Instability of peaked traveling wave solutions of the b-family of Camassa-Holm equations

Aigerim Madiyeva

Department of Mathematics and Statistics, McMaster University, Canada

Peakons in the b-CH equations

***b*-family of the generalized Camassa-Holm equations** (b-CH)

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}. \quad (1)$$

- a model for unidirectional wave propagation on shallow water
- Camassa-Holm (CH) equation for b = 2

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \quad (2)$$

- Degasperis-Procesi (DP) equation for b = 3

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx} \quad (3)$$

- integrable only for b = 2 and b = 3

Peaked traveling wave solution:

$$u(x,t) = ce^{-|x-ct|} \qquad x \in \mathbb{R}$$

on a line and

$$u(x,t) = c\frac{\cosh(\pi - |x-ct|)}{\cosh(\pi)} \qquad x \in \mathbb{T}$$

on the periodic domain $\mathbb{T} := [-\pi, \pi]$.

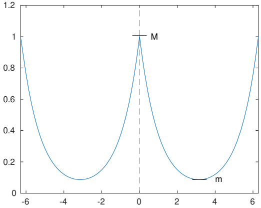

Figure 1: The graph of φ on $[-2\pi, 2\pi]$ with maxima at M and minima at m.

Main problem considered in this thesis is the question of stability or instability of peaked travelling waves $u(t,x) = c\varphi(x-t)$

Orbital Stability

Definition

We say that φ is orbitally stable in Banach space X if for every $\varepsilon > 0$, there exist $\nu > 0$ such that for every $u_0 \in X$ with

$$\|u_0 - \varphi\|_X < \nu, \tag{4}$$

the unique global solution $u \in C(\mathbb{R}, X)$ to the initial-value problem for the b-CH equation (10) with $u(0,\cdot) = u_0$ satisfies

$$\inf_{\xi \in \mathbb{R}} \|u(t,\cdot) - \varphi(\cdot - \xi)\|_X < \varepsilon, \ \forall t > 0. \tag{5}$$

Otherwise, it is orbitally unstable.

Assumption

The peakon φ is orbitally stable in $H^1(\mathbb{R})$.

Main result

The peakon φ is orbitally unstable in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$.

For $b < 1$, Assumption cannot be satisfied because the peakons are unstable in $H^1(\mathbb{R})$. (D.D. Holm et al. (2003), A.N.W. Hone (2003), E. G. Charalampidis et al. (2020))

Theorem 1

Fix $b > 1$ and assume Assumption. For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H^1(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ satisfying

$$\|u_0 - \varphi\|_{H^1} + \|u_0' - \varphi'\|_{L^\infty} < \delta, \tag{6}$$

such that the local solution $u \in C([0, T), H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}))$ to the b-CH equation with $u(0, \cdot) = u_0$ and $T > t_0$ satisfies

$$\|u_x(t_0, \cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty} > 1, \tag{7}$$

where $\xi(t) \in \mathbb{R}$ is the maximum point of peak of $u(t, \cdot)$. Moreover, there exist u_0 such that the maximal existence time T is finite.

Let $\varphi \in H^1(\mathbb{R})$ be the Green function satisfying

$$(1-\partial_x^2)\varphi = 2\delta_0, \quad x \in \mathbb{R} \tag{8}$$

with δ_0 being Dirac delta distribution centered at $x=0$. The Green function φ can be expressed explicitly in the form

$$\varphi(x) = e^{-|x|}, \quad x \in \mathbb{R}, \tag{9}$$

Then, the b-CH equation (1) can be written in the weak form:

$$u_t + uu_x + \frac{1}{4}\varphi' * \left(bu^2 + (3-b)u_x^2\right) = 0, \tag{10}$$

where $(f * g)(x) := \int f(x-y)g(y)dy$ is the convolution operator.
Consider solutions of the Cauchy problem:

$$\begin{cases} u_t + uu_x + Q[u](x) = 0 \\ u|_{t=0} = u_0 \end{cases} \tag{11}$$

- If $u \in H^1(\mathbb{R})$, then $Q[u] \in C^0(\mathbb{R})$.
- If $u \in W^{1,\infty}(\mathbb{R})$, then $Q[u]$ is Lipschitz on $\mathbb{R}$.

Definition

We say that $u \in C([0,T), H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}))$ is a weak solution to the initial-value problem (11) for some maximal existence time $T > 0$ if

$$\int_0^T \int_{\mathbb{R}} \left(u\psi_t + \frac{1}{2}u^2\psi_x - Q[u]\psi \right) dxdt + \int_{\mathbb{R}} u_0(x)\psi(0,x)dx = 0 \tag{12}$$

is satisfied for every test function $\psi \in C^1([0,T] \times \mathbb{R})$ such that $\psi(T,\cdot) = 0$.

Assume that u is a weak solution to the Cauchy problem (11) and there exists $\xi(t) \in$ R for $t \in [0,T)$ such that $u(t,\cdot) \in C^1_{\xi(t)}$ for $t \in [0,T)$. Then, $\xi(t) \in C^1(0,T)$ satisfies

$$\frac{d\xi}{dt} = u(t,\xi(t)), \quad t \in (0;T). \tag{13}$$

where

$$C^1_\xi := \{u \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}) : \quad u_x \in C(\mathbb{R}\backslash\{\xi\})\}. \tag{14}$$

Consider a decomposition of $u(t;x)$:

$$u(t,x)=\varphi(x-t-a(t))+v(t,x-t-a(t)),\ \ t\in[0,T),\ x\in\mathbb{R}$$

and

$$\frac{da}{dt}=v(t,0),\ \ t\in(0,T).$$

Evolution problem:

$$\begin{cases} v_t=(1-\varphi)v_x+(v(0)-v)\varphi'+(v(0)-v)v_x-2Q[v,\varphi](x)-Q[v](x), \\ v|_{t=0}=v_0 \end{cases} \tag{15}$$

where

$$Q[v,\varphi](x):=\frac{1}{4}\int_{\mathbb{R}}\varphi'(x-y)\left(bv\varphi+(3-b)v'\varphi'\right)(y)dy.$$

Method of characteristic curves:

$$x = X(t,s), \qquad V(t,s) := v(t, X(t,s))$$

$$\begin{cases} \frac{dX}{dt} = \varphi(X) - \varphi(0) + v(t,X) - v(t,0) \\ X|_{t=0} = s \end{cases}$$

$$\begin{cases} \frac{dV}{dt} = \varphi'(X)(V(t,0) - V(t,X)) - 2Q[v,\varphi](X) - Q[v](X), \\ V|_{t=0} = v_0(s). \end{cases}$$

Existence and Uniqueness of $X(t,s)$ and $V(t,s)$ follows from ODE theory.

$$U(t,s) := v_x(t, X(t,s))$$

$$\begin{cases} \frac{dU}{dt} = (1-b)\varphi'(X)U + b\varphi(X)V + [V_0 - V]\varphi''(X) + \frac{1}{2}bV^2 + \frac{1}{2}(1-b)U^2 \\ \qquad -2P[v,\varphi](X) - P[v](X), \\ U|_{t=0} = v_0'(s), \end{cases}$$

$U_0^+(t) := U(t, 0^+)$ is defined on the right side of the peak.

$$\frac{dU_0^+}{dt} = (b-1)U_0^+ + bV_0 + \frac{1}{2}bV_0^2 + \frac{1}{2}(1-b)(U_0^+)^2 - 2P[v,\varphi](0) - P[v](0). \quad (16)$$

By Sobolev's embedding of H^1 into L^∞, there is a positive constant C such that

$$\|v(t,\cdot)\|_{L^\infty} < C\|v(t,\cdot)\|_{H^1} < C\varepsilon. \quad (17)$$

$\Rightarrow$ *Instability of infinity time*

$\frac{dU_0^+}{dt} \leq (b-1)U_0^+ + C_b\varepsilon.$

$\Rightarrow$ *Blow up in finite time*

$\frac{dU_0^+}{dt} \leq (b-1)U_0^+ + \frac{1}{2}(1-b)(U_0^+)^2 + C_b\varepsilon.$

Instability of peaked periodic waves in CH equation

Peaked periodic wave

$$\varphi(x) = \frac{\cosh(\pi - |x|)}{\sinh(\pi)}, \quad x \in \mathbb{T},$$

The orbital stability results for the peaked periodic waves are available in $H^1(\mathbb{T})$ for the CH equation (J. Lenells, 2004).

Theorem (for $b = 2$)

For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H^1_{\mathrm{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ satisfying

$$\|u_0 - \varphi\|_{H^1(\mathbb{T})} + \|u_0' - \varphi'\|_{L^\infty(\mathbb{T})} < \delta,$$

such that the local solution $u \in C([0,T), H^1_{\mathrm{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T}))$ to the CH equation with the initial data u_0 and $T > t_0$ satisfies

$$\|u_x(t_0,\cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty(\mathbb{T})} \geq 1,$$

where $\xi(t) \in \mathbb{T}$ is a point of peak of the function $u(t,\cdot)$ on $\mathbb{T}$. Moreover, there exist u_0 such that the maximal existence time T is finite.

The article was accepted for publication (SIAM J.Math. Anal.).

Peaked periodic waves in DP ($b = 3$) equation

We do not know whether the peaked periodic waves in DP are orbitally stable in H^1 or L^2 spaces.

Three conserved quantities:

$$E_1(u) = \int_{-\pi}^{\pi} u\, dx, \quad E_2(u) = \int_{-\pi}^{\pi} yv\, dx, \quad E_3(u) = \int_{-\pi}^{\pi} u^3\, dx \tag{18}$$

where

$$y := u - u_{xx}, \quad v := (4 - \partial_x^2)^{-1} u \tag{19}$$

Consider the following variational problem:

$$\inf_{u \in L^2(\mathbb{T})} \left\{ E_2(u) : \quad E_1(u) = e_1, \quad E_3(u) = e_3 \right\}, \tag{20}$$

where e_1 and e_3 are fixed.

Lemma

The travelling wave φ is critical point of the constrained variational problem.

The travelling wave solution φ satisfies the equation

$$-c\varphi' + \varphi\varphi' + \frac{3}{4}\varphi' * \varphi^2 = 0,$$

The component $v_\varphi := (4 - \partial_x^2)^{-1}\varphi$, which solves the travelling wave reduction $v(t,x) = v_\varphi(x - t)$ of the equivalent equation :

$$v_\varphi - v_\varphi'' = \frac{1}{2c}\varphi^2 + b, \tag{21}$$

where $b = -\frac{1}{2}\operatorname{csch}(2\pi)$ and $c = \coth(\pi)$.
Lagrange functional

$$J(u) := E_2(u) + c_1[E_1(u) - e_1] + c_3[E_3(u) - e_3] \tag{22}$$

where c_1 and c_3 are the Lagrange multipliers associated with the two constraints.
Euler–Lagrange equation for J is given by

$$u - 3(4 - \partial_x^2)^{-1}u + \frac{1}{2}c_1 + \frac{3}{2}c_3 u^2 = 0. \tag{23}$$

$u = \varphi$, $v_\varphi := (4 - \partial_x^2)^{-1}\varphi$ and Lagrange multipliers given by $c_1 = \operatorname{csch}(2\pi)$ and $c_3 = -\frac{1}{3\coth(\pi)}$.

The second variation of $J(u)$ at $u = \varphi$ is given by the self-adjoint operator in $L^2(\mathbb{T})$:

$$\mathcal{L} = 1 - 3(4 - \partial_x^2)^{-1} - \frac{\varphi(x)}{c}. \tag{24}$$

This operator is given by the difference of the multiplicative operator $L_0 := 1 - c^{-1}\varphi$ and a compact operator $K := 3(4 - \partial_x^2)^{-1}$.

Lemma

The continuous spectrum of $\mathcal{L}$ is located at $\sigma_c(\mathcal{L}) = [0, 1 - \mathrm{sech}(\pi)]$ and a negative eigenvalue $\lambda_0 < 0$.

Due to the two constraints in (20), we consider $\mathcal{L}$ on the constrained subspace of $L^2(\mathbb{T})$ given by

$$X_s := \{w \in L^2(\mathbb{T}) : \quad \langle w, \varphi^2 \rangle = 0, \quad \langle w, 1 \rangle = 0\}. \tag{25}$$

Theorem

The spectrum of $\mathcal{L}|_{X_s}$ in $X_s \subset L^2(\mathbb{T})$ is positive and 0 is the endpoint of the continuous spectrum.

Summary

- We have shown instability of peaked travelling wave in $W^{1,\infty}$ provided that they are orbitally stable in H^1 for any $b > 1$.
- We have shown instability peaked periodic waves in CH equation based on orbital stability theory of J.Lenells.
- We have studied stability of peaked periodic wave in DP by using conserved quantities. However, the minimizer is degenerate. Therefore, we did not complete the proof of orbital stability.

Thank you for your attention!

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