Rogue wave potentials occurring in the sine-Gordon equation

An analysis of relatively large and sudden events happening in the physical world.

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Abstract

In this thesis we construct rogue waves occurring in the sine-Gordon equation. An algebraic method is used to find explicit solutions to a Lax pair of equations. The Lax pair being studied is compatible with solutions to the sine-Gordon equation. Rotational and librational traveling wave solutions to the sine-Gordon equation are considered in the Lax pair. The Darboux transformation is applied with the Lax pair solutions computed at the rotational and librational waves to generate algebraic solitons and rogue waves, respectively. The rogue waves occur on the end points of the Floquet-Lax spectrum bands and can achieve a magnification factor of at most 3.

1 Introduction

A rogue wave is a wave that appears suddenly with a relatively large amplitude and then disappears without trace. These freak events have been well documented in the physical world. Rogue waves have been observed on the ocean surface [3, 11], in superfluid helium [9] and in microwave cavities [12]. Physicists are currently studying rogue waves in optics because waves in optical fibres share similar mathematics with water waves [20]. For these reasons it is important to study rogue waves as solutions to important equations in mathematical physics. In doing so we may be able to predict, analyze and categorize these seemingly unforeseen events.

A lot of work has already been done in constructing and analyzing rogue wave solutions for the focusing non-linear Schrödinger equation (fNLSE). Rogue waves in the fNLSE can be found by considering the Lax spectrum for the linear Lax equations compatible with solutions to the fNLSE. In [8] Deconick and Segal found a connection between the classical Lax spectrum and stability spectrum for traveling periodic wave solutions of the fNLSE. By using this connection, one can relate rogue waves to modulation instability and the linear stability problem.

In [4, 5] Chen et al use the Darboux transformation with solutions to the spectral problem analyzed in [8] to construct rogue wave solutions of the fNLSE. They were able to find particular analytical solutions appearing on the end points of the spectral bands. The rogue wave solutions are expressed in terms of Jacobi-elliptic functions (see appendix). This method of finding exact spectral parameters and constructing rogue waves on the background of the periodic waves relies on a non-linearization of the Lax equations.

In this thesis we aim to apply a similar method to the sine-Gordon equation. This can be done because the Darboux Transformation works for an entire hierarchy of partial differential equations related to the fNLSE that includes the sine-Gordon equation. A lot of useful work has already been done by Chen and Pelinovsky in [15] in the context of rogue waves arising in the modified Korteweg–de Vries (mKDV) equation. This work is useful to us because the Lax spectral problem for the mKDV equation is identical to the Lax spectral problem for the sine-Gordon equation up to a clever change of variables.

This thesis is structured as follows. First, we introduce the sine-Gordon equation and its traveling waves in both laboratory and characterisitc coordinates. Second, we intoduce the Lax pair of equations that define the spectral problem. Third, we consider the squared eigenfunction relation to find exact spectral parameters corresponding to rogue wave and algebraic solitons on the background of periodic potentials written in terms of Jacobi-elliptic functions. Fourth, we utilize the Darboux Transformation to actually create analytical rogue waves and algebraic solitons of the sine-Gordon equation. The thesis ends with analysis and pictures of the rogue waves and algebraic solitons.

2 Sine-Gordon Equation

2.1 Laboratory Coordinates

The sine-Gordon equation in laboratory coordinates $(x, t) \in \mathbb{R}^2$ is:

$$u_{tt} - u_{xx} + \sin(u) = 0, \tag{2.1}$$

where $u(x, t)$ is a real valued function and the subscripts in x and t denote partial differentiation. Equation (2.1) has many physical applications, it is used in describing the magnetic flux in long superconducting Josephson junctions [16–18], modeling fermions [6], explaining stability structure in galaxies [13, 21, 22], analyzing mechanical vibrations of a ribbon pendulum [23] and much more.

In [14] Lu and Miller introduce the sine-Gordon equation in the form

$$\epsilon u_{tt} - \epsilon u_{xx} + \sin(u) = 0, \tag{2.2}$$

where ϵ is an arbitrarily small positive parameter and the initial value problem is considered. This scaled version of the sine-Gordon equation is useful in the theory of crystal dislocations [10], superconducting Josephson junctions [19], vibrations of DNA molecules [25] and quantum field theory [6]. Lu and Miller analyze a sequence of solutions u_n associated with a sequence ϵ_n converging to zero that satisfy the initial value problem. The set of solutions $\{u_n\}$, indexed by n, is called the fluxon condensate. The case when $\epsilon = 1$, which is (2.1), explains "defects" in the fluxon condensate. These defects can also be viewed as rogue waves on an elliptic function background [14].

2.2 Traveling Wave Reduction

Traveling wave solutions of the sine-Gordon equation are of the form

$$u(x,t) = f(x - ct), \tag{2.3}$$

where c is a real valued constant often refered to as the wave speed and $f : \mathbb{R} \to \mathbb{R}$. Substituting (2.3) into (2.1) creates an Ordinary Differential Equation (ODE) reduction of the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = (c^2 - 1)f'' + \sin(f) = 0 \tag{2.4}$$

where the prime corresponds to differentiation in $\rho := x - ct$.

Proposition 1. *The quantity*

$$E(f(\rho), f'(\rho)) := \frac{1}{2}(c^2 - 1)(f'(\rho))^2 + 1 - \cos(f(\rho)) \tag{2.5}$$

is constant in ρ. E is referred to as the "total energy" [7].

Proof.

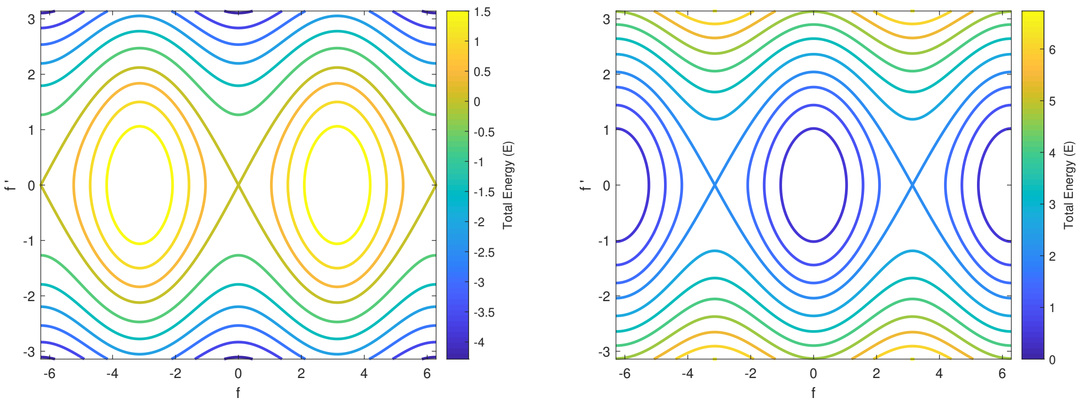
$$\begin{aligned}\frac{dE}{d\rho} &= (c^2 - 1)f'f'' + \sin(f)f' \\ &= f' \left[\, (c^2 - 1)f'' + \sin(f) \, \right] \\ &= 0,\end{aligned}$$

where the last equality is due to (2.4). □

The level sets of the energy (2.5) correspond to first order ODEs whose solutions are traveling waves of the sine-Gordon equation. Figure 1 contains Matlab plots of these aforementioned level sets.

When $c^2 > 1$ (Superliminal motion) there are three different cases for $f \in [-\pi, \pi]$. When $E \in (0, 2)$ the level curve is a periodic orbit centered around (0,0). When $E = 2$ there are two heteroclinic orbits connecting $(-\pi, 0)$ to $(\pi, 0)$. $E > 2$ yields rotational orbits. The superliminal patterns are illustrated on the (f, f')-plane in figure 1b.

When $c^2 < 1$ (Subliminal motion), it is evident from figure 1a that the transformation $f \mapsto f + \pi$ maps the phase portrait for subliminal motion to the phase portrait



(a) Subliminal: $c^2 < 1$ (b) Superliminal: $c^2 > 1$

Figure 1: Level sets of the energy function (2.5) for traveling waves (2.3) of the sine-Gordon equation in laboratory coordinates (2.1).

for superliminal motion. From this point on we will only consider the superliminal case, $c^2 > 1$.

When $c^2 > 1$ and $E \in (0, 2)$ the total energy can be integrated from $f = 0$ to $f = f_0$, where f_0 is the turning point at which $\cos(f_0) - 1 + E = 0$. This region corresponds to one quadrature of the periodic orbit around $(f, f') = (0, 0)$.

Proposition 2. *Let T_{sup} denote the period of the superliminal traveling periodic wave for $E \in (0, 2)$. Then,*

$$T_{sup} = 4\, K\left(\frac{E}{2}\right)\sqrt{c^2 - 1} \tag{2.6}$$

where $K(\cdot)$ is the complete elliptic integral of the first kind (see Appendix A) .

Proof. Separating $(f')^2 = \frac{2}{c^2-1}\left(\cos(f) - 1 + E\right)$ and integrating this aforementioned quadrature yields:

$$\Rightarrow \int_0^{\frac{T_{sup}}{4}} \frac{\sqrt{2}}{\sqrt{c^2-1}} dz = \int_0^{f_0} \frac{df}{\sqrt{\cos(f) - 1 + E}}$$

$$\Rightarrow \frac{T_{sup}}{4} \frac{\sqrt{2}}{\sqrt{c^2-1}} = \int_0^{f_0} \frac{df}{\sqrt{-2\sin^2(\frac{f}{2}) + E}}.$$

Notice that $0 = E - 2\sin^2(\frac{f_0}{2})$ by a standard trigonometric identity so that $f_0 = 2\sin^{-1}(\sqrt{\frac{E}{2}})$. Now consider the change of variables: $2\theta = f$ and $m = \frac{2}{E}$. Clearly $m \in (1, \infty)$ since $E \in (0, 2)$. These change of variables imply:

$$\Rightarrow \frac{T_{sup}}{4}\frac{\sqrt{2}}{\sqrt{c^2-1}} = \frac{2}{\sqrt{E}}\int_0^{\sin^{-1}(\sqrt{m^{-1}})}\frac{d\theta}{\sqrt{1-m\sin^2\theta}}.$$

Letting $x = \sin\theta$ and then $y = \sqrt{m}x$ gives a further simplification of the integral:

$$\begin{aligned}
&\Rightarrow \frac{T_{sup}}{4}\frac{\sqrt{2}}{\sqrt{c^2-1}} = \frac{2}{\sqrt{E}}\int_0^{\sqrt{m^{-1}}}\frac{dx}{\sqrt{(1-x^2)(1-mx^2)}}\\
&\Rightarrow \frac{T_{sup}}{4}\frac{\sqrt{2}}{\sqrt{c^2-1}} = \frac{2}{\sqrt{mE}}\int_0^{1}\frac{dy}{\sqrt{(1-\frac{y^2}{m})(1-y^2)}}\\
&\Rightarrow \frac{T_{sup}}{4}\frac{1}{\sqrt{c^2-1}} = K\left(\frac{E}{2}\right)
\end{aligned}$$

This yields the expression (2.6). □

Proposition 3. *Solutions to the quadrature (2.5) for $c^2 > 1$ can be written in the form*

$$\cos(f(\rho)) = 1 + \beta sn^2(h\rho, k), \tag{2.7}$$

where α, β, h and k are constants depending on c and E according to table 1 and sn is a Jacobi elliptic function (see appendix A) [7].

Name	E	β	h	k
Rotational	$E > 2$	-2	$\sqrt{\frac{E}{2(c^2-1)}}$	$\sqrt{\frac{2}{E}}$
Librational	$0 < E < 2$	$-E$	$\sqrt{\frac{1}{(c^2-1)}}$	$\sqrt{\frac{E}{2}}$

Table 1: Coefficients of the exact solutions (2.7)

Proof. I will begin by verifying entries in the first row of table 1. Differentiating (2.7) and then squaring both sides yields

$$(f'(\rho))^2 = \frac{4(-2)^2 h^2 sn^2(h\rho,k)cn^2(h\rho,k)dn^2(h\rho,k)}{\sin^2(f(\rho))}. \tag{2.8}$$

Substituting equations (2.8) and (2.7) into the RHS of (2.5) and then replacing some of the constants with their values in the first row of table 1 means that

$$\frac{1}{2}(c^2-1)(f'(\rho))^2 + 1 - \cos(f(\rho)) = \frac{4Esn^2(h\rho,k)cn^2(h\rho,k)dn^2(h\rho,k)}{\sin^2(f(\rho))} + 2sn^2(h\rho,k).$$

By using (A.1) and (A.2) in the appendix, it is evident from equation (2.7) that

$$\begin{aligned}\sin^2(f) &= 1 - cos^2(f)\\ &= 4sn^2(h\rho;k)cn^2(h\rho;k)\end{aligned} \tag{2.9}$$

so that equation (2.8) becomes

$$(f'(\rho))^2 = 4h^2 dn^2(h\rho,k) \tag{2.10}$$

and then

$$\begin{aligned}\frac{1}{2}(c^2-1)(f'(\rho))^2 + 1 - \cos(f(\rho)) &= Edn^2(h\rho,k) + 2sn^2(h\rho,k)\\ &= E,\end{aligned} \tag{2.11}$$

which is equation (2.5) for rotational waves.

I will now verify entries in the second row of table 1. Again, differentiating (2.7) and squaring both sides, then replacing $\sin^2(f)$ with $1-\cos^2(f)$, then replacing $\cos(f)$ with equation (2.7) and then substituting entries in the second row of table 1, I obtain

$$\begin{aligned}(f'(\rho))^2 &= \frac{4E^2h^2sn^2(h\rho,k)cn^2(h\rho,k)dn^2(h\rho,k)}{1-\cos^2(f(\rho))}\\ &= \frac{4E^2h^2sn^2(h\rho,k)cn^2(h\rho,k)dn^2(h\rho,k)}{-\beta(\beta sn^4(h\rho,k)+2sn^2(h\rho,k))}\\ &= \frac{-4Eh^2cn^2(h\rho,k)dn^2(h\rho,k)}{-2(1-k^2sn^2(h\rho,k))},\end{aligned}$$

that is,

$$(f'(\rho))^2 = 2Eh^2 cn^2(h\rho, k). \tag{2.12}$$

Substituting equations (2.12) and (2.7) into the RHS of equation (2.5) implies

$$\begin{aligned} \frac{1}{2}(c^2-1)(f'(\rho))^2 + 1 - \cos(f(\rho)) &= Ecn^2(h\rho;k) + Esn^2(h\rho;k) \\ &= E, \end{aligned}$$

that is equation (2.5) for librational waves.

□

2.3 Lorentz transformation and c-invariant solutions

The Lorentz transformation for $c^2 > 1$,

$$\hat{x} = \frac{x - ct}{\sqrt{c^2-1}} \ , \ \ \hat{t} = \frac{t - cx}{\sqrt{c^2-1}} \tag{2.13}$$

can be used to show that traveling wave solutions of the sine-Gordon equation are invariant with respect to the wave speed c. Applying the chain rule to $u(x,t)$ with (2.13) gives

$$u_{xx} = \frac{1}{c^2-1}\left(u_{\hat{x}\hat{x}} - 2cu_{\hat{x}\hat{t}} + c^2 u_{\hat{t}\hat{t}}\right) \tag{2.14}$$

and

$$u_{tt} = \frac{1}{c^2-1}\left(u_{\hat{t}\hat{t}} - 2cu_{\hat{t}\hat{x}} + c^2 u_{\hat{x}\hat{x}}\right), \tag{2.15}$$

where the subscripts in $\hat{t}$ and $\hat{x}$ denote partial derivatives. Letting

$$u(x,t) = \pi + \hat{u}(\hat{x},\hat{t}) \tag{2.16}$$

implies that

$$\sin(u) = -\sin(\hat{u}). \tag{2.17}$$

Substituting equations (2.14), (2.15) and (2.17) into the sine-Gordon equation yields

$$\hat{u}_{\hat{t}\hat{t}} - \hat{u}_{\hat{x}\hat{x}} + \sin(\hat{u}) = 0, \tag{2.18}$$

which has the same form as equation (2.1). Substituting a solution of the form $\hat{u} = \hat{f}(\hat{x}) - \pi$ into (2.18) yields the ordinary differential equation

$$\hat{f}'' + \sin(\hat{f}) = 0, \tag{2.19}$$

which is equivalent to setting $c^2 - 1 = 1$ in equation (2.4). This generates traveling wave solutions of the sine-Gordon equation of the form $u(x,t) = \hat{f}(\frac{x-ct}{\sqrt{c^2-1}})$ with $c^2 > 1$.

2.4 Change of Coordinates $(x,t) \rightarrow (\xi,\eta)$

I will now introduce a convenient change of variables. Consider the characteristic coordinates (ξ,η) defined by

$$\xi = \frac{1}{2}(x+t) \ \ , \ \ \eta = \frac{1}{2}(x-t) \tag{2.20}$$

with inverse transform

$$x = \xi + \eta \ \ , \ \ t = \xi - \eta. \tag{2.21}$$

Applying the chain rule to $u(x,t)$ with (2.20) gives

$$u_{xx} = \frac{1}{4}\left(u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta}\right) \tag{2.22}$$

and

$$u_{tt} = \frac{1}{4}\left(u_{\xi\xi} - 2u_{\eta\xi} + u_{\eta\eta}\right), \tag{2.23}$$

where the subscripts in η and ξ denote partial derivatives. Substituting (2.23) and (2.22) into equation (2.1) produces the sine-Gordon equation in characteristic coordinates

$$u_{\xi\eta} = \sin(u). \tag{2.24}$$

Now consider a traveling wave in (ξ, η) given by $u(\xi, \eta) = \tilde{f}(\xi - \eta)$ where $\tilde{f}$: $\mathbb{R} \to \mathbb{R}$. Substitution of $\tilde{f}(\xi - \eta)$ into (2.24) yields

$$\tilde{f}'' + \sin(\tilde{f}) = 0, \tag{2.25}$$

where the prime in this case indicates a derivative in $\xi - \eta$. Comparing equation (2.25) with (2.19) means that we can set $\tilde{f} = \hat{f}$. From this point onwards we will refer to solutions of equation (2.19) as "the traveling wave" and denote it by f. Unless specified otherwise, we will take f to be in the traveling frame $z := \xi - \eta$.

Notice also that equation (2.25) is equivalent to setting $c^2 - 1 = 1$ in equation (2.4). Therefore, equations (2.10) and (2.12) imply that

$$(f'(z))^2 = \frac{4}{k^2} dn^2(\frac{z}{k}; k) \tag{2.26}$$

for the rotational traveling wave and

$$(f'(z))^2 = 4k^2 cn^2(z; k) \tag{2.27}$$

for the librational traveling wave.

At this point it may seem as though the characteristic coordinates are redundant because we are left analyzing the same ODE to generate traveling waves. The importance of these new variables will become apparent in the following sections. In particular, characteristic coordinates convert the linear Lax equations in laboratory coordinates to a more simple set of equations analogous to those in [15].

3 Lax pair

3.1 Lax pair of sine-Gordon in laboratory coordinates

In [7] Deconick et al used a pair of Lax equations that are compatible for solutions of the sine-Gordon equation in laboratory coordinates. The compatibility condition, $\chi_{xy} = \chi_{yx}$, of the following Lax pair of linear equations, (3.1) and (3.2), is equation (2.1):

$$\frac{\partial}{\partial x}\chi = \begin{bmatrix} \frac{-i\gamma}{2} + \frac{i\cos(u)}{8\gamma} & \frac{i\sin(u)}{8\gamma} - \frac{1}{4}(u_x + u_t) \\ \frac{i\sin(u)}{8\gamma} + \frac{1}{4}(u_x + u_t) & \frac{i\gamma}{2} - \frac{i\cos(u)}{8\gamma} \end{bmatrix} \chi \tag{3.1}$$

and

$$\frac{\partial}{\partial t}\chi = \begin{bmatrix} \frac{-i\gamma}{2} - \frac{i\cos(u)}{8\gamma} & -\frac{i\sin(u)}{8\gamma} - \frac{1}{4}(u_x + u_t) \\ -\frac{i\sin(u)}{8\gamma} + \frac{1}{4}(u_x + u_t) & \frac{i\gamma}{2} + \frac{i\cos(u)}{8\gamma} \end{bmatrix} \chi. \tag{3.2}$$

Here $\gamma \in \mathbb{C}$ is a spectral parameter of (2.1) and $\chi = (p, q)^T$ is an eigenfunction in variables (x, t). The Lax spectrum of (2.1) is defined as the set of admissable values of γ. This compatability structure is essential because it gives us the tools necessary to apply the Darboux Transformation. The Darboux transformation, which I will introduce in greater detail later, generates a new solution to the sine-Gordon equation when given a solution u to the sine-Gordon equation and its corresponding solution χ to the Lax equations (3.1) and (3.2) with a particular value of spectral parameter γ.

This Lax pair is also useful for analyzing the stability spectrum of periodic solutions. Deconick et al found a connection between the Lax spectrum, the set of admissible values of γ, and the stability spectrum in [7]. The inherited integrabilty from a compatabile Lax pair is also essential in the development of algebraic methods for finding exact expressions of rogue waves and solving initial value problems.

3.2 Lax pair in characteristic coordinates

Proposition 4. *The following Lax pair is compatible for solutions of the sine-Gordon equation in characteristic coordinates (2.24):*

$$\frac{\partial}{\partial \xi}\begin{bmatrix} p \\ q \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \lambda & -u_\xi \\ u_\xi & -\lambda \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} \tag{3.3}$$

$$\frac{\partial}{\partial \eta}\begin{bmatrix} p \\ q \end{bmatrix} = \frac{1}{2\lambda}\begin{bmatrix} \cos(u) & \sin(u) \\ \sin(u) & -\cos(u) \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} \tag{3.4}$$

*where $\lambda \in \mathbb{C}$ is the spectral parameter of (2.24) and $\chi = (p, q)^T$ is an eigenfunction in variables (ξ, η). The Lax spectrum of (2.24) is defined as the set of admissable values of λ. Equation (3.3) is referred to as the sine-Gordon spectral problem (**SGSP**).*

Proof. Differentiating the first entry of (3.3) in η and then substituting (3.4) yields

$$\Rightarrow p_\xi = \frac{1}{2}(\lambda p - u_\xi q)$$
$$\Rightarrow p_{\xi\eta} = \frac{\lambda p_\eta}{2} - \frac{u_{\xi\eta} q + u_\xi q_\eta}{2}$$
$$\Rightarrow p_{\xi\eta} = \frac{p\cos(u)}{4} + \frac{q\sin(u)}{4} - \frac{u_{\xi\eta}q}{2} - \frac{u_\xi p\sin(u)}{4\lambda} + \frac{u_\xi q\cos(u)}{4\lambda}. \tag{3.5}$$

On the other hand, differentiating the first entry of (3.4) in ξ and then substituting (3.3) yields

$$\Rightarrow p_\eta = \frac{1}{2\lambda}(p\cos(u) + q\sin(u))$$
$$\Rightarrow p_{\eta\xi} = \frac{1}{2\lambda}(p_\xi\cos(u) - p\sin(u)u_\xi + q_\xi\sin(u) + q\cos(u)u_\xi)$$
$$\Rightarrow p_{\eta\xi} = \frac{p\cos(u)}{4} + \frac{u_\xi q\cos(u)}{4\lambda} - \frac{pu_\xi\sin(u)}{4\lambda} - \frac{q\sin(u)}{4}. \tag{3.6}$$

Therefore (3.5) is equal to (3.6) if and only if $u_{\eta\xi} = \sin(u)$.

Differentiating the second entry of (3.3) in η and then substituting (3.4) yields

$$\Rightarrow q_\xi = \frac{1}{2}(u_\xi p - \lambda q)$$
$$\Rightarrow q_{\xi\eta} = \frac{u_{\xi\eta}p}{2} + \frac{u_\xi p_\eta}{2} - \frac{\lambda q_\eta}{2}$$
$$\Rightarrow q_{\xi\eta} = \frac{u_{\xi\eta}p}{2} + \frac{u_\xi p\cos(u)}{4\lambda} + \frac{u_\xi q\sin(u)}{4\lambda} - \frac{p\sin(u)}{4} + \frac{q\cos(u)}{4}. \tag{3.7}$$

On the other hand, differentiating the second entry of (3.4) in ξ and then substituting (3.3) yields

$$\Rightarrow q_\eta = \frac{p\sin(u)}{2\lambda} - \frac{q\cos(u)}{2\lambda}$$
$$\Rightarrow q_{\eta\xi} = \frac{p_\xi\sin(u)}{2\lambda} + \frac{p\cos(u)u_\xi}{2\lambda} - \frac{q_\xi\cos(u)}{2\lambda} + \frac{q\sin(u)u_\xi}{2\lambda}$$
$$\Rightarrow q_{\eta\xi} = \frac{p\sin(u)}{4} + \frac{u_\xi q\sin(u)}{4\lambda} + \frac{pu_\xi\cos(u)}{4\lambda} + \frac{q\cos(u)}{4}. \tag{3.8}$$

Similarily (3.7) is equal to (3.8) if and only if $u_{\xi\eta} = \sin(u)$.

□

3.3 A transformation between the Lax spectra

Proposition 5. *The spectral parameter γ in laboratory coordinates, (3.1)-(3.2), is related to the spectral parameter λ in characteristic coordinates, (3.3)-(3.4), by*

$$\lambda = -2i\gamma. \tag{3.9}$$

Proof. Consider the Lax system in characteristic coordinates (3.3) - (3.4). Applying the chain rule and substituting (3.9) in place of λ tells us that

$$\begin{aligned}
\frac{\partial}{\partial x}\chi &= \frac{1}{2}\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\chi \\
&= \left(\frac{1}{4}\begin{bmatrix} \lambda & -u_\xi \\ u_{xi} & -\lambda \end{bmatrix} + \frac{1}{4\lambda}\begin{bmatrix} \cos(u) & \sin(u) \\ \\ sin(u) & -\cos(u) \end{bmatrix}\right)\chi \\
&= \begin{bmatrix} \frac{-i\gamma}{2} + \frac{i\cos(u)}{8\gamma} & \frac{-(u_x+u_t)}{4} + \frac{i\sin(u)}{8\gamma} \\ \frac{(u_x+u_t)}{4} + \frac{i\sin(u)}{8\gamma} & \frac{i\gamma}{2} - \frac{i\cos(u)}{8\gamma} \end{bmatrix}\chi
\end{aligned} \tag{3.10}$$

and also

$$\begin{aligned}
\frac{\partial}{\partial t}\chi &= \frac{1}{2}\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)\chi \\
&= \left(\frac{1}{4}\begin{bmatrix} \lambda & -u_\xi \\ u_\xi & -\lambda \end{bmatrix} - \frac{1}{4\lambda}\begin{bmatrix} \cos(u) & \sin(u) \\ \sin(u) & -\cos(u) \end{bmatrix}\right)\chi \\
&= \begin{bmatrix} \frac{-i\gamma}{2} - \frac{i\cos(u)}{8\gamma} & \frac{-(u_x+u_t)}{4} - \frac{i\sin(u)}{8\gamma} \\ \frac{(u_x+u_t)}{4} - \frac{i\sin(u)}{8\gamma} & \frac{i\gamma}{2} + \frac{i\cos(u)}{8\gamma}. \end{bmatrix}\chi.
\end{aligned} \tag{3.11}$$

Equations (3.10) and (3.11) match (3.1) and (3.2). This means that (3.9) represents a mapping from the Lax spectrum in variable γ to the Lax spectrum in variable λ. □

4 Algebraic Method

4.1 What is an algebraic method?

The purpose of the algebraic method is to relate the nonlinear PDE and its Lax pair in order to obtain an explicit expression for particular eigenvalues of the Lax equations. The eigenvalues that we will derive explicitly correspond to the end points of the spectral bands of the Floquet spectrum, which will be elaborated on in section 4.5.

In [4, 5] Chen et al use an algebraic method to find particular eigenvalues for a spectral problem related to the fNLSE. Their method involves non-linearizing the Lax equations that are compatible for solutions of the fNLSE into two Hamiltonian systems and considering a squared eigenfunction relation between eigenfunctions of the spectral problem and a solution of the fNLSE. The Hamiltonian systems are then written as another Lax equation that yields a polynomial function whose roots are exact eigenvalues of the Lax spectrum. They complete the algebraic method by relating constants of this polynomial to parameters of Jacobi elliptic functions. They found that these eigenvalues correspond to endpoints of the spectral bands of the Lax spectrum. This algebraic method relies heavily on the squared eigenfunction relation and symplectic structure of the fNLSE. Finally, they use the spectral parameters and their associated eigenfunctions to construct rogue wave solutions to the fNLSE.

In [15] Chen and Pelinovsky use an analogous algebraic method by considering the symplectic structure and squared eigenfunction relation for the mKDV equation. Once again they were able to write particular eigenvalues in terms of parameters of Jacobi-elliptic functions that define traveling waves. These eigenvalues for the mKDV also correspond to end points of the associated Lax spectrum. Finally, they construct rogue waves and algebraic solitons from the eigenvalues and their associated eigenfunctions.

One of the Lax equations in [15] for the mKDV equation is almost identical to equation (3.3). This resemblance suggests that a similar algebraic method can be used for the sine-Gordon equation. We will refer to the term $-u_\xi$, where u us a solution to the sine-Gordon equation (2.24) as the **potential**. For the traveling waves it follows that the potential is $-u_\xi = -f'(\xi - \eta)$.

4.2 Hamiltonian of Linear Lax Equation

Assume that (p_1, q_1) is a solution to equation (3.3) for a fixed eigenvalue of λ_1. We proceed as in [15], which contains the analogous spectral problem for the mKDV equation, by looking for admissible eigenvalues λ_1 of the SGSP for which

$$-u_\xi = p_1^2 + q_1^2. \tag{4.1}$$

First I will use (4.1) to non-linearize the SGSP into a Hamiltonian system with Hamiltonian $H(p_1, q_1)$. The Hamiltonian system is generated by canonical equations of motion:

$$\frac{dq_1}{d\xi} = -\frac{\partial H}{\partial p_1} \tag{4.2}$$

$$\frac{dp_1}{d\xi} = \frac{\partial H}{\partial q_1} \tag{4.3}$$

so that equations (3.3) and (4.1) determine H from

$$\begin{cases} 2\frac{\partial H}{\partial q_1} = \lambda_1 p_1 + (p_1^2 + q_1^2) q_1 \\ 2\frac{\partial H}{\partial p_1} = (p_1^2 + q_1^2) p_1 + \lambda_1 q_1. \end{cases} \tag{4.4}$$

Integrating the first equation of system (4.4) in q_1 gives

$$2H(p_1, q_1) = \lambda p_1 q_1 + \frac{1}{4}(p_1^2 + q_1^2)^2 + r(p_1), \tag{4.5}$$

where r is some function of p_1. This means that $2\frac{\partial H}{\partial p_1} = \lambda q_1 + (p_1^2 + q_1^2)p + r'(p_1)$ which implies that $r'(p_1) = 0$ when compared to (4.4). Without loss of generality I can set $r(p_1) = 0$ and introduce a constant F_0 so that

$$2H(p_1, q_1) = \lambda_1 p_1 q_1 + \frac{1}{4}(p_1^2 + q_1^2)^2 := \frac{1}{4}F_0. \tag{4.6}$$

This nonlinearization of the SGSP will be very useful in the development of the algebraic method. We note that the Hamiltonian introduced here shows a lot of resemblance to the Hamiltonian in [15] for the mKDV equation.

4.3 Computing λ_1

In this subsection we will find an explicit expression for λ_1 in terms of the elliptic modulus k for the travelling waves. In the next subsection I will take this explicit expression and show that the eigenfunctions exist in the traveling frame $z := \xi - \eta$. This will complete the algebraic method.

Proposition 6. *Suppose that* (λ_1, p_1, q_1) *is a solution to the SGSP with traveling periodic potential,*$-f'(\xi-\eta)$*, that satisfies (4.1), then*

$$E = \lambda_1^2 + \frac{F_0}{2} + 1. \tag{4.7}$$

Proof. Differentiating equation (2.25) in $z = \xi - \eta$ yields

$$f''' + \cos(f) f' = 0. \tag{4.8}$$

Comparing (4.8) with (2.5) eliminates $\cos(f)$ and produces the third order equation

$$f''' = f'(E-1) - \frac{1}{2}(f')^3. \tag{4.9}$$

Next I will differentiate (4.1) up to the third order.

$$\begin{aligned}
&\Rightarrow -f'' = 2p_1p_1' + 2q_1q_1' \\
&\Rightarrow -f'' = p_1(\lambda_1 p_1 - q_1 f') + q_1(p_1 f' - \lambda_1 q_1) \quad by\ (3.3) \\
&\Rightarrow -f'' = \lambda_1(p_1^2 - q_1^2) \\
&\Rightarrow -f''' = \lambda_1(p_1 2p_1' - q_1 2q_1') \\
&\Rightarrow -f''' = \lambda_1(p_1(\lambda_1 p_1 - q_1 f') - q_1(p_1 f' - \lambda_1 q_1)) \quad by\ (3.3) \\
&\Rightarrow f''' = \lambda_1^2 f' + 2\lambda_1 f' p_1 q_1 \quad by\ (4.1)\ ,
\end{aligned}$$

where the prime on the eigenfunction denotes differentiation in ξ. We can eliminate the dependence on p_1 and q_1 by substituting equation (4.6) into the above expression deriving

$$f''' = \lambda_1^2 f' + \frac{F_0}{2} f' - \frac{1}{2}(f')^3. \tag{4.10}$$

Comparing (4.10) with (4.9) completes the proof.

□

We have verified the following identities:

$$\begin{cases} \frac{1}{4}(p_1^2 + q_1^2)^2 + \lambda_1 p_1 q_1 = \frac{1}{4}F_0 \\ p_1^2 + q_1^2 = -f' \\ \lambda_1(p_1^2 - q_1^2) = -f''. \end{cases} \tag{4.11}$$

This list of useful equations will be important in the algebra and analysis that follows.

We now require another relation between parameters λ_1, F_0 and E if we wish to close an expression for λ_1. I can achieve this by developing another third order expression for f. To do so I will consider the following matrices and corresponding Lax equation for the Hamiltonian:

$$Q(\lambda) = \begin{bmatrix} \lambda & p^2 + q^2 \\ -p^2 - q^2 & -\lambda \end{bmatrix} \tag{4.12}$$

$$W(\lambda) = \begin{bmatrix} W_{11}(\lambda) & W_{12}(\lambda) \\ W_{12}(-\lambda) & -W_{11}(-\lambda) \end{bmatrix} \tag{4.13}$$

where:

$$W_{11}(\lambda) = 1 - \frac{p_1 q_1}{\lambda - \lambda_1} + \frac{p_1 q_1}{\lambda + \lambda_1} \tag{4.14}$$

$$= 1 - \frac{F_0 - (f')^2}{2(\lambda^2 - \lambda_1^2)} \tag{4.15}$$

$$W_{12}(\lambda) = \frac{p_1^2}{\lambda - \lambda_1} + \frac{q_1^2}{\lambda + \lambda_1} \tag{4.16}$$

$$= \frac{-\lambda f' - f''}{\lambda^2 - \lambda_1^2}. \tag{4.17}$$

The expressions (4.15) and (4.17) for W_{11} and W_{12} were derived from (4.14) and (4.16) using (4.11). Pelinovsky and Chen consider very similar matrices to Q and W in [15]. We will proceed from here as in [15] by using Q and W to create a new Lax equation compatible for the SGSP and then compute the determinant of W.

Proposition 7. *The SGSP with constraint (4.1) is satisfied if and only if the following Lax equation is satisfied for every* $\lambda \neq \pm\lambda_1$*:*

$$2\frac{d}{d\xi}W(\lambda) = Q(\lambda)W(\lambda) - W(\lambda)Q(\lambda), \tag{4.18}$$

where λ *is the spectral parameter of the SGSP.*

Proof. I will only show this for elements (1,1) and (1,2). Symmetry will handle the other two components.

For (1,1) we have that

$$\begin{aligned}
2\frac{d}{d\xi}W_{11}(\lambda) &= -\frac{2p_1'q_1 + 2p_1q_1'}{\lambda - \lambda_1} + \frac{2p_1'q_1 + 2p_1q_1'}{\lambda + \lambda_1} \\
&= -\frac{(\lambda_1 p_1 + (p_1^2 + q_1^2)q_1)q_1 + p_1(p_1(p_1^2 + q_1^2) + \lambda_1 q_1)}{\lambda - \lambda_1} \\
&+ \frac{(\lambda_1 p_1 + (p_1^2 + q_1^2)q_1)q_1 + p_1(p_1(p_1^2 + q_1^2) + \lambda_1 q_1)}{\lambda + \lambda_1} \\
&= \frac{q_1^4}{\lambda + \lambda_1} - \frac{p_1^4}{\lambda + \lambda_1} + \frac{p_1^4}{\lambda - \lambda_1} - \frac{q_1^4}{\lambda - \lambda_1}
\end{aligned}$$

On the other hand from the (1,1) element of the RHS of (4.18) it follows that

$$\begin{aligned}
&\lambda W_{11}(\lambda) + (p_1^2 + q_1^2)W_{12}(-\lambda) - (\lambda W_{11}(\lambda) - (p_1^2 + q_1^2)W_{12}(\lambda)) \\
&= (p_1^2 + q_1^2)\left(W_{12}(-\lambda) + W_{12}(\lambda)\right) \\
&= (p_1^2 + q_1^2)(\frac{-p_1^2}{\lambda + \lambda_1} - \frac{q_1^2}{\lambda - \lambda_1} + \frac{p_1^2}{\lambda - \lambda_1} + \frac{q_1^2}{\lambda + \lambda_1}) \\
&= \frac{q_1^4}{\lambda + \lambda_1} - \frac{p_1^4}{\lambda + \lambda_1} + \frac{p_1^4}{\lambda - \lambda_1} - \frac{q_1^4}{\lambda - \lambda_1}.
\end{aligned}$$

Hence the two expressions are identical to each other. For (1,2) we have that

$$\begin{aligned}
2\frac{d}{d\xi}W_{12}(\lambda) &= 2\left(\frac{2p_1p_1'}{\lambda - \lambda_1} + \frac{2q_1q_1'}{\lambda + \lambda_1}\right) \\
&= 2\left(\frac{2p_1p_1'}{\lambda - \lambda_1} + \frac{2q_1q_1'}{\lambda + \lambda_1}\right) \\
&= 2\left(\frac{p_1(\lambda_1 p_1 + (p_1^2 + q_1^2)q_1)}{\lambda - \lambda_1} - \frac{q_1(p_1(p_1^2 + q_1^2) + \lambda_1 q_1)}{\lambda + \lambda_1}\right) \\
&= \frac{2\,\lambda_1\,p_1^2}{\lambda - \lambda_1} + \frac{2\,p_1\,q_1^3}{\lambda - \lambda_1} + \frac{2\,p_1^3\,q_1}{\lambda - \lambda_1} - \frac{2\,\lambda_1\,q_1^2}{\lambda + \lambda_1} - \frac{2\,p_1\,q_1^3}{\lambda + \lambda_1} - \frac{2\,p_1^3\,q_1}{\lambda + \lambda_1}
\end{aligned}$$

The (1,2) element from the right hand side of (4.18) yields

$$2\,\lambda\,\left(\frac{q_1^2}{\lambda+\lambda_1}+\frac{p_1^2}{\lambda-\lambda_1}\right)-\left(2\,p_1^2+2\,q_1^2\right)\,\left(\frac{p_1\,q_1}{\lambda+\lambda_1}-\frac{p_1\,q_1}{\lambda-\lambda_1}+1\right)$$
$$=\frac{2\,\lambda\,p_1^2}{\lambda-\lambda_1}-2\,q_1^2-2\,p_1^2+\frac{2\,p_1\,q_1^3}{\lambda-\lambda_1}+\frac{2\,p_1^3\,q_1}{\lambda-\lambda_1}+\frac{2\,\lambda\,q_1^2}{\lambda+\lambda_1}-\frac{2\,p_1\,q_1^3}{\lambda+\lambda_1}-\frac{2\,p_1^3\,q_1}{\lambda+\lambda_1}$$
$$=\frac{2\,\lambda\,p_1^2}{\lambda-\lambda_1}-2\,\frac{q_1^2(\lambda+\lambda_1)}{\lambda+\lambda_1}-2\,\frac{p_1^2(\lambda-\lambda_1)}{\lambda-\lambda_1}+\frac{2\,p_1\,q_1^3}{\lambda-\lambda_1}+\frac{2\,p_1^3\,q_1}{\lambda-\lambda_1}+\frac{2\,\lambda\,q_1^2}{\lambda+\lambda_1}-\frac{2\,p_1\,q_1^3}{\lambda+\lambda_1}-\frac{2\,p_1^3\,q_1}{\lambda+\lambda_1}$$
$$=\frac{2\,\lambda_1\,p_1^2}{\lambda-\lambda_1}+\frac{2\,p_1\,q_1^3}{\lambda-\lambda_1}+\frac{2\,p_1^3\,q_1}{\lambda-\lambda_1}-\frac{2\,\lambda_1\,q_1^2}{\lambda+\lambda_1}-\frac{2\,p_1\,q_1^3}{\lambda+\lambda_1}-\frac{2\,p_1^3\,q_1}{\lambda+\lambda_1}$$

Again, the two expressions are identical to each other. □

Proposition 8. *Suppose that* (λ_1, p_1, q_1) *is a solution to the SGSP with traveling potential,* $f(\xi-\eta)$*, that satisfies (4.1), then*

$$F_0^2 = 4E(E-2). \tag{4.19}$$

Proof. The determinant of $W(\lambda)$ from (4.13) is computed as

$$\begin{aligned}
det[W(\lambda)] &= -[W(\lambda)]^2 - W_{12}(\lambda)W_{12}(-\lambda)\\
&= \frac{-\lambda^2+{\lambda_1}^2+4\,\lambda_1\,p_1\,q_1+p_1^4+2\,p_1^2\,q_1^2+q_1^4}{\lambda^2-{\lambda_1}^2}\\
&= -1+\frac{4\lambda_1 p_1 q_1+(p_1^2+q_1^2)^2}{\lambda^2-\lambda_1^2}\\
&= -1+\frac{F_0}{\lambda^2-\lambda_1^2}.
\end{aligned}$$

This means that the determinant only admits simple poles. Using the f, f', f'' formulations of $W_{11}(\lambda)$ and $W_{12}(\lambda)$ as (4.15) and (4.17) respectively instead of the eigenfunction formulations (4.14) and (4.16) we arrive at the following expression for the determinant:

$$det[W(\lambda)] = -\left(\frac{F_0-(f')^2}{2\,\lambda^2-2\,{\lambda_1}^2}-1\right)^2-\frac{((f'')+\lambda\,(f'))\,((f'')-\lambda\,(f'))}{\left(\lambda^2-{\lambda_1}^2\right)^2}$$

$$= -\frac{(F_0 - (f')^2 - 2(\lambda^2 - \lambda_1^2))^2}{4(\lambda^2 - \lambda_1^2)^2} - \frac{4((f'') + \lambda(f'))((f'') - \lambda(f'))}{4(\lambda^2 - \lambda_1^2)^2}$$
$$= \frac{-F_0^2 + 4\,F_0\,\lambda^2 - 4\,F_0\,{\lambda_1}^2 + 2\,F_0\,(f')^2 - 4\,\lambda^4 + 8\,\lambda^2\,{\lambda_1}^2}{4(\lambda^2 - \lambda_1^2)^2}$$
$$- \frac{-4\,{\lambda_1}^4 + 4\,{\lambda_1}^2\,(f')^2 - (f')^4 - 4\,(f'')^2}{4(\lambda^2 - \lambda_1^2)^2}$$

Therefore, the determinant will have double poles at $\pm\lambda_1$ unless

$$-F_0^2 + (2F_0 + 4\lambda_1^2)(f')^2 - (f')^4 = 4(f'')^2. \tag{4.20}$$

We can proceed further by subsituting (2.25) and (2.5) into a simple trigonometric identity

$$\sin^2(f) + \cos^2(f) = (f'')^2 + \left(\frac{1}{2}(f')^2 + 1 - E\right)\left(\frac{1}{2}(f')^2 + 1 - E\right) = 1$$

so that

$$4(f'')^2 = -(f')^4 + 4E(f')^2 - 4(f')^2 + 8E - 4E^2. \tag{4.21}$$

Comparing (4.20) with (4.21) and using (4.7) completes the proof.

□

Proposition 9. *Suppose that (λ_1, p_1, q_1) is a solution to the SGSP with traveling potential $-f'(\xi - \eta)$ that satisfies (4.1), then*

$$\lambda_1^2 = E \mp \sqrt{E(E-2)} - 1 \tag{4.22}$$

and

$$F_0 = \pm 2\sqrt{E(E-2)} \tag{4.23}$$

where E can be written in terms of the elliptic modulus k: $E = \frac{2}{k^2}$ for the rotational traveling wave and $E = 2k^2$ for the librational traveling wave.

Proof. This follows immediately after taking the square root from equation (4.19) and substituting it into equation (4.7). □

Substituting $k^2 = \frac{2}{E}$ into equation (4.22) creates a more compact expression for the eigenvalues corresponding to rotational waves:

$$\begin{aligned} \lambda_{1R} &= \pm\sqrt{\frac{2 - k^2 \pm 2\sqrt{1-k^2}}{k^2}} \\ &= \pm\frac{1 \pm \sqrt{1-k^2}}{k}, \end{aligned} \tag{4.24}$$

where the second choice of sign corresponds to the sign choice after taking the square root of equation (4.19).

Substituting $k^2 = \frac{E}{2}$ for librational waves into (4.22) creates a more compact expression for λ_1 for librational waves:

$$\begin{aligned} \lambda_{1L} &= \pm\sqrt{2k^2 - 1 \pm 2ik\sqrt{1-k^2}} \\ &= \pm(k \pm i\sqrt{1-k^2}), \end{aligned} \tag{4.25}$$

where the second choice of sign depends on the sign choice after taking the square root of equation (4.19).

4.4 Traveling Eigenfunctions

Proposition 10. *Let $u(\xi, \eta) = f(\xi - \eta)$ be the traveling wave solution to the sine-Gordon equation (2.24) and (λ_1, p_1, q_1) a solution to the Lax pair (3.3) - (3.4) with $-f' = p_1^2 + q_1^2$. Then $\varphi = \varphi(\xi - \eta)$ where $\varphi = (p_1, q_1)$.*

Proof. The first equation in (3.4) together with (2.24), (2.5) and then (4.7) implies

$$\begin{aligned} &\Rightarrow 2\lambda_1 p_{1\eta} = p_1\left(\frac{1}{2}(f')^2 + 1 - E\right) - q_1 f'' \\ &\Rightarrow 2\lambda_1 p_{1\eta} = p_1\left(\frac{1}{2}(f')^2 - \lambda_1^2 - \frac{F_0}{2}\right) - q_1 f'' \end{aligned}$$

Now using the identies in (4.11) yields

$$\begin{aligned} &\Rightarrow 2\lambda_1 p_{1\eta} = -2\lambda_1 p_1^2 q_1 + (p_1^2 - q_1^2)\lambda_1 q_1 - \lambda_1^2 p_1 \\ &\Rightarrow 2p_{1\eta} = -p_1\lambda_1 - (p_1^2 + q_1^2)q_1 \\ &\Rightarrow 2p_{1\eta} = -p_1\lambda_1 + f' q_1. \end{aligned}$$

Comparing this with (3.3) means that $-p_{1\eta} = p_{1\xi}$ with a general solution $p_1 = p_1(\xi - \eta)$. The second equation in (3.4) together with (2.24), (2.5) and then (4.7) implies

$$\Rightarrow 2\lambda_1 q_{1\eta} = -q_1 \left(\frac{1}{2}(f')^2 + 1 - E\right) - p_1 f''$$
$$\Rightarrow 2\lambda_1 q_{1\eta} = -q_1 \left(\frac{1}{2}(f')^2 - \lambda_1^2 - \frac{F_0}{2}\right) - p_1 f''$$

Now using the identies in (4.11) yields

$$\Rightarrow 2\lambda_1 q_{1\eta} = \lambda_1(p_1^2 - q_1^2)p_1 + 2\lambda_1 p_1 q_1^2 + q_1\lambda_1^2$$
$$\Rightarrow 2q_{1\eta} = q_1\lambda_1 + (p_1^2 + q_1^2)p_1$$
$$\Rightarrow 2q_{1\eta} = q_1\lambda_1 - (f')p_1$$

Comparing this with (3.3) means that $-q_{1\eta} = q_{1\xi}$ with a general solution $q_1 = q_1(\xi - \eta)$. □

4.5 A numerical construction of the Lax spectrum

If the entries of the SGSP are periodic with the same period L then Floquet's Theorem guarantees that bounded solutions of the linear equation (3.3) can be represented in the form

$$\begin{pmatrix} p_1(\xi) \\ q_1(\xi) \end{pmatrix} = \begin{pmatrix} \breve{p}_1(\xi) \\ \breve{q}_1(\xi) \end{pmatrix} e^{i\mu\xi} , \tag{4.26}$$

where $\breve{p}_1(\xi + L) = \breve{p}_1(\xi)$, $\breve{q}_1(\xi + L) = \breve{q}_1(\xi)$ and $\mu \in [-\frac{\pi}{L}, \frac{\pi}{L}]$. The spectrum is invariant with respect to the other coordinate η, so we are allowed to set $\eta = 0$. Substituting (4.26) into the SGSP (3.3) yields the eigenvalue problem

$$\begin{pmatrix} 2\frac{d}{d\xi} + 2i\mu & f' \\ f' & -2\frac{d}{d\xi} - 2i\mu \end{pmatrix} \begin{pmatrix} \breve{p}_1 \\ \breve{q}_1 \end{pmatrix} = \lambda \begin{pmatrix} \breve{p}_1 \\ \breve{q}_1 \end{pmatrix} . \tag{4.27}$$

The numerical scheme involves discretizing the eigenfunction domain $[0, L]$ and admissable μ values $[-\frac{\pi}{L}, \frac{\pi}{L}]$ so that (4.27) becomes an eigenvalue and eigenvector problem for each μ that can be handled using Matlab's eig() function. The derivative operator $\frac{d}{d\xi}$ is replaced with a 12^{th} order finite difference matrix. The union of each set of eigenvalues associated for each μ defines the periodic Lax spextrum.

Interestingly, the end points of the spectral bands correspond to (4.22).

Figure 2 has the numerically constructed Lax spectra for the rotational and librational traveling waves using certain values of k. The code used to generate these figures can be found in Appendix B. The parameter N defines the resolution of the eigenfunction domain and the parameter $floqnum$ defines the resolution of the parameter μ.

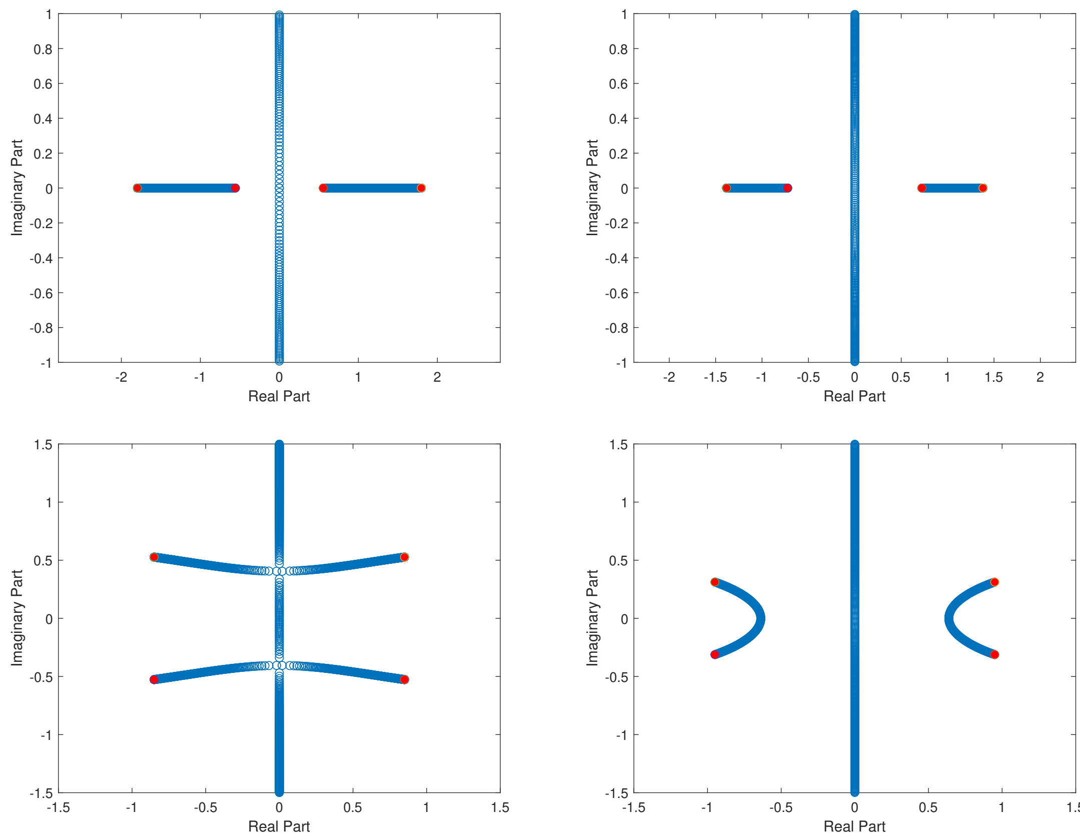


Figure 2: The periodic spectrum of the SGSP (4.27) using the rotational (top) and librational (bottom) waves as the potentials with $k = 0.85$ (left) and $k = 0.95$ (right). Red dots represent eigenvalues (4.22).

5 New solutions to the sine-Gordon equation

5.1 Darboux Transformation (DT)

The one-fold Darboux transformation for the sine-Gordon equation is

$$\hat{w} = w + \frac{4\lambda pq}{p^2 + q^2}, \tag{5.1}$$

where (λ, p, q) is a solution to the SGSP with compatible potential $w := -u_\xi$. The DT generates a new potential $\hat{w} = -\hat{u}_\xi$ to the linear Lax equations (3.3) and (3.4) where $\hat{u}$ is a new solution to the sine-Gordon equation [15]. Applying (5.1) with (λ_1, p_1, q_1) found in the algebraic method of section 4 yields

$$\begin{aligned}\hat{w} &= w + \frac{4\lambda_1 p_1 q_1}{p_1^2 + q_1^2} \\ &= w + \frac{F_0 - w^2}{w} \\ &= \frac{F_0}{w},\end{aligned} \tag{5.2}$$

where equations (4.11) were used. Equation (5.2) is valid for the rotational waves because the Jacobi-elliptic dn function is never zero. Therefore, expression (2.26) together with the Jacobi elliptic identity (A.3), expression (4.23) and the identities for k in table 1 allow us to re-write equation (5.2) for the rotational traveling waves as

$$\begin{aligned}\hat{w}_R &= \frac{F_0}{w} \\ &= \pm\frac{\sqrt{E(E-2)}k}{\sqrt{1-k^2}} dn(\frac{z}{k} + K(k); k) \\ &= \pm\sqrt{\frac{4}{k^2}\left(\frac{1-k^2}{k^2}\right)}\sqrt{\frac{k^2}{1-k^2}} dn(\frac{z}{k} + K(k); k) \\ &= \pm\frac{2}{k} dn(\frac{z}{k} + K(k); k),\end{aligned} \tag{5.3}$$

which is simply a translated and reflected version of the rotational potential $w = \frac{2}{k} dn(\frac{z}{k}; k)$ that we started with. The choice of sign in equation (5.3) is the same as the choice of sign in (4.23).

The term F_0 is purely imaginary for the librational traveling wave. In addition, the denominator in equation (5.2) may vanish at some points if w is the cn librational potential. The two-fold Darboux transformation is required to generate new potentials with librational waves.

Let (p_1, q_1, λ_1) and (p_2, q_2, λ_2) be solutions to the system (3.3) and (3.4) with $\lambda_1 \neq \lambda_2$ and $\lambda_1 \neq -\lambda_2$, then the two-fold Darboux transformation takes the form:

$$\hat{w} = w + \frac{4(\lambda_1^2 - \lambda_2^2)[\lambda_1 p_1 q_1 (p_2^2 + q_2^2) - \lambda_2 p_2 q_2 (p_1^2 + q_1^2)]}{(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2) - 2\lambda_1\lambda_2[4p_1 q_1 p_2 q_2 + (p_1^2 - q_1^2)(p_2^2 - q_2^2)]}, \tag{5.4}$$

where $w = -u_\xi$ and $\hat{w} = -\hat{u}_\xi$ with $\hat{u}$ being a new solution to the sine-Gordon equation [15]. F_0 for librational waves is purely imaginary so that $F_0 = -\bar{F}_0$. Therfore, the two-fold DT for librational traveling waves with (λ_1, p_1, q_1) as in (4.11) and $(\lambda_2, p_2, q_2) = (\bar{\lambda}_1, \bar{p}_1, \bar{q}_1)$ becomes

$$\begin{aligned}
\hat{w}_L &= w + \frac{4(\lambda_1^2 - \lambda_2^2)[\lambda_1 p_1 q_1 (p_2^2 + q_2^2) - \lambda_2 p_2 q_2 (p_1^2 + q_1^2)]}{(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2) - 2\lambda_1\lambda_2[4p_1 q_1 p_2 q_2 + (p_1^2 - q_1^2)(p_2^2 - q_2^2)]} \\
&= w + \frac{(\lambda_1^2 - \lambda_2^2)[(F_0 - w^2)\bar{w} - (\bar{F}_0 - \bar{w}^2)w]}{(\lambda_1^2 + \lambda_2^2)w\bar{w} - 2[\frac{1}{4}(F_0 - w^2)(\bar{F}_0 - w^2) + (w')(\bar{w}')]} \qquad (5.5) \\
&= w + \frac{F_0(\bar{F}_0 - F_0)w}{(2E - 2)w^2 - 2[\frac{1}{4}(-F_0^2 + w^4) + (w')^2]} \\
&= w + \frac{F_0(\bar{F}_0 - F_0)w}{(F_0 + 2\lambda_1^2)w^2 - \frac{1}{2}(-F_0^2 + w^4) - 2(w')^2} \quad by \quad (4.7) \\
&= w + \frac{F_0(\bar{F}_0 - F_0)w}{F_0^2} \quad by \ (4.20) \\
&= -w.
\end{aligned}$$

We were able to replace $\bar{w}$ with w because w is purely real. The new solution $\hat{w}_L$ is simply a reflected version of the previous solution w.

5.2 Second Linearly Independent Eigenfunctions of SGSP

Applying the Darboux Transformations with (λ_1, p_1, q_1) does not generate new solutions to the sine-Gordon equation since $\hat{w}$ is either a translated or reflected version of w. Another solution to the Lax equations with the same λ_1 is required if we want to avoid a trivial transformation. New linearly independent Lax eigenfunctions were constructed in [4, 5, 15] by considering the Wronksian. We will use second linearly independent eigenfunctions for the rotational waves that are similar to those found in [4, 5] for the fNLSE:

$$\begin{aligned} \hat{p}_1 &= p_1\phi_R - \frac{q_1}{p_1^2 + q_1^2} \\ \hat{q}_1 &= q_1\phi_R + \frac{p_1}{p_1^2 + q_1^2}, \end{aligned} \tag{5.6}$$

here the Wronskian between (p_1, q_1) and $(\hat{p}_1, \hat{q}_1)$ is normalized to 1. I have introduced the function $\phi_R : (\xi, \eta) \to \mathbb{C}$ for rotational waves.

The representation (5.6) is non-singular for the rotational waves because $w = p_1^2 + q_1^2 = -f'(\xi - \eta)$ with sign-definite f. On the other hand, the representation (5.6) is singular for librational waves because $w = -f'(\xi - \eta)$ may vanish in some points. In order to avoid singularities in eigenfunctions for librational waves we will consider the second linearly independent eigenfunctions like those in [15] used for the mKDV equation of the form

$$\begin{aligned} \hat{p}_1 &= \frac{\phi_L - 1}{q_1} \\ \hat{q}_1 &= \frac{\phi_L + 1}{p_1}, \end{aligned} \tag{5.7}$$

where the Wronskian is normalized to 2. Here the denominators are nonzero everywhere because if either p_1 or q_1 vanish in some points, the last two equations of system (4.11) yield a contradiction with real f and complex λ_1. I have introduced the function $\phi_L : (\xi, \eta) \to \mathbb{C}$ for librational waves.

In Sections 6 and 7, I will find expressions for ϕ_R and ϕ_L by substituting (5.6) and (5.7) into the Lax equations with $\lambda = \lambda_1$. While the eigenfunctions (p_1, q_1) are bounded and periodic, we should expect that ϕ is non-periodic and unbounded, so that the solution $(\hat{p}_1, \hat{q}_1)$ is non-periodic and unbounded hence a trivial transformation is avoided. For this reason the function ϕ determines the growth of the rogue wave constructed after $(\hat{p_1}, \hat{q_1})$ is substituted into the DT.

6 The growth of rotational waves

6.1 Computing ϕ_R

Proposition 11. *The function ϕ_R determined by (5.6) is given by:*

$$\phi_R(\xi, \eta) = -\int_0^{\xi-\eta} \frac{F_0}{2(f')^2} dz + \frac{\xi+\eta}{2}, \tag{6.1}$$

where f is the rotational wave with $-f' = p_1^2 + q_1^2$ and F_0 is given by (4.23).

Proof. Since (λ_1, p_1, q_1) is a solution to the linear Lax equations for potential $u_\xi = f'$ it follows that

$$2\frac{\partial}{\partial \xi}\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & -f' \\ f' & -\lambda_1 \end{bmatrix}\begin{bmatrix} p_1 \\ q_1 \end{bmatrix}. \tag{6.2}$$

In order for $\hat{p}_1$ and $\hat{q}_1$ from (5.6) to be eigenfunctions of the Lax equation we want

$$2\frac{\partial}{\partial \xi}\begin{bmatrix} \hat{p}_1 \\ \hat{q}_1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & -f' \\ f' & -\lambda_1 \end{bmatrix}\begin{bmatrix} \hat{p}_1 \\ \hat{q}_1 \end{bmatrix}. \tag{6.3}$$

Differentiating (5.6) in ξ produces the equation

$$2\frac{\partial \hat{p}_1}{\partial \xi} == 2p_{1\xi}\phi_R + 2p_1\phi_{R\xi} - \frac{2q_{1\xi}}{(p_1^2+q_1^2)} + \frac{2q_1(2p_1p_{1\xi} + 2q_1q_{1\xi})}{(p_1^2+q_1^2)^2}.$$

Using (6.2) and (4.11) we can continue on and obtain:

$$2\frac{\partial \hat{p}_1}{\partial \xi} == (\lambda_1 p_1 - f'q_1)\phi_R + 2p_1\phi_{R\xi} - \frac{p_1 f' - \lambda_1 q_1}{(p_1^2+q_1^2)} - \frac{2q_1(f'')}{(p_1^2+q_1^2)^2}. \tag{6.4}$$

Using (6.3) and then (5.6) we also have that

$$\begin{aligned} 2\frac{\partial \hat{p}_1}{\partial \xi} &= \lambda_1\hat{p}_1 - f'\hat{q}_1 \\ &= \lambda_1\left(p_1\phi_R - \frac{q_1}{p_1^2+q_1^2}\right) - f'\left(q_1\phi_R + \frac{p_1}{p_1^2+q_1^2}\right). \end{aligned} \tag{6.5}$$

Setting (6.4) equal to (6.5) it follows that

$$\Rightarrow 0 = 2p_1\phi_{R\xi} + \frac{2\lambda_1 q_1}{p_1^2+q_1^2} - \frac{2q_1 f''}{(p_1^2+q_1^2)^2}.$$

Now using (4.11) implies that

$$\Rightarrow 0 = \phi_{R\xi} + \frac{2\lambda_1 p_1 q_1}{(p_1^2+q_1^2)^2}$$
$$\Rightarrow 0 = \phi_{R\xi} + \frac{1}{(f')^2}\left(\frac{F_0-(f')^2}{2}\right).$$

Therefore, the expression for ϕ_ξ is equal to

$$\phi_{R\xi} = -\frac{1}{2}\left(\frac{F_0-(f')^2}{(f')^2}\right). \tag{6.6}$$

Again since (λ_1, p_1, q_1) is a solution to the linear Lax system for rotational waves it follows that

$$2\lambda_1\frac{\partial}{\partial\eta}\begin{bmatrix}p_1\\ q_1\end{bmatrix} = \begin{bmatrix}\cos(f) & \sin(f)\\ \sin(f) & -\cos(f)\end{bmatrix}\begin{bmatrix}p_1\\ q_1\end{bmatrix}. \tag{6.7}$$

In order for $\hat{p}$ and $\hat{q}$ from (5.6) to be eigenfunctions of the Lax equation we want

$$2\lambda_1\frac{\partial}{\partial\eta}\begin{bmatrix}\hat{p}_1\\ \hat{q}_1\end{bmatrix} = \begin{bmatrix}\cos(f) & \sin(f)\\ \sin(f) & -\cos(f)\end{bmatrix}\begin{bmatrix}\hat{p}_1\\ \hat{q}_1\end{bmatrix}. \tag{6.8}$$

Differentiating (5.6) in η produces the equation

$$2\lambda_1\frac{\partial\hat{p}_1}{\partial\eta} = 2\lambda_1 p_{1\eta}\phi_R + 2\lambda_1 p_1\phi_{R\eta} - \frac{2\lambda_1 q_{1\eta}}{(p_1^2+q_1^2)} + \frac{2q_1(2\lambda_1 p_{1\eta}p_1 + 2\lambda_1 q_{1\eta}q_1)}{(p_1^2+q_1^2)^2}.$$

Using equation (6.7) I can continue the above calculations by replacing $2\lambda_1 q_{1\eta}$ and $2\lambda_1 p_{1\eta}$,

$$\begin{aligned}2\lambda_1\frac{\partial\hat{p}_1}{\partial\eta} &= (p_1\cos(f)+q_1\sin(f))\phi_R + 2\lambda_1 p_1\phi_{R\eta} - \frac{(p_1\sin(f)-q_1\cos(f))}{(p_1^2+q_1^2)}\\ &+\frac{2q_1((p_1\cos(f)+q_1\sin(f))p_1 + (p_1\sin(f)-q_1\cos(f))q_1)}{(p_1^2+q_1^2)^2}.\end{aligned} \tag{6.9}$$

On the other hand using equations (5.6) and (6.8) we arrive at

$$
\begin{aligned}
2\lambda_1 \frac{\partial \hat{p}_1}{\partial \eta} &= \hat{p}_1 \cos(f) + \hat{q}_1 \sin(f) \\
&= \left(p_1 \phi_R - \frac{q_1}{p_1^2 + q_1^2} \right) \cos(f) + \left(q_1 \phi_R + \frac{p_1}{p_1^2 + q_1^2} \right) \sin(f). \qquad (6.10)
\end{aligned}
$$

Setting (6.9) equal to (6.10) leaves us with

$$
\Rightarrow \lambda_1 \phi_{R\eta} + \frac{2q_1 p_1 \cos(f) + (q_1^2 - p_1^2) \sin(f)}{(p_1^2 + q_1^2)^2} = 0.
$$

Therefore, the expression for $\phi_{R\eta}$ is equal to

$$
\lambda_1 \phi_{R\eta} = \frac{(p_1^2 - q_1^2) \sin(f) - 2q_1 p_1 \cos(f)}{(p_1^2 + q_1^2)^2}. \qquad (6.11)
$$

Equation (6.11) can be simplfied using expressions (4.11),

$$
\Rightarrow 2\lambda_1^2 \phi_{R\eta} = \frac{-2f'' \sin(f) - F_0 \cos(f) + (f')^2 \cos(f)}{(f')^2}.
$$

We can proceed further by recalling formulae (2.25) and (2.5) to replace $\sin(f)$ and $\cos(f)$

$$
\begin{aligned}
\Rightarrow 2\lambda_1^2 \phi_{R\eta} &= \frac{2(f'')^2 - F_0(\frac{1}{2}(f')^2 + 1 - E) + (f')^2(\frac{1}{2}(f')^2 + 1 - E)}{(f')^2} \\
\Rightarrow 2\lambda_1^2 \phi_{R\eta} &= \frac{2(f'')^2}{(f')^2} - \frac{F_0}{2} - \frac{F_0}{(f')^2} + \frac{F_0 E}{(f')^2} + \frac{(f')^2}{2} + 1 - E. \qquad (6.12)
\end{aligned}
$$

To deal with f'' we consider the following simple trigonometric identity together with (2.25) and (2.5):

$$
\sin^2(f) + \cos^2(f) = (f'')^2 + \left(\frac{1}{2}(f')^2 + 1 - E \right) \left(\frac{1}{2}(f')^2 + 1 - E \right) = 1.
$$

This implies that

$$\Rightarrow 2\frac{(f'')^2}{(f')^2} = -\frac{(f')^2}{2} + 2E - 2 + \frac{4E}{(f')^2} - \frac{2E^2}{(f')^2}. \tag{6.13}$$

Substituting (6.13) into (6.12) yields

$$2\lambda^2\phi_{R\eta} = \left(E - 1 - \frac{F_0}{2}\right) + \frac{1}{(f')^2}\left(4E - 2E^2 - F_0 + F_0 E\right). \tag{6.14}$$

Therefore ϕ_R is found from the following system of PDEs:

$$\begin{cases} 2\lambda_1^2\phi_{R\eta} = \left(E - 1 - \frac{F_0}{2}\right) + \frac{1}{(f')^2}\left(4E - 2E^2 - F_0 + F_0 E\right) \\ \phi_{R\xi} = -\frac{1}{2}\left(\frac{F_0 - (f')^2}{(f')^2}\right). \end{cases} \tag{6.15}$$

The connection formulae (4.7) and (4.19) allow us to simplify the first equation in (6.15) so that

$$\begin{cases} \phi_{R\eta} = \frac{F_0}{2(f')^2} + \frac{1}{2} \\ \phi_{R\xi} = -\frac{F_0}{2(f')^2} + \frac{1}{2} \end{cases} \tag{6.16}$$

and these two partial derivatives can be combined into

$$\phi_{R\eta} + \phi_{R\xi} = 1. \tag{6.17}$$

so that

$$\phi_R(\xi, \eta) = \eta + g(\xi - \eta), \tag{6.18}$$

for some function g to be determined. Taking a derivative in ξ yields

$$\phi_{R\xi} = g'(\xi - \eta). \tag{6.19}$$

Comparing (6.19) with the second equation in (6.16) and integrating from 0 to $z = \xi - \eta$ completes the proof of the explicit expression (6.1).

□

6.2 Analytical Properties of ϕ_R

The term ϕ_R is going to dictate the non-periodic dynamics of newly formed potentials from the one-fold DT because all of the other terms in the second linearly independent eigenfunction (5.6) are bounded, smooth and periodic. We will see in later sections that the new potentials constructed with the second eigenfunctions reach their Jacobi-elliptic backgrounds as $|\phi_R|$ grows to infinity. We will prove in this subsection that $|\phi_R|$ grows to infinity along trajectories moving away from a particular line in the (ξ, η) plane.

Substituting (2.26) into (6.1) gives :

$$\phi_R(\xi, \eta) = \frac{\xi + \eta}{2} - \frac{F_0 k^2}{8} \int_0^{\xi - \eta} \frac{dz}{dn^2(\frac{z}{k}; k)} \tag{6.20}$$

In the following lemmas, I will study the behaviour of $\phi_R(\xi, \eta)$ as a function of (ξ, η).

Lemma 1. $\phi_R(\xi, \eta) \in C^\infty(\mathbb{R})$

Proof. This result is obvious since $\frac{\xi+\eta}{2} \in C^\infty(\mathbb{R}^2)$ and $\frac{1}{dn^2(\frac{z}{k};k)} \in C^\infty(\mathbb{R})$ □

The integrand $\frac{1}{dn^2(\frac{z}{k};k)}$ can be written as a Fourier series because it is smooth and L-periodic for some constant L:

$$\frac{1}{dn^2(\frac{z}{k}, k)} = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n z}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n z}{L}\right), \tag{6.21}$$

where $L = 2kK(k)$ is the smallest period and a_0 is a constant.

Using this Fourier representation of the integrand it follows that

$$\phi_R(\xi, \eta) = \frac{\xi + \eta}{2} - \frac{F_0 a_0 k^2 (\xi - \eta)}{8} + \Phi_R(\xi - \eta) \tag{6.22}$$

$$= \xi \left(\frac{1}{2} - \frac{F_0 a_0 k^2}{8}\right) + \eta \left(\frac{1}{2} + \frac{F_0 a_0 k^2}{8}\right) + \Phi_R(\xi - \eta), \tag{6.23}$$

where $\Phi_R(z)$ is the integral of the Fourier series with zero mean value, hence it is periodic and bounded. This means that $\phi_R(\xi,\eta) = \Phi_R(\xi-\eta)$ when $(\xi,\eta) \in \Omega$ where I have defined Ω to be the line

$$\Omega := \left\{ (\xi,\eta) \in \mathbb{R}^2 \;\;:\;\; \xi\left(\frac{1}{2} - \frac{F_0 a_0 k^2}{8}\right) + \eta\left(\frac{1}{2} + \frac{F_0 a_0 k^2}{8}\right) = 0 \right\}. \qquad (6.24)$$

Let $d_2(u,v)$ denote the standard euclidean distance between two points in $\mathbb{R}^2$ and define for any $u \in \mathbb{R}^2$:

$$d(\Omega,u) := \inf\left\{ d_2(v,u) \;\mid\; v \in \Omega \right\}. \qquad (6.25)$$

Lemma 2. *Let $\nu(s) = (\nu_1(s), \nu_2(s))$ be some curve in $\mathbb{R}^2$ parametrized by $s \in \mathbb{R}$ and suppose that $d(\Omega,\nu(s)) \le B_1 \;\; \forall s \in \mathbb{R}$ and constant $B_1 \ge 0$. It follows that $|\phi_R(\nu(s))| \le B_2$ for some constant $B_2 > 0$ and $\forall s \in \mathbb{R}$.*

Proof. Since Ω is closed and $d(\Omega,\nu(s)) \le B_1$ we can write $\nu(s) = (\breve{\xi}(s), \breve{\eta}(s)) + (\beta_1(s), \beta_2(s))$, where $d_2\left((\beta_1(s),\beta_2(s)),0\right) = d(\Omega,\nu(s)) \le B_1$ and $(\breve{\xi}(s),\breve{\eta}(s)) \in \Omega$ is the point for which the infimum of the distances in the definition of $d(\Omega,\nu(s))$ occurs, this point exists in Ω because Ω is closed. Therefore $|\beta_1(s)|, |\beta_2(s)| \le B_1$ for all s and

$$\begin{aligned}
\Rightarrow |\phi_R(\nu(s))| &= \left| (\breve{\xi}(s) + \beta_1(s))\left(\frac{1}{2} - \frac{F_0 a_0 k^2}{8}\right) + (\breve{\eta}(s) + \beta_2(s))\left(\frac{1}{2} + \frac{F_0 a_0 k^2}{8}\right) + \Phi_R(\nu(s)) \right| \\
&= \left| \beta_1(s)\left(\frac{1}{2} - \frac{F_0 a_0 k^2}{8}\right) + \beta_2(s)\left(\frac{1}{2} + \frac{F_0 a_0 k^2}{8}\right) + \Phi_R(\nu(s)) \right| \\
&\le |B_1|\left|\frac{1}{2} - \frac{F_0 a_0 k^2}{8}\right| + |B_1|\left|\frac{1}{2} + \frac{F_0 a_0 k^2}{8}\right| + |\Phi_R(\nu(s))|.
\end{aligned}$$

□

Lemma 2 is also true when $d(\Omega^{||},\nu(s)) \le B_1$, where $\Omega^{||}$ defines any line in (ξ,η) parallel Ω and $B_1 \ge 0$.

Lemma 3. *$|\phi_R(\xi'+\delta,\eta')| \to \infty$ as $\delta \to \pm\infty$ uniformly in $(\xi',\eta') \in \Omega$ in the sense that : $\forall\, M > 0$, $\exists\, \epsilon > 0$ such that $|\phi_R(\xi' \pm \kappa, \eta')| > M$ for every (ξ',η') whenever $\kappa > \epsilon$. Likewise, $|\phi_R(\xi',\eta'+\delta)| \to \infty$ as $\delta \to \pm\infty$ uniformly in $(\xi',\eta') \in \Omega$.*

Proof. Pick an arbitrary $(\xi', \eta') \in \Omega$ and subtitute it into the absolute value of (6.23) :

$$|\phi(\xi', \eta')| = \left| \xi' \left(\frac{1}{2} - \frac{F_0 a_0}{8E} \right) + \eta' \left(\frac{1}{2} + \frac{F_0 a_0}{8E} \right) + \Gamma(z) \right| \tag{6.26}$$

Perturbing this result in ξ by δ gives :

$$\begin{aligned}
|\phi(\xi' + \delta, \eta')| &= \left| (\xi' + \delta) \left(\frac{1}{2} - \frac{F_0 a_0 k^2}{8} \right) + \eta' \left(\frac{1}{2} + \frac{F_0 a_0 k^2}{8} \right) + \Phi_R(z) \right| \\
&= \left| \xi' \left(\frac{1}{2} - \frac{F_0 a_0 k^2}{8} \right) + \eta' \left(\frac{1}{2} + \frac{F_0 a_0 k^2}{8} \right) + \delta \left(\frac{1}{2} - \frac{F_0 a_0 k^2}{8} \right) + \Phi_R(z) \right| \\
&= \left| 0 + \delta \left(\frac{1}{2} - \frac{F_0 a_0 k^2}{8} \right) + \Phi_R(z) \right| \quad since \ \ (\xi', \eta') \in \Omega \\
&= \left| \delta \left(\frac{1}{2} - \frac{F_0 a_0 k^2}{8} \right) + \Phi_R(z) \right| \\
&= |\delta| \ \left| k_1 + \frac{\Phi_R(z)}{\delta} \right|
\end{aligned}$$

where I have convienently defined $k_1 = \left(\frac{1}{2} - \frac{F_0 a_0 k^2}{8} \right)$. Since Φ_R is bounded this clearly diverges to plus infinity as $\delta \to \pm\infty$. This divergence is independent on the choice of (ξ', η') because $|\delta||k_1 + \frac{\Gamma(z)}{\delta}|$ only depends on constants and bounded Φ_R. Likewise, perturbing $|\phi(\xi', \eta')|$ in η by δ gives the other result.

□

Lemma 4. $\forall\ M > 0$, $\exists\ \epsilon > 0$ *such that* $|\phi(\xi^*, \eta^*)| > M$ *whenever* $(\xi^*, \eta^*) \in \mathbb{R}^2 \backslash \Delta_\epsilon$ *where :*

$$\Delta_\epsilon = \left\{ \bigcup_{(\xi', \eta') \in \Omega} [(\xi' - \epsilon, \xi' + \epsilon) \times \eta'] \right\}. \tag{6.27}$$

Also, $\forall\ M > 0$, $\exists\ \epsilon > 0$ *such that* $|\phi(\xi^*, \eta^*)| > M$ *whenever* $(\xi^*, \eta^*) \in \mathbb{R}^2 \backslash \Lambda_\epsilon$ *where :*

$$\Lambda_\epsilon = \left\{ \bigcup_{(\xi', \eta') \in \Omega} [\xi' \times (\eta' - \epsilon, \eta' + \epsilon)] \right\}. \tag{6.28}$$

Proof. This is just a restatement of Lemma 3.

□

Theorem 1. *Let $\sigma(s) : [0, \infty) \to \mathbb{R}^2$ be a curve such that $d(\Omega, \sigma(s)) \to \infty$ as $s \to \infty$, than $|\phi(\sigma(s))| \to \infty$ as $s \to \infty$.*

Proof. First pick $M > 0$. Using this same M construct Λ_ϵ and Δ_ϵ from Lemma 4. Since $d(\Omega, \sigma(s)) \to \infty$ as $s \to \infty$ it follows that $\sigma(s)$ will eventually be contained in $\mathbb{R}^2 \backslash \Lambda_\epsilon$ or $\mathbb{R}^2 \backslash \Delta_\epsilon$.

□

We will see in later sections that algebraic solitons propogate along Ω.

7 The growth of librational waves

7.1 Computing ϕ_L

Proposition 12. *The function ϕ_L defined in (5.7) is given by:*

$$\phi_L(\xi, \eta) = (F_0 - (f')^2) \left(\frac{\eta}{2\lambda_1} + \int_0^{\xi - \eta} \frac{2\lambda_1 (f')^2 dz}{(F_0 - (f')^2)^2} \right), \tag{7.1}$$

where f is the librational wave with $-f' = p_1^2 + q_1^2$ and F_0 is given by (4.23).

Proof. Consider again (6.2) and (6.3) but with $\hat{p}_1$ and $\hat{q}_1$ given by (5.7). Applying product and quotient rule to (5.7) and then substituting (6.2) gives

$$\begin{aligned} 2\frac{\partial \hat{p}_1}{\partial \xi} &= \frac{2(\phi_{L\xi} q_1 - (\phi_L - 1) q_{1\xi})}{q_1^2} \\ &= \frac{2\phi_{L\xi} q_1 - (\phi_L - 1)(p_1 f' - \lambda_1 q_1)}{q_1^2} \\ &= \frac{2\phi_{L\xi} p_1 q_1 - \phi_L p_1^2 f' + \phi_L \lambda_1 p_1 q_1 + p_1^2 f' - \lambda_1 p_1 q_1}{p_1 q_1^2}. \end{aligned} \tag{7.2}$$

On the other hand from (5.7) and (6.3) we have that

$$\begin{aligned} 2\frac{\partial \hat{p}_1}{\partial \xi} &= \lambda_1 \hat{p}_1 - f' \hat{q}_1 \\ &= \frac{p_1 q_1 \lambda_1 \phi_L - p_1 q_1 \lambda_1 - q_1^2 f' \phi_L - q_1^2 f'}{p_1 q_1^2} \end{aligned} \tag{7.3}$$

Setting (7.2) equal to (7.3) yields

$$\Rightarrow 2\phi_{L\xi}p_1q_1 - \phi_L p_1^2 f' + \phi_L\lambda_1 p_1 q_1 + p_1^2 f' - \lambda_1 p_1 q_1 = p_1q_1\lambda_1\phi_L - p_1q_1\lambda_1 - q_1^2 f'\phi_L - q_1^2 f'$$

$$\Rightarrow 2\phi_{L\xi} = f'\phi_L\left(\frac{p_1}{q_1} - \frac{q_1}{p_1}\right) - f'\left(\frac{p_1}{q_1} + \frac{q_1}{p_1}\right)$$

$$\Rightarrow 2\phi_{L\xi} = -f'\phi_L\left(\frac{q_1^2 - p_1^2}{p_1q_1}\right) - f'\left(\frac{p_1^2 + q_1^2}{p_1q_1}\right) \tag{7.4}$$

Substitution of the useful identities (4.11) into the above result produces

$$\Rightarrow 2\phi_{L\xi} = (-f')\phi\left(\frac{\frac{f''}{\lambda_1}}{\frac{F_0-(f')^2}{4\lambda_1}}\right) + \left(\frac{(f')^2}{\frac{F_0-(f')^2}{4\lambda_1}}\right)$$

$$\Rightarrow \phi_\xi = \left(\frac{2(f')(f'')\phi}{(f')^2 - F_0}\right) - \left(\frac{2\lambda_1(f')^2}{(f')^2 - F_0}\right) \;. \tag{7.5}$$

Differentiating (5.7) and then substituting (6.7) yields

$$\begin{aligned} 2\lambda_1\frac{\partial \hat{p_1}}{\partial \eta} &= \frac{2\lambda_1(\phi_{L\eta}q_1 - (\phi_L - 1)q_{1\eta})}{q_1^2} \\ &= \frac{2\lambda_1\phi_{L\eta}q_1 - (p_1\sin(f) - q_1\cos(f))(\phi_L - 1)}{q_1^2} \\ &= \frac{2\lambda_1\phi_{L\eta}p_1q_1 - p_1^2\phi_L\sin(f) + p_1^2\sin(f) + p_1q_1\phi_L\cos(f) - p_1q_1\cos(f)}{p_1q_1^2}. \end{aligned} \tag{7.6}$$

On the other hand, from (6.8) and (5.7) we have that

$$\begin{aligned} 2\lambda_1\frac{\partial \hat{p}_1}{\partial \eta} &= \hat{p}_1\cos(f) + \hat{q}_1\sin(f) \\ &= \left(\frac{\phi_L - 1}{q_1}\right)\cos(f) + \left(\frac{\phi_L + 1}{p_1}\right)\sin(f) \\ &= p_1q_1\left(\frac{\phi_L - 1}{p_1q_1^2}\right)\cos(f) + q_1^2\left(\frac{\phi_L + 1}{p_1q_1^2}\right)\sin(f). \end{aligned} \tag{7.7}$$

Setting (7.7) equal to (7.6) generates

$$\phi_{L\eta} = \frac{(q_1^2 - p_1^2)\sin(f) + (p_1^2 + q_1^2)\phi_L \sin(f)}{2\lambda_1 p_1 q_1}.$$

Then using (2.25) and (4.11) simplifies $\phi_{L\eta}$ to

$$\lambda_1 \phi_{L\eta} = \frac{2(f'')^2 - 2\lambda_1 (f')\phi_l f''}{((f')^2 - F_0)}. \tag{7.8}$$

Thus, ϕ_L is found from the system of PDEs given by (7.5) and (7.8).

System (7.5) and (7.8) can be simplfied further with the transformation

$$\phi_L = [F_0 - (f')^2]\Upsilon. \tag{7.9}$$

Differentiating (7.9) in ξ and comparing it to (7.5) yields

$$\Upsilon_\xi = \frac{2\lambda_1 (f')^2}{(F_0 - (f')^2)^2}. \tag{7.10}$$

Differentiating (7.9) in η and comparing it with (7.8) yields

$$\lambda_1 \Upsilon_\eta = -\frac{2(f'')^2}{(F_0 - (f')^2)^2}. \tag{7.11}$$

If follows from (4.7), (4.19) and (4.21) that

$$\lambda_1(\Upsilon_\xi + \Upsilon_\eta) = \frac{1}{2}, \tag{7.12}$$

which implies that $\Upsilon(\xi, \eta) = \frac{\eta}{2\lambda_1} + g(\xi - \eta)$ for some function g to be determined. Substituting this into (7.5) yields

$$g'(\xi - \eta) = \frac{2\lambda_1 (f')^2}{(F_0 - (f')^2)^2},$$

so that

$$g(\xi - \eta) = \int_0^{\xi - \eta} \frac{2\lambda_1 (f')^2}{(F_0 - (f')^2)^2}.$$

Therefore, Υ can be written explicitly as

$$\Upsilon(\xi,\eta) = \frac{\eta}{2\lambda_1} + \int_0^{\xi-\eta} \frac{2\lambda_1 (f')^2 dz}{(F_0 - (f')^2)^2}. \tag{7.13}$$

Reverting back to ϕ_L via change of coordinate (7.9) completes the proof. $\square$

7.2 Analytical Properties of ϕ_L

The term ϕ_L is going to dictate the non-periodic dynamics of newly formed potentials from the two-fold DT because all of the other terms in the second linearly independent eigenfunction (5.7) are bounded, smooth and periodic. We will see in later sections that the new potentials constructed with the second eigenfunctions reach their Jacobi-elliptic backgrounds as $|\phi_L|$ grows to infinity. We will prove in this subsection that $|\phi_L|$ has the shape of a wavey cone that grows to infinity along trajectories moving away from $(\xi,\eta) = (0,0)$.

Lemma 5. *The function $\phi_L(\xi,\eta)$ grows linearly in $|x| + |t|$ as $|x| + |t| \to \infty$ for every $k \in (0,1)$.*

Proof. Let $w = -f'$, factoring out $\frac{1}{2\lambda_1}$ in the second term of equation (7.1) returns

$$\phi_L(\xi,\eta) = \left(\frac{iH - w^2}{2\lambda_1}\right)\left(\eta + 4\int_0^{\xi-\eta} \frac{\lambda_1^2 w^2}{(w^2 - iH)^2}\right), \tag{7.14}$$

where $F_0 = iH$ and $H = \pm 2\sqrt{E(2-E)} \in \mathbb{R}$. Since $\frac{iH - w^2}{2\lambda_1}$ is bounded and periodic we can restrict our analysis to

$$\begin{aligned}\tilde{\phi}(\xi,\eta) &= \eta + 4\int_0^{\xi-\eta} \frac{\lambda_1^2 w^2 dz}{(w^2 - iH)^2} \\ &= \eta + 4\int_0^{\xi-\eta} \frac{\left[(E-1) \mp i\sqrt{E(2-E)}\right]\left[w^2 \pm 2i\sqrt{E(2-E)}\right]^2 w^2}{(w^4 + H^2)^2} dz\end{aligned}$$

where we have rationalized the denominator using its complex conjugate and substituted equation (4.22) to achieve the second equality. Substituting $w = -kcn(z;k)$ and $2k^2 = E$ for librational traveling waves and taking the imaginary part yields

$$
\begin{aligned}
Im[\tilde{\phi}] &= \pm 4\sqrt{E(2-E)}\int_0^{\xi-\eta}\frac{4w^2(E-1)-(w^4-4E(2-E))}{(w^4+H^2)^2}w^2dz \\
&= \pm 4\sqrt{E(2-E)}\int_0^{\xi-\eta}\frac{16k^2[1-k^2+(2k^2-1)cn^2(z;k)-k^2cn^4(z;k)]}{(w^4+H^2)^2}w^2dz \\
&= \pm 4\sqrt{E(2-E)}\int_0^{\xi-\eta}\frac{16k^2sn^2(z;k)dn^2(z;k)}{(w^4+H^2)^2}w^2dz.
\end{aligned}
$$

The integrand is clearly sign definite for $k \in (0,1)$. This means that $|\tilde{\phi}|^2 = Re[\tilde{\phi}]^2 + Im[\tilde{\phi}]^2$ will only exhibit bounded growth away from $(0,0)$ if $\xi - \eta = c_1$ for some constant $c_1 \in \mathbb{R}$. But $Re[\tilde{\phi}]$ grows linearly in η along the line $\xi - \eta = c_1$. $\square$

8 Algebraic solitons on rotational waves

In this section we will derive an explicit formula for algebraic solitons arising on the background of rotational waves. I should note that algebraic solitons on a periodic background are not rogue waves. The solitons of the sine-Gordon equation are obtained using the one-fold DT with the second eigenfunctions (5.6) for the particular eigenvalues λ_{1R} (4.24) obtained in the algebraic method. We also compute the magnification of the algebraic solitons compared with the rotational waves.

8.1 Deriving algebraic solitons using DT

From this point onwards we will denote $w = -f'$ to specify the potential. Using the DT (5.1) with the second eigenfunction (5.6) for rotational waves corresponding to the eigenvalue $\lambda_1 = \lambda_{1R}$ and using (4.11) yields

$$
\begin{aligned}
\hat{w} &= w + \frac{4\lambda_1\hat{p}_1\hat{q}_1}{\hat{p}_1^2+\hat{q}_1^2} \\
&= w + \frac{4\lambda_1(p_1q_1\phi_R^2w^2 + wp_1^2\phi_R - wq_1^2\phi_R - p_1q_1)}{\phi_R^2w^2p_1^2 + q_1^2 + p_1^2 + q_1^2\phi_R^2w^2} \\
&= w + \frac{w^2(F_0-w^2)\phi_R^2 + 4w'\phi_R w - (F_0-w^2)}{w(\phi_R^2w^2+1)}
\end{aligned} \tag{8.1}
$$

where $\hat{w} = -\hat{u}_\xi$ and $\hat{u}$ is a new solution to the sine-Gordon equation (2.24).
We claim that the new solution (8.1) corresponds to an algebraic soliton on the

background of the rotational wave. Indeed, we confirmed in section 6.2 that ϕ_R is bounded and periodic along the line Ω so that (8.1) is bounded and periodic along Ω. We verified numerically in figure 3 that (8.1) achieves its maximum amplitude periodically along Ω.

Figure 3 is an illustration of the algebraic solitons propagating along Ω for two particular values of k and two particular sign choices of F_0. The pictures were generated using the Matlab code presented in Appendix C. We see numerically that the solution surface $|\hat{w}|$ achieves its maximum at $(\xi, \eta) = (0, 0)$ and is repeated along Ω.

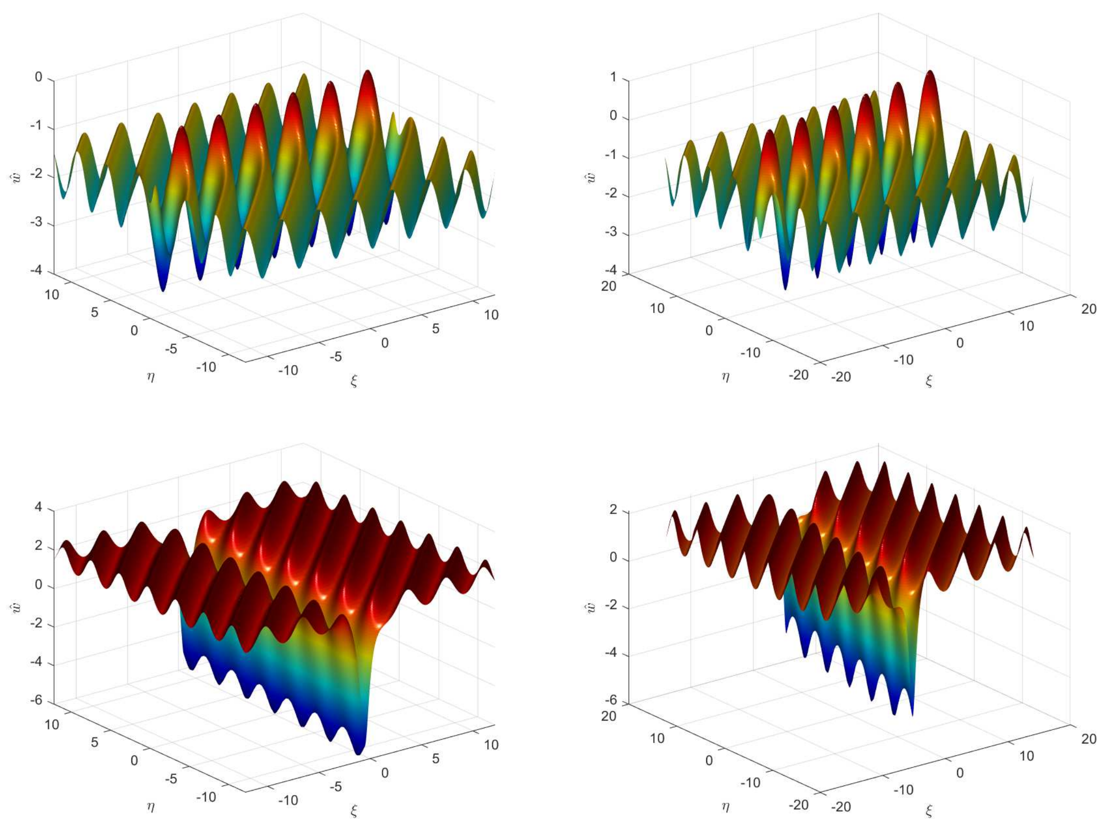


Figure 3: Algebraic solitons generated from the one-fold Darboux transformation using the traveling rotational waves as potentials with $k = 0.85$ (left) and $k = 0.95$ (right) for $\lambda_{1R} = \frac{1-\sqrt{1-k^2}}{k}$(top) and $\lambda_{1R} = \frac{1+\sqrt{1-k^2}}{k}$ (bottom).

8.2 The background of the algebraic solitons

In this subsection I will determine the behaviour of the rotational rogue wave (8.1) away from Ω. We proved in Theorem 1 that moving away from Ω corresponds to growing ϕ_R. Taking this limit shows that the background of (8.1) is

$$\begin{aligned}
\lim_{|\phi_R|\to\infty} \hat{w} &= \lim_{|\phi_R|\to\infty} \left(w + \frac{w^2(F_0 - w^2)\phi_R^2 + 4w'\phi_R w - (F_0 - w^2)}{\phi_R^2 w^3 + w} \right) \\
&= \lim_{|\phi_R|\to\infty} \left(w + \frac{w^2(F_0 - w^2) + \frac{4w'w}{\phi_R} - \frac{(F_0 - w^2)}{\phi_R^2}}{w^3 + \frac{w}{\phi_R^2}} \right) \\
&= \left(w + \frac{w^2(F_0 - w^2)}{w^3} \right) \\
&= \frac{F_0}{w} \; ,
\end{aligned}$$

and we showed with equation (5.3) that this is a translated version of the rotational wave w. Hence, (8.1) is built on top of a rotational traveling wave background. It is also evident in figure 3 that the surfaces decay to traveling waves away from the propogating solitons.

8.3 The Magnification of the Algebraic Solitons

In this subsection I will compute the magnification of the potential (8.1). It is clear from figure 3 that the maximum of $|\hat{w}|$ occurs at $(0,0)$ and is periodically repeated along Ω. At $(\xi, \eta) = (0,0)$ it follows that $w = -\frac{2}{k} dn(0;k) = -\frac{2}{k}$ and $\phi_R(0,0) = 0$ so that:

$$\begin{aligned}
\hat{w}(0,0) &= w(0,0) + \frac{w(0,0)^2(F_0 - w(0,0)^2)\phi_R(0,0) + 4w'(0,0)w(0,0)\phi_R(0,0) - (F_0 - w(0,0)^2)}{\phi_R(0,0)w(0,0)^3 + w(0,0)} \\
&= w(0,0) - \frac{(F_0 - w(0,0)^2)}{w(0,0)} \\
&= -\frac{4}{k} \pm \frac{2}{k}\sqrt{1 - k^2}.
\end{aligned}$$

The magnification of algebraic solitions relative to the rotational wave is defined by :

$$M := \frac{\sup_{(\xi,\eta)\in\mathbb{R}^2} |\hat{w}|}{\sup_{(\xi,\eta)\in\mathbb{R}^2} |w|}. \tag{8.2}$$

Since dn achieves a maximum value of 1 over $\mathbb{R}$ it is clear that $\sup_{(\xi,\eta)\in\mathbb{R}^2} |w| = \frac{2}{k}$ for the rotational waves. This means that the magnification for the algebraic solitons is

$$\begin{aligned} M(k) &= \frac{|\hat{w}(0,0)|}{|w(0,0)|} \\ &= \frac{|-\frac{4}{k} \pm \frac{2}{k}\sqrt{1-k^2}|}{\frac{2}{k}} \\ &= 2 \mp \sqrt{1-k^2}, \end{aligned} \tag{8.3}$$

where the choice in sign corresponds to the sign choice after taking the square root of equation (4.19). This coincides with the magnification factor of the rogue waves on the background of the dn-periodic waves of the mKdV and NLS equations [5, 15].

9 Rogue waves on librational waves

In this section we will derive an explicit formula for rogue waves arising on the background of librational waves. These exact solutions of the sine-Gordon equation are obtained using the two-fold DT with the second eigenfunctions for the particular eigenvalues λ_{1L} (4.25) obtained in the algebraic method. We also compute the magnification of the rogues waves compared with the librational waves.

9.1 Deriving rogue waves using DT

Using the DT (5.4) with the second eigenfunctions (5.7) for librational waves coresponding to the eigenvalues $\lambda_1 = \lambda_{1L}$ and $\lambda_2 = \bar{\lambda}_1$ and using (4.11) along with its complex conjugate, $(p_2, q_2) = (\bar{p}_1, \bar{q}_1)$, yields:

$$\begin{aligned} \hat{w} &= w + \frac{4(\lambda_1^2 - \lambda_2^2)[\lambda_1 \hat{p}_1 \hat{q}_1 (\hat{p}_2^2 + \hat{q}_2^2) - \lambda_2 \hat{p}_2 \hat{q}_2 (\hat{p}_1^2 + \hat{q}_1^2)]}{(\lambda_1^2 + \lambda_2^2)(\hat{p}_1^2 + \hat{q}_1^2)(\hat{p}_2^2 + \hat{q}_2^2) - 2\lambda_1\lambda_2[4\hat{p}_1\hat{q}_1\hat{p}_2\hat{q}_2 + (\hat{p}_1^2 - \hat{q}_1^2)(\hat{p}_2^2 - \hat{q}_2^2)]} \\ &= w + \frac{F_1}{F_2} \end{aligned} \tag{9.1}$$

where

$$F_1 = (\lambda_1^2 - \lambda_2^2)\Bigg[(F_0 - w^2)(\phi_L^2 - 1)(\bar{\phi}_L^2\bar{w} + \bar{w} + 2\bar{\phi}_L(q_2^2 - p_2^2)\ ) - (\bar{F}_0 - \bar{w}^2)(\bar{\phi}_L^2 - 1)(\phi_L^2 w + w + 2\phi_L(q_1^2 - p_1^2)\ )\Bigg] \tag{9.2}$$

and

$$\begin{aligned} F_2 = {} & (\lambda_1^2 + \lambda_2^2)(\phi_L^2 w + w + 2\phi_L(q_1^2 - p_1^2))(\bar{\phi}_L^2\bar{w} + \bar{w} + 2\bar{\phi}_L(q_2^2 - p_2^2)) \\ & - 2\Bigg[\frac{1}{4}(\phi_L^2 - 1)(\bar{\phi}_L^2 - 1)(F_0 - w^2)(\bar{F}_0 - \bar{w}^2) \\ & + (\phi_L^2 w' + w' - 2\lambda_1\phi_L w)(\bar{\phi}_L^2\bar{w}' + \bar{w}' - 2\lambda_2\bar{\phi}_L\bar{w})\Bigg]. \end{aligned} \tag{9.3}$$

The difference in squared eigenfunctions, $q_1^2 - p_1^2$ and $q_2^2 - p_2^2$, can be rewritten in terms of w since $w^2 = (f')^2$ and $-f'' = \lambda_1(p_1^2 - q_1^2)$.

We claim that the new solution (9.1) corresponds to an isolated rogue wave on the background of the librational wave. In the remaining parts of this section we will verify this claim. We verified numerically in figure 4 that the rogue wave achieves its maximum at the origin and that this maximum is a localized event. In the next two subsections we will confirm that the rogue wave is built on the librational wave background and that the magnification factor exceeds 2.

Figure 4 is an illustration of the surfaces (9.1) for two particular values of k and two sign choices of F_0. The pictures were generated using the Matlab code presented in the Appendix C. It is evident with a first glance of these figures that (9.1) represents an isolated rogue wave phenomenon.

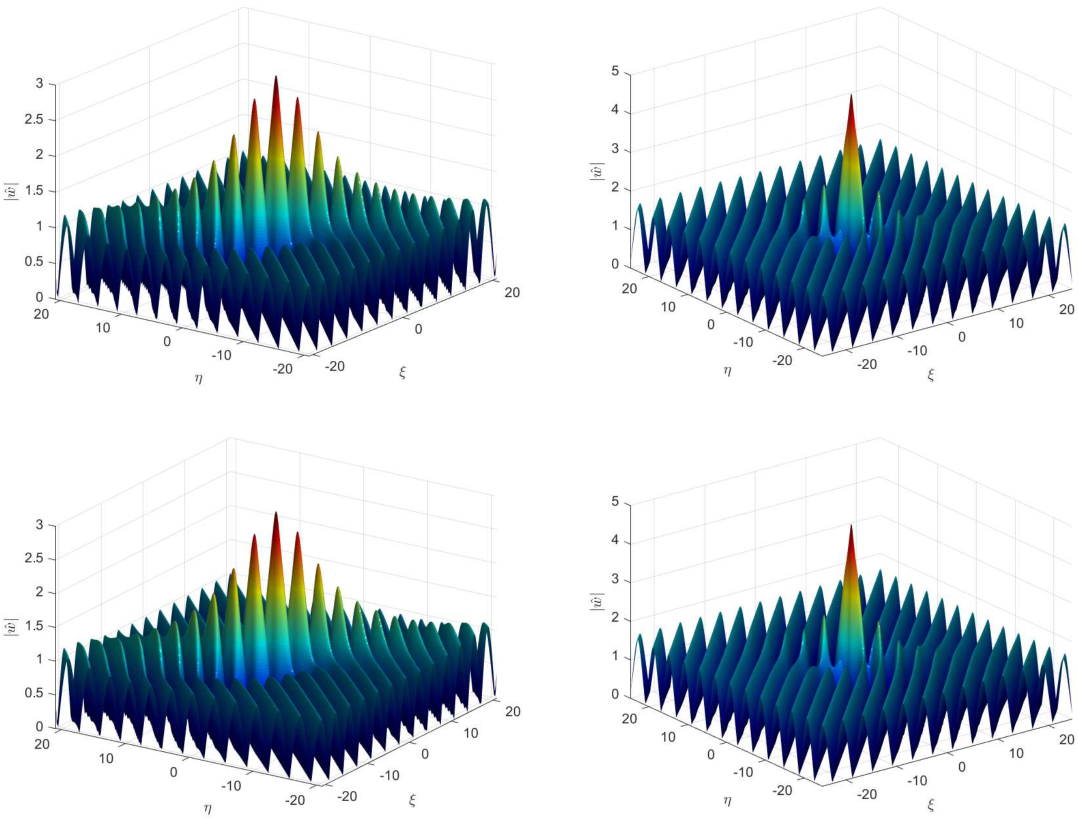


Figure 4: Rogue wave potentials generated using the two fold darboux transformation and the librational traveling wave with $k = 0.5$ (left) and $k = 0.8$ (right) and $\lambda_{1L} = (k - i\sqrt{1-k^2})$ (top) and $\lambda_{1L} = (k + i\sqrt{1-k^2})$ (bottom).

9.2 The background of the rogue waves

The rogue wave (9.1) reaches its background away from $(\xi, \eta) = (0, 0)$. Thanks to lemma 5 we know that $|\phi_L| \to \infty$ along trajectories moving far away from $(0, 0)$. Consider the two quotients $\widetilde{F}_1 = \frac{F_1}{\phi_L^2 \bar{\phi}_L^2}$ and $\widetilde{F}_2 = \frac{F_2}{\phi_L^2 \bar{\phi}_L^2}$. Clearly $\frac{F_1}{F_2} = \frac{\widetilde{F}_1}{\widetilde{F}_2}$ so that $\lim_{|\phi_L| \to \infty} \frac{F_1}{F_2} = \lim_{|\phi_L| \to \infty} \frac{\widetilde{F}_1}{\widetilde{F}_2}$. I will now inspect the limits of $\widetilde{F}_1$ and $\widetilde{F}_2$ separately. Some simple algebraic manipulation gives us

$$\lim_{|\phi_L| \to \infty} \widetilde{F}_1 = (\lambda_1^2 - \lambda_2^2) \left[(F_0 - w^2)(\bar{w}) - (\bar{F}_0 - \bar{w}^2) w \right]$$

and

$$\lim_{|\phi_L|\to\infty} \widetilde{F}_2 = (\lambda_1^2 + \lambda_2^2) w\bar{w} - 2\left[\frac{1}{4}(F_0 - w^2)(\bar{F}_0 - \bar{w}^2) + (w')(\bar{w}')\right].$$

From here it is clear that $\lim_{|\phi_L|\to\infty}\left(w + \frac{\widetilde{F}_1}{\widetilde{F}_2}\right) = \lim_{|\phi|\to\infty}\left(w + \frac{F_1}{F_2}\right)$ is equal to equation (5.5). Performing the same computations that follow equation (5.5) means that $\lim_{|\phi_L|\to\infty}\left(w + \frac{\widetilde{F}_1}{\widetilde{F}_2}\right) = -w$. Hence, we conclude that the solution (9.1) is built on the background of librational waves.

9.3 The rogue wave magnification

For the librational waves it follows that $w(0,0) = -2k$ and $\phi_L(0,0) = 0$. This implies that F_1 and F_2 from (9.2) and (9.3) at $(0,0)$ are given by

$$F_1(0,0) = -64k^3(k^2-1) \tag{9.4}$$

and

$$F_2(0,0) = 16k^2(k^2-1), \tag{9.5}$$

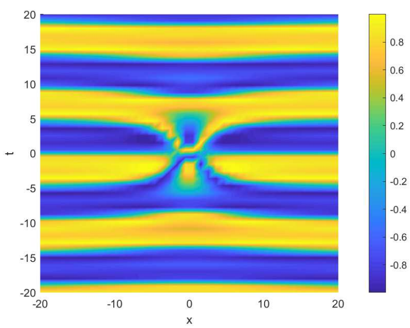
so that

$$|\hat{w}(0,0)| = 6k. \tag{9.6}$$

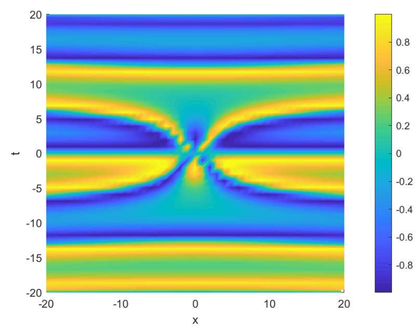
We showed numerically in figure 4 that the solution surface achieves its maximum at $|\hat{w}(0,0)|$. The maximum of $|w|$ for the librational waves is $2k$. This means that the magnification of the rogue waves with respect to librational travelling waves is $M = \frac{6k}{2k} = 3$. This coincides with the result derived for the rogue waves on the background of cn-periodic waves of the mKdV and NLS equations in [15] and [5]. Clearly this magnification exceeds 2 so that (9.1) represents a rogue wave on the background of librational waves.

9.4 Defects in the fluxon condensate

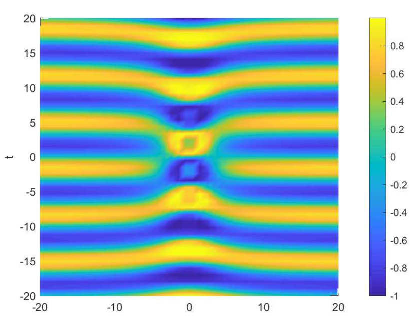
In [14] Lu and Miller constructed a similar rogue wave solution which defines defects in the fluxon condensate. In figure 5 we plot $\sin(\hat{u})$ for the rogue wave (9.1) for different values of k. Numerically differentiating the rogue wave potential in η with a forward difference allowed us to retrieve $\sin(\hat{u})$, since $-\hat{u}_{\xi\eta} = \sin(\hat{u})$. Our surface plots of $\sin(\hat{u})$ are very similar to the figures in appendix D of [14].



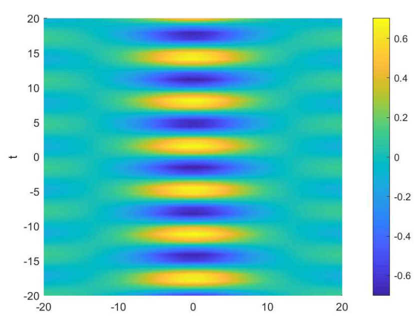
(a) $k = \sin(\frac{3\pi}{8})$



(b) $k = \sin(\frac{11\pi}{24})$



(c) $k = \sin(\frac{\pi}{6})$



(d) $k = \sin(\frac{\pi}{24})$

Figure 5: $\sin(\hat{u})$ for librational rogue waves for various k values.

This confirms that rogue waves constructed on a background of librational waves in the sine-Gordon equation correspond to defects in the fluxon condensate.

9.5 Changing the integration constant in rogue wave growth

Here we modify the growth of the rogue wave by altering the constant of integration in equation (7.1). This modification is implemented by fixing a $C_0 \in \mathbb{R}$ and then redefining ϕ_L as

$$\phi_L(\xi,\eta) = (F_0 - w^2)\left(\frac{\eta}{2\lambda_1} + \int_{C_0}^{\xi-\eta} \frac{2\lambda_1 w^2 dz}{(F_0 - w^2)^2}\right). \tag{9.7}$$

The new growth term (9.7) is used to create the second linearly independent eigenfunctions and then the two fold DT is applied to create the rogue potential $\hat{w}$ with respect to this new growth term. The background of these new rogue waves is still $-w$ since the modulus of expression (9.7) grows as $|\xi| + |\eta| \to \infty$. The proof is similar to that of Lemma 5 but one must realize that the growth surface $|\phi_L|$ periodically grows away from the point $(\xi,\eta) = (C_0, 0)$ instead of $(\xi,\eta) = (0,0)$.

We are interested to see what happens with the maximum of the rogue wave surface $|\hat{w}|$ as we vary C_0. We show numerically in figure 6 that the rogue wave reaches its highest magnification when $C_0 = 0$ or multiples of the period of the librational wave $L = 2K(k)$, where $K(\cdot)$ is the complete elliptic integral of the first kind. We are able to restrict our analysis to $0 \leq C_0 \leq L$ by considering the transformation: $C_0 = L + \hat{C}_0$, $\eta = \hat{\eta} - L$ and $\xi = \hat{\xi} - L$.

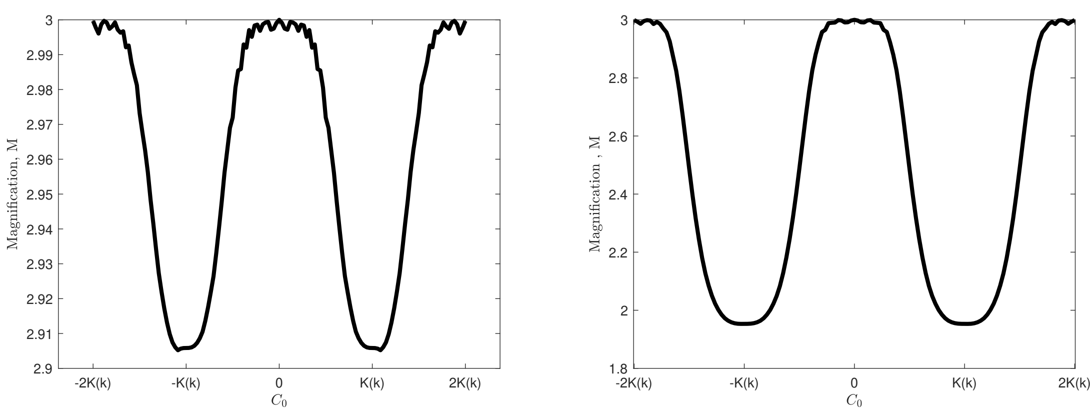


Figure 6: The magnification of the rogue wave vs C_0 for $k = 0.5$ (left) and $k = 0.8$ (right)

10 Concluding Remarks

We were able to generate rogue waves in the sine-Gordon equation using an algebraic method and the Darboux transformation. The algebraic method yields solutions (λ, p, q) to the Lax system with traveling wave potential satisfying a squared eigenfunction relation, $-u_\xi = p^2 + q^2$. The eigenvalues extracted from this method are located at the end points of the spectral bands of the Floquet spectrum. This was shown numerically in figures 2 and 3.

The Darboux transformation was applied to the solutions of the Lax pair occuring at the end points of the aforementioned spectral bands. The one fold Darboux transformation for the rotational waves produced algebraic solitons propagating along a straight line. The two fold Darboux transformation with librational wave solutions produced a proper rogue wave.

The growth term, ϕ, introduced in the second linearly independent eigenfunctions characterizes the algebraic solitons and the rogue waves. In particular, the surface of ϕ ought to grow away from the point of integration in order for a proper rogue wave to be formed. The growth term was originally taken to be zero at $(\xi, \eta) = (0, 0)$. Integrating ϕ with this initial condition creates a rogue wave with the highest possible magnification factor. We showed numerically that changing this integration constant decreases the magnification of the rogue wave relative to librational waves.

The sine-Gordon equation is rich in many physical applications including describing the magnetic flux in long superconducting Josephson junctions [16–18], modeling fermions [6], explaining stability structure in galaxies [13, 21, 22] and analyzing mechanical vibrations of a ribbon pendulum [23]. Our results predict the occurence of rogue wave behaviour in these physical systems. This is useful information for physicists studying freak events in the natural world. Our work also verifies that the algebraic method works for an entire class of partial differential equations that share a similar spectral problem. This is because the same algebraic method has been successful with the fNLSE and mKDV equations.

There are a lot of open problems that remain in this thesis. It was not addressed if the oscillations occuring near the peaks of figure 6 are numerical noise or inherent to the system. We did not analytically study the maxiumum behaviour of the rogue waves and algebraic solitons. We do not have a complete picture of the Lax spectrum, the algebraic method was only successful at retrieving the end points of the spectral bands. The other eigenvalues along the spectral bands corespond to more rogue waves. Lastly, a more solid definition of what it means to be a rogue wave should be developed.

A Formulation of Jacobian Elliptic functions

Useful information regarding Jacobi elliptic functions was taken from [1,2,24,26]. Jacobi elliptic functions are derived from the inversion of the elliptic integral of the first kind,

$$F(\tau, k) = \int_0^\tau \frac{dt}{\sqrt{1 - k^2 \sin^2 t}},$$

where $k \in (0, 1)$ is the elliptic modulus. The complete elliptic integral of the is defined as $K(k) = F(\frac{\pi}{2}, k)$.

From here we can define the three basic Jacobi elliptic functions,

$$\text{sn}(v, k) = \sin \tau \ ,$$
$$\text{cn}(v, k) = \cos \tau$$

and

$$\text{dn}(v, k) = \sqrt{1 - k^2 \text{sn}^2(v, k)}.$$

The basic Jacobi elliptic functions are smooth. sn and cn are periodic with period $L = 4K(k)$ while dn is periodic with period $L = 2K(k)$. They also satisfy

$$\text{sn}^2 v + \text{cn}^2 v = 1 \tag{A.1}$$

$$k^2 \text{sn}^2 v + \text{dn}^2 v = 1 \tag{A.2}$$

$$\frac{\sqrt{1 - k^2}}{dn(y; k)} = dn(y + K(k); k) \tag{A.3}$$

and have derivatives

$$\frac{d \ \text{sn}v}{dv} = \text{cn}v \ \text{dn}v$$
$$\frac{d \ \text{cn}v}{dv} = -\text{sn}v \ \text{dn}v$$
$$\frac{d \ \text{dn}v}{dv} = -k^2 \text{sn}v \ \text{cn}v.$$

B Spectral Code (Matlab)

B.1 Rotational Waves

```
% SGSP rotational spectrum
tic
close all; clear all ; clc ;

% free parameters in the SGSP for rotational waves
% k is elliptic modulus , N is periodic domain resolution for
% eigenfunctions and floqnum is the resolution of floquet exponents
k=0.95; N = 105; floqnum = 101 ;

E = 2/(k^2) ; % Parameter E from table 1
K = ellipke(k^2); % K is complete elliptic integral of first kind
T = 2*K*k ; % period of the dn wave

% exact eigenvalues from algebraic method
lambda_exact1dn = sqrt( E - sqrt(E*(E-2)) -1 ) ;
lambda_exact2dn = sqrt( E + sqrt(E*(E-2)) -1 ) ;
theta = linspace(-pi/T,pi/T,floqnum); % floquet exponent domain
spectrum = []; % initializing spectrum as empty vector

xend = T/2; % discretizing eigenfunction domain
len = 2*N-1;
Xcomplete = linspace(-xend,xend,len+1);
X = Xcomplete(2:end);

[elipsn,elipcn,elipdn] = ellipj((1/k).*X,k^2); % jacobi elliptic functions
A2dn = (2/k).*diag(elipdn);  % dn potential for sine-gordon

% derivative matrix!
h = X(2)-X(1);
LEN = length(X) ;
g = zeros(1,LEN) ;
hh = zeros(1,LEN) ;
g(2) = (-6/7); g(3) = (15/56); g(4) = (-5/63) ; g(5) = (1/56); g(6) = (-1/385);
g(7) = (1/5544) ;
```

```
g(LEN)=(6/7); g(LEN-1)=(-15/56);g(LEN-2)=(5/63); g(LEN-3)=(-1/56);
g(LEN-4)=(1/385); g(LEN-5)=(-1/5544) ;
hh=(-1)*g;

for i=1:length(theta)
% Construction of derivative operator matrix
A1 = (1/h)*toeplitz(g,hh) ;
A1 = 2*A1 + 2*eye(size(A1))*sqrt(-1)*theta(i) ;
% eigenvalue problem matrix defined by numerical construction of Lax
SpecMat = [A1 A2dn; A2dn -A1];
% Calculating the spectrum of each matrix.
lambda=eig(SpecMat);
% Updating spectrum with eigenvalues for current floquet parameter
spectrum=[spectrum ;lambda];
clear A1 SpecMat lambda
end

% plotting the rotational spectrum
figure(11)
plot(real(spectrum) , imag(spectrum),'o')
hold on
xlabel('Real Part')
ylabel('Imaginary Part')
plot(real(lambda_exact1dn),imag(lambda_exact1dn),'o','MarkerFaceColor' ,'r')
plot(real(lambda_exact2dn),imag(lambda_exact2dn),'o', 'MarkerFaceColor', 'r')
plot(real(-conj(lambda_exact1dn)),imag(-conj(lambda_exact1dn)),'o',
'MarkerFaceColor', 'r')
plot(real(-conj(lambda_exact2dn)),imag(-conj(lambda_exact2dn)),'o',
'MarkerFaceColor', 'r')
axis([-(lambda_exact2dn+1) (lambda_exact2dn+1) -1 1])
xlabel('Real Part')
ylabel('Imaginary Part')
hold off;
toc
```

B.2 Librational Waves

```
% SGSP Librational Spectrum
```

```
tic
close all; clear all ; clc ;

% free parameters in the SGSP for the librational wave
% k is elliptic modulus parameter, N is resolution of periodic
% eigenfunction domain and floqnum is the resolution of floquet exponents
k=0.95; N = 105; floqnum = 105 ;

E = 2*(k^2) ; % parameter E in table 1 for librational waves
K = ellipke(k^2); % K is complete elliptic integral of first kind
T = 4*K; % period of the cn wave

lambda_exact1cn = sqrt( E - sqrt(E*(E-2)) -1 ) ; % exact eigenvalues from
%algebraic method
lambda_exact2cn = sqrt( E + sqrt(E*(E-2)) -1 ) ;

theta = linspace(-pi/T, pi/T, floqnum); %<- resolution of floquet exponent  !
spectrum = [];

xend = T/2;
% The number of points in the domain
len = 2*N-1;

% The complete domain is made up of 'len+1' equal spaced points with
% +/- xend as endpoints. The first point of the domain is deleted.
Xcomplete = linspace(-xend,xend,len+1);
X = Xcomplete(2:end);

% Calculating the distance between adjacent points in the domain.
h = X(2)-X(1);

% Finding the elliptic function values at each point in domain.
[elipsn,elipcn,elipdn] = ellipj(X,k^2);

% cn potential of SGSP
A2cn = diag((2*k).*elipcn);

% Creating differentiation matrix.
```

```
LEN = length(X) ;
g = zeros(1,LEN) ;
hh = zeros(1,LEN) ;
g(2) = (-6/7); g(3) = (15/56); g(4) = (-5/63) ; g(5) = (1/56); g(6) = (-1/385);
g(7) = (1/5544) ;
g(LEN)=(6/7); g(LEN-1)=(-15/56);g(LEN-2)=(5/63); g(LEN-3)=(-1/56);
g(LEN-4)=(1/385); g(LEN-5)=(-1/5544) ;
hh=(-1)*g;
filteredeigz = [];

% constructing the numerical eigenvalue matrix for each floquet exponent
for i=1:length(theta)

% Construction operator matrix
A1 = (1/h)*toeplitz(g,hh) ;
A1 = A1 + eye(size(A1))*sqrt(-1)*theta(i) ; A1=2*A1;
% matrix defined by numerical construction of SGSP
SpecMat = [A1 A2cn; A2cn -A1] ;

% Calculating the spectrum of each matrix.
lambda=eig(SpecMat);

% Updating spectrum with eigenvalues for current theta(i)
spectrum=[spectrum ;lambda];
clear A1 SpecMat lambda

end
% plotting the spectrum
figure(11)
plot(real(spectrum) , imag(spectrum),'o')
hold on
xlabel('Real Part')
ylabel('Imaginary Part')
plot(real(lambda_exact1cn),imag(lambda_exact1cn),'o', 'MarkerFaceColor', 'r')
plot(real(lambda_exact2cn),imag(lambda_exact2cn),'o', 'MarkerFaceColor', 'r')
plot(real(-conj(lambda_exact1cn)),imag(-conj(lambda_exact1cn)),'o',
'MarkerFaceColor', 'r')
plot(real(-conj(lambda_exact2cn)),imag(-conj(lambda_exact2cn)),'o',
```

```
'MarkerFaceColor', 'r')
axis([-1.5 1.5 -1.5 1.5])
hold off;
```

C Darboux Transformation Code (Matlab)

C.1 Rotational Background

```
close all; clear all ; clc ;
tic
% Free parameters
% elliptic modulus parameter, resolution of xi-eta domain, length of
%domains,ploting limit
k = 0.85 ; N = 41;repp = 3.4;elim = 24;

E = (2/k^2) ; % energy of rotational waves in table 1
K = ellipke(k^2); % K is elliptic integral of first kind
T = 2*K*k ; % period of the dn wave
F0 = -sqrt( 4*E*(E-2) ) ; % F_0 from hamiltonian
%lambda1 = sqrt(E + sqrt(E*(E-2)) - 1)
lambda1a = ((1+sqrt(1-k^2))/(k))
lambda1b = sqrt(E - F0/2 - 1)
% defining xi-eta domain

xidomain = union(linspace(-repp*T,0,4*N),linspace(0,repp*T,4*N) ) ;
etadomain = union(linspace(-repp*T,0,4*N),linspace(0,repp*T,4*N)) ;

% growth , w and w' for rotational waves
phiR = zeros(length(xidomain),length(etadomain));
w = zeros(length(xidomain),length(etadomain));
wprime = zeros(length(xidomain),length(etadomain));
% computing growth at each point
for i=1:length(xidomain)
for j=1:length(etadomain)
% computing integral in rotational growth
if ((xidomain(i) - etadomain(j))>0)
zdom = linspace(0 , (xidomain(i) - etadomain(j)) , N ) ;
```

```
[elipsn,elipcn,elipdn] = ellipj((1/k).*zdom,k^2);
zrange = (1./(elipdn.^2)) ;
val1 = trapz(zdom,zrange) ;
phiR(i,j) = (( xidomain(i) + etadomain(j) )/2) - ((F0*k^2)/8)*val1 ;
end

if ((xidomain(i) - etadomain(j))<0)
zdom = linspace((xidomain(i) - etadomain(j)), 0 , N ) ;
[elipsn,elipcn,elipdn] = ellipj((1/k)*zdom,k^2);
zrange = (1./(elipdn.^2)) ;
val1 = -trapz(zdom,zrange) ;
phiR(i,j) = (( xidomain(i) + etadomain(j) )/2) - ((F0*k^2)/(8))*val1 ;
end

if ((xidomain(i) - etadomain(j))==0)
phiR(i,j) = (( xidomain(i) + etadomain(j) )/2) ;
end

end
end
% ploting growth for rotational waves
figure(1)
surf(xidomain,etadomain,abs(phiR)')

% computing w and w' at each point for rotational waves
for i=1:length(xidomain)
for j=1:length(etadomain)

z = xidomain(i) - etadomain(j) ;
[elipsn,elipcn,elipdn] = ellipj((1/k)*z,k^2);
w(i,j) = -(2/k)*elipdn ;
wprime(i,j) = 2*elipsn*elipcn;

end
end
% initializing algebraic soliton surface
what = zeros(length(xidomain),length(etadomain));
```

```
% one-fold DT for rotational waves
for i=1:length(xidomain)
for j=1:length(etadomain)

num = (F0 - w(i,j)^2)*(w(i,j)^2)*(phiR(i,j)^2) +
4*wprime(i,j)*phiR(i,j)*w(i,j)-(F0 - w(i,j)^2);
den = (phiR(i,j)^2)*(w(i,j)^3) + w(i,j) ;
what(i,j) = w(i,j) + (num/den);

end
end

% plotting algebraic solitons
figure(3)
surf(xidomain,etadomain,what')
ylabel('\eta')
xlabel('\xi')
zlabel('$\hat{w}$','Interpreter','latex')
shading interp
colormap jet
alpha 1;
camlight('headlight');
lighting phong;
material shiny;
```

C.2 Librational Background

```
% Rogue Wave Potentials from librational waves
tic
close all; clear all; clc
% Free parameters
k = 0.8; N = 51; % elliptic modulus and xi/eta resolution

E = 2*(k^2) ; % energy from table 1
K = ellipke(k^2); % K is elliptic integral of first kind, E2 of second
T = 4*K; % period of the cn wave
F0 = sqrt( 4*E*(E-2) ) ; % F_0 from hamiltonian
```

```
% creating the xi-eta domain
xidomain = union(linspace(-3.1*T,0,3*N),linspace(0,3.1*T,3*N) ) ;
etadomain = union(linspace(-3.1*T,0,4*N) , linspace(0,3.1*T,3*N) );

% eigenvalue and its conjugate from algebraic method
lambda1 = sqrt( E - (F0/2) - 1 ) ;
lambda2 = conj(lambda1) ;
lambdacheck = k - sqrt(-1)*sqrt(1-k^2)
lambda1b = sqrt(E - F0/2 - 1)
% initializing useful functions : growth,w,w' and q1^2-p1^2
phiL = zeros(length(xidomain),length(etadomain));
w = zeros(length(xidomain),length(etadomain));
wprime = zeros(length(xidomain),length(etadomain));
q1mp12 = zeros(length(xidomain),length(etadomain));
% for each point in the xi-eta domain
for i=1:length(xidomain)
for j=1:length(etadomain)
% computing integration term in growth function
if ((xidomain(i) - etadomain(j))>0)
zdom = linspace(0 , (xidomain(i) - etadomain(j)) , N ) ;
[elipsn,elipcn,elipdn] = ellipj(zdom,k^2);
zrange = (4*k*k*elipcn.^2)./((F0 - 4*k*k*elipcn.^2).^2) ;
val1 = trapz(zdom,zrange) ;
end

if ((xidomain(i) - etadomain(j))<0)
zdom = linspace((xidomain(i) - etadomain(j)), 0 , N ) ;
[elipsn,elipcn,elipdn] = ellipj(zdom,k^2);
zrange = (4*k*k*elipcn.^2)./((F0 - 4*k*k*elipcn.^2).^2) ;
val1 = -trapz(zdom,zrange) ;
end

if ((xidomain(i) - etadomain(j))==0)
val1=0;
end

[elipsn2,elipcn2,elipdn2] = ellipj(xidomain(i) - etadomain(j),k^2);
```

```
% assigning values to useful functions
w(i,j) = -2*k*elipcn2 ;
wprime(i,j) = 2*k*elipsn2*elipdn2 ;
phiL(i,j) = (F0-w(i,j)^2)*( (etadomain(j)/(2*lambda1)) + 2*lambda1*val1 ) ;
q1mp12(i,j) = -2*k*elipsn2*elipdn2/lambda1 ;

end
end
% plotting the growth (wavey cone)
figure(1)
surf(xidomain,etadomain,abs(phiL)')
shading interp
colormap jet
alpha 1;
camlight('headlight');
lighting phong;
material shiny;

% computing the rogue-wave (two fold DT at each point in xi-eta domain)
% \hat{w}
what = zeros(length(xidomain),length(etadomain));
for i=1:length(xidomain)
for j=1:length(etadomain)

F1a = (F0 - w(i,j)^2)*(phiL(i,j)^2 - 1)*( conj( w(i,j)*phiL(i,j)^2 + w(i,j)
+2*q1mp12(i,j)*phiL(i,j) ) ) ;
F1b = conj(F1a);
F2a = w(i,j)*phiL(i,j)^2 + w(i,j) + 2*(phiL(i,j))*q1mp12(i,j);
F2b = (1/4)*(phiL(i,j)^2 - 1)*conj((phiL(i,j)^2 -
1))*(F0-w(i,j)^2)*conj((F0-w(i,j)^2));
F2c = (wprime(i,j)*phiL(i,j)^2 + wprime(i,j) -
2*lambda1*phiL(i,j)*w(i,j))*conj((wprime(i,j)*phiL(i,j)^2 + wprime(i,j) -
2*lambda1*phiL(i,j)*w(i,j)));

F1 = (lambda1^2-lambda2^2)*(F1a-F1b);
F2 = (lambda1^2 + lambda2^2)*F2a*conj(F2a) - 2*(F2b+F2c);
what(i,j) = w(i,j) + (F1/F2) ;
```

```
end
end
toc

% plotting the rogue wave
figure(2)
surf(xidomain,etadomain,abs(what)')
xlabel('\xi')
ylabel('\eta')
zlabel('$| \hat{w} |$','Interpreter','latex')
shading interp
colormap jet
alpha 1;
camlight('headlight');
lighting phong;
material shiny;
```

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