_3) = (1.25, 1, 0.5)$ in Figure 6. The spectral parameter ζ is defined by (1.14) with $\gamma = 0.5K$. For the choice of ζ in (ζ_2, ζ_1), breathers represent the depression (dark) profile propagating on the traveling periodic background to the left in the reference frame $\xi = x + ct$ moving with the wave speed c since $c_s > c$ in (1.31).

In the limit $\zeta_3 \to 0$, the elliptic wave is reduced to the snoidal profile by the Landen transformation [3, Example 3.2]. Dark breathers on the snoidal profile were constructed explicitly in [37, Theorem 1]. The dark breathers also move to the left in the reference frame $\xi = x + ct$ moving with the wave speed c. see Figure 1 in [37].

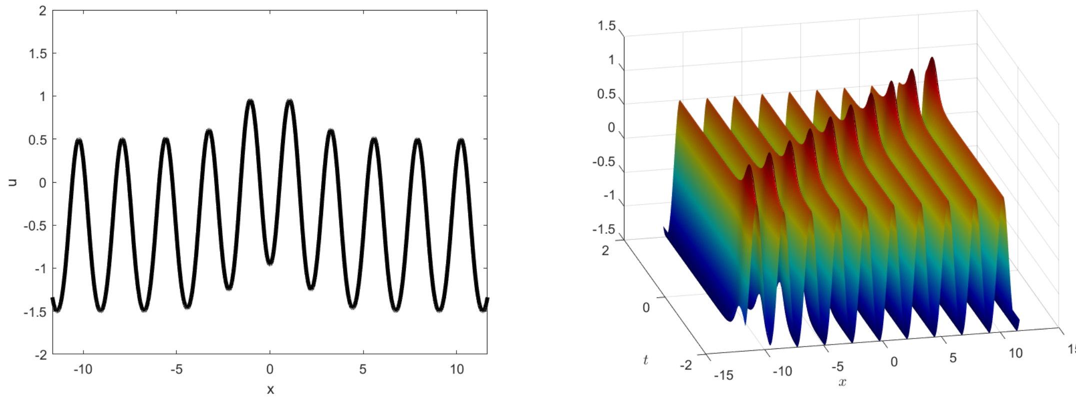

FIGURE 3. The snapshow versus x for $t = 0$ (left) and the solution surface versus (ξ, t) (right) for the bright breather with $(\zeta_1, \zeta_2, \zeta_3) = (2, 1, 0.5)$.

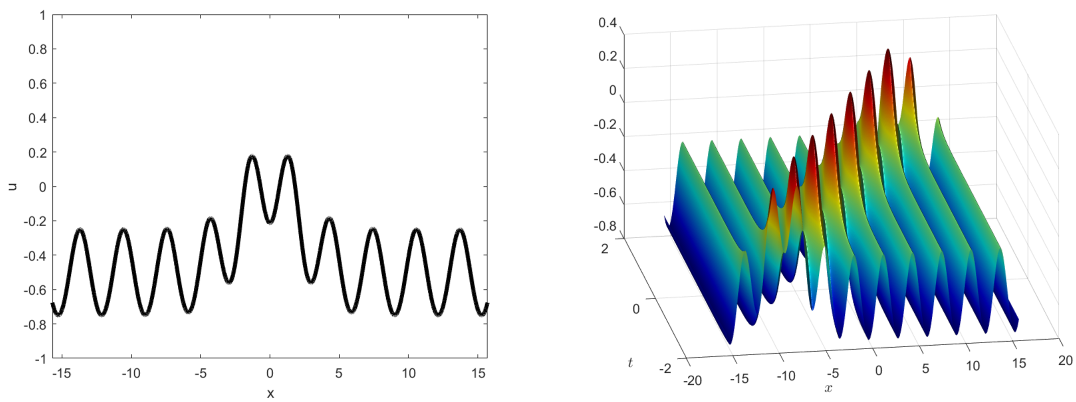

FIGURE 4. The same as Figure 3 but for $(\zeta_1, \zeta_2, \zeta_3) = (2, 1, 0.5)$.

1.3. **Background of the problem.** Breathers for the Korteweg–de Vries (KdV) equation were first constructed in the pioneering work [29] based on the dressing method. Algebro-geometric constructions of breathers were developed in [21] with a rather abstract approach based on commutation identities. Another method of constructing such solutions for the KP hierarchy was developed in [5, 38] with the degeneration of Grassmanians, see also [32].

Inspired by the recent studies of the soliton gas in [13, 23, 24], the breather solutions on the elliptic wave background appear to be the central objects for interactions of solitons in the limit of infinitely many solitons. Consequently, the same solutions of the KdV equation were comprehensively studied in two independent works [6, 26]. It is shown in [26] that the bright profiles are related to the spectral parameter chosen below the Lax spectrum and the dark profiles are related to the choice in a spectral gap of a finite length.

The same bright and dark profiles of breathers are found in the nonlocal Benjamin–Ono (BO) equation [10], where the traveling wave is expressed in terms of trigonometric rather

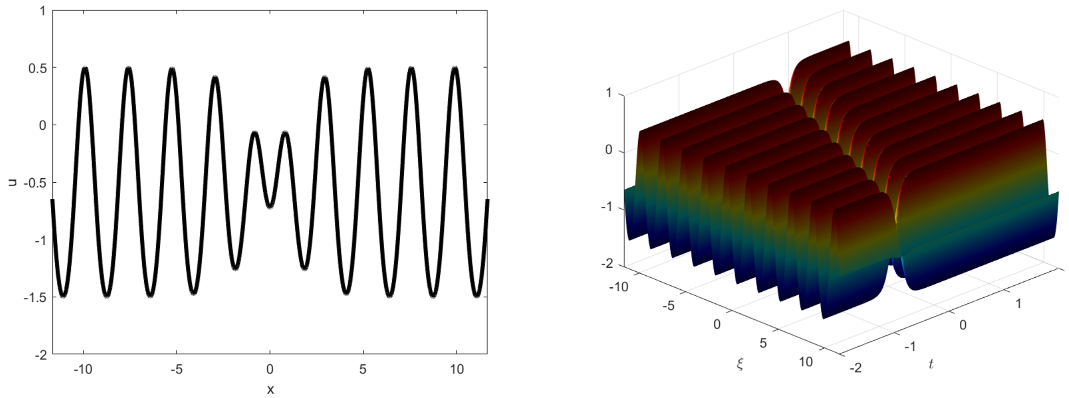

FIGURE 5. The snapshow versus x for $t = 0$ (left) and the solution surface versus (ξ, t) (right) for the dark breather with $(\zeta_1, \zeta_2, \zeta_3) = (2, 1, 0.5)$.

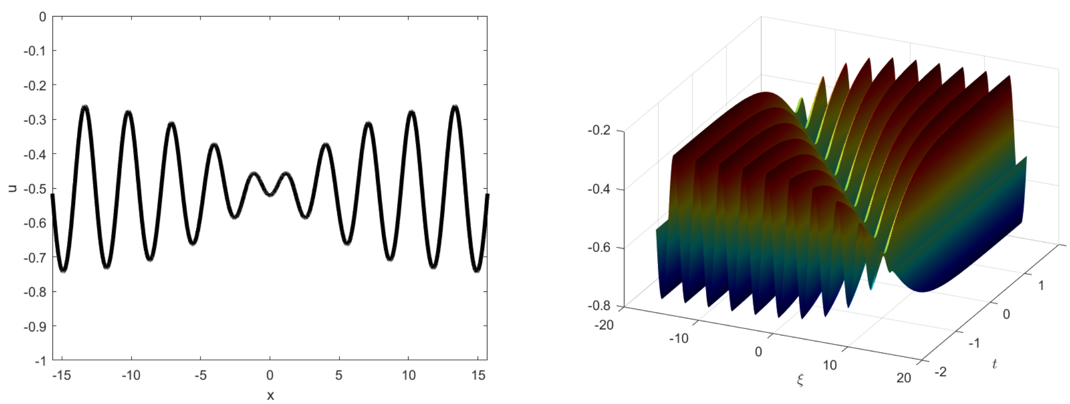

FIGURE 6. The same as Figure 5 but for $(\zeta_1, \zeta_2, \zeta_3) = (2, 1, 0.5)$.

than elliptic functions but the Lax spectrum and the choice of the spectral parameter are similar to the case of the KdV equation. For the nonlocal derivative nonlinear Schrödinger NLS equation, it is shown in [11] that only dark profiles exist in the defocusing case, whereas both bright and dark profiles coexist in the focusing case. Similarly to the BO equation, the traveling wave is still expressed by trigonometric functions and has a very similar Lax spectrum but the spectral parameter for the breathers can be chosen inside the continuous spectrum, contrary to the cases of KdV and BO equations.

For the defocusing cubic NLS equation, dark profiles of breathers were constructed for the snoidal elliptic wave in [39, 40], see also [36]. It was found recently in [27] that both bright and dark profiles of breathers exist for the elliptic wave with a non-trivial phase, which is still expressed by the genus-one elliptic functions. The elliptic wave background is modulationally unstable for the focusing cubic NLS equation and formations of rogue waves and time-decaying and space-periodic breathers complicate dynamics of perturbations [7, 12, 15].

For the defocusing mKdV equation, the results of [3] and this study shows that breathers propagating on the general elliptic wave may have bright, dark, and kink profiles. This is very different to the only dark profiles found on the snoidal elliptic wave in the symmetric case [37]. For the focusing mKdV equation, one family of traveling waves (generalizing the dnoidal profiles) is modulationally stable and the other family of traveling waves (generalizing the cnoidal profiles) is modulationally unstable, see [14] and the references therein. Consequently, bright breathers were observed on the dnoidal profiles [24, 25] and rogue waves were constructed on the cnoidal profiles [8, 9], but the comprehensive study of breathers in the focusing mKdV equation is still open.

For the focusing mKdV equation, stability of the elliptic profiles and breathers on their background were only developed in the symmetric case [33, 34]. The main obstacle has been again the explicit characterization of the eigenfunctions of the Lax operators. Only recently in [35], the eigenfunctions are constructed by degeneration of the Riemann theta functions of genus two, very similar to the general elliptic theory in [20]. One of the possible application of our work is to construct the explicit eigenfunctions of the Lax operators for the focusing mKdV equation in terms of the elliptic functions.

Finally, we mention another form used to characterize eigenfunctions of the Lax operators related to the elliptic degeneration of the Riemann theta functions of genus two [30, 31]. The elliptic functions are characterized as quotient of products of Jacobi theta functions centered at the poles and zeros of the elliptic function. Although this form can also be obtained for the eigenfunctions of the Lax system (1.2), we show in Section 5 below that the poles and zeros of the elliptic eigenfunctions are defined by using the branch point singularities, and therefore, the factorization formulas are not as explicit as the representation of eigenfunctions in terms of elliptic functions obtained in Theorem 1.

1.4. **Methods and organization of the paper.** Theorem 1 is proven in Sections 2 and 3. The main idea for the proof comes from an application of the Miura transformation

$$v(x,t) = u^2(x,t) \pm u_x(x,t), \tag{1.32}$$

which relates solutions $u(x,t)$ of the mKdV equation (1.1) to solutions $v(x,t)$ of the KdV equation $v_t + 6vv_x + v_{xxx} = 0$. A similar transformation maps potentials of the spectral problem (1.3) to potentials of the stationary Schrödinger equation. By using Weierstrass' elliptic functions, we show that the Schrödinger equation for the general elliptic wave (1.8) becomes the Lamé equation with a single-gap elliptic potential, solutions of which are well-known, see [28, p.395] and [22, Section 3]. . This yields the representation of the eigenfunctions of the spectral problem (1.3) as a linear combination of two solutions of the Schrödinger equation. The coefficients of this linear combination are uniquely computed by using expansions of elliptic functions near poles in the complex plane. This is achieved in Section 2 with the complex parameter $\gamma \in [0,K] \times [0,iK']$ to represent the eigenfunctions in the form (1.16) and (1.17).

The pre-image of the mapping $[0,K]\times[0,iK']\ni\gamma\to\zeta\in[0,\infty)$ shown in Figure 1 is computed in Section 3, where we also compute the time-dependent form (1.15) of eigenfunctions. The mapping $[0,K]\times[0,iK']\ni\gamma\to\mu\in\mathbb{C}$ is obtained in the explicit form (1.18) also by using relations to Weierstrass' elliptic functions and computation at a single point $x_0\in\mathbb{C}$. We also show that $\mu\in\mathbb{R}$ if $\zeta\in(-\zeta_1,-\zeta_2)\cup(-\zeta_3,\zeta_3)\cup(\zeta_2,\zeta_1)$, which are the three gaps of the Lax spectrum (1.10) associated with the elliptic wave.

Breathers in the form (1.26) are studied in Section 4, where we justify the conditions (1.27) and (1.28) as well as the choice (1.30) for the breather solution to be real-valued and bounded. For the positive half-gap $(0,\zeta_3)$, where bright breathers of Figures 3 and 4 are constructed, we only need to rewrite the eigenfunctions of Theorem 1 for $\gamma=iK'+\delta$ with $\delta\in(2\alpha,K)$ and choose the norming constants conveniently. However, for the positive gap (ζ_2,ζ_1), where dark breathers of Figures 5 and 6 are constructed, we use $\gamma\in(0,K)$ but change x to $iK'+x$. This transformation maps the bounded profile ϕ of the elliptic wave to the singular profile but also maps the singular profile of the breather solution $u=u(x,t)$ into a bounded profile.

Finally, in Section 5, we discuss the factorization formula for elliptic eigenfunctions of the Lax system (1.2) associated with the elliptic wave in the factorized forms (1.12) or (1.13). We show that the poles and zeros of the elliptic eigenfunctions are defined by using the branch pole singularities, which cannot be unfolded with the use of elliptic functions.

1.5. **Notations for elliptic functions.**

We use Jacobi's theta functions:

$$\begin{cases} \theta_1(y)=2\sum_{n=1}^{\infty}(-1)^{n-1}q^{(n-\frac{1}{2})^2}\sin(2n-1)y,\\ \theta_4(y)=1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2ny,\end{cases}$$

where $q:=e^{-\frac{\pi K'}{K}}$ with K being the complete elliptic integral for elliptic modulus k and K' being the same for $k'=\sqrt{1-k^2}$. For notational convenience, we use

$$H(x)=\theta_1(y),\quad \Theta(x)=\theta_4(y),\quad \text{with}\quad y=\frac{\pi x}{2K}. \tag{1.33}$$

Jacobi's theta functions (1.33) are related to the elliptic function sn, cn, and dn due to

$$\operatorname{sn}(x)=\frac{H(x)}{\sqrt{k}\Theta(x)},\quad \operatorname{cn}(x)=\frac{\sqrt{k'}H(x+K)}{\sqrt{k}\Theta(x)},\quad \operatorname{dn}(x)=\frac{\sqrt{k'}\Theta(x+K)}{\Theta(x)}. \tag{1.34}$$

These elliptic functions are related by the fundamental relations

$$\operatorname{sn}^2(x)+\operatorname{cn}^2(x)=1,\quad \operatorname{dn}^2(x)+k^2\operatorname{sn}^2(x)=1,\quad \operatorname{dn}^2(x)-k^2\operatorname{cn}^2(x)=(k')^2. \tag{1.35}$$

Functions H, sn, and cn are $2K$-antiperiodic and $2iK'$-periodic, whereas functions Θ and dn are $2K$-periodic and $2iK'$-periodic. Further properties of elliptic functions can be consulted in [17].

2. Eigenfunctions of the spectral problem (1.3)

We derive the explicit representations (1.16) and (1.17) of elliptic eigenfunctions in Theorem 1. To do so, we first give the relation between the elliptic wave (1.8) and Weierstrass' elliptic functions (Section 2.1), which is used in the derivation. Then, we proceed with the Miura transformation (1.32) to derive the explicit solutions of the Lamé equation with a single-gap elliptic potential (Section 2.2). Finally, we obtain the unique expressions for the linear superpositions of explicit solutions of the Lamé equation from expansion of elliptic functions near poles in the complex plane (Section 2.3).

2.1. **Relation to Weierstrass' elliptic function.** We recall from [2, 41], reviewed in [3], that the elliptic wave (1.8) is related to Weierstrass' elliptic function

$$\wp(x) = e_3 + \frac{\nu^2}{\mathrm{sn}^2(\nu x, k)}, \quad \nu = \sqrt{e_1 - e_3}, \quad k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}, \tag{2.1}$$

where $e_3 < e_2 < e_1$ are given by the relations

$$\left\{ \begin{array}{l} e_1 = \frac{1}{3}(\zeta_1^2 + \zeta_2^2 - 2\zeta_3^2), \\ e_2 = \frac{1}{3}(\zeta_1^2 - 2\zeta_2^2 + \zeta_3^2), \\ e_3 = \frac{1}{3}(-2\zeta_1^2 + \zeta_2^2 + \zeta_3^2), \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} e_1 - e_2 = \zeta_2^2 - \zeta_3^2, \\ e_1 - e_3 = \zeta_1^2 - \zeta_3^2, \\ e_2 - e_3 = \zeta_1^2 - \zeta_2^2. \end{array} \right. \tag{2.2}$$

Weierstrass' function $\wp$ is periodic with periods 2ω and $2\omega'$, where

$$\omega = \frac{K}{\sqrt{e_1 - e_3}}, \quad \omega' = \frac{iK'}{\sqrt{e_1 - e_3}}. \tag{2.3}$$

By [3, Lemma 3.4], there exists $v \in [-\omega, \omega] \times [-\omega', \omega']$ such that

$$\frac{c}{6} = \wp(v), \quad \frac{b}{2} = \wp'(v). \tag{2.4}$$

By using $v \in [-\omega, \omega] \times [-\omega', \omega']$, the elliptic functions ϕ, ϕ', and ϕ^2 are related to Weierstrass' function $\wp$ as follows:

$$\phi(x) = \frac{1}{2} \frac{\wp'(x - \frac{v}{2}) + \wp'(x + \frac{v}{2})}{\wp(x - \frac{v}{2}) - \wp(x + \frac{v}{2})} = \zeta\left(x + \frac{v}{2}\right) - \zeta\left(x - \frac{v}{2}\right) - \zeta(v), \tag{2.5}$$

$$\phi'(x) = \wp\left(x - \frac{v}{2}\right) - \wp\left(x + \frac{v}{2}\right), \tag{2.6}$$

and

$$\phi^2(x) = \wp\left(x + \frac{v}{2}\right) + \wp\left(x - \frac{v}{2}\right) + \wp(v). \tag{2.7}$$

where ζ is Weierstrass' zeta function. Moreover, by [3, Lemma 3.6], we have the correspondence between $v \in [-\omega, \omega] \times [-\omega', \omega']$ in (2.4) and $\alpha \in (0, K)$ in (1.11) given by

$$\frac{v}{2} = -\frac{iK' + \alpha}{\sqrt{e_1 - e_3}}. \tag{2.8}$$

2.2. **Miura transformation.** We consider here the spectral problem (1.3) with $u(x,t) = \phi(x+ct)$ and use x in place of $x+ct$. Squaring the spectral problem (1.3) yields

$$\begin{pmatrix} -\partial_x^2 + \phi^2 & \phi' \\ \phi' & -\partial_x^2 + \phi^2 \end{pmatrix} \psi = \zeta^2 \psi, \tag{2.9}$$

where $\psi = (p,q)^T$ is the eigenfunction and $\zeta \in \mathbb{R}$ is the spectral parameter. By folding (2.9) in variables $\psi_\pm = p \pm q$, we get two uncoupled Schrödinger equations

$$\left(-\partial_x^2 + \phi^2 \pm \phi'\right) \psi_\pm = \zeta^2 \psi_\pm. \tag{2.10}$$

By using (2.6) and (2.7), we obtain

$$\phi^2(x) \pm \phi'(x) = \wp(v) + 2\wp\left(x \mp \frac{v}{2}\right). \tag{2.11}$$

In view of (2.8), we obtain from (2.1) and (2.2) that

$$\begin{aligned} \wp\left(x \mp \frac{v}{2}\right) &= e_3 + \frac{e_1 - e_3}{\mathrm{sn}^2(\sqrt{e_1 - e_3}(x \mp \frac{v}{2}))} \\ &= \frac{1}{3}(-2\zeta_1^2 + \zeta_2^2 + \zeta_3^2) + \frac{\zeta_1^2 - \zeta_3^2}{\mathrm{sn}^2(\nu x \pm iK' \pm \alpha)} \\ &= \frac{1}{3}(-2\zeta_1^2 + \zeta_2^2 + \zeta_3^2) + (\zeta_1^2 - \zeta_2^2)\,\mathrm{sn}^2(\nu x \pm \alpha), \end{aligned} \tag{2.12}$$

where we have used (1.22) as well as the definition of ν and k in (1.9). Since

$$\wp(v) = \frac{c}{6} = \frac{1}{3}(\zeta_1^2 + \zeta_2^2 + \zeta_3^2), \tag{2.13}$$

we finally obtain from (2.10), (2.11), and (2.12):

$$\left(-\partial_x^2 - 2(\zeta_1^2 - \zeta_2^2)\,\mathrm{cn}^2(\nu x \pm \alpha)\right) \psi_\pm = (\zeta^2 - \zeta_1^2 + \zeta_2^2 - \zeta_3^2)\psi_\pm. \tag{2.14}$$

Recall from [28, p.395] that the general solution of the Lamé equation

$$-\varphi''(z) + 2k^2\,\mathrm{sn}^2(z)\varphi(z) = \eta\varphi(z) \tag{2.15}$$

is given by

$$\varphi(z) = c_1 \frac{H(z+\gamma)}{\Theta(z)} e^{-zZ(\gamma)} + c_2 \frac{H(z-\gamma)}{\Theta(z)} e^{zZ(\gamma)}, \quad Z(\gamma) := \frac{\Theta'(\gamma)}{\Theta(\gamma)},$$

where $c_1, c_2 \in \mathbb{C}$ are arbitrary coefficients and $\gamma \in [-K,K] \times [-iK', iK']$ is defined from the spectral parameter $\eta \in \mathbb{R}$ by the dispersion relation $\eta = k^2 + \mathrm{dn}^2(\gamma)$.

Since $z = \nu x \pm \alpha$, we obtain by comparing (2.14) with (2.15) that

$$\psi_\pm(x) = 2c_1^\pm \frac{H(\nu x + \gamma \pm \alpha)}{\Theta(\nu x \pm \alpha)} e^{-\nu x Z(\gamma)} + 2c_2^\pm \frac{H(\nu x - \gamma \pm \alpha)}{\Theta(\nu x \pm \alpha)} e^{\nu x Z(\gamma)}, \tag{2.16}$$

where $c_1^\pm, c_2^\pm \in \mathbb{C}$ are arbitrary coefficients, the factor 2 is used for convenience, and $\gamma \in [0,K] \times [0, iK']$ is defined from the spectral parameter $\zeta \in \mathbb{R}$ by the dispersion relation

$$\zeta^2 = \zeta_1^2 - (\zeta_1^2 - \zeta^2)\,\mathrm{sn}^2(\gamma) = \zeta_2^2 + (\zeta_1^2 - \zeta^2)\,\mathrm{cn}^2(\gamma) = \zeta_3^2 + (\zeta_1^2 - \zeta_3^2)\,\mathrm{dn}^2(\gamma), \tag{2.17}$$

where the fundamental elliptic relations (1.35) have been used. Lemma 1 below clarifies the admissible values of $\gamma \in [0, K] \times [0, iK']$ from the dispersion relation (2.17).

2.3. **Coefficients of the linear superposition.** Since $\psi_\pm = p \pm q$, we get two linearly independent solutions $\psi = (p_1, q_1)$ and $\psi = (p_2, q_2)$ of the spectral problem (1.3) from solutions (2.16). The first solution is

$$\begin{pmatrix} p_1(x) \\ q_1(x) \end{pmatrix} = \begin{pmatrix} c_1^+ \frac{H(\nu x+\gamma+\alpha)}{\Theta(\nu x+\alpha)} + c_1^- \frac{H(\nu x+\gamma-\alpha)}{\Theta(\nu x-\alpha)} \\ c_1^+ \frac{H(\nu x+\gamma+\alpha)}{\Theta(\nu x+\alpha)} - c_1^- \frac{H(\nu x+\gamma-\alpha)}{\Theta(\nu x-\alpha)} \end{pmatrix} e^{-\nu x Z(\gamma)} \tag{2.18}$$

and the second solution is

$$\begin{pmatrix} p_2(x) \\ q_2(x) \end{pmatrix} = \begin{pmatrix} c_2^+ \frac{H(\nu x-\gamma+\alpha)}{\Theta(\nu x+\alpha)} + c_2^- \frac{H(\nu x-\gamma-\alpha)}{\Theta(\nu x-\alpha)} \\ c_2^+ \frac{H(\nu x-\gamma+\alpha)}{\Theta(\nu x+\alpha)} - c_2^- \frac{H(\nu x-\gamma-\alpha)}{\Theta(\nu x-\alpha)} \end{pmatrix} e^{\nu x Z(\gamma)}. \tag{2.19}$$

Since there exist only two linearly-independent solutions of the spectral problem (1.3), the parameters (c_1^+, c_1^-) and (c_2^+, c_2^-) cannot be arbitrary but must be constant proportional to scalar arbitrary parameters.

Example 2. *If $\zeta = 0$, we have $\gamma = 2\alpha + iK'$, see Figure 1. Relations (1.19) imply that*

$$\frac{\Theta'(\gamma)}{\Theta(\gamma)} = \frac{\Theta'(2\alpha + iK')}{\Theta(2\alpha + iK')} = -\frac{i\pi}{2K} + \frac{H'(2\alpha)}{H(2\alpha)}.$$

The expression (2.18) yields (1.20) for $\gamma = 2\alpha + iK'$ if $c_1^+ = 0$ since

$$\frac{H(\nu x + \gamma - \alpha)}{\Theta(\nu x - \alpha)} e^{-\nu x \frac{\Theta'(\gamma)}{\Theta(\gamma)}} = \frac{H(\nu x + \alpha + iK')}{\Theta(\nu x - \alpha)} e^{-\nu x \frac{\Theta'(2\alpha+iK')}{\Theta(2\alpha+iK')}} = ie^{\frac{\pi K'}{4K}} \frac{\Theta(\nu x + \alpha)}{\Theta(\nu x - \alpha)} e^{-\nu x \frac{H'(2\alpha)}{H(2\alpha)}}.$$

The expression (2.19) yields (1.21) for $\gamma = 2\alpha + iK'$ if $c_2^- = 0$ since

$$\frac{H(\nu x - \gamma + \alpha)}{\Theta(\nu x + \alpha)} e^{\nu x \frac{\Theta'(\gamma)}{\Theta(\gamma)}} = \frac{H(\nu x - \alpha - iK')}{\Theta(\nu x + \alpha)} e^{\nu x \frac{\Theta'(2\alpha+iK')}{\Theta(2\alpha+iK')}} = -ie^{\frac{\pi K'}{4K}} \frac{\Theta(\nu x - \alpha)}{\Theta(\nu x + \alpha)} e^{\nu x \frac{H'(2\alpha)}{H(2\alpha)}}.$$

The proportionality factors in both expressions are not relevant due to the scalar multiplication of eigenfunctions. Thus, if $\zeta = 0$, then $c_1^+ = 0$ and $c_2^- = 0$.

To find the relations between c_1^+ and c_1^- in the first solution (2.18) for other values of ζ, we compute the pole contributions of the elliptic solutions of the spectral problem (1.3) at

$$x = \frac{v}{2} = -\frac{\alpha + iK'}{\nu}.$$

Since $\wp(x) = \frac{1}{x^2} + \tilde{\wp}(x)$ as $x \to 0$ with analytic $\tilde{\wp}(x)$ near $x = 0$, it follows from (2.5) that

$$\phi(x) = -\frac{1}{x - \frac{v}{2}} + \mathcal{O}\left(x - \frac{v}{2}\right), \quad \text{as } x \to \frac{v}{2}. \tag{2.20}$$

By using (1.34) for $H(x) = \sqrt{k}\,\mathrm{sn}(x)\Theta(x)$, we obtain the asymptotic expansion:

$$H(x) = \sqrt{k}\Theta(0)x + \mathcal{O}(x^3), \quad \text{as } x \to 0.$$

In view of (1.19) and (2.8), this implies that

$$\Theta(\nu x+\alpha) = -i\nu\sqrt{k}\Theta(0)e^{\frac{\pi K'}{4K}}e^{\frac{i\pi\nu}{2K}\left(x-\frac{v}{2}\right)}\left(x-\frac{v}{2}\right)+\mathcal{O}\left(x-\frac{v}{2}\right)^3, \quad \text{as } x\to\frac{v}{2}. \tag{2.21}$$

By using (2.18) and (2.21), we obtain as $x\to\frac{v}{2}$:

$$\begin{aligned}
&\frac{H(\nu x+\gamma+\alpha)}{\Theta(\nu x+\alpha)}e^{-\nu x Z(\gamma)}\\
&=\frac{i}{\nu\sqrt{k}\Theta(0)}e^{-\frac{\pi K'}{4K}}e^{(\alpha+iK')Z(\gamma)}e^{-\nu\left(x-\frac{v}{2}\right)\left[Z(\gamma)+\frac{i\pi}{2K}\right]}\frac{H\left(\nu\left(x-\frac{v}{2}\right)+\gamma-iK'\right)}{x-\frac{v}{2}}+\mathcal{O}\left(x-\frac{v}{2}\right)\\
&=\frac{1}{\nu\sqrt{k}\Theta(0)}e^{(\alpha+iK')Z(\gamma)+\frac{i\pi\gamma}{2K}}e^{-\nu\left(x-\frac{v}{2}\right)Z(\gamma)}\frac{\Theta\left(\nu\left(x-\frac{v}{2}\right)+\gamma\right)}{x-\frac{v}{2}}+\mathcal{O}\left(x-\frac{v}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{H(\nu x+\gamma-\alpha)}{\Theta(\nu x-\alpha)}e^{-\nu x Z(\gamma)} &= e^{(\alpha+iK')Z(\gamma)}\frac{H(\gamma-2\alpha-iK')}{\Theta(-2\alpha-iK')}+\mathcal{O}\left(x-\frac{v}{2}\right),\\
&= e^{(\alpha+iK')Z(\gamma)+\frac{i\pi\gamma}{2K}}\frac{\Theta(\gamma-2\alpha)}{H(-2\alpha)}+\mathcal{O}\left(x-\frac{v}{2}\right).
\end{aligned}$$

In order to substitute the meromorphic parts into $p' = i\zeta p+\phi q$, we deduce that

$$p(x) = \frac{c_1^+}{\nu\sqrt{k}\Theta(0)}e^{(\alpha+iK')Z(\gamma)+\frac{i\pi\gamma}{2K}}\frac{\Theta(\gamma)}{x-\frac{v}{2}}+\mathcal{O}(1), \quad \text{as } x\to\frac{v}{2}, \tag{2.22}$$

which yields

$$p'(x) = -\frac{c_1^+}{\nu\sqrt{k}\Theta(0)}e^{(\alpha+iK')Z(\gamma)+\frac{i\pi\gamma}{2K}}\frac{\Theta(\gamma)}{\left(x-\frac{v}{2}\right)^2}+\mathcal{O}(1), \quad \text{as } x\to\frac{v}{2}. \tag{2.23}$$

On the other hand, we get

$$\begin{aligned}
q(x) = &\frac{c_1^+}{\nu\sqrt{k}\Theta(0)}e^{(\alpha+iK')Z(\gamma)+\frac{i\pi\gamma}{2K}}e^{-\nu\left(x-\frac{v}{2}\right)Z(\gamma)}\frac{\Theta\left(\nu\left(x-\frac{v}{2}\right)+\gamma\right)}{x-\frac{v}{2}}\\
&- c_1^- e^{(\alpha+iK')Z(\gamma)+\frac{i\pi\gamma}{2K}}\frac{\Theta(\gamma-2\alpha)}{H(-2\alpha)}+\mathcal{O}\left(x-\frac{v}{2}\right), \quad \text{as } x\to\frac{v}{2}.
\end{aligned} \tag{2.24}$$

The double pole in $p'(x)$ given by (2.23) cancels out with the double pole in $\phi(x)q(x)$ given by the product of (2.20) and (2.24). The simple pole from $i\zeta p(x)$ given by (2.22) must cancel the simple pole in $\phi(x)q(x)$, due to a relation between c_1^+ and c_1^-. Using (2.20), (2.22), (2.23), and (2.24) in $p' = i\zeta p+\phi q$, we obtain

$$i\zeta\frac{c_1^+\Theta(\gamma)}{\nu\sqrt{k}\Theta(0)}-\frac{c_1^+}{\sqrt{k}\Theta(0)}\left[-\Theta(\gamma)Z(\gamma)+\Theta'(\gamma)\right]+c_1^-\frac{\Theta(\gamma-2\alpha)}{H(-2\alpha)}=0,$$

after cancelling the constant factor $e^{(\alpha+iK')Z(\gamma)+\frac{i\pi\gamma}{2K}}$. Since $\Theta'(\gamma)=\Theta(\gamma)Z(\gamma)$, this yields the linear equation

$$i\zeta\frac{c_1^+\Theta(\gamma)}{\nu\sqrt{k}\Theta(0)}-c_1^-\frac{\Theta(2\alpha-\gamma)}{H(2\alpha)}=0. \tag{2.25}$$

The same equation follows also from $q'=-i\zeta q+\phi p$ as $x\to\frac{v}{2}$.

Repeating the derivation as $x\to-\frac{v}{2}$, where

$$\begin{cases}\phi(x)=\frac{1}{x+\frac{v}{2}}+\mathcal{O}\left(x+\frac{v}{2}\right),\\ \Theta(\nu x-\alpha)=i\nu\sqrt{k}\Theta(0)e^{\frac{\pi K'}{4K}}e^{-\frac{i\pi\nu}{2K}\left(x+\frac{v}{2}\right)}\left(x+\frac{v}{2}\right)+\mathcal{O}\left(x+\frac{v}{2}\right)^3,\end{cases}\quad \text{as } x\to-\frac{v}{2},$$

we obtain similarly another linear equation

$$c_1^+\frac{\Theta(2\alpha+\gamma)}{H(2\alpha)}+i\zeta\frac{c_1^-\Theta(\gamma)}{\nu\sqrt{k}\Theta(0)}=0. \tag{2.26}$$

The determinant of the linear system (2.25) and (2.26) is zero if

$$\zeta^2=\frac{\nu^2k\Theta^2(0)\Theta(2\alpha-\gamma)\Theta(2\alpha+\gamma)}{\Theta^2(\gamma)H^2(2\alpha)}. \tag{2.27}$$

Since $H(x)=\sqrt{k}\,\mathrm{sn}(x)\Theta(x)$ and

$$\Theta^2(0)\Theta(2\alpha-\gamma)\Theta(2\alpha+\gamma)=\Theta^2(2\alpha)\Theta^2(\gamma)-H^2(2\alpha)H^2(\gamma),$$

we use the relation $\nu=\zeta_1\,\mathrm{sn}(2\alpha)$ from (1.23) and obtain from (2.27):

$$\begin{aligned}\zeta^2&=\nu^2k\left[\frac{\Theta^2(2\alpha)}{H^2(2\alpha)}-\frac{H^2(\gamma)}{\Theta^2(\gamma)}\right]\\ &=\nu^2\left[\frac{1}{\mathrm{sn}^2(2\alpha)}-k^2\,\mathrm{sn}^2(\gamma)\right]\\ &=\zeta_1^2-(\zeta_1^2-\zeta_3^2)k^2\,\mathrm{sn}^2(\gamma)\\ &=\zeta_3^2+(\zeta_1^2-\zeta_3^2)\,\mathrm{dn}^2(\gamma),\end{aligned}$$

which recovers the dispersion relation (2.17). Therefore, one of the two equations in the linear system (2.25) and (2.26) is redundant.

Recall from Example 2 that $\gamma=2\alpha+iK'$ for $\zeta=0$, which yields $c_1^+=0$ since $\Theta(2\alpha-\gamma)=0$ and $\Theta(2\alpha+\gamma)\neq0$. Therefore, we use (2.26) of the linear system (2.25) and (2.26) to define up to the constant multiplication factor:

$$c_1^+=-i\zeta,\quad c_1^-=\frac{\nu\sqrt{k}\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)H(2\alpha)}=\zeta_1\frac{\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)\Theta(2\alpha)},$$

where we have used again that $\nu=\zeta_1\,\mathrm{sn}(2\alpha)$. This gives the first solution (2.18) in the final form (1.16).

Similarly, we expand the second solution (2.19) as $x\to\pm\frac{v}{2}$. Here, we use the fact that (2.19) is obtained from (2.18) by replacing γ to $-\gamma$. Since the poles $\pm\frac{v}{2}$ are independent of

γ, we obtain the linear system of equations for (c_2^+, c_2^-) by replacing γ to $-\gamma$ in (2.25) and (2.26):

$$i\zeta \frac{c_2^+ \Theta(\gamma)}{\nu\sqrt{k}\Theta(0)} - c_2^- \frac{\Theta(2\alpha+\gamma)}{H(2\alpha)} = 0. \tag{2.28}$$

and

$$c_2^+ \frac{\Theta(2\alpha-\gamma)}{H(2\alpha)} + i\zeta \frac{c_2^- \Theta(\gamma)}{\nu\sqrt{k}\Theta(0)} = 0. \tag{2.29}$$

Again, it follows from Example 2 for $\zeta = 0$ that $c_2^- = 0$ since $\Theta(2\alpha - \gamma) = 0$ and $\Theta(2\alpha+\gamma) \neq 0$. Therefore, we use (2.28) of the linear system (2.28) and (2.29) to define up to the constant multiplication factor:

$$c_2^+ = \frac{\nu\sqrt{k}\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)H(2\alpha)} = \zeta_1 \frac{\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)\Theta(2\alpha)}, \quad c_2^- = i\zeta.$$

This gives the second solution (2.19) in the final form (1.17).

3. Time evolution of the eigenfunctions

We derive the time evolution (1.15) of the elliptic eigenfunctions in Theorem 1. To do so, we first inspect the relationship between the spectral parameter $\zeta \in \mathbb{R}$ and the shift parameter $\gamma \in [0, K] \times [0, iK']$ given by the dispersion relation (1.14) (Section 3.1). Then, we separate variables in the Lax system (1.2) and obtain the characteristic polynomial for the traveling waves (Section 3.2), which is also solved by the elliptic functions. Finally, we derive the unique expression (1.18) for μ in terms of the shift parameter γ (Section 3.3).

3.1. **Dispersion relation for eigenfunctions.** The following lemma defines the pre-image of the interval $[0, \infty)$ under the mapping $\gamma \to \zeta$ given by the dispersion relation (1.14). The pre-image is shown as the path in the complex γ plane in Figure 1.

Lemma 1. *The pre-image of the path $\infty \to \zeta_1 \to \zeta_2 \to \zeta_3 \to 0$ for $\zeta \in (0, \infty)$ is given by the path $(0, iK') \to (0, 0) \to (K, 0) \to (K, iK') \to (2\alpha, iK')$ in the complex γ-plane.*

Proof. For $\zeta \in [\zeta_1, \infty)$, we use $\gamma = i\gamma'$ with $\gamma' \in [0, K']$ and rewrite (1.14) as

$$\zeta^2 = \zeta_3^2 + (\zeta_1^2 - \zeta_3^2)\frac{\mathrm{dn}^2(\gamma'; k')}{\mathrm{cn}^2(\gamma'; k')}, \tag{3.1}$$

where $k' = \sqrt{1-k^2}$. Since $\mathrm{cn}^2(K'; k') = 0$, the image of $\gamma = iK'$ is $\zeta = \infty$. Since $\mathrm{cn}^2(0; k') = \mathrm{dn}^2(0; k') = 1$, the image of $\gamma = 0$ is $\zeta = \zeta_1$. For $\gamma' \in (0, K')$, we have $\zeta \in (\zeta_1, \infty)$.

For $\zeta \in [\zeta_2, \zeta_1]$, we use $\gamma \in [0, K]$. It follows from

$$\zeta^2 = \zeta_3^2 + (\zeta_1^2 - \zeta_3^2)\,\mathrm{dn}^2(\gamma), \tag{3.2}$$

that the image of $\gamma = 0$ is $\zeta = \zeta_1$ and the image of $\gamma = K$ is $\zeta = \zeta_2$ since $\zeta^2 = \zeta_3^2 + (\zeta_1^2 - \zeta_3^2)(1 - k^2) = \zeta_2^2$. For $\gamma \in (0, K)$, we have $\zeta \in (\zeta_2, \zeta_1)$.

For $\zeta \in [\zeta_3, \zeta_2]$, we use $\gamma = K + i\gamma'$ with $\gamma' \in [0, K']$ and rewrite (1.14) as

$$\zeta^2 = \zeta_3^2 + (\zeta_1^2 - \zeta_3^2)\frac{1-k^2}{\mathrm{dn}^2(i\gamma';k)} = \zeta_3^2 + (\zeta_2^2 - \zeta_3^2)\frac{\mathrm{cn}^2(\gamma';k')}{\mathrm{dn}^2(\gamma';k')}. \tag{3.3}$$

Since $\mathrm{cn}^2(0;k') = \mathrm{dn}^2(0;k') = 1$, the image of $\gamma = K$ is $\zeta = \zeta_2$. Since $\mathrm{cn}^2(K';k') = 0$, the image of $\gamma = K + iK'$ is $\zeta = \zeta_3$. For $\gamma' \in (0, K')$, we have $\zeta \in (\zeta_3, \zeta_2)$.

For $\zeta \in [0, \zeta_3]$, we use $\gamma = \delta + iK'$ with $\gamma \in [0, K]$ and rewrite (1.14) as

$$\zeta^2 = \zeta_3^2 - (\zeta_1^2 - \zeta_3^2)\frac{\mathrm{cn}^2(\delta)}{\mathrm{sn}^2(\delta)}. \tag{3.4}$$

Since $\mathrm{cn}^2(K) = 0$, the image of $\gamma = K + iK'$ is $\zeta^2 = \zeta_3^2$. By (1.23), we have

$$\mathrm{cn}(2\alpha) = \frac{\zeta_3}{\zeta_1}, \quad \mathrm{sn}(2\alpha) = \frac{\nu}{\zeta_1},$$

and since $\nu^2 = \zeta_1^2 - \zeta_3^2$, the image of $\gamma = 2\alpha + iK'$ is $\zeta = 0$. For $\delta \in (0, K)$, we have $\zeta \in (0, \zeta_3)$. □

Expressions (3.1) and (3.3) are useful in the spectral bands $[\zeta_1, \infty)$ and $[\zeta_3, \zeta_2]$, whereas expressions (3.2) and (3.4) are useful in the spectral gaps (ζ_2, ζ_1) and $[0, \zeta_3)$. We will only use the latter expressions for the purpose of this work. The following lemma gives a useful relation between the dispersion relation (2.17) and Weierstrass' function $\wp$.

Lemma 2. *Let $a \in [0, \omega] \times [0, \omega']$ be defined by*

$$\gamma = iK' + \nu a. \tag{3.5}$$

The dispersion relation (2.17) can be rewritten as

$$\zeta^2 = \wp(v) - \wp(a). \tag{3.6}$$

Proof. By using (2.1), (2.2), and (2.13), we obtain

$$\begin{aligned}\zeta^2 &= \wp(v) - \wp(a) \\ &= \frac{1}{3}(\zeta_1^2 + \zeta_2^2 + \zeta_3^2) - \frac{1}{3}(-2\zeta_1^2 + \zeta_2^2 + \zeta_3^2) - \frac{\zeta_1^2 - \zeta_3^2}{\mathrm{sn}^2(\gamma \mp iK')} \\ &= \zeta_1^2 - (\zeta_1^2 - \zeta_2^2)\,\mathrm{sn}^2(\gamma),\end{aligned}$$

which is equivalent to (2.17). □

3.2. **Characteristic polynomial for traveling waves.** The variables (x, t) in the Lax system (1.2) with the traveling wave $u(x, t) = \phi(x + ct)$ can be separated in the form $\varphi(x, t) = \psi(x + ct)e^{\mu t}$. With the separation of variables, the Lax system (1.2) is split into the spectral problem

$$\frac{d}{dx}\psi = \begin{pmatrix} i\zeta & \phi \\ \phi & -i\zeta \end{pmatrix}\psi \tag{3.7}$$

and the linear algebraic system

$$\mu\psi + c\begin{pmatrix} i\zeta & \phi \\ \phi & -i\zeta \end{pmatrix}\psi = \begin{pmatrix} 4i\zeta^3 + 2i\zeta\phi^2 & 4\zeta^2\phi - 2i\zeta\phi' + 2\phi^3 - \phi'' \\ 4\zeta^2\phi + 2i\zeta\phi' + 2\phi^3 - \phi'' & -4i\zeta^3 - 2i\zeta\phi^2 \end{pmatrix}\psi, \quad (3.8)$$

where all functions depend on only one variable which stands for the traveling wave coordinate $x + ct$. Since (3.8) is the linear algebraic system, admissible values of μ are found from the characteristic equation

$$\begin{vmatrix} 4i\zeta^3 + 2i\zeta\phi^2 - ic\zeta - \mu & 4\zeta^2\phi - 2i\zeta\phi' + 2\phi^3 - \phi'' - c\phi \\ 4\zeta^2\phi + 2i\zeta\phi' + 2\phi^3 - \phi'' - c\phi & -4i\zeta^3 - 2i\zeta\phi^2 + ic\zeta - \mu \end{vmatrix} = 0. \quad (3.9)$$

Expanding the determinant in (3.9) and using (1.5) and (1.6), we obtain

$$\mu^2 + 16P(\zeta) = 0, \quad (3.10)$$

where the characteristic polynomial $P(\zeta)$ can be written in the form:

$$\begin{aligned} P(\zeta) &= \zeta^6 - \frac{c}{2}\zeta^4 + \frac{1}{16}(c^2 - 8d)\zeta^2 - \frac{b^2}{16} \\ &= (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_3^2), \end{aligned}$$

due to the parameterization (1.7). It follows from (3.8) that the quotient $\rho = q/p$ for the eigenfunction $\psi = (p, q)^T$ satisfies

$$\rho = -\frac{4i\zeta^3 + i\zeta(2\phi^2 - c) - \mu}{4\zeta^2\phi - 2i\zeta\phi' - b} = \frac{4\zeta^2\phi + 2i\zeta\phi' - b}{4i\zeta^3 + i\zeta(2\phi^2 - c) + \mu}. \quad (3.11)$$

Substituting (2.13) and (3.6) into (3.10) and using (2.2), we obtain

$$\begin{aligned} \mu^2 &= 16(\zeta_1^2 + \wp(a) - \wp(v))(\zeta_2^2 + \wp(a) - \wp(v))(\zeta_3^2 + \wp(a) - \wp(v)) \\ &= 16(\wp(a) - e_1)(\wp(a) - e_2)(\wp(a) - e_3) \\ &= 4(\wp'(a))^2, \end{aligned}$$

where we used the first-order quadrature $(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ for Weierstrass' elliptic function $\wp$. It remains to prove that $\mu = -2\wp'(a)$.

3.3. **Explicit expression for μ.** First, we note from (1.16) that $p_1(x_0) = q_1(x_0)$ at

$$x_0 := \frac{\alpha - \gamma}{\nu} = -a - \frac{v}{2},$$

where we have used (2.8) and (3.5). By using the quotient (3.11) with $\rho(x_0) = 1$ for $x_0 = -a - \frac{v}{2}$, we obtain

$$\begin{aligned} \mu &= 4\zeta^2\phi(x_0) - b + 2i\zeta\left(2\zeta^2 + \phi^2(x_0) - \frac{c}{2} - \phi'(x_0)\right) \\ &= 4\zeta^2(\zeta(-a) - \zeta(-a - v) - \zeta(v)) - 2\wp'(v) + 4i\zeta(\zeta^2 + \wp(-a) - \wp(v)) \\ &= 4(\wp(v) - \wp(a))(\zeta(a + v) - \zeta(a) - \zeta(v)) - 2\wp'(v), \end{aligned}$$

where we have used expressions (2.4), (2.5), (2.6), (2.7), and (3.6). By using [16, 8.177],

$$\zeta(a+v) - \zeta(a) - \zeta(v) = \frac{1}{2}\frac{\wp'(a) - \wp'(v)}{\wp(a) - \wp(a)},$$

we obtain $\mu = -2\wp'(a)$. Since $a = (\gamma + iK')/\nu$, we use (1.22) and (2.1) to rewrite the expression for μ in the form

$$\mu = -2\wp'(a) = \frac{4\nu^3 \operatorname{cn}(\nu a)\operatorname{dn}(\nu a)}{\operatorname{sn}^3(\nu a)} = -4\nu^3 k^2 \operatorname{sn}(\gamma)\operatorname{cn}(\gamma)\operatorname{dn}(\gamma), \tag{3.12}$$

which yields (1.18). The explicit expression (3.12) enables a unique parameterization of the square root singularity in the definition (3.10) of μ. We give explicit expressions for μ in the gaps (ζ_2, ζ_1) and $(0, \zeta_3)$ to show that the different branches of the square root are chosen in between the two gaps.

For $\zeta \in (\zeta_2, \zeta_1)$, we have $\gamma \in (0, K)$. The expression (3.12) is equivalent to

$$\mu = -4\sqrt{(\zeta_1^2 - \zeta^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_3^2)}, \quad \zeta \in (\zeta_2, \zeta_1), \tag{3.13}$$

due to (2.17).

For $\zeta \in (0, \zeta_3)$, we write $\gamma = \delta + iK'$ with $\delta \in (2\alpha, K)$. The expression (3.12) can be rewritten as

$$\mu = -4\nu^3 k^2 \operatorname{sn}(\delta + iK')\operatorname{cn}(\delta + iK')\operatorname{dn}(\delta + iK') = \frac{4\nu^3 \operatorname{cn}(\delta)\operatorname{dn}(\delta)}{\operatorname{sn}^3(\delta)}, \tag{3.14}$$

which is equivalent to

$$\mu = 4\sqrt{(\zeta_1^2 - \zeta^2)(\zeta_2^2 - \zeta^2)(\zeta_3^2 - \zeta^2)}, \tag{3.15}$$

since the dispersion relation (2.17) can be rewritten as

$$\zeta^2 = \zeta_1^2 - \frac{(\zeta_1^2 - \zeta_3^2)}{\operatorname{sn}^2(\delta)} = \zeta_2^2 - \frac{(\zeta_1^2 - \zeta_3^2)\operatorname{dn}^2(\delta)}{\operatorname{sn}^2(\delta)} = \zeta_3^2 - \frac{(\zeta_1^2 - \zeta_3^2)\operatorname{cn}^2(\delta)}{\operatorname{sn}^2(\delta)}.$$

The sign of the square root branch has been changed between (3.13) and (3.15). When $\zeta \to 0$, we have $\mu \to 4\zeta_1\zeta_2\zeta_3$ from (3.15), in which case the expressions (1.15) with (1.20) and (1.21) coincide with (2.27)–(2.28) in [3].

4. Breathers on the elliptic wave background

We use a linear combination of the two eigenfunctions in Theorem 1 in order to construct breathers by using the Darboux transformation (1.25). This leads to the expression (1.26), where we intend to show for the half-gap $(0, \zeta_3)$ and the gap (ζ_2, ζ_1) separately that the choice (1.30) produces a bounded and real-valued solution of the mKdV equation (1.1). Sections 4.1 and 4.2 report the corresponding results in the two gaps, from which breather solutions in Figures 3–4 and 5–6 were constructed.

4.1. **Breathers in the half gap** $(0, \zeta_3)$**.** By Lemma 1, we use the parameterization $\gamma = iK' + \delta$ with $\delta \in (2\alpha, K)$ and associate the spectral parameter ζ with δ by using (3.4). The eigenfunctions (1.16) and (1.17) change due to the relations (1.19). Both terms in the linear superposition for either (1.16) or (1.17) have the same constant multiplicative factor, which we do not write to redefine the eigenfunctions in the form

$$\begin{pmatrix} p_1(x) \\ q_1(x) \end{pmatrix} = \begin{pmatrix} -i\zeta \frac{\Theta(\nu x+\delta+\alpha)}{\Theta(\nu x+\alpha)} + \zeta_1 \frac{\Theta(0)H(2\alpha+\delta)}{H(\delta)\Theta(2\alpha)} \frac{\Theta(\nu x+\delta-\alpha)}{\Theta(\nu x-\alpha)} \\ -i\zeta \frac{\Theta(\nu x+\delta+\alpha)}{\Theta(\nu x+\alpha)} - \zeta_1 \frac{\Theta(0)H(2\alpha+\delta)}{H(\delta)\Theta(2\alpha)} \frac{\Theta(\nu x+\delta-\alpha)}{\Theta(\nu x-\alpha)} \end{pmatrix} e^{-\nu x \frac{H'(\delta)}{H(\delta)}} \tag{4.1}$$

and

$$\begin{pmatrix} p_2(x) \\ q_2(x) \end{pmatrix} = \begin{pmatrix} \zeta_1 \frac{\Theta(0)H(2\alpha+\delta)}{H(\delta)\Theta(2\alpha)} \frac{\Theta(\nu x-\delta+\alpha)}{\Theta(\nu x+\alpha)} + i\zeta \frac{\Theta(\nu x-\delta-\alpha)}{\Theta(\nu x-\alpha)} \\ \zeta_1 \frac{\Theta(0)H(2\alpha+\delta)}{H(\delta)\Theta(2\alpha)} \frac{\Theta(\nu x-\delta+\alpha)}{\Theta(\nu x+\alpha)} - i\zeta \frac{\Theta(\nu x-\delta-\alpha)}{\Theta(\nu x-\alpha)} \end{pmatrix} e^{\nu x \frac{H'(\delta)}{H(\delta)}} \tag{4.2}$$

All elliptic functions are real-valued for real $\delta \in (2\alpha, K)$, which justify the conditions (1.27) and (1.28). The time variable in (4.1) and (4.2) is set at $t = 0$. The dependence on time t is given by (1.15) with μ given by (3.14).

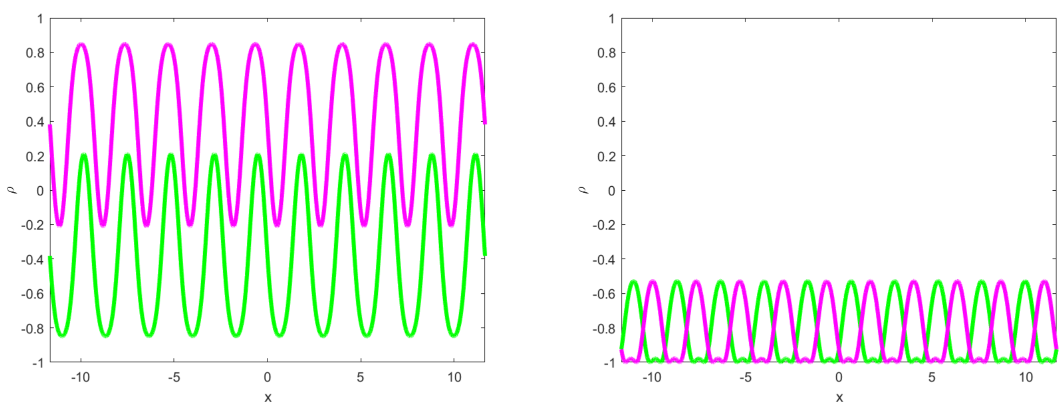

FIGURE 7. Re(ρ) versus x (left) and Im(ρ) versus x (right), where $\rho = q/p$ is the quotient of the two components of eigenfunctions $\psi = (p, q)^T$ of the spectral problem (3.7) with the elliptic potential ϕ for $(\zeta_1, \zeta_2, \zeta_3) = (2, 1, 0.5)$. The spectral parameter ζ is defined by (3.4) with $\delta = \frac{1}{2}(2\alpha + K)$. The green line shows ρ for the first solution (4.1) and the magenta line shows ρ for he second solution (4.2).

For the numerical check, we plot in Figure 7 the quotient $\rho = q/p$ (real part is shown on the left panel and imaginary part is on the right panel) for the first solution (4.1) (green line) and the second solution (4.2) (magenta line). The quotient ρ is computed in two different ways: by using the elliptic functions (3.11) with μ given by (3.14) and by using the explicit solutions (4.1) and (4.2). The difference between the two computational formulas is found within the machine precision error.

The breather solutions are obtained from (1.26) with the choice (1.30), where $\kappa = \frac{H'(\delta)}{H(\delta)}$ is defined by (1.29). The breather speed c_s is given by (1.31) and κ also defines the spatial decay rate of the breather at infinity. Since $\kappa > 0$ and $\mu > 0$, we have $c_s < c$.

4.2. **Breathers in the gap** (ζ_2, ζ_1)**.** By Lemma 1, we use the parameter $\gamma \in (0, K)$ and associate the spectral parameter ζ by using (3.2). The eigenfunctions (1.16) and (1.17) can be used directly since all elliptic functions are real-valued for real $\gamma \in (0, K)$.

For the numerical check, we plot in Figure 8 the quotient $\rho = q/p$ (real part is shown on the left panel and imaginary part is on the right panel) for the first solution (1.16) (green line) and the second solution (1.17) (magenta line). The quotient ρ is computed in two different ways: by using the elliptic functions (3.11) with μ given by (3.12) and by using the explicit solutions (1.16) and (1.17). The difference is again found within the machine precision error.

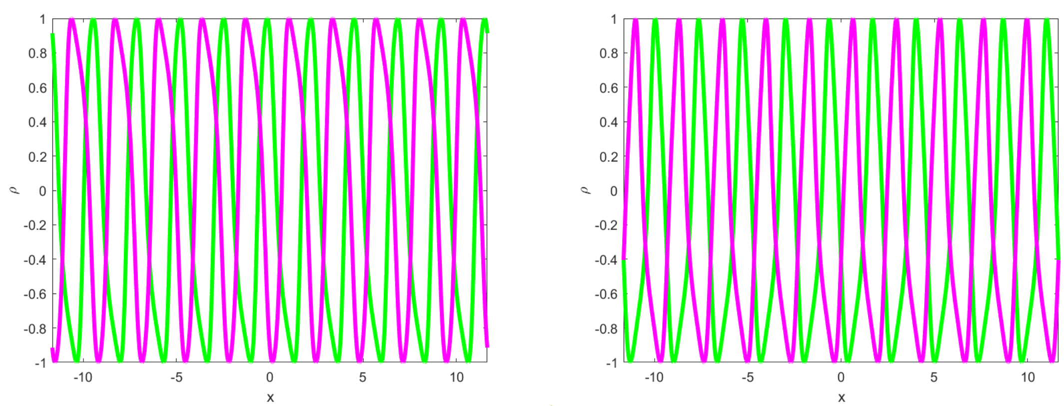

FIGURE 8. $\text{Re}(\rho)$ versus x (left) and $\text{Im}(\rho)$ versus x (right), where $\rho = q/p$ is the quotient of the two components of eigenfunctions $\psi = (p, q)^T$ of the spectral problem (3.7) with the elliptic potential ϕ for $(\zeta_1, \zeta_2, \zeta_3) = (2, 1, 0.5)$. The spectral parameter ζ is defined by (3.2) with $\gamma = \frac{1}{2}K$. The green line shows ρ for the first solution (1.16) and the magenta line shows ρ for he second solution (1.17).

If we proceed with the breather solution based on the Darboux transformation (1.25) with eigenfunctions (1.16) and (1.17), we get singular solutions for real $x \in \mathbb{R}$. However, based on [26, 37], we change $x \to x + iK'$ in order to get bounded breather solutions from the singular solutions given by (1.25). In application to the elliptic wave with the profile ϕ, the transformation $x \to x + iK'$ leaves ϕ real-valued but transforms the bounded solution (1.8) into the singular solution,

$$\phi(x) = \frac{2(\zeta_1 + \zeta_3)(\zeta_2 + \zeta_3)k^2 \,\text{sn}^2(\nu x)}{(\zeta_1 + \zeta_3)k^2 \text{sn}^2(\nu x, k) - (\zeta_1 - \zeta_2)} - \zeta_1 - \zeta_2 - \zeta_3, \tag{4.3}$$

where we have used (1.22). In application to the eigenfunctions (1.16) and (1.17), the transformation $x \to x + iK'$ yields up to the constant multiplicative factor:

$$\begin{pmatrix} p_1(x) \\ q_1(x) \end{pmatrix} = \begin{pmatrix} -i\zeta \frac{\Theta(\nu x+\gamma+\alpha)}{H(\nu x+\alpha)} + \zeta_1 \frac{\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)\Theta(2\alpha)} \frac{\Theta(\nu x+\gamma-\alpha)}{H(\nu x-\alpha)} \\ -i\zeta \frac{\Theta(\nu x+\gamma+\alpha)}{H(\nu x+\alpha)} - \zeta_1 \frac{\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)\Theta(2\alpha)} \frac{\Theta(\nu x+\gamma-\alpha)}{H(\nu x-\alpha)} \end{pmatrix} e^{-\nu x Z(\gamma)} \tag{4.4}$$

and

$$\begin{pmatrix} p_2(x) \\ q_2(x) \end{pmatrix} = \begin{pmatrix} \zeta_1 \frac{\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)\Theta(2\alpha)} \frac{\Theta(\nu x-\gamma+\alpha)}{H(\nu x+\alpha)} + i\zeta \frac{\Theta(\nu x-\gamma-\alpha)}{H(\nu x-\alpha)} \\ \zeta_1 \frac{\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)\Theta(2\alpha)} \frac{\Theta(\nu x-\gamma+\alpha)}{H(\nu x+\alpha)} - i\zeta \frac{\Theta(\nu x-\gamma-\alpha)}{H(\nu x-\alpha)} \end{pmatrix} e^{\nu x Z(\gamma)}, \tag{4.5}$$

where we have used (1.19). All elliptic functions in (4.4) and (4.5) are real-valued for real $\gamma \in (0, K)$, which justify the conditions (1.27) and (1.28). The time variable in (4.4) and (4.5) is set at $t = 0$. The dependence on time t is given by (1.15) with μ given by (3.13).

The breather solutions are obtained from (1.26) with the choice (1.30), where $\kappa = Z(\gamma)$ is defined by (1.29). Although ϕ given by (4.3) is singular for real $x \in \mathbb{R}$, the solution u in (1.26) with the choice (1.30) is bounded for real $x \in \mathbb{R}$. The breather speed c_s is given by (1.31) and κ also defines the spatial decay rate of the breather at infinity. Since $\kappa > 0$ and $\mu < 0$, we have $c_s > c$.

5. Factorization of elliptic eigenfunctions

We have obtained eigenfunctions of the spectral problem (3.7) in the explicit form (1.16) and (1.17). The spectral parameter $\zeta \in \mathbb{R}$ is represented by the shift parameter $\gamma \in [0, K] \times [0, iK']$ in Lemma 1, which also uniquely parameterizes the eigenfunctions (1.16) and (1.17). This explicit representation of eigenfunctions is now compared with the factorization formulas discussed in [3] and used in other works [30, 31]. We obtain the factorization formulas in Section 5.1. Furthermore, we show that the poles and zeros of the elliptic eigenfunctions are defined by using the branch pole singularities, which cannot be unfolded with the use of elliptic functions. This is done for the simpler cases of rational and hyperbolic degenerations of the elliptic functions in Sections 5.2 and 5.3 respectively.

5.1. **Derivation of the factorization formulas.** By using the quotient (3.11) with a unique choice for μ in (3.12), we introduce zeros of numerators and denominators in (3.11) for $b = 4\zeta_1\zeta_2\zeta_3$ and $c = 2(\zeta_1^2 + \zeta_2^2 + \zeta_3^2)$ in the general case with $0 < \zeta_3 < \zeta_2 < \zeta_1$. In particular, we prove in Lemma 3 below that there exist exactly two symmetric pairs of four roots in x labeled as $\{\pm x_1, \pm x_2\} \in [-\omega, \omega] \times [-\omega', \omega']$ of

$$2i\zeta \left[\phi^2(x) + 2\zeta^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2\right] + \mu = 0 \tag{5.1}$$

and exactly two symmetric pairs of four roots in x labeled as $\{\pm x_1^*, \pm x_2^*\} \in [-\omega, \omega] \times [-\omega', \omega']$ of

$$2i\zeta \left[\phi^2(x) + 2\zeta^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2\right] - \mu = 0, \tag{5.2}$$

where ω and ω' are given by (2.3). Furthermore, we prove in Lemma 4 that two roots from $\{\pm x_1, \pm x_2\}$ and two roots from $\{\pm x_1^*, \pm x_2^*\}$ are four roots in x of

$$\zeta^2 \phi(x) - \frac{i}{2}\zeta\phi'(x) - \zeta_1\zeta_2\zeta_3 = 0 \tag{5.3}$$

in $[-\omega,\omega]\times[-\omega',\omega']$, whereas the complement from the two sets of roots give four roots in x of

$$\zeta^2\phi(x)+\frac{i}{2}\zeta\phi'(x)-\zeta_1\zeta_2\zeta_3=0. \tag{5.4}$$

For convenience, we denote roots of (5.1) and (5.2), which are simultaneously roots of (5.3) by $\{x_1,x_2\}$ and $\{x_1^*,x_2^*\}$ respectively. Since ϕ is even and ϕ' is odd in x, the symmetry between (5.3) and (5.4) implies that roots of (5.4) are given by $\{-x_1,-x_2\}$ and $\{-x_1^*,-x_2^*\}$.

Lemma 3. *There exist exactly two symmetric pairs of four roots in $[-\omega,\omega]\times[-\omega',\omega']$ with respect to x for either (5.1) or (5.2).*

Proof. For simplicity, we deal with solutions of (5.1). By using the analytical representation (2.7) with (2.4), we can rewrite (5.1) in the equivalent form

$$2i\zeta\left[\wp\left(x+\frac{v}{2}\right)+\wp\left(x-\frac{v}{2}\right)+2\zeta^2-\frac{2}{3}(\zeta_1^2+\zeta_2^2+\zeta_3^2)\right]+\mu=0.$$

By using (2.1), (2.2), and (2.8), we obtain

$$2i\zeta(\zeta_1^2-\zeta_2^2)\left[\mathrm{sn}^2(z+\alpha)+\mathrm{sn}^2(z-\alpha)\right]+4i\zeta(\zeta^2-\zeta_1^2)+\mu=0, \tag{5.5}$$

where $z=\nu x$. With the help of the addition formula

$$\mathrm{sn}(u\pm v)=\frac{\mathrm{sn}(u)\,\mathrm{cn}(v)\,\mathrm{dn}(v)\pm\mathrm{sn}(v)\,\mathrm{cn}(u)\,\mathrm{dn}(u)}{1-k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)},$$

we rewrite (5.5) in the form

$$4i\zeta(\zeta_1^2-\zeta_2^2)\frac{\mathrm{sn}^2(z)\,\mathrm{cn}^2(\alpha)\,\mathrm{dn}^2(\alpha)+\mathrm{sn}^2(\alpha)\,\mathrm{cn}^2(z)\,\mathrm{dn}^2(z)}{[1-k^2\,\mathrm{sn}^2(z)\,\mathrm{sn}^2(\alpha)]^2}+4i\zeta(\zeta^2-\zeta_1^2)+\mu=0.$$

Eliminating $\mathrm{sn}(\alpha)$, $\mathrm{cn}(\alpha)$, and $\mathrm{dn}(\alpha)$ from (1.24) yields

$$4i\zeta(\zeta_1-\zeta_2)(\zeta_1+\zeta_3)\frac{(\zeta_2+\zeta_3)^2\,\mathrm{sn}^2(z)+(\zeta_1^2-\zeta_3^2)\,\mathrm{cn}^2(z)\,\mathrm{dn}^2(z)}{[(\zeta_1+\zeta_3)-(\zeta_1-\zeta_2)\,\mathrm{sn}^2(z)]^2}+4i\zeta(\zeta^2-\zeta_1^2)+\mu=0. \tag{5.6}$$

By using the fundamental relations (1.35), equation (5.6) yields a bi-quadratic equation for $\mathrm{sn}^2(z)$ with two roots. For each root of $\mathrm{sn}^2(z)$, there exist exactly two solutions of $\mathrm{sn}^2(z)=w\in\mathbb{C}$ in $[-K,K]\times[-iK',iK']$, see Proposition 3.1 in [3]. This construction yields exactly four solutions of (5.1) for x in $[-\omega,\omega]\times[-\omega',\omega']$. Since $\mathrm{sn}^2(z)$ is even in z, the four roots of (5.1) form two symmetric pairs. The proof for (5.2) is identical. □

Lemma 4. *Two roots of (5.1) and two roots of (5.2) are exactly four roots of (5.3) in $[-\omega,\omega]\times[-\omega',\omega']$ with respect to x.*

Proof. By squaring (5.3) and using (1.6), we obtain

$$\begin{aligned}&-\zeta^2(\phi')^2=4(\zeta^2\phi-\zeta_1\zeta_2\zeta_3)^2\\&=-\zeta^2(\phi^4-2(\zeta_1^2+\zeta_2^2+\zeta_3^2)\phi^2+8\zeta_1\zeta_2\zeta_3\phi+\zeta_1^4+\zeta_2^4+\zeta_3^4-2(\zeta_1^2\zeta_2^2+\zeta_1^2\zeta_3^2+\zeta_2^2\zeta_3^2)).\end{aligned}$$

This yields a bi-quadratic equation

$$4\zeta^2 \left(\phi^2 + 2\zeta^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2\right)^2 + \mu^2 = 0, \tag{5.7}$$

where we have used (3.10) with $P(\zeta) = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_3^2)$. Extracting the square roots of (5.7) yields (5.1) and (5.2). Since ϕ is even and ϕ' is odd in x, the roots of (5.3) do not form symmetric pairs. This implies that two of the four roots of (5.3) are given by roots of (5.1) and the other two roots of (5.3) are given by roots of (5.2). □

The first quotient $\rho = \frac{q_1}{p_1}$.

Poles of both numerator and denominator in (3.11) coincide with the poles of $\phi^2(x)$ and $\phi'(x)$, which are double poles at the same locations $\pm\frac{v}{2}$, see (2.6) and (2.7). Since both the numerator and the denominator in each quotient of (3.11) are elliptic functions, have only four zeros in $[-\omega, \omega] \times [-\omega', \omega']$, and their double poles coincide at $\pm\frac{v}{2}$, the factorization formula for elliptic functions (Proposition 3.3 in [3]) implies for both quotients in (3.11) that

$$\begin{aligned}\rho(x) &= C\frac{H(\nu(x+x_1^*))H(\nu(x+x_2^*))\cancel{H(\nu(x-x_1^*))H(\nu(x-x_2^*))}}{H(\nu(x-x_1))H(\nu(x-x_2))\cancel{H(\nu(x-x_1^*))H(\nu(x-x_2^*))}} \\ &= C\frac{H(\nu(x+x_1^*))H(\nu(x+x_2^*))\cancel{H(\nu(x+x_1))H(\nu(x+x_2))}}{H(\nu(x-x_1))H(\nu(x-x_2))\cancel{H(\nu(x+x_1))H(\nu(x+x_2))}},\end{aligned} \tag{5.8}$$

for some constant $C \in \mathbb{R}$. To obtain C explicitly, we consider the quotients in (3.11) at the pole singularity as $x \to -\frac{v}{2}$, that is, $x \to \nu^{-1}(iK' + \alpha)$. By using (2.20) and (5.8), we obtain

$$\lim_{x \to \nu^{-1}(iK'+\alpha)} \rho(x) = -1 = C e^{-\frac{i\pi\nu}{2K}(x_1+x_2+x_1^*+x_2^*)} \frac{\Theta(\alpha+\nu x_1^*)\Theta(\alpha+\nu x_2^*)}{\Theta(\alpha-\nu x_1)\Theta(\alpha-\nu x_2)}. \tag{5.9}$$

This yields

$$\rho(x) = -e^{\frac{i\pi\nu}{2K}(x_1+x_2+x_1^*+x_2^*)} \frac{H(\nu(x+x_1^*))H(\nu(x+x_2^*))\Theta(\alpha-\nu x_1)\Theta(\alpha-\nu x_2)}{H(\nu(x-x_1))H(\nu(x-x_2))\Theta(\alpha+\nu x_1^*)\Theta(\alpha+\nu x_2^*)}, \tag{5.10}$$

which is the quotient $\rho = \frac{q_1}{p_1}$ of the first solution $\psi = (p_1, q_1)^T$. Note that the two poles and two zeros of the elliptic function ρ in (5.8) are related by

$$x_1 + x_2 + x_1^* + x_2^* = 0 \text{ mod } (2\omega, 2\omega').$$

The second quotient $\rho = \frac{q_2}{p_2}$.

We change μ to $-\mu$ in (3.11) and use the same notations for the sets of roots in Lemmas 3 and 4. This yields

$$\begin{aligned}\rho(x) &= C\frac{H(\nu(x+x_1))H(\nu(x+x_2))\cancel{H(\nu(x-x_1))H(\nu(x-x_2))}}{H(\nu(x-x_1^*))H(\nu(x-x_2^*))\cancel{H(\nu(x-x_1))H(\nu(x-x_2))}} \\ &= C\frac{H(\nu(x+x_1))H(\nu(x+x_2))\cancel{H(\nu(x+x_1^*))H(\nu(x+x_2^*))}}{H(\nu(x-x_1^*))H(\nu(x-x_2^*))\cancel{H(\nu(x+x_1^*))H(\nu(x+x_2^*)}},\end{aligned} \tag{5.11}$$

for some constant $C \in \mathbb{R}$. To obtain C explicitly, we consider the singular behavior as $x \to \frac{v}{2}$, that is, $x \to -\nu^{-1}(iK' + \alpha)$, for which

$$\lim_{x \to -\nu^{-1}(iK'+\alpha)} \rho(x) = 1 = Ce^{\frac{i\pi\nu}{2K}(x_1+x_2+x_1^*+x_2^*)} \frac{\Theta(\alpha - \nu x_1)\Theta(\alpha - \nu x_2)}{\Theta(\alpha + \nu x_1^*)\Theta(\alpha + \nu x_2^*)}. \tag{5.12}$$

This yields

$$\rho(x) = e^{-\frac{i\pi\nu}{2K}(x_1+x_2+x_1^*+x_2^*)} \frac{H(\nu(x+x_1))H(\nu(x+x_2))\Theta(\alpha+\nu x_1^*)\Theta(\alpha+\nu x_2^*)}{H(\nu(x-x_1^*))H(\nu(x-x_2^*))\Theta(\alpha-\nu x_1)\Theta(\alpha-\nu x_2)}, \tag{5.13}$$

which is the quotient $\rho = \frac{q_2}{p_2}$ of the second solution $\psi = (p_2, q_2)^T$. Note that the limit (5.12) is different from the limit (5.9) and this yields to the asymmetry between (5.10) and (5.13) in the sense that the two expressions are not obtained by replacing $\{x_1, x_2\}$ with $\{x_1^*, x_2^*\}$ and vice versa.

Factorized form for (p_1, q_1).
If ρ is defined, the components of $\psi = (p, q)^T$ can be found from the first-order equations:

$$\partial_x p = (i\zeta + \phi\rho)p, \quad \partial_x q = (\phi\rho^{-1} - i\zeta)q. \tag{5.14}$$

By using the second quotient in (3.11) in the first equation of system (5.14), we write

$$\partial_x \log p_1 = i\zeta + 4\frac{\zeta^2\phi^2 + \frac{i}{2}\zeta\phi\phi' - \zeta_1\zeta_2\zeta_3\phi}{2i\zeta(\phi^2 + 2\zeta^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2) + \mu}. \tag{5.15}$$

For simplicity of notations, we define

$$\upsilon^2 := \zeta_1^2 + \zeta_2^2 + \zeta_3^2 - 2\zeta^2 - \frac{\mu}{2i\zeta}.$$

In order to compute integrals explicitly, we recall the linear fractional tranformation between the elliptic profile ϕ and the Weierstrass' function $\wp$ (Lemma 3.4 in [3]):

$$\phi(x) = \frac{\alpha_1\wp(x) + \beta_1}{\gamma_1\wp(x) + \delta_1}, \tag{5.16}$$

where

$$\begin{aligned}
\alpha_1 &= \zeta_2 + \zeta_3 - \zeta_1, \\
\beta_1 &= \frac{1}{3}\zeta_1(\zeta_1^2 - 2\zeta_2^2 - 2\zeta_3^2 - 3\zeta_2\zeta_3) + \frac{1}{3}(\zeta_2 + \zeta_3)(2\zeta_1^2 - \zeta_2^2 - \zeta_3^2 + 3\zeta_2\zeta_3), \\
\gamma_1 &= 1, \\
\delta_1 &= \zeta_1(\zeta_2 + \zeta_3) - \zeta_2\zeta_3 - \frac{1}{3}(\zeta_1^2 + \zeta_2^2 + \zeta_3^2).
\end{aligned}$$

By using (5.16), 5.141.5 on p. 626 in [16], and 8.193.2-3 on p. 880 in [16], we obtain

$$\begin{aligned}\int \frac{dx}{\phi(x) \pm \upsilon} &= \int \frac{\gamma_1 \wp(x) + \delta_1}{(\alpha_1 \pm \upsilon\gamma_1)\wp(x) + (\beta_1 \pm \upsilon\delta_1)} dx \\ &= \frac{\gamma_1 x}{\alpha_1 \pm \upsilon\gamma_1} + \frac{\beta_1\gamma_1 - \alpha_1\delta_1}{(\alpha_1 \pm \upsilon\gamma_1)^2 \wp'(x_\pm)} \left[\log \frac{H(\nu(x + x_\pm))}{H(\nu(x - x_\pm))} - 2\nu x \frac{H'(\nu x_\pm)}{H(\nu x_\pm)} \right],\end{aligned}$$

where $x_\pm$ are roots of $\phi(x_\pm) \pm \upsilon = 0$ respectively. We note that

$$\phi'(x_\pm) = \frac{\alpha_1\delta_1 - \beta_1\gamma_1}{(\gamma_1\wp(x_\pm) + \delta_1)^2} \wp'(x_\pm) = \frac{(\alpha_1 \pm \upsilon\gamma_1)^2}{\alpha_1\delta_1 - \beta_1\gamma_1} \wp'(x_\pm).$$

The choice of the sign in $\pm x_+$ and $\pm x_-$ do not change the outcome of integration. Since $\phi^2(x) = \upsilon^2$ is equivalent to (5.1), roots of which have been denoted as $\{\pm x_1, \pm x_2\}$, we can further fix the sign of $x_\pm$ to require them to be roots of (5.3), that is, $x_1 = x_+$ and $x_2 = x_-$. As a result, we have

$$i\zeta\phi'(x_\pm) = 2(\zeta^2\phi(x_\pm) - \zeta_1\zeta_2\zeta_3) = 2(\mp\upsilon\zeta^2 - \zeta_1\zeta_2\zeta_3).$$

By using these expressions, we compute the integrals in (5.15) as follows:

$$\begin{aligned}\log p_1 &= i\zeta x + \int \frac{\phi\phi'}{\phi^2 - \upsilon^2} dx + \frac{2i\zeta_1\zeta_2\zeta_3}{\zeta} \int \frac{\phi}{\phi^2 - \upsilon^2} dx - 2i\zeta \int \frac{\phi^2}{\phi^2 - \upsilon^2} dx \\ &= -i\zeta x + \frac{1}{2}\log(\phi^2 - \upsilon^2) \\ &\quad + \frac{i\zeta_1\zeta_2\zeta_3}{\zeta} \int \left[\frac{1}{\phi + \upsilon} + \frac{1}{\phi - \upsilon} \right] dx + i\zeta\upsilon \int \left[\frac{1}{\phi + \upsilon} - \frac{1}{\phi - \upsilon} \right] dx \\ &= -i\zeta x + \frac{1}{2}\log(\phi^2 - a^2) + \frac{1}{2}\phi'(x_+) \int \frac{dx}{\phi + a} + \frac{1}{2}\phi'(x_-) \int \frac{dx}{\phi - a} \\ &= -i\zeta x + \frac{1}{2}\log(\phi^2 - \upsilon^2) + \frac{\gamma_1 x}{2} \left[\frac{\phi'(x_+)}{\alpha_1 + \upsilon\gamma_1} + \frac{\phi'(x_-)}{\alpha_1 - \upsilon\gamma_1} \right] \\ &\quad - \frac{1}{2}\log \frac{H(\nu(x + x_1))H(\nu(x + x_2))}{H(\nu(x - x_1))H(\nu(x - x_2))} + \nu x \left[\frac{H'(\nu x_1)}{H(\nu x_1)} + \frac{H'(\nu x_2)}{H(\nu x_2)} \right].\end{aligned}$$

Since $\phi^2(x) - \upsilon^2$ is an elliptic function with roots given by roots of (5.1), we use Lemma 3 and write it in the factorization form

$$\phi^2(x) - \upsilon^2 = C \frac{H(\nu(x - x_1))H(\nu(x - x_2))H(\nu(x + x_1))H(\nu(x + x_2))}{\Theta^2(\nu x - \alpha)\Theta^2(\nu x + \alpha)}, \tag{5.17}$$

where C is a constant. By exponentiating, we obtain the explicit formula for p

$$\begin{aligned} p_1 &= C_1 e^{sx} \frac{\sqrt{H(\nu(x-x_1))H(\nu(x-x_2))\cancel{H(\nu(x+x_1))}\,\cancel{H(\nu(x+x_2))}}}{\Theta(\nu x-\alpha)\Theta(\nu x+\alpha)} \\ &\qquad \times \frac{\sqrt{H(\nu(x-x_1))H(\nu(x-x_2))}}{\sqrt{\cancel{H(\nu(x+x_1))}\,\cancel{H(\nu(x+x_2))}}} \\ &= C_1 e^{sx} \frac{H(\nu(x-x_1))H(\nu(x-x_2))}{\Theta(\nu x-\alpha)\Theta(\nu x+\alpha)}, \end{aligned} \tag{5.18}$$

where C_1 is arbitrary constant and

$$\begin{aligned} s &:= -i\zeta + \frac{2\gamma_1(\upsilon^2\gamma_1\zeta^2 - \alpha_1\zeta_1\zeta_2\zeta_3)}{i\zeta(\alpha_1^2-\gamma_1^2\upsilon^2)} + \frac{\nu H'(\nu x_1)}{H(\nu x_1)} + \frac{\nu H'(\nu x_2)}{H(\nu x_2)} \\ &= i\frac{4\zeta^4 - 4\zeta^2(\zeta_2+\zeta_3-\zeta_1)^2 - i\zeta\mu + 4\zeta_1\zeta_2\zeta_3(\zeta_2+\zeta_3-\zeta_1)}{4\zeta^3 + 4\zeta(\zeta_2\zeta_3 - \zeta_1\zeta_2 - \zeta_1\zeta_3) - i\mu} + \frac{\nu H'(\nu x_1)}{H(\nu x_1)} + \frac{\nu H'(\nu x_2)}{H(\nu x_2)}. \end{aligned} \tag{5.19}$$

By using the quotient (5.10) for ρ, we hence obtain the explicit formula for $q_1 = p_1\rho$ as

$$q_1 = C_1 C e^{sx} \frac{H(\nu(x+x_1^*))H(\nu(x+x_2^*))}{\Theta(\nu x-\alpha)\Theta(\nu x+\alpha)}, \tag{5.20}$$

where the constants C is defined uniquely in (5.9). By choosing

$$C_1 = \frac{e^{-\frac{i\pi\nu}{2K}(x_1+x_2)}}{\Theta(\alpha-\nu x_1)\Theta(\alpha-\nu x_2)},$$

we obtain the explicit expressions for the first solution in the form:

$$\begin{pmatrix} p_1(x) \\ q_1(x) \end{pmatrix} = \begin{pmatrix} \frac{H(\nu(x-x_1))H(\nu(x-x_2))}{\Theta(\nu x-\alpha)\Theta(\nu x+\alpha)\Theta(\alpha-\nu x_1)\Theta(\alpha-\nu x_2)} e^{-\frac{i\pi\nu}{2K}(x_1+x_2)} \\ -\frac{H(\nu(x+x_1^*))H(\nu(x+x_2^*))}{\Theta(\nu x-\alpha)\Theta(\nu x+\alpha)\Theta(\alpha+\nu x_1^*)\Theta(\alpha+\nu x_2^*)} e^{\frac{i\pi\nu}{2K}(x_1^*+x_2^*)} \end{pmatrix} e^{sx}. \tag{5.21}$$

Factorized form for (p_2, q_2).
We replace μ by $-\mu$ in (5.15). This replaces υ^2 with

$$(\upsilon^*)^2 := \zeta_1^2 + \zeta_2^2 + \zeta_3^2 - 2\zeta^2 + \frac{\mu}{2i\zeta}.$$

Since $\phi^2(x) - (\upsilon^*)^2$ is an elliptic function with roots given by roots of (5.2), we use Lemma 3 and write it in the factorization form

$$\phi^2(x) - (\upsilon^*)^2 = C^* \frac{H(\nu(x-x_1^*))H(\nu(x-x_2^*))H(\nu(x+x_1^*))H(\nu(x+x_2^*))}{\Theta^2(\nu x-\alpha)\Theta^2(\nu x+\alpha)}, \tag{5.22}$$

where C^* is another constant. Computations remain the same with $\{\pm x_1, \pm x_2\}$ replaced by $\{\pm x_1^*, \pm x_2^*\}$ so that roots of $\phi(x_\pm^*) \pm \upsilon^* = 0$ are placed in the correspondence with $x_1^* = x_+^*$ and $x_2^* = x_-^*$. As result of similar computations, we obtain

$$p_2 = C_2 e^{s^* x} \frac{H(\nu(x-x_1^*))H(\nu(x-x_2^*))}{\Theta(\nu x-\alpha)\Theta(\nu x+\alpha)} \tag{5.23}$$

and

$$q_2 = C_2 C e^{s^* x} \frac{H(\nu(x+x_1))H(\nu(x+x_2))}{\Theta(\nu x - \alpha)\Theta(\nu x + \alpha)}, \tag{5.24}$$

where C_2 is arbitrary constant, the constant C is defined uniquely in (5.12), and

$$s^* := i\frac{4\zeta^4 - 4\zeta^2(\zeta_2+\zeta_3-\zeta_1)^2 + i\zeta\mu + 4\zeta_1\zeta_2\zeta_3(\zeta_2+\zeta_3-\zeta_1)}{4\zeta^3 + 4\zeta(\zeta_2\zeta_3 - \zeta_1\zeta_2 - \zeta_1\zeta_3) + i\mu} + \frac{\nu H'(\nu x_1^*)}{H(\nu x_1^*)} + \frac{\nu H'(\nu x_2^*)}{H(\nu x_2^*)}. \tag{5.25}$$

We prove in Lemma 5 that $s^* = -s$. Hence, by choosing

$$C_2 = \frac{e^{\frac{i\pi\nu}{2K}(x_1^* + x_2^*)}}{\Theta(\alpha + \nu x_1^*)\Theta(\alpha + \nu x_2^*)},$$

we obtain the explicit expressions for the second solution in the form:

$$\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} \frac{H(\nu(x-x_1^*))H(\nu(x-x_2^*))}{\Theta(\nu x-\alpha)\Theta(\nu x+\alpha)\Theta(\alpha+\nu x_1^*)\Theta(\alpha+\nu x_2^*)} e^{\frac{i\pi\nu}{2K}(x_1^*+x_2^*)} \\ \frac{H(\nu(x+x_1))H(\nu(x+x_2))}{\Theta(\nu x-\alpha)\Theta(\nu x+\alpha)\Theta(\alpha-\nu x_1)\Theta(\alpha-\nu x_2)} e^{-\frac{i\pi\nu}{2K}(x_1+x_2)} \end{pmatrix} e^{-sx}. \tag{5.26}$$

Lemma 5. *It follows that $s + s^* = 0$, where s and s^* are given by (5.19) and (5.25).*

Proof. We recall that the Wronskian determinant of the two solutions of the linear system (1.2) is independent of (x,t). By using (5.9), (5.12), (5.18), (5.20), (5.23), and (5.24), we obtain for the Wronskian W of the two solutions:

$$\begin{aligned} W = {} & e^{(s+s^*)x} \frac{H(\nu(x-x_1))H(\nu(x-x_2))H(\nu(x+x_1))H(\nu(x+x_2))}{\Theta^2(\nu x-\alpha)\Theta^2(\nu x+\alpha)\Theta^2(\alpha-\nu x_1)\Theta^2(\alpha-\nu x_2)} e^{-\frac{i\pi\nu}{K}(x_1+x_2)} \\ & + e^{(s+s^*)x} \frac{H(\nu(x-x_1^*))H(\nu(x-x_2^*))H(\nu(x+x_1^*))H(\nu(x+x_2^*))}{\Theta^2(\nu x-\alpha)\Theta^2(\nu x+\alpha)\Theta^2(\alpha+\nu x_1^*)\Theta^2(\alpha+\nu x_2^*)} e^{\frac{i\pi\nu}{K}(x_1^*+x_2^*)}. \end{aligned}$$

By using (5.17) and (5.22), this can be rewritten in the form

$$\begin{aligned} W = {} & e^{(s+s^*)x} \frac{\phi^2(x) - \upsilon^2}{C\Theta^2(\alpha-\nu x_1)\Theta^2(\alpha-\nu x_2)} e^{-\frac{i\pi\nu}{K}(x_1+x_2)} \\ & + e^{(s+s^*)x} \frac{\phi^2(x) - (\upsilon^*)^2}{C^*\Theta^2(\alpha+\nu x_1^*)\Theta^2(\alpha+\nu x_2^*)} e^{\frac{i\pi\nu}{K}(x_1^*+x_2^*)}. \end{aligned}$$

We show that the coefficient in front of $e^{(s+s^*)x}\phi^2(x)$ vanishes identically. Indeed, by using (1.19), we compute from (5.17) and (5.22) as $\nu x \to iK' + \alpha$ that

$$(\zeta_1 - \zeta_2 - \zeta_3)^2 \frac{\Theta^4(\alpha)}{H^4(\beta)} \Theta^2(\alpha-\beta)\Theta^2(\alpha+\beta) = C\Theta(\alpha-\nu x_1)\Theta(\alpha-\nu x_2)\Theta(\alpha+\nu x_1)\Theta(\alpha+\nu x_2),$$

$$(\zeta_1 - \zeta_2 - \zeta_3)^2 \frac{\Theta^4(\alpha)}{H^4(\beta)} \Theta^2(\alpha-\beta)\Theta^2(\alpha+\beta) = C^*\Theta(\alpha-\nu x_1^*)\Theta(\alpha-\nu x_2^*)\Theta(\alpha+\nu x_1^*)\Theta(\alpha+\nu x_2^*).$$

The coefficient in front of $e^{(s+s^*)x}\phi^2(x)$ in W is zero if and only if

$$\frac{\Theta(\alpha+\nu x_1)\Theta(\alpha+\nu x_2)}{\Theta(\alpha-\nu x_1)\Theta(\alpha-\nu x_2)} e^{-\frac{i\pi}{K}\nu(x_1+x_2)} + \frac{\Theta(\alpha-\nu x_1^*)\Theta(\alpha-\nu x_2^*)}{\Theta(\alpha+\nu x_1^*)\Theta(\alpha+\nu x_2^*)} e^{\frac{i\pi}{K}\nu(x_1^*+x_2^*)} = 0.$$

Thi relation follows by taking the limit $\rho(x)$ in (5.10) as $x \to -\nu^{-1}(iK'+\alpha)$ or $\rho(x)$ in (5.11) as $x \to \nu^{-1}(iK'+\alpha)$:

$$-1 = e^{\frac{i\pi\nu}{K}(x_1+x_2+x_1^*+x_2^*)} \frac{\Theta(\alpha-\nu x_1)\Theta(\alpha-\nu x_2)\Theta(\alpha-\nu x_1^*)\Theta(\alpha-\nu x_2^*)}{\Theta(\alpha+\nu x_1)\Theta(\alpha+\nu x_2)\Theta(\alpha+\nu x_1^*)\Theta(\alpha+\nu x_2^*)}.$$

Hence we get

$$W = -e^{(s+s^*)x}\left[\frac{\upsilon^2 e^{-\frac{i\pi\nu}{K}(x_1+x_2)}}{C\Theta^2(\alpha-\nu x_1)\Theta^2(\alpha-\nu x_2)} + \frac{(\upsilon^*)^2 e^{\frac{i\pi\nu}{K}(x_1^*+x_2^*)}}{C^*\Theta^2(\alpha+\nu x_1^*)\Theta^2(\alpha+\nu x_2^*)}\right].$$

The Wronskian W is independent of x if and only if $s+s^*=0$. □

Example 3. *As $\zeta \to 0$, we recover the exact solutions (1.20) and (1.21) from (5.21) and (5.26) by using*

$$\begin{aligned} x_1 &= \frac{iK'-\alpha}{\nu} = \frac{\upsilon}{2} + 2iK', \\ x_2 &= \frac{-iK'-\alpha}{\nu} = \frac{\upsilon}{2}, \\ x_1^* &= \frac{iK'+\alpha}{\nu} = -\frac{\upsilon}{2}, \\ x_2^* &= \frac{-iK'+\alpha}{\nu} = -\frac{\upsilon}{2} - 2iK', \end{aligned}$$

which agrees with the fact that roots of (5.1) and (5.2) approach to the double poles of $\phi^2(x)$ as $\zeta \to 0$. Since $\mu \to 4\zeta_1\zeta_2\zeta_3$ as $\zeta \to 0$, we use (5.19) and obtain $s \to s_0$ as $\zeta \to 0$ with

$$s_0 = \zeta_1 - \zeta_2 - \zeta_3 - 2Z'(\alpha) = -\nu\frac{H'(2\alpha)}{H(2\alpha)},$$

where we have used Lemma 4.2 in [3] for the second equality. This agrees with (4.1) and (4.2) for $\delta = 2\alpha$. By using (1.19), we substitute values of x_1, x_2, x_1^*, x_2^* into (5.21) and obtain

$$p_1(x) = -q_1(x) = \frac{\Theta(\nu x+\alpha)}{\Theta(\nu x-\alpha)}\frac{e^{\frac{i\pi\alpha}{K}}}{H^2(2\alpha)}e^{-\nu x\frac{H'(2\alpha)}{H(2\alpha)}},$$

which is equivalent to (1.20) up to the scalar multiplication. Similarly, we substitute values of x_1, x_2, x_1^*, x_2^* into (5.26) and obtain

$$p_2(x) = q_2(x) = \frac{\Theta(\nu x-\alpha)}{\Theta(\nu x+\alpha)}\frac{e^{\frac{i\pi\alpha}{K}}}{H^2(2\alpha)}e^{\nu x\frac{H'(2\alpha)}{H(2\alpha)}},$$

which is equivalent to (1.21) up to the scalar multiplication.

5.2. **Rational solutions.** If $e_1 = e_2 = e_3$, then $\wp(x) = x^{-2}$, which implies from (2.5) that

$$\phi(x) = -\frac{1}{v} - \frac{4v}{4x^2 - v^2}, \tag{5.27}$$

with $v \in \mathbb{C}$ being the only parameter of the solution. It is clear that the rational solution (5.27) is not important for applications since the solution is not real-valued if $v \in \mathbb{C}\backslash\mathbb{R}$ and it is singular for real x if $v \in \mathbb{R}$. This case is included for illustrations of expressions for roots $\{\pm x_1, \pm x_2\}$ and $\{\pm x_1^*, \pm x_2^*\}$ of equations (5.1) and (5.2), respectively.

It follows from (2.4) with $\wp(x) = x^{-2}$ that

$$c = \frac{6}{v^2}, \quad b = -\frac{4}{v^3}.$$

By using (5.27) in (1.6) with the given values of c and b, we also obtain

$$d = -\frac{3}{2v^4}.$$

It follows from (1.7) with expressions for b, c, d that $\zeta_1 = -v^{-1}$ and $\zeta_2 = \zeta_3 = v^{-1}$. By using parameter $a \in \mathbb{C}$, we get from (3.6) and (3.12) that

$$\zeta^2 = \frac{a^2 - v^2}{a^2 v^2}, \quad \mu = \frac{4}{a^3}.$$

Equation (5.1) is rewritten explicitly as

$$\frac{16v^2}{(4x^2 - v^2)^2} + \frac{8}{4x^2 - v^2} - \frac{2}{a^2} - \frac{2i}{a^3\zeta} = 0. \tag{5.28}$$

In order to scale $v \in \mathbb{C}$ out, we use the scaled variables

$$x = \frac{v}{2}y, \quad \zeta = \frac{z}{v}, \quad a = \frac{v}{\sqrt{1 - z^2}}. \tag{5.29}$$

Equation (5.28) in scaled variables (5.29) can be rewritten in the form

$$\frac{16}{(y^2 - 1)^2} + \frac{8}{y^2 - 1} - 2(1 - z^2)\left(1 + iz^{-1}\sqrt{1 - z^2}\right) = 0, \tag{5.30}$$

from which we obtain two pairs of roots labeled as $\pm y_+$ and $\pm y_-$, where

$$y_\pm = \sqrt{1 + \frac{4}{\pm\sqrt{1 + 2(1 - z^2)(1 + iz^{-1}\sqrt{1 - z^2})} - 1}}. \tag{5.31}$$

Similarly, we obtain from (5.2) an equation

$$\frac{16}{(y^2 - 1)^2} + \frac{8}{y^2 - 1} - 2(1 - z^2)\left(1 - iz^{-1}\sqrt{1 - z^2}\right) = 0, \tag{5.32}$$

with two pairs of roots labeled as $\pm y_+^*$ and $\pm y_-^*$, where

$$y_\pm^* = \sqrt{1 + \frac{4}{\pm\sqrt{1 + 2(1 - z^2)(1 - iz^{-1}\sqrt{1 - z^2})} - 1}}. \tag{5.33}$$

Expressions (5.31) and (5.33) have square root singularities at $z = 0$ and $z = \pm 1$.

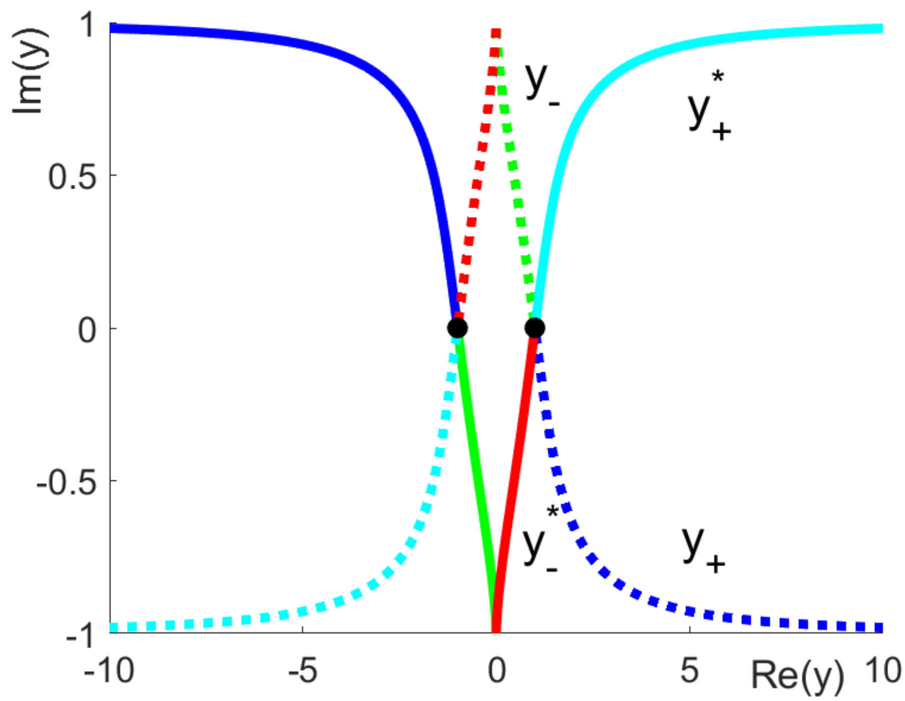

FIGURE 9. Pairs of roots (5.31) and (5.33) on the complex y-plane parameterized by $z \in [0, 1]$. The solid lines correspond to roots of (5.30) and (5.32), which are simultaneously roots of (5.34). The dashed lines correspond to roots of (5.30) and (5.32), which are simultaneously roots of (5.35).

Let us fix $z \in (0, 1)$. Figure 9 show pairs of roots $\pm y_+$, $\pm y_-$, $\pm y_+^*$, $\pm y_-^*$ on the complex y-plane parameterized by $z \in (0, 1)$. The roots converge to ± 1 as $z \to 0$ (shown by black dots). Two pairs of roots converge to i as $z \to 1$ and two pairs of roots diverge to infinity as $z \to 1$. The solid lines show the roots of (5.30) and (5.32), which are simultaneously roots of

$$\frac{8izy}{(y^2 - 1)^2} + \frac{4z^2}{y^2 - 1} - (1 - z^2) = 0, \tag{5.34}$$

which is obtained from (5.3) with the scaling (5.29). Similarly, the dashed lines show the sign-reflected roots, which are roots of

$$-\frac{8izy}{(y^2 - 1)^2} + \frac{4z^2}{y^2 - 1} - (1 - z^2) = 0, \tag{5.35}$$

obtained from (5.4) with the scaling (5.29). Hence, based on the numerical data and the scaled transformation (5.29), we have

$$x_1 = -\frac{v}{2} y_+, \quad x_2 = -\frac{v}{2} y_-, \quad x_1^* = \frac{v}{2} y_+^*, \quad x_2^* = \frac{v}{2} y_-^*.$$

We also note that $y^*_\pm = \overline{y}_\pm$, which suggests that $x^*_{1,2} = -\overline{x}_{1,2}$.

5.3. **Hyperbolic solutions.** If $0 < \zeta_3 = \zeta_2 < \zeta_1$, we have $k = 1$ from (1.9). Since $K = \infty$ and $K' = \frac{\pi}{2}$, the fundamental rectangle $[-K, K] \times [-iK', iK']$ becomes a horizontal strip of the width π in the vertical direction. Equation (5.6) becomes

$$4i\zeta(\zeta_1^2 - \zeta_2^2)\frac{\zeta_2^2\cosh^2(2z) + \zeta_1^2 - 2\zeta_2^2}{(\zeta_1 + \zeta_2\cosh(2z))^2} + 4i\zeta(\zeta^2 - \zeta_1^2) + \mu = 0,$$

which can be expanded as

$$\zeta_2^2\left(\zeta^2 - \zeta_2^2 + \frac{\mu}{4i\zeta}\right)\cosh^2(2z) + 2\zeta_1\zeta_2\left(\zeta^2 - \zeta_1^2 + \frac{\mu}{4i\zeta}\right)\cosh(2z) + \zeta^2\zeta_1^2 - 3\zeta_1^2\zeta_2^2 + 2\zeta_2^4 + \frac{\mu\zeta_1^2}{4i\zeta} = 0.$$

Since $\mu^2 = -16(\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)^2$, multiplying by $\left(\zeta^2 - \zeta_2^2 - \frac{\mu}{4i\zeta}\right)$ yields after straightforward computations:

$$\left(\zeta_1\zeta_2(\zeta^2 - \zeta_2^2)\cosh(2z) + \zeta_2^2(\zeta^2 - \zeta_1^2) + \frac{\mu\zeta^2(\zeta_1^2 - \zeta_2^2)}{4i\zeta(\zeta^2 - \zeta_2^2)}\right)^2 + (\zeta_1^2 - \zeta_2^2)^2\left(\zeta^2(\zeta_1^2 - 2\zeta_2^2) + \frac{2\mu\zeta^2\zeta_2^2}{4i\zeta(\zeta^2 - \zeta_2^2)}\right) = 0.$$

We consider $\zeta \in (0, \zeta_2)$ and select $\mu = 4(\zeta_2^2 - \zeta^2)\sqrt{\zeta_1^2 - \zeta^2}$ from (3.15). Solving the previous equation for $\cosh(2z)$ yields two solutions:

$$\cosh(2z) = \frac{-\zeta_2^2(\zeta_1^2 - \zeta^2) + i\zeta(\zeta_1^2 - \zeta_2^2)\sqrt{\zeta_1^2 - \zeta^2} \pm (\zeta_1^2 - \zeta_2^2)\sqrt{\zeta^2(2\zeta_2^2 - \zeta_1^2) - 2i\zeta\zeta_2^2\sqrt{\zeta_1^2 - \zeta^2}}}{\zeta_1\zeta_2(\zeta_2^2 - \zeta^2)}. \tag{5.36}$$

Roots of (5.36) solve equation (5.1). Hence, they are denoted by $\pm z_1$ and $\pm z_2$ in $(-\infty, \infty) \times \left[-\frac{i\pi}{2}, \frac{i\pi}{2}\right]$. Similarly, roots of

$$\cosh(2z) = \frac{-\zeta_2^2(\zeta_1^2 - \zeta^2) - i\zeta(\zeta_1^2 - \zeta_2^2)\sqrt{\zeta_1^2 - \zeta^2} \pm (\zeta_1^2 - \zeta_2^2)\sqrt{\zeta^2(2\zeta_2^2 - \zeta_1^2) + 2i\zeta\zeta_2^2\sqrt{\zeta_1^2 - \zeta^2}}}{\zeta_1\zeta_2(\zeta_2^2 - \zeta^2)}, \tag{5.37}$$

solve equation (5.2). Hence, they are denoted by $\pm z_1^*$ and $\pm z_2^*$ in $(-\infty, \infty) \times \left[-\frac{i\pi}{2}, \frac{i\pi}{2}\right]$. We detect numerically which of the four roots of (5.36) and (5.37) are simultaneously roots of (5.3). They are labeled as z_1, z_2, z_1^*, z_2^*. The sign-reflected roots $-z_1$, $-z_2$, $-z_1^*$, $-z_2^*$ of (5.36) and (5.37) are simultaneously roots of (5.4). The correspondence between the roots of z and roots of x is $z = \nu x$, where $\nu = \sqrt{\zeta_1^2 - \zeta_2^2}$.

Figure 10 displays roots z_1, z_2, z_1^*, z_2^* of (5.36) and (5.37) for $\zeta_1 = 1$ and $\zeta_2 = \zeta_3 = 0.5$. The roots are parameterized by $\zeta \in (0, \zeta_2)$. The big dots of the same color show roots at $\zeta = \zeta_2$ and $\zeta = 0$. Dashed horizontal lines in magenta shows the boundary of the vertical strip at $\pm iK' = \pm\frac{\pi i}{2}$. For $\zeta = 0$, four pairs of roots are located at $\pm iK' \pm \alpha$. As $\zeta \to \zeta_2$, four roots diverge to infinity since $K = \infty$ and two pairs of roots coalesce on $i\mathbb{R}$.

The solid lines show the roots of (5.1) and (5.2) which are simultaneously roots of (5.3), that is, roots labeled as $\{z_1, z_2, z_1^*, z_2^*\}$. Consequently, the dashed lines show the reflected roots $\{-z_1, -z_2, -z_1^*, -z_2^*\}$ which are roots of (5.1) and (5.2) which are simultaneously roots of (5.4). We can see that the roots satisfy the symmetry $z_{1,2}^* = -\bar{z}_{1,2}$, as in the case of rational solutions.

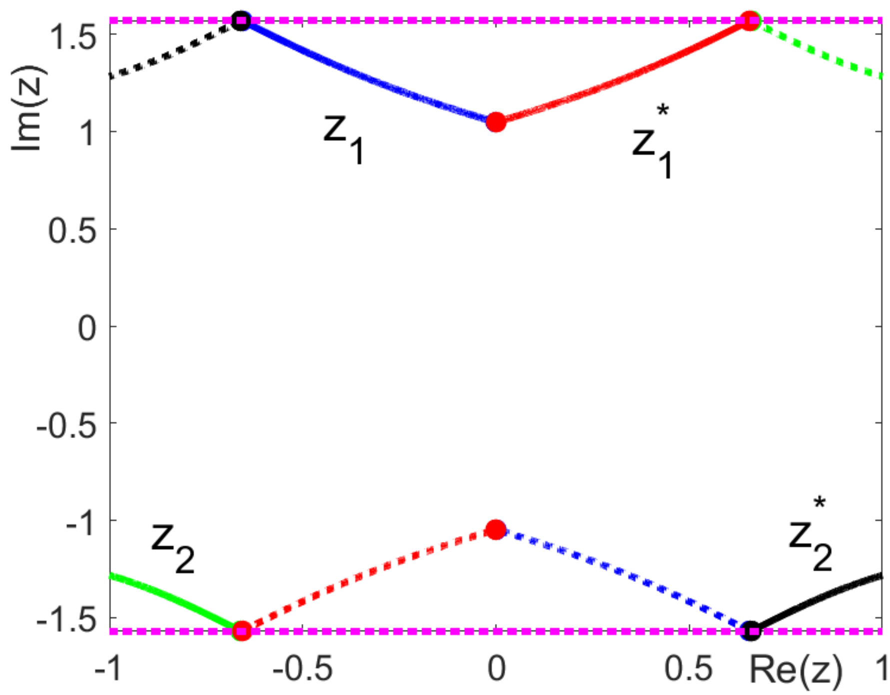

FIGURE 10. Roots $\{\pm z_1, \pm z_2\}$ of (5.36) and $\{\pm z_1^*, \pm z_2^*\}$ of (5.37) on the complex plane for $\zeta \in (0, \zeta_2)$ for $\zeta_1 = 1$ and $\zeta_2 = \zeta_3 = 0.5$.

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(D. E. Pelinovsky) Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada, L8S 4K1
Email address: pelinod@mcmaster.ca

(R. Weikard) Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294–1241 USA
Email address: weikard@uab.edu