# Instability of peaked travelling wave solutions of the $b$-family of Camassa-Holm equations

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*A Thesis Submitted to the School of Graduate Studies in the Partial Fulfillment of the Requirements for the Master of Science Degree*

McMaster University
Department of Mathematics & Statistics
Master of Science (2021)

TITLE: Instability of peaked travelling wave solutions
of the $b$-family of Camassa-Holm equations
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SUPERVISOR: Dmitry Pelinovsky,
NUMBER OF PAGES: v, 45

## Abstract

The aim of this work is to study instability of peaked travelling waves of the $b$-family of the generalized Camassa-Holm equations. This family includes the integrable Camassa–Holm and Degasperis–Procesi equations for $b = 2$ and $b = 3$ respectively. We first review previous results on the existence and stability of peaked travelling waves on the infinite line and in the periodic domain. Next, we prove instability of the peaked solitary waves under suitable assumptions. The instability is obtained by the methods of characteristics and comparison theory for differential equations. We give some precise results on instability of the peaked periodic waves in the Camassa–Holm equation. Finally, we review open problems in the stability theory of peaked periodic waves in the Degasperis–Procesi equation.

## Acknowledgements

I would like to express my deepest gratitude to my wonderful supervisor, Professor Dmitry Pelinovsky, for his most valuable suggestions and advice, for his patience, and for the generous assistance that he provided in the development of this thesis. I learned a lot from his willingness to tackle a wide variety of problems. I feel really lucky to be able to work under his direction.

I am further grateful to Professors Lia Bronsard and Bartosz Protas who accepted to be the external examiners of this thesis.

Also, I am grateful for the companionship of my friends in the mathematics department, with whom I shared the difficulties and excitement of such an adventure.

Finally, I would like to acknowledge with gratitude my lovely family for giving me great moral support and encouragement during these years.

# Contents

# Chapter 1

# Introduction

The main topic of this thesis is the following $b$-family of the generalized Camassa–Holm equations (which we call $b$-CH):

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}. \tag{1.1}$$

The family generalizes the classical cases of the Camassa–Holm (CH) equation for $b = 2$ and the Degasperis–Procesi (DP) equation for $b = 3$. The CH equation has first appeared in the work of Fokas and Fuchssteiner [20] in the study of bi-Hamiltonian structure of the Korteweg–de Vries equation. It was later introduced by Camassa and Holm [5] in hydrodynamical applications as a model for unidirectional wave propagation on shallow water. Nowadays, this equation is commonly referred to as the CH equation. The hydrodynamical relevance of the CH equation as a model for shallow water waves was also discussed in [6, 9, 34]. The CH equation can be interpreted geometrically in terms of geodesic flows on the diffeomorphism group [36, 47]. The DP equation can also be regarded as a model for nonlinear shallow water dynamics with asymptotic accuracy equal to the CH equation [18]. Dullin, Gottwald and Holm [19] demonstrated that the DP equation can be obtained as a model for unidirectional water wave propagation in shallow water by an appropriate Kodama transformation and introduced the $b$-family of the CH equations.

It is interesting to note that the equations in the $b$-family are integrable only for $b = 2$ and $b = 3$, according to the following integrability tests: the Wahlquist-Estabrook prolongation method, the Painleve analysis, and the symmetry reductions [32, 33, 45].

One of the most intriguing properties of the generalized Camassa–Holm equations is the occurrence of wave breaking. This means that even smooth initial data may lead to blow-up in finite time, which happens in such a way that the solution stays bounded but its slope develops a singularity.

Another remarkable property of the generalized Camassa–Holm equations is the existence of peaked travelling waves called *peakons*. Peakons are non-smooth solitons and they were discovered in the original paper [5]. Peakons travelling at the constant speed $c \neq 0$ are explicitly given by

$$u(x,t) = ce^{-|x-ct|} \qquad x \in \mathbb{R} \tag{1.2}$$

on a line and

$$u(x,t) = c\frac{\cosh(\pi - |x-ct|)}{\cosh(\pi)} \qquad x \in \mathbb{T} \tag{1.3}$$

on the periodic domain $\mathbb{T} := [-\pi, \pi]$, where we normalize the period to $2\pi$. Because these solutions have peaks at their crests, they have to be interpreted as suitable weak solutions of the CH equation [12]. More generally, one can also consider multi-peakon solutions on the line:

$$u(x,t) = \sum_{n=1}^{N} p_n(t)e^{-|x-q_n(t)|}, \tag{1.4}$$

which are finite sums of single peakons for each fixed time $t$. The representation (1.4) is a weak solution of the CH equation, provided that the functions $p$ and $q$ satisfy the finite-dimensional Hamiltonian system given by [3]

$$\frac{dq_k}{dt} = \sum_{j=1}^{n} p_j e^{-|q_k-q_j|}, \qquad \frac{dp_k}{dt} = (b-1)p_k \sum_{j=1}^{n} p_j sgn(q_k - q_j)e^{-|q_k-q_j|}.$$

Generalizations of the CH equation with multi-peakons were constructed in [1, 2]. Collisions of peakons and anti-peakons were studied in [4, 26, 29, 40, 43].

In order to introduce the weak form of the $b$-CH equation (1.1), we rewrite it in the form:

$$(1 - \partial_x^2)(u_t + uu_x) + \frac{1}{2}\partial_x \left(bu^2 + (3-b)u_x^2\right) = 0 \tag{1.5}$$

Let $\varphi$ be the Green function of $(1 - \partial_x^2)$ normalized by $(1 - \partial_x^2)\varphi = 2\delta_0$, where $\delta_0$ is the Dirac delta distribution centered at $x = 0$. Then, the $b$-CH equation (1.5) can be written in the convolution form:

$$u_t + uu_x + \frac{1}{4}\varphi' * \left(bu^2 + (3-b)u_x^2\right) = 0, \tag{1.6}$$

where $(f * g)(x) := \int f(x-y)g(y)dy$ is the convolution operator. It is clear that $\varphi \in H^s$ for $s < \frac{3}{2}$, where $H^s$ is the Sobolev space of squared integrable distributions equipped with the norm $\|f\|_{H^s} := \|\langle\cdot\rangle^s \hat{f}\|_{L^2}$ with $\langle x\rangle := \sqrt{1+x^2}$ and $\hat{f}$ being the Fourier transform of $f$.

When constructing models described by partial differential equations, one should make sure that the initial–value (Cauchy) problem is well-posed. Therefore, we recall the notion of well-posedness in the sense of Hadamard [25]. We say that the $b$-CH equation (1.6) is locally well-posed in $H^s$ if the following three conditions hold:

(i) For any initial data $u(0) \in H^s$, there exists a time $T = T_{u(0)} > 0$ and a solution $u \in C([0,T]; H^s)$ to the initial-value problem.

(ii) This solution $u$ is unique in the space $C([0,T]; H^s)$.

(iii) The solution map $u(0) \to u(t)$ is continuous. More precisely, if $u_n(0)$ is a sequence of initial data converging to $u(0)$ in $H^s$ and if $u_n(t) \in C([0,T_n]; H^s)$ is the solution to the Cauchy problem with initial data $u_n(0)$, then there is $T \in (0, \min_{n\in N} T_n)$ such that the solutions $u_n(t)$ for all $t \in [0,T]$ satisfies

$$\lim_{n\to\infty} \sup_{0\le t\le T} \|u_n(t) - u(t)\|_{H^s} = 0.$$

The initial-value problem for the $b$-CH equation (1.6) was shown to be locally well-posed in $H^s$ for $s > 3/2$ [10, 11]. On the other hand, the initial value problem is ill-posed in $H^s$ for $s < 3/2$ due to the lack of continuous dependence [26]. It was proven recently in [23] that the initial-value problem is also ill-posed in $H^s$ for $s = 3/2$.

Although the initial-value problem is not well-posed in $H^1$ where peaked waves and the conservation laws are defined, it is still locally well-posed in $H^1 \cap W^{1,\infty}$, where $W^{1,\infty}$ is the space with a bounded first derivative [17, 42].

Stability of peaked travelling waves has been considered in the literature. Orbital stability of peakons in $H^1$ on the line was shown for the CH equation ($b = 2$) in [13, 14] by using the three conservation laws of the CH equation:

$$F_1(u) = \int u dx, \quad F_2(u) = \int (u^2 + u_x^2) dx, \quad F_3(u) = \int u(u^2 + u_x^2) dx \tag{1.7}$$

The stability results in $H^1$ were extended in the periodic domain $\mathbb{T}$ in [38, 39].

Stability of peakons for the DP equation ($b = 3$) in $L^2$ on the line was further obtained in [41] by using the conservation laws of the DP equation:

$$E_1(u) = \int u\, dx, \quad E_2(u) = \int yv\, dx, \quad E_3(u) = \int u^3\, dx \tag{1.8}$$

where $y = (1 - \partial_x^2)u$ and $v = (4 - \partial_x^2)^{-1}u$. Note that the second conservation law for $E_2(u)$ is equivalent to $\|u\|_{L^2}^2$ due to Parseval's equality and Fourier transform:

$$E_2(u) = \int yv\, dx = \int \frac{1+\xi^2}{4+\xi^2} |\hat{u}(\xi)|^2\, d\xi \sim \|\hat{u}\|_{L^2}^2 = \|u\|_{L^2}^2. \tag{1.9}$$

Also note that the first conservation law $F_1(u)$ in (1.7) and $E_1(u)$ in (1.8) are only needed in the periodic domain $\mathbb{T}$ and are not needed on the line $\mathbb{R}$. The stability result in $L^2$ was not extended in the periodic domain $\mathbb{T}$ so far.

Asymptotic stability of multi-peakons of the CH equation in the class of $H^1$ functions $u$ with $y := (1-\partial_x^2)u$ of a non-negative finite measure is proven in [48]. Asymptotic stability of trains of peakons and anti-peakons with $y$ being a sign-indefinite finite measure was constructed recently in [49].

The recent work [50] shows that, although the peakons are orbitally stable in $H^1$, they are still unstable with respect to perturbations in $W^{1,\infty}$. This was done for the CH equation on the line in [50] and for a similar (Novikov) equation in [8]. As a part of this thesis, we published [44], where the growth of perturbations to the peaked periodic wave was established in $W^{1,\infty}$ for the CH equation.

*The main result of this thesis* is the proof that peaked travelling waves in the $b$-CH equation are unstable with respect to perturbations in $W^{1,\infty}$. Even if the solution remains close to the peakon in the $H^1$ norm, then the perturbation still grows in the $W^{1,\infty}$ norm. Moreover, we show that the slope of the peaked perturbation to the peaked travelling wave may grow to infinity and may reach infinity in a finite time. In other words, there exists a time when the profile of the solution steepens gradually, leading to a vertical slope or gradient catastrophe.

We will next explain the organization of this thesis.

In Chapter 2, we define weak solutions to the $b$-CH equation in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$. For solutions with a single peak at $\xi(t)$ on $\mathbb{R}$, we prove that the single peak propagates with the local characteristic speed so that

$$\frac{d\xi}{dt} = u(t, \xi(t)). \tag{1.10}$$

This allows us to define the precise form of the evolution equations for the peaked perturbations to the peaked wave. Next, we consider the nonlinear dynamics of these perturbations by using the method of characteristics. Although the $H^1$ norm of the perturbation does not grow in the nonlinear evolution, we prove that the $W^{1,\infty}$ norm of the perturbation can grow and even blow up in a finite time as follows:

$$u_x(t, x) \to -\infty \quad \text{at some } x \in \mathbb{R} \quad \text{as } t \to T^-. \tag{1.11}$$

The wave breakdown criterion (1.11) is natural for the inviscid Burgers equation

$$u_t + uu_x = 0, \tag{1.12}$$

which contributes to the local part of the $b$-CH equation (1.6). The precise blow-up rate was derived for the strong solutions in $H^3(\mathbb{R})$ in [10]. We show that the wave breakdown (1.11) also occurs as a result of the nonlinear instability of peakons in the $b$-CH equations.

In the case of $b = 3$, we will control peaked perturbations by using the orbital stability result in $L^2$ rather than in $H^1$ due to the results of [41]. Although the $L^\infty$ norm of perturbations is not controlled by the orbital stability result in $L^2$, we will extend the method of characteristics to study both the wave amplitude and the wave slope at the peak's location in order to prove the same wave breakdown criterion (1.11) for $b = 3$.

In Chapter 3, we extend the main result to the instability of peaked periodic waves on the periodic domain $\mathbb{T}$ in the setting of the CH equation ($b = 2$). In the periodic domain, we need to control the mean value of the perturbation, which can be done due to the first conservation law $F_1(u)$ in (1.7).

In Chapter 4, we address stability of peaked periodic waves in the setting of the DP equation ($b = 3$). No stability result was available in $L^2$ in the periodic domain $\mathbb{T}$, compared to the result of [41] on the line $\mathbb{R}$. In order to obtain such stability result, we will consider minimization of the conserved quantity $E_2(u)$ subject to the fixed values of $E_1(u)$ and $E_3(u)$ in (1.8). This approach is similar to the characterization of the stability of peaked periodic waves for the CH equation ($b = 2$) in [37], where the author shows that the peaked waves give a unique constrained minimum of energy.

We will verify that the peaked periodic wave is a critical point of the action functional for the constrained minimization problem. Next, we consider the second variation and prove that it is positive under the constraints of fixed $E_1(u)$ and $E_3(u)$. However, the zero eigenvalue is not bounded away from the continuous spectrum of the second variation, hence the critical point is degenerate and the second variation test on minimization fails. It is still unclear how to prove that the peaked periodic wave is a constrained energy minimizer for the DP equation in $L^2$.

# Chapter 2

# Peakons in the $b$-CH equations

## 2.1 Main result

Let $\varphi \in H^1(\mathbb{R})$ be the Green function satisfying

$$(1 - \partial_x^2)\varphi = 2\delta_0, \quad x \in \mathbb{R} \tag{2.1}$$

with $\delta_0$ being Dirac delta distribution centered at $x = 0$. The Green function $\varphi$ can be expressed explicitly in the form

$$\varphi(x) = e^{-|x|}, \quad x \in \mathbb{R}, \tag{2.2}$$

which shows that $\varphi$ is a piecewise $C^1$ function with the maximum at $M := \varphi(0) = 1$.

The Green function $\varphi$ determines also the travelling periodic wave solution $u(t,x) = c\varphi(x - ct)$ to the $b$-CH equation (1.6), where $c$ is the wave speed. For convenience, we set $c = 1$ in all formulas. Before stating our instability results for the $b$-CH equation precisely, we give the definition of orbital stability in energy space $X = H^1(\mathbb{R})$ where the peaked waves of the $b$-CH equation are defined.

**Definition 2.1.** *We say that $\varphi$ is orbitally stable in energy space $X$ if for every $\varepsilon > 0$, there exist $\nu > 0$ such that for every $u_0 \in X$ with*

$$\|u_0 - \varphi\|_X < \nu, \tag{2.3}$$

*the unique global solution $u \in C(\mathbb{R}, X)$ to the initial-value problem for the b-CH equation (1.6) with $u(0,\cdot) = u_0$ satisfies*

$$\inf_{\xi \in \mathbb{R}} \|u(t,\cdot) - \varphi(\cdot - \xi)\|_X < \varepsilon, \ \forall t > 0 \tag{2.4}$$

The above definition of orbital stability states that any solution starting close to peakons $\varphi$ remains close to some translate of $\varphi$ in the norm $\|\cdot\|_X$, at any later time. For the main result stated here, we assume orbital stability of the peakon $\varphi$.

**Assumption 2.1.** *The peakon $\varphi$ is orbitally stable in $H^1(\mathbb{R})$.*

**Remark 2.1.** *Assumption 2.1 was proven for the CH equation ($b = 2$) by A. Constantin and W.A. Strauss [14] with the precise choice $\nu := (\frac{\varepsilon}{3})^4$.*

**Remark 2.2.** *In the case of the DP equation ($b = 3$), it was shown by Zh.Lin and Y.Liu [41] that the orbital stability of peakons holds in $L^2(\mathbb{R})$ with a choice of $\nu := c\varepsilon^{1/4}$ for some positive constant $c > 0$. The difference from the orbital stability in $H^1(\mathbb{R})$ is due to the conserved quantities in (1.8), which do not allow to control the peakon's perturbation in $H^1(\mathbb{R})$.*

**Remark 2.3.** *Assumption 2.1 does not hold for $b < 1$ because the peakons are known to be unstable from early numerical results [30, 31] and the recent analytical results in [7].*

The main result of this chapter is to show that the $H^1(\mathbb{R})$-orbitally stable peakons are strongly unstable with respect to perturbations in the $W^{1,\infty}(\mathbb{R})$ norm. The following theorem represents the main result.

**Theorem 2.1.** *Fix $b > 1$ and assume Assumption 2.1. For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H^1(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ satisfying*

$$\|u_0 - \varphi\|_{H^1} + \|u_0' - \varphi'\|_{L^\infty} < \delta, \tag{2.5}$$

*such that the local solution $u \in C([0, T), H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}))$ to the b-CH equation (1.6) with $u(0, \cdot) = u_0$ and $T > t_0$ satisfies*

$$\|u_x(t_0, \cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty} > 1, \tag{2.6}$$

*where $\xi(t) \in \mathbb{R}$ is the maximum point of peak of $u(t, \cdot)$. Moreover, there exist $u_0$ such that the maximal existence time $T$ is finite.*

Since Assumption 2.1 may not hold in $H^1(\mathbb{R})$ for $b = 3$ since orbital stability of peakons in the DP equation was proven in $L^2(\mathbb{R})$ only [41], we give a modified version of Theorem 2.1 in the case $b = 3$.

**Theorem 2.2.** *Fix $b = 3$. For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ satisfying*

$$\|u_0 - \varphi\|_{L^2 \cap L^\infty} + \|u_0' - \varphi'\|_{L^\infty} < \delta, \tag{2.7}$$

*such that the local solution $u \in C([0,T), L^2(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}))$ to the b-CH equation (1.6) with $u(0,\cdot) = u_0$ and $T > t_0$ satisfies*

$$\|u_x(t_0,\cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty} > 1, \tag{2.8}$$

*where $\xi(t) \in \mathbb{R}$ is the maximum point of peak of $u(t,\cdot)$. Moreover, there exist $u_0$ such that the maximal existence time $T$ is finite.*

The proof of Theorems 2.1 and 2.2 is based on several recent developments. A similar theorem for peakons of the CH equation ($b = 2$) was proven in [50]. Analogous study was performed for peakons in a different model (the Novikov equation) in [8], where the local well-posedness result of the initial-value problem in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ was not previously available.

In the context of the peaked periodic waves, linear instability of peaked periodic waves was obtained for a different model (the reduced Ostrovsky equation) in [21]. More recently, spectral instability for perturbations in $L^2(\mathbb{T})$ was obtained in [22] for the reduced Ostrovsky equation with either quadratic or cubic nonlinearities. Neither local well-posedness nor the nonlinear instability was considered for the reduced Ostrovsky equation in [21, 22]. Finally, our work in [44] provided the proof of instability of peaked periodic waves in the CH equation ($b = 2$).

## 2.2 Preliminary results

Let us consider the following Cauchy problem of the $b$-CH equation

$$\begin{cases} u_t + uu_x + Q[u](x) = 0 \\ u|_{t=0} = u_0 \end{cases} \tag{2.9}$$

where $Q[u](x)$ is defined as follows

$$Q[u](x) := \frac{1}{2}\int_{\mathbb{R}} \varphi'(x-y) q[u](y) dy, \quad q[u] := \frac{b}{2}u^2 + \frac{3-b}{2}(u')^2, \quad x \in \mathbb{R}. \tag{2.10}$$

The following lemma describes properties of $Q[u]$ depending on the class of functions for $u$.

**Lemma 2.1.** *If $u \in H^1(\mathbb{R})$, then $Q[u] \in C^0(\mathbb{R})$. If in addition, $u \in W^{1,\infty}(\mathbb{R})$, then $Q[u]$ is Lipschitz on $\mathbb{R}$.*

*Proof.* The integration in (2.10) can be split as a sum of two terms:

$$Q[u](x) = \frac{1}{2}\left[\int_{-\infty}^{x} e^{x} q[u](y)dy - \int_{x}^{\infty} e^{-x} q[u](y)dy\right],$$

Since $q[u]$ is absolutely integrable if $u \in H^1(\mathbb{R})$, each integral is continuous on $\mathbb{R}$. If in addition, $q[u]$ is bounded, then $Q[u]$ is Lipschitz on $\mathbb{R}$. $\square$

**Definition 2.2.** *We say that $u \in C([0,T), H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}))$ is a weak solution to the initial-value problem (2.9) for some maximal existence time $T > 0$ if*

$$\int_0^T \int_{\mathbb{R}} \left( u\psi_t + \frac{1}{2}u^2\psi_x - Q[u]\psi \right) dxdt + \int_{\mathbb{R}} u_0(x)\psi(0,x)dx = 0 \tag{2.11}$$

*is satisfied for every test function $\psi \in C^1([0,T] \times \mathbb{R})$ such that $\psi(T,\cdot) = 0$.*

We consider the class of peaked wave solutions with a single peak on $\mathbb{R}$ placed at the point $x = \xi(t)$ for every $t \in [0,T)$. Hence we introduce the following notation:

$$C^1_\xi := \{u \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}) : \quad u_x \in C(\mathbb{R}\backslash\{\xi\})\}. \tag{2.12}$$

The following lemma shows that the single peak moves with its local characteristic speed.

**Lemma 2.2.** *Assume that $u$ is a weak solution to the Cauchy problem (2.9) and there exists $\xi(t) \in \mathbb{R}$ for $t \in [0,T)$ such that $u(t,\cdot) \in C^1_{\xi(t)}$ for $t \in [0,T)$. Then, $\xi(t) \in C^1(0,T)$ satisfies*

$$\frac{d\xi}{dt} = u(t,\xi(t)), \quad t \in (0;T). \tag{2.13}$$

*Proof.* Integrating (2.11) by parts for $x < \xi(t)$ and $x > \xi(t)$ on $\mathbb{R}$ and using the fact that $u(t,\cdot) \in C^0(\mathbb{R})$ and $u(t,\cdot) \in C^1_{\xi(t)}$ for $t \in [0,T)$, we obtain the following equations piecewise outside the peak's location:

$$u_t(t,x) + u(t,x)u_x(t,x) + Q[u](t,x) = 0, \quad \pm[x - \xi(t)] > 0, \quad t \in (0,T). \tag{2.14}$$

By Lemma 2.1, $Q[u]$ is a continuous function of $x$ on $\mathbb{R}$ for $t \in [0,T)$, hence it follows from (2.14) that

$$[u_t]^+_- + u(t,\xi(t))[u_x]^+_- = 0, \quad t \in (0,T), \tag{2.15}$$

where

$$[v]^+_- = \lim_{x\to\xi(t)^+} v(t,x) - \lim_{x\to\xi(t)^-} v(t,x)$$

is the jump of $v$ across the peak location at $x = \xi(t)$. On the other hand, since $u(t,\cdot) \in C^1_{\xi(t)}$ for $t \in [0,T)$, we differentiate $u(t,\xi(t))$ continuously on both sides from $x = \xi(t)$ and define

$$\dot{u}^\pm(t) := \lim_{x\to\xi(t)^\pm} \left[ u_t + \frac{d\xi}{dt}u_x \right], \quad t \in (0,T). \tag{2.16}$$

Since $u(t,\xi(t))$ is continuous for $t \in [0,T)$, we have $\dot{u}^+(t) = \dot{u}^-(t)$ almost everywhere for $t \in (0,T)$. Therefore, it follows from (2.16) that

$$[u_t]_-^+ + \frac{d\xi}{dt}[u_x]_-^+ = 0, \quad \text{a.e. } t \in (0,T). \tag{2.17}$$

Since $[u_x]_-^+ \neq 0$ if $u \notin C^1(\mathbb{R})$, then it follows from (2.15) and (2.17) that $\xi(t)$ satisfies (2.13) almost everywhere for $t \in (0,T)$. Since $u \in C([0,T) \times \mathbb{R})$ due to Sobolev embedding of $H^1(\mathbb{R})$ into $C^0(\mathbb{R})$, then $u(t,\xi(t)) \in C^0(0,T)$, so that equation (2.13) is satisfied everywhere for $t \in (0,T)$ and $\xi \in C^1(0,T)$. □

As a corollary of Lemma 2.2, if $u(t,x) = \varphi(x-ct)$ is the travelling peakon, then $c = \varphi(0) = 1$. The following lemma proves that the Green function $\varphi$ represents the travelling peakon.

**Lemma 2.3.** *Let $\varphi$ be the Green function (2.2). Then, it satisfies the nonlocal equation for the travelling wave solution $u(t,x) = \varphi(x-t)$ of the b-CH equation (1.6):*

$$-\varphi + \frac{1}{2}\varphi^2 + \frac{1}{4}\varphi * (b\varphi^2 + (3-b)(\varphi')^2) = 0 \tag{2.18}$$

*where the nonlocal equation is piecewise $C^1$ on both sides from the peak at $x=0$.*

*Proof.* Substituting $u(t,x) = \varphi(x-t)$ into (1.6) and integrating in $x$ yields the nonlocal equation with the integration constant $d$:

$$-\varphi + \frac{1}{2}\varphi^2 + \frac{1}{4}\varphi * (b\varphi^2 + (3-b)(\varphi')^2) = d.$$

Substituting $(\varphi')^2 = \varphi^2$ for $x \in \mathbb{R}\backslash\{0\}$, which follows from (2.2), yields

$$-\varphi + \frac{1}{2}\varphi^2 + \frac{3}{4}\varphi * \varphi^2 = d$$

In order to verify the validity of this equation and to compute the explicit values of $d$, we consider $x \in (0,\infty)$, for which we can use the expression $\varphi(x) = e^{-x}$. Then,

$$\begin{aligned}\varphi * \varphi^2 &= \int_{-\infty}^0 e^{x+y+2y}dy + \int_0^x e^{-x+y-2y}dy + \int_x^\infty e^{x-y-2y}dy \\ &= \frac{4}{3}e^{-x} - \frac{2}{3}e^{-2x}\end{aligned}$$

so that

$$-\varphi + \frac{1}{2}\varphi^2 + \frac{3}{4}\varphi * \varphi^2 = 0,$$

which shows that $d=0$. Computations for $x \in (-\infty,0)$ are similar. □

By using Lemmas 2.2 and 2.3, we shall now derive the evolution equations for peaked perturbations near the peaked wave. We are looking for a weak solution $u \in C([0,T), H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}))$ to the CH equation in the form (2.11), for which there exists $\xi(t) = t + a(t) \in \mathbb{R}$ for $t \in [0,T)$ such that $u(t,\cdot) \in C^1_{\xi(t)}$ for $t \in [0,T)$. We present the solution in the form:

$$u(t,x) = \varphi(x - t - a(t)) + v(t, x - t - a(t)), \quad t \in [0,T), \; x \in \mathbb{R} \tag{2.19}$$

where $a(t)$ is the deviation of the peak position from its unperturbed position moving with the unit speed and $v(t,x)$ is the perturbation to the peakon $\varphi$. By Lemma 2.2, $a \in C^1(0;T)$ satisfies the equation

$$\frac{da}{dt} = v(t,0), \quad t \in (0,T). \tag{2.20}$$

Substituting (2.19) and (2.20) into the initial-value problem (2.9) yields the following evolution problem for the peaked perturbation $v$:

$$\begin{cases} v_t = (1-\varphi)v_x + (v(0) - v)\varphi' + (v(0) - v)v_x - 2Q[v,\varphi](x) - Q[v](x), \\ v|_{t=0} = v_0 \end{cases} \tag{2.21}$$

where

$$Q[v,\varphi](x) := \frac{1}{4}\int_{\mathbb{R}} \varphi'(x-y)\left(bv\varphi + (3-b)v'\varphi'\right)(y)dy. \tag{2.22}$$

In the derivation of (2.21), we have used the stationary equation (2.18) piecewise on both sides from the peak and replaced $x - t - a(t)$ by $x$ due to the translational invariance of the system (2.9).

## 2.3 Method of characteristics

Here we analyze the initial-value problem (2.21). The local well-posedness theory will be developed through the method of characteristics piecewise for $x \in R$ on both sides of the peak at $x = 0$. Therefore, we define the family of characteristic coordinates $X(t,s)$ satisfying the initial-value problem:

$$\begin{cases} \frac{dX}{dt} = \varphi(X) - \varphi(0) + v(t,X) - v(t,0) \\ X|_{t=0} = s \end{cases} \tag{2.23}$$

Along each characteristic curve parameterized by $s$, let us define $V(t,s) := v(t, X(t,s))$. It follows from (2.21) and (2.23) that $V(t,s)$ on each characteristic curve $x = X(t,s)$ satisfies the initial-value problem:

$$\begin{cases} \frac{dV}{dt} = \varphi'(X)(V(t,0) - V(t,X)) - 2Q[v,\varphi](X) - Q[v](X), \\ V|_{t=0} = v_0(s). \end{cases} \tag{2.24}$$

The following lemma transfers well-posedness theory for differential equations to the existence, uniqueness, and smoothness of the family of characteristic coordinates and the solution surface.

**Lemma 2.4.** *Assume $v_0 \in H^1 \cap C_0^1$. There exists the maximal existence time $T > 0$ (finite or infinite) such that the family of characteristic coordinates $[0,T) \times \mathbb{R} \ni (t,s) \mapsto X \in \mathbb{R}$ for (2.23) and the solution surface $[0,T) \times \mathbb{R} \ni (t,s) \mapsto V \in \mathbb{R}$ to (2.24) exist and are unique as long as $v_x(t,\cdot) \in L^\infty(\mathbb{R})$ for $t \in [0,T)$. Moreover, $X$ and $V$ are $C^1$ in $t$ and $C_0^1$ in $s$ for every $(t,s) \in [0,T) \times \mathbb{R}$.*

*Proof.* A simple extension of the proof of Lemma 2.1 implies that if $f \in H^1 \cap C_0^1$, then $Q[f] \in C_0^1$. Every $C_0^1$ function is Lipschitz continuous at $x = 0$. In addition, since $v(t,\cdot) \in H^1(\mathbb{R})$ for every $t > 0$, then $v(t,\cdot) \in L^\infty(\mathbb{R})$, hence the function $v(t,\cdot)$ is globally Lipschitz continuous when it is locally Lipschitz continuous. Since $\varphi$ have the same properties on $\mathbb{R}$, the right-hand-sides of systems (2.23) and (2.24) are global Lipschitz continuous functions of $X$ as long as the solution $v(t,\cdot) \in H^1$ remains $v(t,\cdot) \in C_0^1$ for $t \in [0,T)$ with some (finite or infinite) $T > 0$. Existence and uniqueness of the classical solutions $X(\cdot,s) \in C^1(0,T)$ and $V(\cdot,s) \in C^1(0,T)$ for every $s \in \mathbb{R}$ follows from the ODE theory. By the continuous dependence theorem, $X(t,\cdot) \in C_0^1$ and $V(t,\cdot) \in C_0^1$ for every $t \in [0,T)$.

Let us now show that $v(t,\cdot) \in C_0^1$ for $t \in [0,T)$ if $v_x(t,\cdot) \in L^\infty(\mathbb{R})$ for $t \in [0,T)$. By differentiating (2.23) piecewise for $s > 0$ and $s < 0$, we obtain

$$\begin{cases} \frac{dX_s}{dt} = \left[\varphi'(X) + v_x(t,X)\right] X_s, \quad t \in (0,T), \\ X_s|_{t=0} = 1. \end{cases} \tag{2.25}$$

with the solution

$$X_s(t,s) = \exp\left( \int_0^t \left[\varphi'(X(t',s)) + v_x(t,X(t',s))\right] dt' \right). \tag{2.26}$$

If $v_x(t,\cdot) \in L^\infty(\mathbb{R})$ for $t \in [0,T)$, then $X_s(t,s) > 0$ for $t \in [0,T)$ piecewise for $s > 0$ and $s < 0$, hence the change of coordinates $(t,s) \to (t,X)$ is a $C_0^1$ invertible transformation. As a result, $V(t,\cdot) \in C_0^1$ implies that $v(t,\cdot) \in C_0^1$ for $t \in [0,T)$. □

**Remark 2.4.** *Since $\varphi(0) = 1$ and $v(t,\cdot) \in C^0(\mathbb{R})$ for every $t \in \mathbb{R}^+$, $X = 0$ is a critical point of the initial-value problem (2.23). Therefore, the unique solution of Lemma 2.4 for $s = 0$ satisfies $X(t,0) = 0$. This limiting characteristic curve separates the family of characteristic curves with $s > 0$ and $s < 0$.*

In order to control $v_x(t,\cdot) \in L^\infty(\mathbb{R})$ for $t \in [0,T)$ needed in the condition of Lemma 2.4, we differentiate (2.21) in $x$ and obtain

$$\begin{cases} v_{xt} = (1-\varphi)v_{xx} + (1-b)\varphi' v_x + b\varphi v + (v|_{x=0} - v)\varphi'' + (v|_{x=0} - v)v_{xx} \\ \qquad + \frac{1}{2}bv^2 + \frac{1}{2}(1-b)v_x^2 - 2P[v,\varphi] - P[v], \\ v_x|_{t=0} = v_0', \end{cases} \tag{2.27}$$

where we have used (2.1) and have defined $P[v]$ and $P[v,\varphi]$ from $Q[v]$ and $Q[v,\varphi]$ by replacing $\varphi'(x-y)$ by $\varphi(x-y)$ in the convolution integrals.

It follows from (2.23) and (2.27) that $U(t,s) := v_x(t,X(t,s))$ on each characteristic curve satisfies the initial-value problem:

$$\left\{ \begin{array}{l} \frac{dU}{dt} = (1-b)\varphi'(X)U + b\varphi(X)V + [V_0 - V]\varphi''(X) + \frac{1}{2}bV^2 + \frac{1}{2}(1-b)U^2 \\ \qquad -2P[v,\varphi](X) - P[v](X), \\ U|_{t=0} = v_0'(s), \end{array} \right. \tag{2.28}$$

where $V_0(t) := V(t,0)$.

The proof of Theorem 2.1 follows from the analysis of the system (2.28) on the right side of the peak at $X=0$.

## 2.4 Proof of instability in Theorem 2.1

Although $\varphi'(X)$ has a jump discontinuity at $X=0$, the regions $\mathbb{R}^+$ and $\mathbb{R}^-$ for $s$ are separated from each other due to the fact that the limiting characteristic curve at $X=0$ corresponds to the critical point of the initial-value problem (2.23). As a result, we consider the initial-value problem (2.28) separately for $s>0$ and $s<0$.

Let $U_0^+(t) := U(t,0^+)$ be defined on the right side of the peak at $X=0$. Taking the limit $s\to 0^+$ in (2.28) yields the differential equation

$$\frac{dU_0^+}{dt} = (b-1)U_0^+ + bV_0 + \frac{1}{2}bV_0^2 + \frac{1}{2}(1-b)(U_0^+)^2 - 2P[v,\varphi](0) - P[v](0). \tag{2.29}$$

Due to the decomposition (2.19), we can rewrite the initial bound (2.5) in the form:

$$\|v_0\|_{H^1} + \|v_0'\|_{L^\infty} < \delta, \tag{2.30}$$

where $\delta > 0$ is arbitrary small parameter. We first show that there exists $t_0 \in (0;T)$ and $v_0 \in C_0^1 \subset H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ such that the unique solution $v \in C^1([0;T), C_0^1)$ to the initial-value problem (2.21) constructed by Lemma 2.4 satisfies

$$\|v_x(t_0,\cdot)\|_{L^\infty} \geq 1. \tag{2.31}$$

Here and in what follows, $T>0$ is the maximal existence time of the unique solution $v \in C^1([0;T), C_0^1)$ to the initial-value problem (2.21) constructed by Lemma 2.4.

By Assumption 2.1, for every small $\varepsilon > 0$, there exists $\nu(\varepsilon) > 0$ such that if $\|v_0\|_{H^1(\mathbb{R})} < \nu(\varepsilon)$, then $\|v(t,\cdot)\|_{H^1(\mathbb{R})} < \varepsilon$ for every $t \in [0,T)$. By Sobolev's embedding of $H^1$ into $L^\infty$, there is a positive constant $C$ such that

$$\|v(t,\cdot)\|_{L^\infty} < C\|v(t,\cdot)\|_{H^1} < C\varepsilon. \tag{2.32}$$

By using the bound (2.32), we can control $V_0(t)$ as $|V_0(t)| \leq C\varepsilon$. In addition, we can control $P[v](0)$ and $P[v,\varphi](0)$ as $|P[v](0)| \leq C_b\varepsilon^2$ and $|P[v,\varphi](0)| \leq C_b\varepsilon$, where $C_b$ is the positive constant that may depend on parameter $b$. Using these bounds in the differential equation (2.29) yields

$$\frac{dU_0^+}{dt} \leq (b-1)U_0^+ + \frac{1}{2}(1-b)(U_0^+)^2 + C_b\varepsilon. \tag{2.33}$$

If $b > 1$ as in Theorem 2.1, we can simplify (2.33) for every $\varepsilon \in (0;1]$ to the form:

$$\frac{dU_0^+}{dt} \leq (b-1)U_0^+ + C_b\varepsilon. \tag{2.34}$$

Let us now assume that the initial data $v_0 \in C_0^1$ satisfies

$$\lim_{x\to 0^+} v_0'(x) = -\|v_0'\|_{L^\infty} = -\frac{2C_b\varepsilon}{b-1}. \tag{2.35}$$

The initial bound (2.30) is consistent with (2.35) if for every small $\delta > 0$, the small value of $\varepsilon$ satisfies

$$\nu(\varepsilon) + \frac{2C_b\varepsilon}{b-1} < \delta,$$

which just specifies small $\varepsilon$ in terms of small $\delta$. By integrating (2.34) and using (2.35), we obtain

$$U_0^+(t) \leq e^{(b-1)t}\left[U_0^+(0) + \frac{C_b\varepsilon}{b-1}\right] = -\frac{C_b\varepsilon}{b-1}e^{(b-1)t}, \quad t \in [0,T). \tag{2.36}$$

Hence, for every small $\varepsilon > 0$ there exists sufficiently large

$$t_1 := -\frac{\log(C_1\varepsilon) - \log(b-1)}{b-1} > 0,$$

such that $U_0^+(t_1) \leq -1$. If $t_1 < T$, the bound (2.31) is true with some $t_0 \in (0, t_1]$ since

$$\|v_x(t,\cdot)\|_{L^\infty} \geq \|U(t,\cdot)\|_{L^\infty(0,\infty)} \geq |U_0^+(t)|, \qquad t \in [0,T). \tag{2.37}$$

If $t_1 \geq T$, then $T$ is the finite maximal existence time in Lemma 2.4 for the solution

$$X(\cdot,s); V(\cdot,s) \in C^1([0,T)); \quad s \in [0,\infty).$$

By (2.26) and Lemma 2.4, it means necessarily that $\lim_{t\to T^-} \|v_x(t,\cdot)\|_{L^\infty(\mathbb{R})} = \infty$, so that there exists $t_0 \in (0,T)$ such that the bound (2.31) is true.

This argument gives the proof of Theorem 2.1. It remains to show that there exists $v_0 \in C_0^1$ such that the maximal existence time $T$ can be finite.

Let $U_0(\epsilon)$ be the negative root of the quadratic equation

$$(b-1)U - \frac{1}{2}(b-1)U^2 + C_b\varepsilon = 0 \tag{2.38}$$

It is clear that $U_0(\varepsilon) \in (-\frac{C_b\varepsilon}{b-1}, 0)$. Assume that the initial data $v_0 \in C_0^1$ satisfies

$$\lim_{x\to 0^+} v_0'(x) = -||v_0'||_{L^\infty} = 2U_0(\varepsilon). \tag{2.39}$$

The initial bound (2.30) is consistent with (2.39) if for every small $\delta > 0$, the small value of $\varepsilon$ satisfies

$$\nu(\varepsilon) + 2|U_0(\varepsilon)| < \delta, \tag{2.40}$$

which again specifies small $\varepsilon$ in terms of small $\delta$. If $U^+(0) = 2U_0(\varepsilon)$, then the differential equation (2.29) for $U^+$ implies that $\frac{d}{dt}U^+(0) < 0$, hence $U^+(t) < U^+(0)$ for small positive $t$ and the map $t \to U(t)$ is monotonically decreasing.

Let $\overline{U}$ be the supersolution that satisfies:

$$\frac{d\overline{U}}{dt} = (b-1)\overline{U} - \frac{1}{2}(b-1)\overline{U}^2 + C_b\varepsilon. \tag{2.41}$$

with $\overline{U}(0) = 2U_0(\varepsilon)$. It follows by the comparison theory for differential equations that $U^+(t) \leq \overline{U}(t)$ for every $t > 0$ for which $U^+(t)$ exists. Since there exists a finite $\overline{T} > 0$ such that $\overline{U}(t) \to -\infty$ as $t \to \overline{T}^-$, then there exists $T \in (0, \overline{T}]$ such that $U^+(t) \to -\infty$ as $t \to T^-$. This shows that there exists $v_0 \in C_0^1$ such that the maximal existence time $T$ can be finite.

## 2.5 Proof of instability in Theorem 2.2

In the case of the DP equation ($b = 3$), orbital stability of peakons was proven in $L^2(\mathbb{R})$ [41]. It is unlikely that Assumption 2.1 holds in $H^1(\mathbb{R})$.

We will show here that the proof of instability can still be achieved by using a weaker control of the peaked perturbation to the peakon. In particular, $V_0(t)$ can no longer be controlled by the bound (2.32). On the other hand, there is no need to control the $H^1$ norm of the perturbation $v$ because the nonlocal term in $Q[v]$ in (2.10) does not contain $v_x$ terms if $b = 3$.

Due to the decomposition (2.19), we can rewrite the initial bound (2.7) in the form:

$$||v_0||_{L^2\cap L^\infty} + \|v_0'\|_{L^\infty} < \delta, \tag{2.42}$$

where $\delta > 0$ is arbitrary small parameter.

In order to control $V_0(t)$, we rewrite the differential equation (2.24) for $b = 3$ at $X = 0$ in the form

$$\frac{dV_0}{dt} = \frac{3}{2}\int_{-\infty}^{\infty} \varphi'(y)\varphi(y)v(t,y)dy + \frac{3}{4}\int_{-\infty}^{\infty} \varphi'(y)v(t,y)^2 dy, \tag{2.43}$$

where $\varphi'(-y) = -\varphi'(y)$ has been used. Similarly, we rewrite the differential equation (2.29) for $b = 3$ in the form:

$$\frac{dU_0^+}{dt} = 2U_0^+ + 3V_0 + \frac{3}{2}V_0^2 - (U_0^+)^2 - \frac{3}{2}\int_{-\infty}^{\infty} \varphi(y)^2 v(t,y)dy - \frac{3}{4}\int_{-\infty}^{\infty} \varphi(y)v(t,y)^2 dy, \tag{2.44}$$

where we have used $\varphi(-y) = \varphi(y)$.

Recall from [41] that for every small $\varepsilon > 0$, there exists $\nu(\varepsilon) > 0$ such that

$$\text{if } \|v_0\|_{L^2\cap L^\infty} < \nu(\varepsilon), \text{ then } \|v(t,\cdot)\|_{L^2} < \varepsilon \text{ for every } t \in [0,T).$$

By using Cauchy-Schwartz inequality, we obtain the following bounds:

$$\left|\int_{\mathbb{R}} \varphi'\varphi dy\right| \leq C\|v\|_{L^2}, \quad \left|\int_{\mathbb{R}} \varphi' v^2 dy\right| \leq C\|v\|_{L^2}^2$$

so that it follows from (2.43) that $|V_0'(t)| \leq C_1\varepsilon$ for some $\varepsilon$-independent constant $C_1 > 0$. Combining with the initial bound (2.30) yields at most the linear growth of $V_0(t)$:

$$|V_0(t)| \leq \nu(\varepsilon) + C_1\varepsilon t. \tag{2.45}$$

Similarly, it follows from (2.44) that

$$\frac{dU_0^+}{dt} \leq 2U_0^+ + C_2\varepsilon + 3|V_0(t)| + \frac{3}{2}V_0(t)^2, \tag{2.46}$$

for another $\varepsilon$-independent constant $C_2 > 0$, which yields

$$\frac{d}{dt}(e^{-2t}U_0^+) \leq \left(C_2\varepsilon + 3|V_0(t)| + \frac{3}{2}V_0(t)^2\right)e^{-2t}. \tag{2.47}$$

Integrating (2.47) yields

$$U_0^+(t) \leq (U_0^+(0) + C_3\varepsilon + C_3\nu(\varepsilon))e^{2t} \tag{2.48}$$

for another $\varepsilon$-independent constant $C_2 > 0$, due to the following simple bounds:

$$\int_0^t te^{-2t}dt = \frac{1}{4}(1 - e^{-2t} - 2te^{-2t}) \leq \frac{1}{4}$$

and

$$\int_0^t t^2 e^{-2t} dt = \frac{1}{4}(1 - e^{-2t} - 2te^{-2t} - 2t^2 e^{-2t}) \leq \frac{1}{4}.$$

The rest of the proof of instability holds if $U_0^+(0) = -2C_3(\varepsilon + \nu(\varepsilon))$, which is consistent with the initial bound (2.42) if

$$(1 + 2C_3)\nu(\varepsilon) + 2C_3\varepsilon < \delta,$$

which specifies $\varepsilon$ in terms of $\delta$. Substituting $U_0^+(0)$ into (2.48) yields

$$|U_0^+(t)| \geq (C_3\varepsilon + C_3\nu(\varepsilon))e^{2t} \geq 1,$$

for sufficiently large $t \geq t_0 := \frac{1}{2}|\log(\varepsilon + \nu(\epsilon)) + \log(C_3)|$.

It remains to prove that there is $v_0$ satisfying the bound (2.30) such that the maximal existence time of the local solution $v$ is finite. We would then consider the differential equation (2.44) with the quadratic term in $U_0^+$ and estimate it with the bound (2.45) as follows:

$$\frac{dU_0^+}{dt} \leq 2U_0^+ - (U_0^+)^2 + C_2\left(\nu(\varepsilon) + \varepsilon t + \varepsilon^2 t^2\right), \tag{2.49}$$

where $C_2 > 0$ is $\varepsilon$-independent and we have used the fact from [41] that $\nu(\varepsilon) = c\varepsilon^{1/4} \gg \varepsilon$.

Let us define

$$A(\varepsilon) := C_2 \max_{t \in [0, t_0(\varepsilon)]} \left[\nu(\varepsilon) + \varepsilon t + \varepsilon^2 t^2\right] \tag{2.50}$$

for some $\varepsilon$-dependent value of $t_0(\varepsilon)$ to be determined. The supersolution $\overline{U}$ satisfies the differential equation:

$$\frac{d\overline{U}}{dt} = 2\overline{U} - \overline{U}^2 + A(\varepsilon). \tag{2.51}$$

It follows by the comparison theory for differential equations that if $U_0^+(0) = \overline{U}(0)$, then $U_0^+(t) \leq \overline{U}(t)$ for every $t > 0$ for which $U_0^+(t)$ exists. If there exists a finite $\overline{T} \in (0, t_0(\varepsilon))$ such that $\overline{U}(t) \to -\infty$ as $t \to \overline{T}$, then there exists $T \in (0, \overline{T}]$ such that $U_0^+(t) \to -\infty$ as $t \to \overline{T}$, which implies the finite-time blow-up.

It remains to show that there exist an initial condition $\overline{U}(0)$ and the finite $\overline{T} \in (0, t_0(\varepsilon))$ such that $\overline{U}(t) \to -\infty$ as $t \to \overline{T}$ in the solution of the differential equation (2.51). We drop the dependence of $A$ from $\varepsilon$ for simplicity. The exact solution is given in the implicit form by the separation of variables:

$$t = \int_{\overline{U}(0)}^{\overline{U}} \frac{dx}{(-x^2 + 2x + A)}.$$

Evaluating the integral gives

$$\begin{aligned} t &= -\frac{1}{2\sqrt{1+A}} \int_{\overline{U(0)}}^{\overline{U}} \left( \frac{1}{(x-1-\sqrt{1+A})} - \frac{1}{(x-1+\sqrt{1+A})} \right) dx \\ &= -\frac{1}{2\sqrt{1+A}} \log \left( \frac{(\overline{U}-1-\sqrt{1+A})(\overline{U(0)}-1+\sqrt{1+A})}{(\overline{U(0)}-1-\sqrt{1+A})(\overline{U}-1+\sqrt{1+A})} \right), \end{aligned}$$

from which we derive

$$\overline{U}(t) = \frac{(1+\sqrt{1+A})(\overline{U}(0)-1+\sqrt{1+A}) - (1-\sqrt{1+A})(\overline{U(0)}-1-\sqrt{1+A})e^{-2\sqrt{1+A}t}}{(\overline{U}(0)-1+\sqrt{1+A}) - (\overline{U}(0)-1-\sqrt{1+A})e^{-2\sqrt{1+A}t}}$$

The solution $\overline{U}(t)$ blows up at $t \to \overline{T}$, for which the denominator vanishes. This happens at

$$\overline{T} = -\frac{1}{2\sqrt{1+A}} \log \left( \frac{\overline{U}(0)-1+\sqrt{1+A}}{\overline{U}(0)-1-\sqrt{1+A}} \right)$$

Let us fix $\overline{U}(0) = -A$ and assume that the value of $A$ is small, which means that $A(\varepsilon) \to 0$ as $\varepsilon \to 0$. Expanding $\sqrt{1+A}$ and simplying the expressions yield

$$\overline{T} = -\frac{1}{2} \log \left( \frac{A}{4} \right) + \mathcal{O}(A|\log A|). \tag{2.52}$$

If we pick $t_0(\varepsilon) = |\log\left(\frac{A}{4}\right)|$, then $\overline{T} \in (0, t_0(\varepsilon))$. The consistency with the definition (2.50) implies that $A(\varepsilon) = 2C_2\nu(\varepsilon)$ because $\nu(\varepsilon) = c\varepsilon^{1/4} \gg \varepsilon$ implies that

$$\nu(\varepsilon) \gg \varepsilon t_0(\varepsilon) = \varepsilon \left| \log \left( \frac{C_2\nu(\varepsilon)}{2} \right) \right|.$$

Since $U_0^+(0) = \overline{U}(0) = -A(\varepsilon) = -2C_2\nu(\varepsilon)$, the consistency with the initial bound (2.42) is satisfied if

$$(1+2C_2)\nu(\varepsilon) < \delta,$$

which again specifies $\varepsilon$ in terms of $\delta$. Hence, all bounds are consistent and by the comparison theory, there exists $T \in (0, \overline{T}]$ such that $U_0^+(t) \to -\infty$ as $t \to \overline{T}$.

# Chapter 3

# Peaked periodic waves in CH equation

## 3.1 Main result

We consider the Camassa-Holm (CH) equation given by

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{3.1}$$

in the setting of $\mathbb{T} := [-\pi, \pi]$ subject to the periodic boundary conditions at $\pm\pi$. The CH equation (3.1) can be rewritten in the convolution form

$$u_t + uu_x + \frac{1}{2}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right) = 0, \tag{3.2}$$

where $\varphi'$ denotes piecewise continuous derivative of $\varphi$ in $x$.

The Green function $\varphi$ can be expressed explicitly in the form

$$\varphi(x) = \frac{\cosh(\pi - |x|)}{\sinh(\pi)}, \quad x \in \mathbb{T}, \tag{3.3}$$

which shows that $\varphi$ is a piecewise $C^1$ function with the maximum at $M := \varphi(0) = \coth(\pi)$ and the minima at $m := \varphi(\pm\pi) = \operatorname{csch}(\pi)$. The central peak of $\varphi$ is located at $x = 0$ with $\varphi'(0^\pm) = \mp 1$, from which $\varphi$ is monotonically decreasing towards the turning points at $x = \pm\pi$ where $\varphi'(\pm\pi) = 0$. The graph of $\varphi$ versus $x$ is shown in the Figure 3.1.

The initial-value problem for the periodic CH equation (3.1) is locally well-posed in the space $H^3_{per}(\mathbb{T})$ [11], $H^s_{per}(\mathbb{T})$ with $s > 3/2$ [16, 28], $C^1_{per}(\mathbb{T})$ [46], and $H^1_{per}(\mathbb{T}) \cap Lip(\mathbb{T})$ [17], where $Lip(\mathbb{T})$ stands for Lipshitz continuous functions. Peaked periodic waves and their perturbations can be considered in the space of lower regularity $H^1_{per}(\mathbb{T}) \cap Lip(\mathbb{T})$.

Assumption 2.1 was proven in [38, 39] as the following theorem.

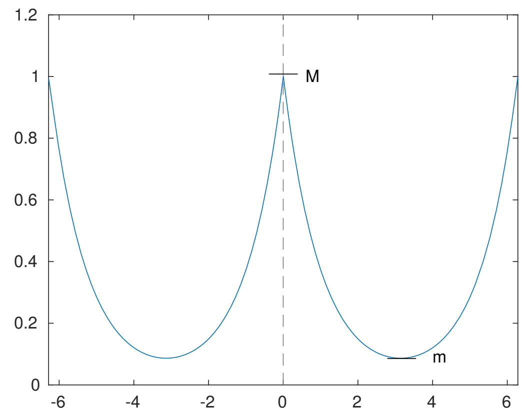


Figure 3.1: The graph of $\varphi$ on $[-2\pi, 2\pi]$ with maxima at $M$ and minima at $m$.

**Theorem 3.1.** *[38, 39] For every $\varepsilon > 0$, there is a $\nu > 0$ such that if $u \in C([0,T), H^1_{\text{per}}(\mathbb{T}))$ is a solution to the CH equation (3.2) with the initial data $u_0$ satisfying*

$$\|u_0 - \varphi\|_{H^1(\mathbb{T})} < \nu, \tag{3.4}$$

*then*

$$\|u(t,\cdot) - \varphi(\cdot - \xi(t))\|_{H^1(\mathbb{T})} < \varepsilon, \quad t \in [0,T), \tag{3.5}$$

*where $\xi(t) \in \mathbb{T}$ is a point where the function $u(t,\cdot)$ attains its maximum on $\mathbb{T}$ and $T > 0$ is either finite or infinite.*

The following theorem reinstates Theorem 2.1 in the periodic domain.

**Theorem 3.2.** *For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ satisfying*

$$\|u_0 - \varphi\|_{H^1(\mathbb{T})} + \|u_0' - \varphi'\|_{L^\infty(\mathbb{T})} < \delta, \tag{3.6}$$

*such that the local solution $u \in C([0,T), H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T}))$ to the CH equation (3.2) with the initial data $u_0$ and $T > t_0$ satisfies*

$$\|u_x(t_0,\cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty(\mathbb{T})} \geq 1, \tag{3.7}$$

*where $\xi(t) \in \mathbb{T}$ is a point of peak of the function $u(t,\cdot)$ on $\mathbb{T}$. Moreover, there exist $u_0$ such that the maximal existence time $T$ is finite.*

In order to prove Theorem 3.2, we will revise the approach of Chapter 2 by simplifying the nonlocal linear term in the initial-value problem (2.27) for the peaked perturbation to the peaked periodic wave and by incorporating the mean term of the periodic perturbation. For $b = 2$, the evolution equation in (2.27) can be rewritten in the form:

$$v_t = (c - \varphi)v_x + (v(0) - v)\varphi' + (v(0) - v)v_x - \varphi' * \left(\varphi v + \frac{1}{2}\varphi' v_x\right) - Q[v], \tag{3.8}$$

where $c = M$ is the speed of the peaked periodic wave $\varphi$ and

$$Q[v](x) := \frac{1}{2}\int_{\mathbb{T}} \varphi'(x-y)\left[v^2 + \frac{1}{2}v_x^2\right](y)dy. \tag{3.9}$$

Section 3.2 gives the proof of Theorem 3.2. Section 3.3 gives additional results on explicit solutions of the linearization of the evolution equation (3.8).

## 3.2 Proof of instability

### 3.2.1 Preliminary results

We begin with simplifying the linear term in the evolution equation (3.8) by using the following elementary result.

**Lemma 3.1.** *Assume that $v \in H^1_{\rm per}(\mathbb{T})$. Then, for every $x \in \mathbb{T}$*

$$[v(0)-v(x)]\varphi'(x)-(\varphi' * \varphi v)(x)-\frac{1}{2}(\varphi' * \varphi' v_x)(x) = \varphi(x)\int_0^x v(y)dy - \frac{1}{2}m^2\sinh(x)\int_{-\pi}^{\pi} v(y)dy.$$

*Proof.* Since integrals of absolutely integrable functions are continuous, the map

$$x \mapsto \varphi' * \left(\varphi v + \frac{1}{2}\varphi' v_x\right)$$

is continuous for every $x \in \mathbb{T}$. Now, $H^1_{\rm per}(\mathbb{T})$ is continuously embedded into the space of continuous and periodic functions on $\mathbb{T}$, hence $v \in C^0_{\rm per}(\mathbb{T})$. Integrating by parts yields the following explicit expression for every $x \in \mathbb{T}$:

$$\begin{aligned} (\varphi' * \varphi' v_x)(x) &= (\varphi'' * \varphi' v)(x) - (\varphi' * \varphi'' v)(x) \\ &= (\varphi * \varphi' v)(x) - 2\varphi'(x)v(x) - (\varphi' * \varphi v)(x) + 2\varphi'(x)v(0), \end{aligned}$$

which yields

$$[v(0) - v(x)]\varphi'(x) - (\varphi' * \varphi v)(x) - \frac{1}{2}(\varphi' * \varphi' v_x)(x) = -\frac{1}{2}(\varphi' * \varphi v)(x) - \frac{1}{2}(\varphi * \varphi' v)(x).$$

Furthermore, we obtain for $x \in (0, \pi]$:

$$\begin{aligned}
& -\frac{1}{2}\left(\varphi * \varphi' v\right)(x) - \frac{1}{2}\left(\varphi' * \varphi v\right)(x) \\
= \quad & -\frac{m^2}{2}\left[\int_{-\pi}^{0} \sinh(x) v(y) dy - \int_{0}^{x} \sinh(2\pi - x) v(y) dy + \int_{x}^{\pi} \sinh(x) v(y) dy\right] \\
= \quad & \varphi(x) \int_{0}^{x} v(y) dy - \frac{m^2}{2} \sinh(x) \int_{-\pi}^{\pi} v(y) dy,
\end{aligned}$$

which completes the proof of the equality for $x \in (0, \pi]$. For $x \in [-\pi, 0)$, the computations are similar:

$$\begin{aligned}
& -\frac{1}{2}\left(\varphi * \varphi' v\right)(x) - \frac{1}{2}\left(\varphi' * \varphi v\right)(x) \\
= \quad & -\frac{m^2}{2}\left[\int_{-\pi}^{x} \sinh(x) v(y) dy + \int_{x}^{0} \sinh(2\pi + x) v(y) dy + \int_{0}^{\pi} \sinh(x) v(y) dy\right] \\
= \quad & \varphi(x) \int_{0}^{x} v(y) dy - \frac{m^2}{2} \sinh(x) \int_{-\pi}^{\pi} v(y) dy.
\end{aligned}$$

The zero value at $x = 0$ is recovered by taking the one-sided limits $x \to 0^{\pm}$ in the previous two expressions. □

By using Lemma 3.1, we simplify the linear terms in the evolution equation (3.8) and write the initial-value problem for $v$ in the form:

$$\left\{ \begin{array}{l} v_t = (c - \varphi) v_x + \varphi w - \pi m^2 \bar{v} \sinh(x) + (v|_{x=0} - v) v_x - Q[v], \\ v|_{t=0} = v_0, \end{array} \right. \tag{3.10}$$

where $w$ and $\bar{v}$ are defined by

$$w(t,x) := \int_0^x v(t,y) dy, \quad \bar{v}(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} v(t,y) dy = \frac{w(t,\pi) - w(t,-\pi)}{2\pi}. \tag{3.11}$$

In order to close the evolution problem (3.10), we also derive the following initial-value problem for $w$:

$$\left\{ \begin{array}{l} w_t = (c - \varphi) w_x + \varphi' w - \pi m^2 \bar{v} [\cosh(x) - 1] - \frac{1}{2}(v|_{x=0} - v)^2 - P[v] + P[v]|_{x=0}, \\ w|_{t=0} = w_0, \end{array} \right. \tag{3.12}$$

where

$$P[v](x) := \frac{1}{2} \int_{\mathbb{T}} \varphi(x-y) \left[v^2 + \frac{1}{2} v_x^2\right](y) dy. \tag{3.13}$$

Indeed, substituting $v = w_x$ into the evolution equation in (3.10) yields

$$w_{tx} = (c - \varphi)w_{xx} + \varphi w - \pi m^2 \bar{v} \sinh(x) + (v|_{x=0} - w_x)w_{xx} - Q[v].$$

By integrating this evolution equation in $x$ and picking the constant of integration from the boundary conditions $w(t, 0) = 0$, we obtain the evolution equation in (3.12).

Next, we prove that $\bar{v}$ is independent of $t$. At the same time, $v|_{x=0}$ does depend on $t$ in the nonlinear evolution of (3.10) due to the nonzero contribution of $Q[v](0)$.

**Lemma 3.2.** *Assume that there exists a solution $v \in C([0, T), C_0^1)$ to the initial-value problem (3.10) Then, $\bar{v}(t) = \bar{v}_0$ for every $t \in [0, T)$.*

*Proof.* We integrate the evolution equation in the system (3.10) for the solution $v \in C([0, T), C_0^1)$ in $x$ on $\mathbb{T}$ and use cancellation of the linear terms in $v$. Then, it is true that $\frac{d}{dt}\bar{v}(t) = 0$ if and only if

$$\int_{\mathbb{T}} Q[v](t, x)dx = \frac{1}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} \varphi'(x - y)q[v](t, y)dydx = 0, \tag{3.14}$$

where both $\varphi'$ and $q[v]$ are absolutely integrable. Interchanging the integrations by Fubini's theorem and integrating $\varphi'$ piecewise on both sides of the peak yields for every $t \in [0, T)$:

$$\begin{aligned} \int_{\mathbb{T}} \int_{\mathbb{T}} \varphi'(x - y)q[v](t, y)dydx &= \int_{\mathbb{T}} q[v](t, y) \left( \int_{\mathbb{T}} \varphi'(x - y)dx \right) dy \\ &= \int_{\mathbb{T}} q[v](t, y) \left( \varphi(\pi - y) - \varphi(-\pi - y) \right) dy \\ &= 0, \end{aligned}$$

due to periodicity of $\varphi \in C^0_{\text{per}}(\mathbb{T})$. Hence, $\bar{v}(t) = \bar{v_0}$ for every $t \in [0, T)$. □

Finally, we derive the evolution equation for $u := v_x$ by differentiating the evolution equation for $v$ in $x$:

$$v_{tx} = (c - \varphi)v_{xx} + \varphi'(w - v_x) + \varphi v - \pi m^2 \bar{v} \cosh(x) + (v|_{x=0} - v)v_{xx} + v^2 - \frac{1}{2}(v_x)^2 - P[v], \tag{3.15}$$

where we have used the relation

$$\partial_x Q[v] = P[v] - v^2 - \frac{1}{2} v_x^2.$$

Hence, we obtain

$$\begin{cases} u_t = (c - \varphi)u_x + \varphi v + \varphi'(w - u) - \pi m^2 \bar{v} \cosh(x) + (v|_{x=0} - v)u_x + v^2 - \frac{1}{2}u^2 - P[v], \\ v|_{t=0} = v_0, \end{cases} \tag{3.16}$$

### 3.2.2 Method of characteristics

We can now develop local well-posedness theory of the initial-value problem (3.10) by means of the method of characteristics. The family of characteristic curves $x = X(t,s)$ satisfies the initial-value problem:

$$\begin{cases} \frac{dX}{dt} = \varphi(X) - \varphi(0) + v(t,X) - v(t,0), \\ X|_{t=0} = s, \end{cases} \tag{3.17}$$

for every $s \in \mathbb{T}$. Assuming that $v(t,\cdot) \in C_0^1$ for every $t \in [0,T)$, we can differentiate (3.17) piecewise in $s \in \mathbb{T}\backslash\{0\}$ and obtain

$$\begin{cases} \frac{d}{dt}\frac{\partial X}{\partial s} = [\varphi'(X) + v_x(t,X)]\,\frac{\partial X}{\partial s}, \\ \frac{\partial X}{\partial s}|_{t=0} = 1. \end{cases} \tag{3.18}$$

with the exact solution

$$\frac{\partial X}{\partial s}(t,s) = \exp\left(\int_0^t [\varphi'(X(t',s)) + v_x(t,X(t',s))]\,dt'\right). \tag{3.19}$$

The peak's locations at $X(t,0) = 0$ and $X(t,2\pi) = 2\pi$ are invariant in the time evolution if $v(t,\cdot) \in C_0^1$ for every $t \in [0,T)$.

Along each characteristic curve $x = X(t,s)$ satisfying (3.17), functions $V(t,s) := v(t,X(t,s))$, $W(t,s) = w(t,X(t,s))$, and $U(t,s) = v_x(t,X(t,s))$ satisfy the following initial-value problems:

$$\begin{cases} \frac{dV}{dt} = \varphi(X)W - \pi m^2\bar{v}\sinh(X) - Q[v](X), \\ V|_{t=0} = v_0(s), \end{cases} \tag{3.20}$$

$$\begin{cases} \frac{dW}{dt} = \varphi'(X)W - \pi m^2\bar{v}[\cosh(X) - 1] + \frac{1}{2}[V^2 - (v|_{X=0})^2] - P[v](X) + P[v](0), \\ W|_{t=0} = w_0(s), \end{cases} \tag{3.21}$$

and

$$\begin{cases} \frac{dU}{dt} = \varphi'(X)[W - U] + \varphi(X)V - \pi m^2\bar{v}\cosh(X) - \frac{1}{2}U^2 + V^2 - P[v](X), \\ U|_{t=0} = v_0'(s). \end{cases} \tag{3.22}$$

The following lemma transfers the local well-posedness theory for differential equations to the initial-value problem (3.10).

**Lemma 3.3.** *For every $v_0 \in C_0^1$, there exists the maximal existence time $T > 0$ (finite or infinite) and the unique solution $v \in C^1([0,T), C_0^1)$ to the initial-value problem (3.10) that depends continuously on $v_0 \in C_0^1$.*

*Proof.* If $v \in C^1_0$, then $Q[v] \in C^0_{\rm per}(\mathbb{T}) \cap {\rm Lip}(\mathbb{T})$ and $P[v] \in C^1_{\rm per}(\mathbb{T})$. Therefore, the nonlocal parts of the initial-value problems (3.20), (3.21), and (3.22) are well-defined and can be considered for $X \in [0, 2\pi]$ that corresponds to $s \in [0, 2\pi]$.

For $s \in [0, 2\pi]$, we rewrite the evolution equations in (3.17), (3.20), (3.21), and (3.22) as the dynamical system

$$\frac{d}{dt}\begin{bmatrix} X \\ V \\ W \\ U \end{bmatrix} = \begin{bmatrix} \varphi(X) - \varphi(0) + V - V|_{s=0} \\ \varphi(X)W - \pi m^2 \bar{v}\sinh(X) - Q[v](X) \\ \varphi'(X)W - \pi m^2 \bar{v}[\cosh(X) - 1] + \frac{1}{2}[V^2 - (V|_{s=0})^2] - P[v](X) + P[v]|_{s=0} \\ \varphi'(X)[W - U] + \varphi(X)V - \pi m^2 \bar{v}\cosh(X) - \frac{1}{2}U^2 + V^2 - P[v](X) \end{bmatrix}$$

subject to the initial condition

$$\begin{bmatrix} X \\ V \\ W \\ U \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} s \\ v_0(s) \\ w_0(s) \\ v_0'(s) \end{bmatrix}$$

and the boundary conditions

$$\begin{cases} X(t,0) = 0, & X(t,2\pi) = 2\pi, \\ V(t,0) = V|_{s=0}, & V(t,2\pi) = V|_{s=0}, \\ W(t,0) = 0, & W(t,2\pi) = 2\pi\bar{v}, \end{cases}$$

where $V|_{s=0}$ satisfies

$$\frac{d}{dt}V\Bigg|_{s=0} = -Q[v](0).$$

Since the vector field of the dynamical system is $C^1$ in $(X, V, W, U)$ on $[0, 2\pi] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, there exists a unique local solution $X(\cdot, s), V(\cdot, s), W(\cdot, s), U(\cdot, s) \in C^1([0, T))$ to the initial-value problem for some maximal existence time $T > 0$. The solution depends continuously on the initial data for every $s \in [0, 2\pi]$. Moreover, since the initial data is $C^1(0, 2\pi)$, then $X(t, \cdot), V(t, \cdot), W(t, \cdot), U(t, \cdot) \in C^1(0, 2\pi)$ for every $t \in [0, T)$. The invertibility of the transformation $[0, 2\pi] \ni s \mapsto X(t, s) \in [0, 2\pi]$ is guaranteed along the solution by

$$\frac{\partial X}{\partial s}(t,s) = \exp\left(\int_0^t [\varphi'(X(t',s)) + U(t,s)]\, dt'\right) > 0. \tag{3.23}$$

which follows from (3.19). Since 0 and $2\pi$ are equilibrium points of (3.17), we have $X(t, 0) = 0$ and $X(t, 2\pi) = 2\pi$. Boundary conditions are preserved along with the solution due to consistence of (3.21) and (3.22) with the main equation (3.20). Due to the boundary conditions, the solution $V(t, \cdot) \in C^1(0, 2\pi)$ is extended to $V(t, \cdot) \in C^1_0$ on $\mathbb{T}$. Due to invertibility of the transformation $[0, 2\pi] \ni s \mapsto X(t, s) \in [0, 2\pi]$, we have $v(t, \cdot) \in C^1_0$ for $t \in [0, T)$ and moreover, $v \in C^1([0, T), C^1_0)$. □

### 3.2.3 Dynamics of peaked perturbations at the peak

The proof of Theorem 3.2 relies on the study of the evolution of $v_x \in C^1([0,T), C^0(\mathbb{T}\backslash\{0\}))$ at the right side of the peak. By Lemma 3.3, we are allowed to define the one-sided limits $U^\pm(t) := \lim_{s\to 0^\pm} U(t,s)$ for $t \in [0,T)$, where $U(t,0^-) = U(t,2\pi^-)$. The functions $U^\pm \in C^1(0,T)$ satisfy for $t \in (0,T)$:

$$\frac{dU^\pm}{dt} = \pm U^\pm + MV|_{s=0} - \pi m^2 \bar{v} - \frac{1}{2}(U^\pm)^2 + (V|_{s=0})^2 - P[v](0), \tag{3.24}$$

which follows from taking the limits $s \to 0^\pm$ in (3.22).

The initial bound (3.6) can be rewritten in the form:

$$\|v_0\|_{H^1(\mathbb{T})} + \|v_0'\|_{L^\infty(\mathbb{T})} < \delta, \tag{3.25}$$

where $v_0 \in C_0^1 \subset H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ and $\delta > 0$ is an arbitrary small parameter. We first show that there exists $t_0 \in (0,T)$ and $v_0 \in C_0^1$ such that the unique local solution $v \in C^1([0,T), C_0^1)$ to the initial-value problem (3.10) constructed by Lemma 3.3 satisfies

$$\|v_x(t_0,\cdot)\|_{L^\infty(\mathbb{T})} \geq 1. \tag{3.26}$$

It follows from $2P[v](0) = \int_{\mathbb{T}} \varphi(-y) q[v](y) dy > 0$ that equation (3.24) for the upper sign can be estimated by

$$\frac{dU^+}{dt} \leq U^+ + MV|_{s=0} - \pi m^2 \bar{v} + (V|_{s=0})^2, \tag{3.27}$$

By Theorem 3.1, for every small $\varepsilon > 0$, there exists $\nu(\varepsilon) > 0$ such that if $\|v_0\|_{H^1(\mathbb{T})} < \nu(\varepsilon)$, then $\|v(t,\cdot)\|_{H^1(\mathbb{T})} < \varepsilon$ for every $t \in [0,T)$. By Sobolev's embedding, there is a positive constant $C$ such that

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{T})} \leq C\|v(t,\cdot)\|_{H^1} < C\varepsilon. \tag{3.28}$$

By using (3.28), we can simplify (3.27) for every $\varepsilon \in (0,1]$ to the form:

$$\frac{dU^+}{dt} \leq U^+ + MC\varepsilon + \pi m^2 C\varepsilon + C^2\varepsilon^2 \leq U^+ + C_1\varepsilon, \tag{3.29}$$

for some $\varepsilon$-independent constant $C_1 > 0$. Let us assume that the initial data $v_0 \in C_0^1$ satisfies

$$\lim_{x\to 0^+} v_0'(x) = -\|v_0'\|_{L^\infty(\mathbb{T})} = -2C_1\varepsilon, \tag{3.30}$$

The initial bound (3.25) is consistent with (3.30) if for every small $\delta > 0$, the small value of $\varepsilon$ satisfies

$$\nu(\varepsilon) + 2C_1\varepsilon < \delta,$$

which just specifies small $\varepsilon$ in terms of small $\delta$. By integrating (3.29) and using (3.30), we obtain

$$U^+(t) \le e^t \left[U^+(0) + C_1\varepsilon\right] = -C_1\varepsilon e^t.$$

Hence, for every small $\varepsilon > 0$ there exists a sufficiently large $t_1 = -\log(C_1\varepsilon)$ such that $U^+(t_1) \le -1$. If $t_1 < T$, the bound (3.26) is true with some $t_0 \in (0, t_1]$ since

$$\|v_x(t,\cdot)\|_{L^\infty(\mathbb{T})} = \|U(t,\cdot)\|_{L^\infty(0,2\pi)} \ge |U^+(t)|, \quad t \in [0,T). \tag{3.31}$$

If $t_1 \ge T$, then $T$ is the finite maximal existence time in Lemma 3.3 for the solution

$$X(\cdot,s), V(\cdot,s), W(\cdot,s), U(\cdot,s) \in C^1([0,T)), \quad s \in [0,2\pi].$$

By (3.23), we have $X \in C^1([0,T), C^1(0,2\pi))$ if and only if $U \in C^1([0,T), C^0(0,2\pi))$. By the bound (3.28), we have $W, V \in C^1([0,T), C^0(0,2\pi))$ with bounded limits of $\|W(t,\cdot)\|_{L^\infty(0,2\pi)}$ and $\|V(t,\cdot)\|_{L^\infty(0,2\pi)}$ as $t \to T^-$. Then, necessarily, if $T < \infty$, we have

$$\|U(t,\cdot)\|_{L^\infty(0,2\pi)} \to \infty \quad \text{as} \quad t \to T^-, \tag{3.32}$$

so that there exists $t_0 \in (0,T)$ such that $\|U(t_0,\cdot)\|_{L^\infty(0,2\pi)} \ge 1$ and the bound (3.26) is true due to (3.31).

Finally, we show that there exists $v_0 \in C_0^1$ such that the maximal existence time $T$ is finite. Due to the bound (3.28), we have for every $\varepsilon \in (0,1)$:

$$\left|MV|_{s=0} - \pi m^2\bar{v} + (V|_{s=0})^2 - P[v](0)\right| \le MC\varepsilon + \pi m^2 C\varepsilon + C^2\varepsilon^2 + \frac{1}{2}M\varepsilon^2 \le C_2\varepsilon,$$

for some $\varepsilon$-independent constant $C_2 > 0$. Let $U_0(\varepsilon)$ be the negative root of the quadratic equation

$$U - \frac{1}{2}U^2 + C_2\varepsilon = 0.$$

It is clear that $U_0(\varepsilon) \in (-C_2\varepsilon, 0)$. Assume that the initial data $v_0 \in C_0^1$ satisfies

$$\lim_{x\to 0^+} v_0'(x) = -\|v_0'\|_{L^\infty(\mathbb{T})} = 2U_0(\varepsilon). \tag{3.33}$$

The initial bound (3.25) is consistent with (3.33) if for every small $\delta > 0$, the small value of $\varepsilon$ satisfies

$$\nu(\varepsilon) + 2|U_0(\varepsilon)| < \delta,$$

which again specifies small $\varepsilon$ in terms of small $\delta$. If $U^+(0) = 2U_0(\varepsilon)$, then differential equation (3.24) for $U^+$ implies that $\frac{d}{dt}U^+ < 0$, hence $U^+(t) < U^+(0)$ for small positive

$t$ and the map $t \mapsto U(t)$ is monotonically decreasing. Let $\overline{U}$ be the supersolution that satisfies:

$$\frac{d\overline{U}}{dt} = \overline{U} - \frac{1}{2}\overline{U}^2 + C_2\varepsilon.$$

with $\overline{U}(0) = U^+(0) = 2U_0(\varepsilon)$. It follows by the comparison theory for differential equations that $U^+(t) \leq \overline{U}(t)$ for every $t > 0$ for which $U^+(t)$ exists. Since there exists a finite $\overline{T} > 0$ such that $\overline{U}(t) \to -\infty$ as $t \to \overline{T}^-$, then there exists $T \in (0, \overline{T}]$ such that $U^+(t) \to -\infty$ as $t \to T^-$, which implies that the blow-up criterion (3.32) is satisfied for finite $T > 0$.

Both parts of Theorem 3.2 have been proven.

## 3.3 Linear evolution of peaked perturbations

Here we consider the linearized initial-value problem which follows from (3.10):

$$\begin{cases} v_t = (c - \varphi)v_x + \varphi w - \pi m^2 \bar{v} \sinh(x), \quad t > 0, \\ v|_{t=0} = v_0. \end{cases} \tag{3.34}$$

The evolution problem (3.34) can be solved explicitly by using the method of characteristics piecewise for $x \in \mathbb{T}$ on both sides of the peak at $x = 0$. The family of characteristic curves $x = X(t, s)$ satisfies the initial-value problem

$$\begin{cases} \frac{dX}{dt} = \varphi(X) - \varphi(0), \\ X|_{t=0} = s, \end{cases} \tag{3.35}$$

for every $s \in \mathbb{T}$. It follows from (3.34) and (3.35) that $V(t, s) := v(t, X(t, s))$ satisfies the initial-value problem:

$$\begin{cases} \frac{dV}{dt} = \varphi(X(t, s))W(t, s) - \pi m^2 \bar{v} \sinh(X(t, s)), \\ V|_{t=0} = v_0(s). \end{cases} \tag{3.36}$$

Along each characteristic curve $x = X(t, s)$ satisfying (3.35), function $W(t, s) := w(t, X(t, s))$ satisfies the initial-value problem which follows from the linearization of the initial-value problem (3.12):

$$\begin{cases} \frac{dW}{dt} = \varphi'(X(t, s))W + \pi m^2 \bar{v}[1 - \cosh(X(t, s))], \\ W|_{t=0} = w_0(s), \end{cases} \tag{3.37}$$

where $w_0(x) := \int_0^x v_0(y)dy$.

The initial-value problems (3.35), (3.37), and (3.36) can be solved explicitly. Separation of variables in (3.35) gives

$$mt = \int_s^X \frac{dx}{\cosh(\pi - x) - \cosh(\pi)} = m \log \frac{(e^{2\pi} - e^X)(1 - e^s)}{(1 - e^X)(e^{2\pi} - e^s)},$$

from which we derive

$$X(t,s) = \log \frac{(e^{2\pi} - e^s) + e^{2\pi - t}(e^s - 1)}{(e^{2\pi} - e^s) + e^{-t}(e^s - 1)}, \quad t \in \mathbb{R}^+, \quad s \in [0, 2\pi], \tag{3.38}$$

such that

$$\lim_{s \to 0^+} X(t,s) = 0 \quad \text{and} \quad \lim_{s \to 2\pi^-} X(t,s) = 2\pi.$$

After lengthy manipulations, we can find that

$$\frac{\partial X}{\partial s}(t,s) = \frac{(e^{2\pi} - 1)^2 e^{s-t}}{[(e^{2\pi} - e^s) + e^{-t}(e^s - 1)][(e^{2\pi} - e^s) + e^{2\pi - t}(e^s - 1)]}, \quad t \in \mathbb{R}^+, \quad s \in [0, 2\pi],$$

such that

$$\lim_{s \to 0^+} \frac{\partial X}{\partial s}(t,s) = e^{-t} \quad \text{and} \quad \lim_{s \to 2\pi^-} \frac{\partial X}{\partial s}(t,s) = e^t.$$

Integrating (3.37) with an integrating factor yields

$$W(t,s) = \frac{\partial X}{\partial s} \left[ w_0(s) + \pi m^2 \bar{v} \int_0^t \frac{1 - \cosh X(t', s)}{\frac{\partial X}{\partial s}(t', s)} dt' \right].$$

By using (3.38), we derive the following simple expression:

$$W(t,s) = \frac{\partial X}{\partial s} \left[ w_0(s) - \pi m^2 \bar{v} (\cosh s - 1)(1 - e^{-t}) \right], \quad t \in \mathbb{R}^+, \quad s \in [0, 2\pi], \tag{3.39}$$

such that

$$\lim_{s \to 0^+} W(t,s) = 0 \quad \text{and} \quad \lim_{s \to 2\pi^-} W(t,s) = w_0(2\pi) = 2\pi \bar{v}.$$

Finally, the solution for $V(t,s)$ is obtained with the chain rule

$$\frac{\partial W}{\partial s}(t,s) = V(t,s) \frac{\partial X}{\partial s}(t,s),$$

from which we derive

$$V(t,s) = v_0(s) - \pi m^2 \bar{v} \sinh s (1 - e^{-t}) + \left[ w_0(s) - \pi m^2 \bar{v} (\cosh s - 1)(1 - e^{-t}) \right] Y(t,s) \tag{3.40}$$

with

$$\begin{aligned} Y(t,s) &:= \frac{\partial}{\partial s}\log\frac{\partial X}{\partial s}(t,s) \\ &= \frac{(1-e^{-t})[\sinh(2\pi-s)+e^{-t}\sinh(s)]}{\cosh(2\pi-s)-1+[\cosh(2\pi)+1-\cosh(2\pi-s)-\cosh(s)]e^{-t}+(\cosh s-1)e^{-2t}}, \end{aligned}$$

such that

$$\lim_{s\to 0^+} V(t,s) = v_0(0) \quad \text{and} \quad \lim_{s\to 2\pi^-} V(t,s) = v_0(2\pi) = v_0(0).$$

The exact solutions to the initial-value problems (3.35), (3.37), and (3.36) can be illustrated with the explicit example of linear instability of peaked perturbations to the peaked periodic wave. The solution $v = v(t,x)$ is available in the parametric form (3.38) and (3.40) with parameter $s$ on $[0, 2\pi]$. The solution is extended periodically to $[-2\pi, 0]$.

Figure 3.2 shows the plots of $v(t,x)$ versus $x$ on $[-2\pi, 2\pi]$ for different values of $t$ for two initial conditions: $v_0(x) = \sin(x)$ (top) and $v_0(x) = \cos(x)$ (bottom), in both cases, $\bar{v} = 0$. These panels give clear illustrations that $v(t,\cdot)$ remains bounded in the $L^\infty$ norm and that the slope of the perturbation grows at the right side of the peak.

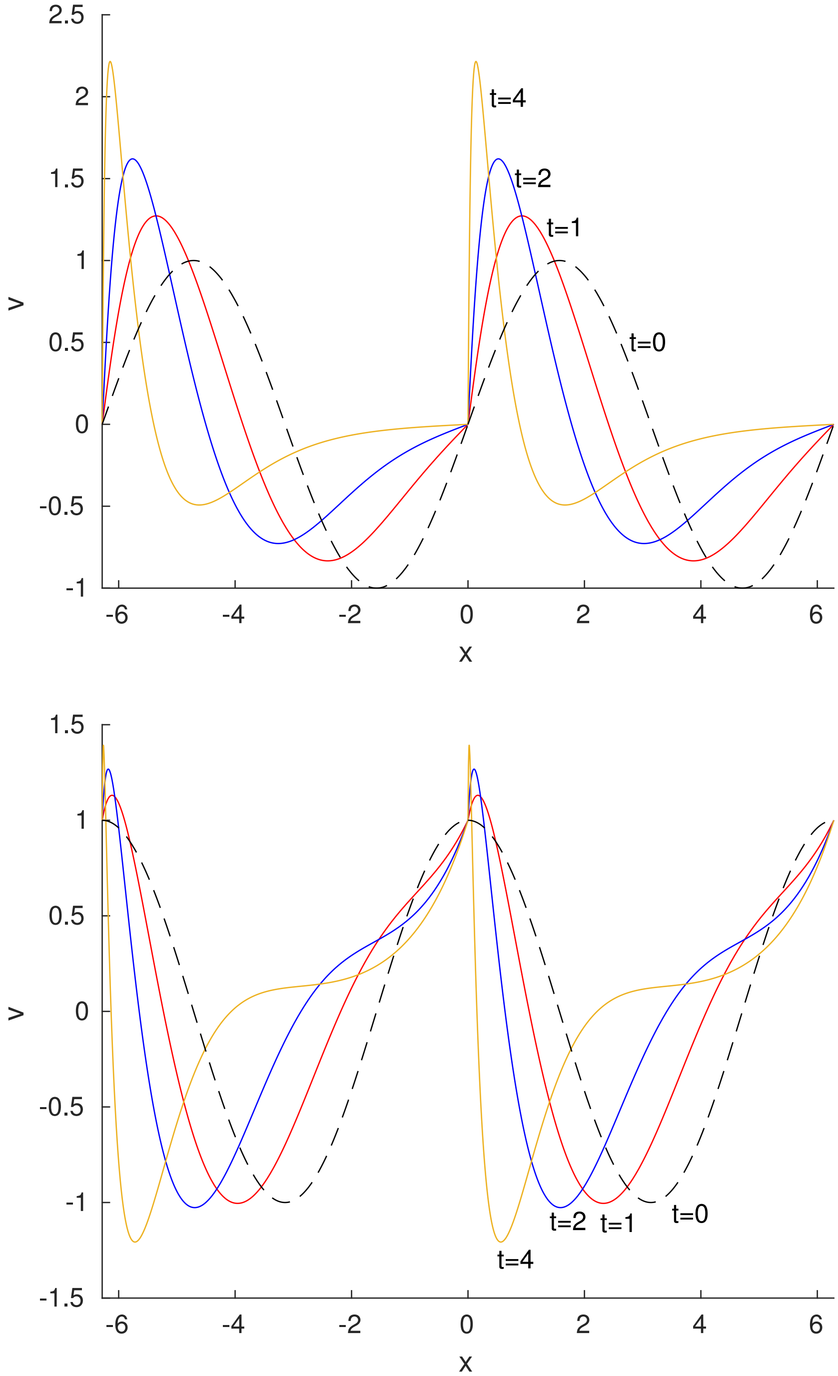


Figure 3.2: The plots of $v(t, x)$ versus $x$ on $[-2\pi, 2\pi]$ for different values of $t$ in the case $v_0(x) = \sin(x)$ (top) and $v_0(x) = \cos(x)$ (bottom).

# Chapter 4

# Peaked periodic waves in DP equation

## 4.1 Main result

We consider the Degasperis–Procesi (DP) equation given by

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{T} \tag{4.1}$$

in the setting of $\mathbb{T} := [-\pi, \pi]$ subject to the periodic boundary conditions at $\pm\pi$. The DP equation (4.1) can be rewritten in the convolution form

$$u_t + uu_x + \frac{3}{4}\varphi' * u^2 = 0. \tag{4.2}$$

Two additional variables are useful for analysis of the DP equation (4.1):

$$y := u - u_{xx}, \quad v := (4 - \partial_x^2)^{-1} u, \tag{4.3}$$

where the linear self-adjoint operator $L_0 := 4 - \partial_x^2$ is invertible with a bounded inverse in $L^2_{per}(\mathbb{T})$. The DP equation (4.1) is equivalently written in the following compact forms:

$$y_t + uy_x + 3yu_x = 0 \tag{4.4}$$

and

$$v_t + uu_x - v_{txx} = 0. \tag{4.5}$$

Local well-posedness of the DP equation (4.2) for initial data in $H^s_{\text{per}}(\mathbb{T})$ with $s > 3/2$ was proved in [52] by using Kato's semigroup theory. (The analogous result on the line $\mathbb{R}$ was proved in [53].) It was shown in [27] that the DP equation (4.2) is ill-posed in

$H^s_{\text{per}}(\mathbb{T})$ with $s < 3/2$, and the ill-posedness is due to the lack of continuous dependence of the solution versus initial condition for $1/2 \leq s < 3/2$. The norm inflation implies that there exist solutions which are initially arbitrarily small and eventually arbitrarily large in the $H^s$ norm in an arbitrarily short time.

The following lemma assures conservation of the three conserved quantities introduced in (1.8) for the local smooth solutions of the DP equation (4.2).

**Lemma 4.1.** *Let $u \in C^0([0,T), H^s_{\text{per}}(\mathbb{T})) \cap C^1([0,T), H^{s-1}_{\text{per}}(\mathbb{T}))$ be the local solution to the DP equation (4.2) for $s > 3/2$ and some $T > 0$. Then, the following quantities*

$$E_1(u) = \int_{-\pi}^{\pi} u\,dx, \quad E_2(u) = \int_{-\pi}^{\pi} yv\,dx, \quad E_3(u) = \int_{-\pi}^{\pi} u^3\,dx \tag{4.6}$$

*are independent of time $t$ on $[0,T)$.*

*Proof.* It follows by integration of (4.2) and periodicity of $u$ that

$$\frac{d}{dt}\int_{-\pi}^{\pi} u\,dx + \frac{3}{4}\int_{-\pi}^{\pi} \varphi' * u^2\,dx = 0,$$

where

$$\frac{3}{4}\int_{-\pi}^{\pi} \varphi' * u^2\,dx = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi'(x-y)u(y)^2\,dy\,dx = \int_{-\pi}^{\pi} u^2(y)\int_{-\pi}^{\pi} \varphi'(x-y)'\,dx)\,dy = 0$$

due to periodicity and continuity of $\varphi$. This yields conservation of $E_1(u)$.

Similarly, we obtain from (4.2) that

$$\frac{1}{3}\frac{d}{dt}\int_{-\pi}^{\pi} u^3\,dx + \frac{3}{4}\int_{-\pi}^{\pi} u^2(\varphi' * u^2)\,dx = 0,$$

where

$$\int_{-\pi}^{\pi} u^2(\varphi' * u^2)\,dx = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi'(x-y)u(x)^2u(y)^2\,dy\,dx = 0$$

due to the oddness of $\varphi'$. This yields conservation of $E_3(u)$.

In order to prove the conservation of $E_2(u)$, we rewrite it in the equivalent form:

$$E_2(u) = \int_{-\pi}^{\pi} (u - u_{xx})v\,dx = \int_{-\pi}^{\pi} (uv + u_xv_x)\,dx,$$

where $v \in C^0([0,T), H^{s+2}_{\text{per}}(\mathbb{T})) \cap C^1([0,T), H^{s+1}_{\text{per}}(\mathbb{T}))$ follows from $v = (4 - \partial_x^2)^{-1}u$ and integration by parts is justified. Multiplying (4.1) and (4.5) by $v$ and $u$ respectively and integrating over $\mathbb{T}$ yields

$$\frac{d}{dt}\int_{-\pi}^{\pi} (uv - u_{xx}v)\,dx + \int_{-\pi}^{\pi} v(4 - \partial_x^2)(uu_x)\,dx = 0$$

where the second term is zero. This yields conservation of $E_2(u)$. □

As is pointed in Chapter 1, no orbital stability results for the peaked periodic waves of the DP equation (4.2) are available in $L^2(\mathbb{T})$, contrary to the results of J. Lenells [38, 39] in $H^1(\mathbb{T})$ for the CH equation or the results of [41] in $L^2(\mathbb{R})$ for the DP equation. Since the proof of instability of the peaked periodic waves in $W^{1,\infty}(\mathbb{T})$ similar to what we did in Chapter 2 for the peakons in $W^{1,\infty}(\mathbb{R})$ is based on the orbital stability assumption, we first need to establish the orbital stability result by using the conserved quantities (4.6).

To address the orbital stability of the peaked periodic waves of the DP equation (4.2) in $L^2(\mathbb{T})$, we consider the following variational problem:

$$\inf_{u \in L^2(\mathbb{T})} \{E_2(u) : \quad E_1(u) = e_1, \quad E_3(u) = e_3\}, \tag{4.7}$$

where $e_1$ and $e_3$ are fixed. The following lemma confirms that the peaked periodic wave $u(t,x) = \varphi(x - ct)$ with $c = \varphi(0)$ given by the Green function (3.3) is indeed a critical point of the constrained variational problem (4.7).

**Lemma 4.2.** *Let $\varphi$ be the Green function (3.3). Then, it is a travelling wave solution $u(t,x) = \varphi(x - ct)$ with $c = \varphi(0)$ of the DP equation (4.2) and a critical point of the constrained variational problem (4.7).*

*Proof.* It follows from (4.2) that the travelling wave solution $u(t,x) = \varphi(x - ct)$ satisfies the nonlocal equation

$$-c\varphi' + \varphi\varphi' + \frac{3}{4}\varphi' * \varphi^2 = 0,$$

which becomes after integration

$$-c\varphi + \frac{1}{2}\varphi^2 + \frac{3}{4}\varphi * \varphi^2 = d, \tag{4.8}$$

where $d$ is the integration constant. Without loss of generality, we shall consider (4.8) for $x \in (0, 2\pi)$, for which one can use the expression $\varphi(x) = m\cosh(\pi - x)$ without the modulus sign. Recall that $m = \varphi(\pm\pi) = \operatorname{csch}(\pi)$ and $M = \varphi(0) = \coth(\pi)$. Since $\varphi * 1 = 2$, we derive

$$\begin{aligned} \varphi * \varphi^2 &= \frac{m^2}{4}\left[\varphi * e^{2\pi - 2x} + 2\varphi * 1 + \varphi * e^{-2\pi + 2x}\right] \\ &= \frac{m^2}{3}\left[3 + 4\cosh(\pi)\cosh(\pi - x) - \cosh(2\pi - 2x)\right], \end{aligned}$$

so that

$$\frac{3}{4}\varphi * \varphi^2 + \frac{1}{2}\varphi^2 = M\varphi + m^2.$$

This expression coincides with (4.8) for $c = M$ and $d = m^2$.

Next, we introduce the component $v_\varphi := (4-\partial_x^2)^{-1}\varphi$, which solves the travelling wave reduction $v(t,x) = v_\varphi(x-ct)$ of the equivalent equation (4.5):

$$-cv_\varphi' + \varphi\varphi' + cv_\varphi''' = 0,$$

which is integrated to the form

$$v_\varphi - v_\varphi'' = \frac{1}{2c}\varphi^2 + b, \tag{4.9}$$

where $b$ is another integration constant.

The explicit expression for $v_\varphi$ is found from the differential equation $4v_\varphi - v_\varphi'' = \varphi$ integrated with periodic boundary conditions $v_\varphi(-\pi) = v_\varphi(\pi)$ and $v_\varphi'(-\pi) = v_\varphi'(\pi)$. Due to the even symmetry of $v_\varphi$, we can replace the periodic boundary conditions with the conditions $v_\varphi'(0) = 0$ and $v_\varphi'(\pi) = 0$. Integrating the differential equation $4v_\varphi - v_\varphi'' = \varphi$ for $x \in (0,\pi)$ yields the expression

$$v_\varphi = C_1 e^{2x} + C_2 e^{-2x} + \frac{\cosh(\pi - x)}{3\sinh(\pi)}. \tag{4.10}$$

From the two boundary conditions, we find that $C_1 = C_2 + \frac{1}{6}$ and $C_1 = C_2 e^{-4\pi}$ so that the explicit solution is

$$v_\varphi(x) = \frac{\cosh(\pi - |x|)}{3\sinh(\pi)} - \frac{\cosh 2(\pi - |x|)}{6\sinh(2\pi)}. \tag{4.11}$$

After substituting (3.3) and (4.11) into (4.9), we obtain $c = \coth(\pi)$ and $b = -\frac{1}{2}\mathrm{csch}(2\pi)$.

It remains to show that the stationary equation (4.9) is the Euler–Lagrange equation for the action functional

$$J(u) := E_2(u) + c_1[E_1(u) - e_1] + c_3[E_3(u) - e_3] \tag{4.12}$$

where $c_1$ and $c_3$ are the Lagrange multipliers associated with the two constraints in the constrained variational problem (4.7). Computing Frechet derivatives gives

$$E_1'(u) = 1, \quad E_3'(u) = 3u^2,$$

and

$$E_2'(u) = 2(1-\partial_x^2)(4-\partial_x^2)^{-1}u = 2u - 6(4-\partial_x^2)^{-1}u,$$

so that the Euler–Lagrange equation for $J$ is given by

$$u - 3(4-\partial_x^2)^{-1}u + \frac{1}{2}c_1 + \frac{3}{2}c_3 u^2 = 0. \tag{4.13}$$

Equation (4.13) coincides with (4.9) for $u = \varphi$ and $v_\varphi := (4-\partial_x^2)^{-1}\varphi$ with the choice of Lagrange multipliers given by $c_1 = -2b$ and $c_3 = -\frac{1}{3c}$. □

The second derivative test on the critical points involves computing the second variation of the action functional $J(u)$ in (4.12). With the Lagrange minimizers $c_1 = -2b$ and $c_3 = -\frac{1}{3c}$, the action functional can be rewritten in the form

$$J(u) = \int_{\mathbb{T}} [u^2 - 3u(4-\partial_x^2)^{-1}u]\, dx - 2b \left[ \int_{\mathbb{T}} u\, dx - e_1 \right] - \frac{1}{3c} \left[ \int_{\mathbb{T}} u^3\, dx - e_3 \right].$$

The second variation of $J(u)$ at $u = \varphi$ is given by the self-adjoint operator in $L^2(\mathbb{T})$:

$$\mathcal{L} = 1 - 3(4-\partial_x^2)^{-1} - \frac{\varphi(x)}{c}. \tag{4.14}$$

Due to the two constraints in (4.7), we consider $\mathcal{L}$ on the constrained subspace of $L^2(\mathbb{T})$ given by

$$X := \{w \in L^2(\mathbb{T}) : \quad \langle w, \varphi^2 \rangle = 0, \quad \langle w, 1 \rangle = 0\}. \tag{4.15}$$

The main result of this chapter is the following theorem which states that the operator $\mathcal{L}|_X$ is positive but not strictly positive. This implies that the peaked periodic wave with the profile $\varphi$ is the degenerate local minimizer of the constrained variational problem (4.7).

**Theorem 4.1.** *The spectrum of $\mathcal{L}|_X$ in $X \subset L^2(\mathbb{T})$ is positive and 0 is the endpoint of the continuous spectrum.*

Section 4.2 contains the proof of Theorem 4.1. Section 4.3 describes further directions in the study.

## 4.2 Proof of Theorem 4.1

### 4.2.1 The spectrum of the operator $\mathcal{L}$ in $L^2(\mathbb{T})$

The linear operator $\mathcal{L}$ in (4.14) is given by the difference of the multiplicative operator $L_0 := 1 - c^{-1}\varphi$ and a compact operator $K := 3(4-\partial_x^2)^{-1}$. We will show here that the spectrum of $\mathcal{L}$ consists of the continuous spectrum on $[0, 1 - \text{sech}(\pi)]$ and a negative eigenvalue $\lambda_0 < 0$.

The following definition reminds us how the spectrum of a linear operator is defined.

**Definition 4.1.** *Let $A$ be a linear operator on a Banach space $X$ with* $\text{dom}(A) \in X$*. The complex plane $\mathbb{C}$ is decomposed into the following two sets:*

1. *The resolvent set*

$$\begin{aligned}\sigma_c(A) = \{&\lambda \in \sigma(A) : \ker(A - \lambda I) = \{0\}, \text{ran}(A-\lambda I) = X, \\ &(A-\lambda I)^{-1} : X \to X \text{ is bounded}\}.\end{aligned}$$

*2. The spectrum*

$$\sigma(A) = \mathbb{C}\backslash\rho(A),$$

*which is further decomposed into the following three disjoint sets:*

- *the point spectrum*

$$\sigma_p(A) = \{\lambda \in \sigma(A) : \ker(A - \lambda I) \neq \{0\}\},$$

- *the residual spectrum*

$$\sigma_r(A) = \{\lambda \in \sigma(A) : \ker(A - \lambda I) \neq \{0\}, \operatorname{ran}(A - \lambda I) \neq X\},$$

- *the continuous spectrum*

$$\begin{aligned}\sigma_c(A) = &\{\lambda \in \sigma(A) : \ker(A - \lambda I) \neq \{0\}, \operatorname{ran}(A - \lambda I) = X, \\ &(A - \lambda I)^{-1} : X \to X \text{ is unbounded}\}.\end{aligned}$$

Also recall from the spectral theorem that if $X$ is a Hilbert space and $A$ is a self-adjoint operator, then $\sigma_r(A) = \varnothing$. If $\mathcal{L}$ in (4.14) is considered in Hilbert space $L^2(\mathbb{T})$ equipped with the standard inner product $\langle \cdot, \cdot \rangle$, then $\mathcal{L}$ is self-adjoint.

The following two lemmas verify the claims that $K$ is compact and that $\sigma_c(\mathcal{L}) = [0, \operatorname{sech}(\pi)]$.

**Lemma 4.3.** *The operator $K$ is compact in $L^2(\mathbb{T})$.*

*Proof.* $L^2(\mathbb{T})$ has an orthogonal Fourier basis given by $\{e^{inx}\}_{n\in\mathbb{Z}}$. Every function $f$ in $L^2(\mathbb{T})$ is uniquely given by the Fourier series $f = \sum_{n\in\mathbb{Z}} C_n e^{inx}$. Consequently, it follows by solving $4v - v'' = 3f$ with the Fourier series that

$$Kf = 3(4 - \partial_x^2)^{-1} f = \sum_{n\in\mathbb{Z}} 3(4 + n^2)^{-1} C_n \, e^{inx}. \tag{4.16}$$

Hence $\sigma(K) = \sigma_p(K) = \{3(4 + n^2)^{-1}, \; n \in \mathbb{Z}\}$. Since the operator $K$ is in the trace class with $\sum_{n\in\mathbb{Z}} 3(4 + n^2)^{-1} < \infty$, the operator $K$ is compact. □

**Remark 4.1.** *It follows from (4.16) and Parceval's identity that*

$$\|Kf\|^2 = 2\pi \sum_{n\in\mathbb{Z}} \frac{9C_n^2}{(4 + n^2)^2} \leq \frac{9}{16} 2\pi \sum_{n\in\mathbb{Z}} C_n^2 = \frac{9}{16}\|f\|^2, \tag{4.17}$$

*so that $\|Kf\| \leq \frac{3}{4}\|f\|$.*

**Lemma 4.4.** *The continuous spectrum of $\mathcal{L}$ is located at $\sigma_c(\mathcal{L}) = [0, 1 - \mathrm{sech}(\pi)]$.*

*Proof.* Since the operator $L_0$ is a multiplication operator on $L^2(\mathbb{T})$ space, then

$$\sigma(L_0) = \sigma_c(L_0) = \mathrm{image}(1 - c^{-1}\varphi) = [1 - M^{-1}m] = [0, 1 - \mathrm{sech}(\pi)], \tag{4.18}$$

where we recall that $c = \max_{x\in\mathbb{T}} \varphi(x) = \varphi(0) = M = \coth(\pi)$ and $\min_{x\in\mathbb{T}} \varphi(x) = \varphi(\pm\pi) = m = \mathrm{csch}(\pi)$. Since the compact operator $K$ is in the trace class, $\sigma_c(\mathcal{L}) = \sigma_c(L_0)$ by Kato's theorem (see Theorem 4.4 on p. 542 in [35]). ☐

In what follows next, we will localize the point spectrum of $\mathcal{L}$ in $L^2(\mathbb{T})$ outside $\sigma_c(\mathcal{L}) = [0, 1-\mathrm{sech}(\pi)]$. The eigenvalue equation $(L-\lambda I)v = 0$ with $\lambda \notin [0, 1-\mathrm{sech}(\pi)]$ is equivalent to the spectral problem

$$v = (1 - c^{-1}\varphi - \lambda)^{-1}Kv. \tag{4.19}$$

Since $K$ is a positive operator and $1 - c^{-1}\varphi - \lambda < 0$ if $\lambda > 1 - \mathrm{sech}(\pi) = \max_{x\in\mathbb{T}}(1 - c^{-1}\varphi(x))$, it is clear that $1 \notin \sigma((1 - c^{-1}\varphi - \lambda)^{-1}K)$, hence no solutions $v \in L^2(\mathbb{T})$ of the spectral problem (4.19) exist for $\lambda > 1 - \mathrm{sech}(\pi)$.

On the other hand, $\mathcal{L}$ is not a positive operator because

$$\mathcal{L}\varphi = \varphi - 3v_\varphi - c^{-1}\varphi^2 = b - \frac{1}{2c}\varphi^2 \tag{4.20}$$

and

$$\langle \mathcal{L}\varphi, \varphi\rangle = bE_1(\varphi) - \frac{1}{2c}E_3(\varphi) = -\frac{2\sinh^2(\pi) + 9}{3\sinh(2\pi)} < 0, \tag{4.21}$$

where we have used that

$$E_1(\varphi) = \int_{-\pi}^{\pi} \frac{\cosh(\pi - |x|)}{\sinh(\pi)}\,dx = 2$$

and

$$E_3(\varphi) = \int_{-\pi}^{\pi} \frac{\cosh^3(\pi - |x|)}{\sinh^3(\pi)}\,dx = \frac{2}{3} + \frac{2}{\sinh^2(\pi)}.$$

Since $\langle \mathcal{L}\varphi, \varphi\rangle < 0$, there must exist solutions $v \in L^2(\mathbb{T})$ of the spectral problem (4.19) for $\lambda < 0$. In order to obtain these eigenvalues, we use $v = K^{-1/2}w$ and introduce the Birman–Schwinger operator (see p. 113-115 in [24]):

$$w = M(\lambda)w, \quad M(\lambda) := K^{1/2}(1 - c^{-1}\varphi - \lambda)^{-1}K^{1/2}, \quad \lambda < 0. \tag{4.22}$$

Since $M(\lambda)$ is a composition of compact and bounded operators, it is a compact operator in $L^2(\mathbb{T})$. It is also a self-adjoint operator in $L^2(\mathbb{T})$. Therefore, the spectrum of $M(\lambda)$ consists of real eigenvalues of the point spectrum. The following lemma describes properties of the Birman–Schwinger operator $M(\lambda)$.

**Lemma 4.5.** *Eigenvalues of $M(\lambda)$ denoted by $\mu(\lambda)$ are $C^1$ functions of $\lambda \in (-\infty, 0)$ satisfying the following properties:*

1. $\mu(\lambda) \to 0$ *as* $\lambda \to -\infty$.
2. $\mu(\lambda) > 0$.
3. $\mu'(\lambda) > 0$.

*Proof.* The smoothness of isolated eigenvalues of a self-adjoint operator $M(\lambda)$ in $L^2(\mathbb{T})$ with smooth coefficients follows by the perturbation theory in [35]. Hence, it remains to prove the three properties of the isolated eigenvalues.

The first property follows from Remark 4.1 and the following estimates:

$$\begin{aligned} \|M(\lambda)w\| &= \|K^{1/2}(1 - c^{-1}\varphi - \lambda)^{-1}K^{1/2}w\| \\ &\leq \frac{\sqrt{3}}{2}\|(1 - c^{-1}\varphi - \lambda)^{-1}K^{1/2}w\| \\ &\leq \frac{\sqrt{3}}{2|\lambda|}\|K^{1/2}w\| \\ &\leq \frac{3}{4|\lambda|}\|w\|, \end{aligned}$$

hence $\|M(\lambda)\| \leq \frac{3}{4|\lambda|}$ and $\|M(\lambda)\| \to 0$ as $\lambda \to -\infty$.

The second property follows from strict positivity of $M(\lambda)$, since $K$ is a positive operator and $1 - c^{-1}\varphi - \lambda > 0$ for $\lambda < 0$.

The third property is obtained from differentiating the spectral problem

$$M(\lambda)w(\lambda) = \mu(\lambda)w(\lambda) \tag{4.23}$$

with respect to $\lambda$, which yields

$$M'(\lambda)w(\lambda) + M(\lambda)w'(\lambda) = \mu'(\lambda)w(\lambda) + \mu(\lambda)w'(\lambda). \tag{4.24}$$

Let us assume for simplicity that $\mu(\lambda)$ is a simple eigenvalue with the only eigenfunction $w(\lambda)$. Multiplying (4.24) by $w(\lambda)$ and integrating in $L^2(\mathbb{T})$ yields

$$\langle M'(\lambda)w(\lambda), w(\lambda)\rangle = \mu'(\lambda)\|w(\lambda)\|^2, \tag{4.25}$$

because $\langle M(\lambda)w'(\lambda), w(\lambda)\rangle = \langle w'(\lambda), M(\lambda)w(\lambda)\rangle = \mu(\lambda)\langle w'(\lambda), w(\lambda)\rangle$. It then follows that

$$\mu'(\lambda) = \frac{\langle M'(\lambda)w(\lambda), w(\lambda)\rangle}{\|w(\lambda)\|^2}, \tag{4.26}$$

where

$$M'(\lambda) := K^{1/2}(1 - c^{-1}\varphi - \lambda)^{-2}K^{1/2} > 0. \tag{4.27}$$

Hence, $\mu'(\lambda) > 0$ and the third property is proven. □

We have shown that there must exist a negative eigenvalue of $\mathcal{L}$, which implies that there exists a negative value of $\lambda$ such that $1 \in M(\lambda)$. By the Birman–Schwinger principle [24], the number of negative eigenvalues of $\mathcal{L}$ coincides with the number of the crossing of eigenvalues $\mu(\lambda)$ of $M(\lambda)$ with the unit level. Figure 4.1 shows that there is only one crossing of eigenvalues $\mu(\lambda)$ of $M(\lambda)$ with the unit level, that is,

$$\#\{\lambda : \mu(\lambda) = 1, \mu \in \sigma_p(M(\lambda))\} = 1.$$

This shows with the assistance of numerical approximations that $\mathcal{L}$ has only one simple negative eigenvalue $\lambda_0 < 0$ with $\lambda_0 \approx -0.4$.

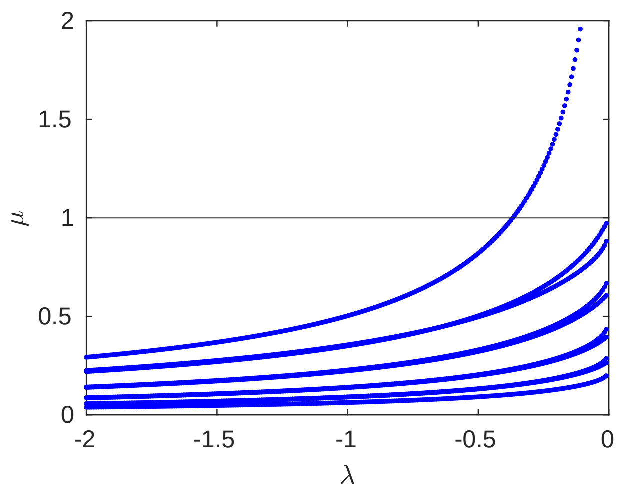


Figure 4.1: Eigenvalues of $M(\lambda)$.

### 4.2.2 Spectrum of the operator $\mathcal{L}|_X$ in $X \subset L^2(\mathbb{T})$

Since the spectrum of $\mathcal{L}$ in $L^2(\mathbb{T})$ is non-positive, $u = \varphi$ is a saddle point of the action functional $J(u)$ in (4.12). However, the two constraints in $X \subset L^2(\mathbb{T})$ due to the conserved quantities restrict the space so that by Rayleigh-Ritz theorem,

$$\lambda_0 := \inf_{v \in L^2(\mathbb{T})} \frac{\langle \mathcal{L}v, v\rangle}{\|v\|^2} \leq \inf_{v \in X \subset L^2(\mathbb{T})} \frac{\langle \mathcal{L}|_X v, v\rangle}{\|v\|^2} =: \lambda_0' \tag{4.28}$$

In order to complete the proof of Theorem 4.1, we need to show that $\lambda_0' \geq 0$. Then, $u = \varphi$ is a constrained minimizer of the action functional $J(u)$ in (4.12).

The following lemma verifies the conditions which ensure that $\lambda_0' \geq 0$.

**Lemma 4.6.** *There exist $a_1, a_3 \in \mathbb{R}$ such that $f = a_1 + a_3\varphi^2$ and $\langle \mathcal{L}^{-1} f, f\rangle < 0$. Consequently, $\lambda_0' \geq 0$ so that $\mathcal{L}|_X \geq 0$.*

*Proof.* It follows from (4.20) that we can take $a_1 = b$ and $a_3 = -1/(2c)$ so that $\mathcal{L}^{-1} f = \varphi$. It follows from (4.21) that

$$\langle \mathcal{L}^{-1} f, f \rangle = bE_1(\varphi) - \frac{1}{2c} E_3(\varphi) = -\frac{2\sinh^2(\pi) + 9}{3\sinh(2\pi)} < 0. \tag{4.29}$$

The existence of $f \in \text{span}(1, \varphi^2)$ such that $\langle \mathcal{L}^{-1} f, f \rangle < 0$ implies positivity of $\mathcal{L}|_X$ by the standard variational theory (see Theorem 4.1 in [51]) since $\mathcal{L}$ has only one simple negative eigenvalue. □

## 4.3 Further directions

Since the continuous spectrum of $\mathcal{L}$ is located on $[0, \text{sech}(\pi)]$ and touches zero, the quadratic forms associated with either $\mathcal{L}$ or $\mathcal{L}|_X$ lack coercivity. Therefore, the local minimizers of the constrained variational problem (4.7) are degenerate and the conserved quantities (4.6) do not imply the orbital stability of the peaked periodic waves of the DP equation in $L^2(\mathbb{T})$, which would be analogous to Theorem 3.1 for the orbital stability of the peaked periodic waves of the CH equation in $H^1_{\text{per}}(\mathbb{T})$. Consequently, we are not able to use the $L^2(\mathbb{T})$ bound on the perturbations to the peaked periodic wave in order to prove the growth of perturbations in $W^{1,\infty}(\mathbb{T})$ similarly to what we did for peakons of the DP equation on the real line. Further study is needed to control the perturbations to the peaked periodic waves of the DP equation in $L^2(\mathbb{T})$.

The difficulty here is very similar to the study of peaked periodic waves in other equations with wave breaking such as the reduced Ostrovsky equation $(u_t + uu_x)_x = u$. It was shown in [21, 22] that the self-adjoint operator related to the conserved quantities and computed at the peaked periodic wave lacks coercivity and this indicates spectral instability of the peaked periodic wave in $L^2(\mathbb{T})$.

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