New integrable semi-discretizations of the coupled nonlinear Schrödinger equations

Sylvie A. Bronsard and Dmitry E. Pelinovsky

Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada, L8S 4K1

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Abstract

We have undertaken an algorithmic search for new integrable semi-discretizations of physically relevant nonlinear partial differential equations. The search is performed by using a compatibility condition for the discrete Lax operators and symbolic computations. We have discovered a new integrable system of coupled nonlinear Schrödinger equations which combines elements of the Ablowitz–Ladik lattice and the triangular–lattice ribbon studied by Vakhnenko. We show that the continuum limit of the new integrable system is given by uncoupled complex modified Korteweg–de Vries equations and uncoupled nonlinear Schrödinger equations.

Keywords: integrable semi-discretizations, derivative nonlinear Schrödinger equation, massive Thirring model, Kaup–Newell spectral problem, Ablowitz–Ladik lattice.

1 Introduction

As was understood long ago, many nonlinear partial differential equations integrable with the inverse scattering transform can be semi-discretized in spatial coordinates in such a way as to preserve integrability. The pioneer example is the Ablowitz–Ladik lattice [2, 3], an integrable semi-discretization of the integrable nonlinear Schrödinger equation. The Ablowitz–Ladik lattice has inspired many groups to search for integrable semi-discretizations of other nonlinear evolution equations, e.g. [4, 9, 12, 18, 19]. The nonlinear ladder equation, the Toda lattice, the discrete modified Korteweg–de Vries equation, the discrete sine–Gordon equation in characteristic coordinates, and the nonlinear self-dual network equations are examples of integrable semi-discrete evolution equations related to the Ablowitz–Kaup–Newel–Segur (AKNS) spectral problem [1].

Other spectral problems have been semi-discretized only very recently. Tsuchida [17] considered nonlinear evolution equations related to the Kaup–Newell spectral problem and constructed integrable semi-discretizations of the derivative nonlinear Schrödinger equation, the Chen–Lee-Liu equation, and the Gerdjikov–Ivanov equations. The coupled Yajima–Oikawa system was semi-discretized by using the Hirota bilinear method in [6]. Generalization of integrable discretizations in the space of two spatial dimensions was considered by Zakharov [24] by using an algebro-geometric approach. The integrable triangular–lattice ribbon was recently studied by Vakhnenko [20, 21] (see also a review in [22]) who further generalized the discrete AKNS spectral problem by including quadratic dependence on the spectral parameter.

In a similar vein, one motivation for our work is to find an integrable semi-discretization of the massive Thirring model (MTM) [10, 13],

$$\left\{ \begin{array}{l} i(u_t + u_x) + v + u|v|^2 = 0, \\ i(v_t - v_x) + u + v|u|^2 = 0, \end{array} \right. \tag{1}$$

which has been used very recently in many studies related to stability of one-dimensional Dirac solitons [5, 7, 14, 15]. Numerical methods based on various spatial semi-discretizations of the MTM were found to suffer from numerical instabilities and artifacts [8, 16]. If we find a semi-discretization which preserves the integrability scheme of the MTM, then the discrete MTM should model stable Dirac solitons without numerical artifacts.

The integrability scheme for the MTM is related to the Kaup–Newell spectral problem [11]. The same spectral problem is also related to the integrability scheme for the derivative nonlinear Schrödinger (dNLS) equation,

$$iu_t + u_{xx} + i(|u|^2 u)_x = 0. \tag{2}$$

In more details, the dNLS equation (2) is the compatibility condition $\vec{\varphi}_{xt} = \vec{\varphi}_{tx}$ for the system of linear equations

$$\vec{\varphi}_x = L(\lambda; u)\vec{\varphi} \quad \text{and} \quad \vec{\varphi}_t = A(\lambda; u)\vec{\varphi}, \tag{3}$$

where λ is a spectral parameter, while $L(\lambda; u)$ and $A(\lambda; u)$ are matrix operators given by

$$L(\lambda; u) = -i\lambda^2\sigma_3 + \lambda \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} \tag{4}$$

and

$$A(\lambda; u) = i(\lambda^2|u|^2 - 2\lambda^4)\sigma_3 + \lambda \begin{pmatrix} 0 & 2\lambda^2 u - |u|^2 u \\ -2\lambda^2\bar{u} + |u|^2\bar{u} & 0 \end{pmatrix} + i\lambda \begin{pmatrix} 0 & u_x \\ \bar{u}_x & 0 \end{pmatrix}, \tag{5}$$

where $\sigma_3 = \mathrm{diag}(1, -1)$ is Pauli's matrix.

Similarly, the MTM system (1) is the compatibility condition $\vec{\varphi}_{xt} = \vec{\varphi}_{tx}$ for the system of linear equations

$$\vec{\varphi}_x = L(\lambda; u, v)\vec{\varphi} \quad \text{and} \quad \vec{\varphi}_t = A(\lambda; u, v)\vec{\varphi}, \tag{6}$$

where λ is a spectral parameter, while $L(\lambda; u, v)$ and $A(\lambda; u, v)$ are matrix operators given by

$$L(\lambda; u, v) = \frac{i}{4}(|u|^2 - |v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} + \frac{i}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\sigma_3 \tag{7}$$

and

$$A(\lambda; u, v) = -\frac{i}{4}(|u|^2 + |v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\sigma_3. \tag{8}$$

Compared to the matrix operators in (4)–(5), both L and A in (7)–(8) depend quadratically on λ and λ^{-1}, which makes analysis of the inverse scattering transform for the MTM sufficiently difficult [23]. By performing a transformation of the physical coordinates x and t to the characteristic coordinates $\xi = x - t$ and $\eta = x + t$, one can rewrite the MTM system (1) and the Lax pair (6) in the form associated with the Kaup–Newell operator L in (4). However, this transformation changes the Cauchy problem for the MTM system (1) to the Goursat problem in characteristic coordinates and vice versa.

Tsuchida [17] obtained semi-discretizations of the dNLS equation (2) and the MTM in characteristic coordinates by using the gauge transformation of the Kaup-Newell spectral problem to the AKNS spectral problem and by searching for a generalized spatial discretization of the AKNS problem. However, as is explained above, these semi-discretizations are not useful in the context of the Cauchy problem for the MTM system in physical coordinates (1).

We have undertaken here a systematic search for the class of semi-discrete matrices L and A with a polynomial dependence on z and z^{-1} up to the quadratic (for L) and quartic (for A) orders, where z is a new spectral parameter. As an outcome of our algorithmic computations, we have obtained a new semi-discretization of the coupled nonlinear Schrödinger equations. This new semi-discretization coincides with the higher-order commuting flow of the triangular–lattice ribbon [20, 21].

The rest of this paper is organized as follows. Section 2 presents the discrete spectral problems for the Ablowitz–Ladik lattice, triangular–lattice ribbon, and the newly derived semi-discretization of the coupled nonlinear Schrödinger equations. Section 3 contains a study of the continuum limit in the new semi-discrete system. Section 4 concludes the paper with a summary.

2 Semi-discretizations of the nonlinear Schrödinger equations

We are looking for an integrable semi-discrete system which appears as a compatibility condition for the system of linear equations

$$\vec{\varphi}_{n+1} = L_n(z)\vec{\varphi}_n \quad \text{and} \quad \frac{d}{dt}\vec{\varphi}_n = A_n(z)\vec{\varphi}_n, \tag{9}$$

where z is a spectral parameter, $n \in \mathbb{Z}$, $t \in \mathbb{R}$, while $L_n(z)$ and $A_n(z)$ are matrix operators containing potentials satisfying the compatibility condition

$$\frac{d}{dt} L_n(z) = A_{n+1}(z) L_n(z) - L_n(z) A_n(z). \tag{10}$$

The Ablowitz–Ladik lattice derived in [2, 3] corresponds to the choice

$$L_n(z) = \left[\begin{array}{cc} z & q_n \\ -\bar{q}_n & z^{-1} \end{array} \right], \quad A_n(z) = \left[\begin{array}{cc} a z^2 + a q_n \bar{q}_{n-1} & a q_n z + \bar{a} q_{n-1} z^{-1} \\ -a \bar{q}_{n-1} z - \bar{a} \bar{q}_n z^{-1} & \bar{a} z^{-2} + \bar{a} q_{n-1} \bar{q}_n \end{array} \right], \tag{11}$$

where $a \in \mathbb{C}$ is arbitrary parameter and the complex-conjugate symmetry is preserved for the complex-valued potential $\{q_n\}_{n\in\mathbb{Z}}$. Substituting (11) into (10) yields the Ablowitz–Ladik lattice

$$\frac{dq_n}{dt} = \alpha(q_{n+1} - q_{n-1})(1 + |q_n|^2) + i\beta(q_{n+1} + q_{n-1})(1 + |q_n|^2), \tag{12}$$

where we have used $a = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$. The α part of this system is also referred to as the discrete modified Korteweg-de Vries equation, while the β part is referred to as the discrete nonlinear Schrödinger equation [1]. The two parts are related by the staggering transformation

$$q_n \mapsto i^n q_n, \quad n \in \mathbb{Z}.$$

The triangular–lattice ribbon derived in [20, 21] corresponds to the choice

$$L_n(z) = \left[\begin{array}{cc} z^2 - \bar{q}_n r_n & q_n z + r_n z^{-1} \\ -\bar{r}_n z - \bar{q}_n z^{-1} & z^{-2} - q_n \bar{r}_n \end{array} \right] \tag{13}$$

and

$$A_n(z) = \left[\begin{array}{cc} a z^2 + a q_n \bar{r}_{n-1} & a q_n z + \bar{a} r_{n-1} z^{-1} \\ -a \bar{r}_{n-1} z - \bar{a} \bar{q}_n z^{-1} & \bar{a} z^{-2} + \bar{a} r_{n-1} \bar{q}_n \end{array} \right], \tag{14}$$

where $a \in \mathbb{C}$ is arbitrary parameter and the complex-conjugate symmetry is preserved for the complex-valued potentials $\{q_n, r_n\}_{n\in\mathbb{Z}}$. Substituting (13) and (14) into (10) yields the triangular–lattice ribbon:

$$\frac{dq_n}{dt} = \alpha(r_n - r_{n-1})(1 + |q_n|^2) + i\beta(r_n + r_{n-1})(1 + |q_n|^2), \tag{15}$$

$$\frac{dr_n}{dt} = \alpha(q_{n+1} - q_n)(1 + |r_n|^2) + i\beta(q_{n+1} + q_n)(1 + |r_n|^2), \tag{16}$$

where we have used $a = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$. The α part of this system is referred to as the nonlinear self-dual network equations [1]. The β part can be transformed to the α part by the staggering transformation

$$q_n \mapsto (-1)^n q_n, \quad r_n \mapsto -i(-1)^n r_n, \quad n \in \mathbb{Z}.$$

Our search of the matrix operators $L_n(z)$ and $A_n(z)$ satisfying the compatibility condition (10) generalizes the choices (11) and (13)–(14). We have considered a general quadratic polynomial in z and z^{-1} for $L_n(z)$ and a general quartic polynomial in z and z^{-1} for $A_n(z)$. By working with the symbolic computation software based on Wolfram's MATHEMATICA, we were able to satisfy the compatibility condition (10) in each order of z and z^{-1} if the matrix operators $L_n(z)$ and $A_n(z)$ are given in the form

$$L_n(z) = \left[\begin{array}{cc} z^2 - \bar{q}_n r_n & q_n z + r_n z^{-1} \\ -\bar{r}_n z - \bar{q}_n z^{-1} & z^{-2} - q_n \bar{r}_n \end{array} \right], \quad A_n(z) = \left[\begin{array}{cc} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{array} \right] \tag{17}$$

with

$$\begin{aligned}
A_{11}(z) &= a z^4 + a q_n \bar{r}_{n-1} z^2 + a \left(q_n \bar{q}_{n-1} + r_n \bar{r}_{n-1} + q_n \bar{q}_{n-1} |r_{n-1}|^2 + |q_n|^2 r_n \bar{r}_{n-1} + q_n^2 \bar{r}_{n-1}^2 \right) \\
&\quad - \bar{a} \bar{q}_n r_{n-1} z^{-2}, \\
A_{12}(z) &= a q_n z^3 + a(r_n + |q_n|^2 r_n - q_n^2 \bar{r}_{n-1}) z + \bar{a}(q_{n-1} + q_{n-1}|r_{n-1}|^2 + \bar{q}_n r_{n-1}^2) z^{-1} + \bar{a} r_{n-1} z^{-3}, \\
A_{21}(z) &= -a \bar{r}_{n-1} z^3 - a(\bar{q}_{n-1} + \bar{q}_{n-1}|r_{n-1}|^2 + q_n \bar{r}_{n-1}^2) z - \bar{a}(\bar{r}_n + |q_n|^2 \bar{r}_n + \bar{q}_n^2 r_{n-1}) z^{-1} - \bar{a} \bar{q}_n z^{-3}, \\
A_{22}(z) &= -a q_n \bar{r}_{n-1} z^2 + \bar{a} \left(q_{n-1} \bar{q}_n + r_{n-1} \bar{r}_n + q_{n-1} \bar{q}_n |r_{n-1}|^2 + |q_n|^2 r_{n-1} \bar{r}_n + \bar{q}_n^2 r_{n-1}^2 \right) \\
&\quad + \bar{a} \bar{q}_n r_{n-1} z^{-2} + \bar{a} z^{-4}.
\end{aligned}$$

The two potentials $\{q_n, r_n\}_{n\in\mathbb{Z}}$ satisfy the lattice differential equations in the form:

$$\begin{aligned} \frac{dq_n}{dt} &= \left[aq_{n+1}(1+|r_n|^2) - \bar{a}q_{n-1}(1+|r_{n-1}|^2) + \bar{q}_n(ar_n^2 - \bar{a}r_{n-1}^2)\right. \\ &\quad \left.+q_n(ar_n\bar{r}_{n-1} - \bar{a}r_{n-1}\bar{r}_n)\right](1+|q_n|^2), \qquad (18) \\ \frac{dr_n}{dt} &= \left[ar_{n+1}(1+|q_{n+1}|^2) - \bar{a}r_{n-1}(1+|q_n|^2) + \bar{r}_n(aq_{n+1}^2 - \bar{a}q_n^2)\right. \\ &\quad \left.+r_n(aq_{n+1}\bar{q}_n - \bar{a}q_n\bar{q}_{n+1})\right](1+|r_n|^2). \qquad (19) \end{aligned}$$

Comparing the matrix operators $L_n(z)$ in (13) and (17), we can see that they are identical to each other. On the other hand, the matrix operator $A_n(z)$ in (17) yields the next commuting flow to the matrix operator $A_n(z)$ in (14). Hence, the new system of lattice differential equations (18)–(19) is the next commuting flow of the triangular–lattice ribbon (15)–(16).

3 Continuum limit of the semi-discrete equations

Here we derive the continuum limit of the semi-discrete equations (18)–(19) and compare them with integrable continuous nonlinear equations. Setting $a = 1$ in (18)–(19) yields

$$\begin{aligned} \frac{dq_n}{dt} &= \left[q_{n+1}(1+|r_n|^2) - q_{n-1}(1+|r_{n-1}|^2) + \bar{q}_n(r_n^2 - r_{n-1}^2)\right. \\ &\quad \left.+q_n(r_n\bar{r}_{n-1} - r_{n-1}\bar{r}_n)\right](1+|q_n|^2), \qquad (20) \\ \frac{dr_n}{dt} &= \left[r_{n+1}(1+|q_{n+1}|^2) - r_{n-1}(1+|q_n|^2) + \bar{r}_n(q_{n+1}^2 - q_n^2)\right. \\ &\quad \left.+r_n(q_{n+1}\bar{q}_n - q_n\bar{q}_{n+1})\right](1+|r_n|^2). \qquad (21) \end{aligned}$$

By taking the asymptotic ansatz

$$\left\{ \begin{array}{l} q_n(t) = \epsilon Q(\epsilon(n+2t), \epsilon^3 t) + \mathcal{O}(\epsilon^3), \\ r_n(t) = \epsilon R(\epsilon(n+2t), \epsilon^3 t) + \mathcal{O}(\epsilon^3), \end{array} \right.$$

we obtain the system of coupled complex modified Korteweg–de Vries equations at the leading order of $\mathcal{O}(\epsilon^4)$:

$$Q_\tau = \frac{1}{3}Q_{\xi\xi\xi} + 2(|Q|^2 + |R|^2)Q_\xi + 2(Q\bar{R} + \bar{Q}R)R_\xi, \qquad (22)$$

$$R_\tau = \frac{1}{3}R_{\xi\xi\xi} + 2(|Q|^2 + |R|^2)R_\xi + 2(Q\bar{R} + \bar{Q}R)Q_\xi, \qquad (23)$$

where $\xi = \epsilon(n+2t)$ and $\tau = \epsilon^3 t$. Although the system of coupled equations (22)–(23) may look as a new integrable system, it has a simple reduction to uncoupled complex modified Korteweg-de Vries equations. Indeed, let $U := Q + R$ and $V := Q - R$. Then, adding and subtracting (22) and (23) yield the following two uncoupled complex modified Korteweg-de Vries equations:

$$U_\tau = \frac{1}{3}U_{\xi\xi\xi} + 2|U|^2 U_\xi, \qquad (24)$$

$$V_\tau = \frac{1}{3}V_{\xi\xi\xi} + 2|V|^2 V_\xi. \qquad (25)$$

Setting $a = i$ in (18)–(19) yields

$$\begin{aligned} \frac{dq_n}{dt} &= i\left[q_{n+1}(1+|r_n|^2) + q_{n-1}(1+|r_{n-1}|^2) + \bar{q}_n(r_n^2 + r_{n-1}^2)\right. \\ &\quad \left.+q_n(r_n\bar{r}_{n-1} + r_{n-1}\bar{r}_n)\right](1+|q_n|^2), \qquad (26) \\ \frac{dr_n}{dt} &= i\left[r_{n+1}(1+|q_{n+1}|^2) + r_{n-1}(1+|q_n|^2) + \bar{r}_n(q_{n+1}^2 + q_n^2)\right. \\ &\quad \left.+r_n(q_{n+1}\bar{q}_n + q_n\bar{q}_{n+1})\right](1+|r_n|^2). \qquad (27) \end{aligned}$$

By taking the asymptotic ansatz

$$\left\{ \begin{array}{l} q_n(t) = e^{2it}\left[\epsilon Q(\epsilon n, \epsilon^2 t) + \mathcal{O}(\epsilon^3)\right], \\ r_n(t) = e^{2it}\left[\epsilon R(\epsilon n, \epsilon^2 t) + \mathcal{O}(\epsilon^3)\right], \end{array} \right.$$

we obtain the system of coupled nonlinear Schrödinger equations at the leading order of $\mathcal{O}(\epsilon^3)$:

$$iQ_\tau + Q_{\xi\xi} + 2|Q|^2 Q + 4|R|^2 Q + 2R^2\bar{Q} = 0, \quad (28)$$

$$iR_\tau + R_{\xi\xi} + 2|R|^2 R + 4|Q|^2 R + 2Q^2\bar{R} = 0, \quad (29)$$

where $\xi = \epsilon n$ and $\tau = \epsilon^2 t$. We can show again that the system of coupled equations (28)–(29) can be reduced to an uncoupled system. By letting $U := Q + R$, $V := Q - R$ and adding and subtracting equations (28) and (29), we obtain the two uncoupled nonlinear Schrödinger equations:

$$iU_\tau + U_{\xi\xi} + 2|U|^2 U = 0, \quad (30)$$

$$iV_\tau + V_{\xi\xi} + 2|V|^2 V = 0. \quad (31)$$

Hence, the new semi-discrete system (18)–(19) is another integrable semi-discretization of the coupled nonlinear Schrödinger equations.

4 Conclusion

We have derived a new integrable system of discrete coupled nonlinear Schrödinger equations by considering quadratic and quartic polynomials in the spectral parameter for the discrete Lax operators satisfying the compatibility condition. The novel system shares many properties with the integrable Ablowitz–Ladik lattice [2, 3] and triangular–lattice ribbon studied by Vakhnenko [20, 21]. It has two continuum reductions which are equivalent to uncoupled modified Korteweg–de Vries and nonlinear Schrödinger equations.

The original goal of our study, finding an integrable semi-discretization of the massive Thirring model (1), has not been reached in our search. Modifications of the quadratic and quartic polynomials in the discrete Lax operators did not produce new integrable system of lattice differential equations. Although the integrable semi-discretizations of the derivative nonlinear Schrödinger equations (2) and the massive Thirring model in characteristic coordinates have been constructed in the literature [17], it still remains an open problem to construct an integrable semi-discretization of the massive Thirring model in physical coordinates.

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