Periodic Waves in the Fractional Modified Korteweg–de Vries Equation

Fábio Natali[1] · **Uyen Le**[2] · **Dmitry E. Pelinovsky**[2,3]

Abstract

Periodic waves in the modified Korteweg–de Vries (mKdV) equation are revisited in the setting of the fractional Laplacian. Two families of solutions in the local case are given by the sign-definite dnoidal and sign-indefinite cnoidal solutions. Both solutions can be characterized in the general fractional case as global minimizers of the quadratic part of the energy functional subject to the fixed L^4 norm: the sign-definite (sign-indefinite) solutions are obtained in the subspace of even (odd) functions. Morse index is computed for both solutions and the spectral stability criterion is derived. We show numerically that the family of sign-definite solutions has a generic fold bifurcation for the fractional Laplacian of lower regularity and the family of sign-indefinite solutions has a generic symmetry-breaking bifurcation both in the fractional and local cases.

Keywords Fractional modified Korteweg–de Vries equation · Periodic waves · Existence · Stability · Bifurcations · Energy minimization

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✉ Dmitry E. Pelinovsky
   dmpeli@math.mcmaster.ca

   Fábio Natali
   fmanatali@uem.br

   Uyen Le
   leu@mcmaster.ca

1   Departamento de Matemática, Universidade Estadual de Maringá, Avenida Colombo 5790, Maringá, PR CEP 87020-900, Brazil

2   Department of Mathematics and Statistics, McMaster University, Hamilton, ON L8S 4K1, Canada

3   Department of Applied Mathematics, Nizhny Novgorod State Technical University, 24 Minin Street, Nizhny Novgorod, Russia 603155

1 Introduction

The purpose of this work is to study existence, variational characterization, and bifurcations of periodic solutions of the following stationary equation:

$$D^\alpha \psi + c\psi + b = 2\psi^3, \tag{1.1}$$

where $\psi(x) : \mathbb{T} \mapsto \mathbb{R}$ is the wave profile on a circle $\mathbb{T} := [-\pi, \pi]$, (c, b) are real parameters, and D^α is the fractional Laplacian on $\mathbb{T}$ defined via Fourier series by

$$\psi(x) = \sum_{n\in\mathbb{Z}} \psi_n e^{inx}, \quad (D^\alpha \psi)(x) = \sum_{n\in\mathbb{Z}} |n|^\alpha \psi_n e^{inx}.$$

Thanks to the scaling transformation for the cubic nonlinearity, the fundamental period of the wave has been scaled to 2π. In addition, we are interested in the simplest periodic waves with the single-lobe profile according to the following definition.

Definition 1.1 We say that the periodic wave satisfying Eq. (1.1) has a single-lobe profile ψ if it admits only one isolated maximum and minimum on $\mathbb{T}$.

Eq. (1.1) arises in the context of the fractional mKdV (modified Korteweg–de Vries) equation written in the form:

$$u_t + 6u^2 u_x - (D^\alpha u)_x = 0, \tag{1.2}$$

with $u(t, x) : \mathbb{R} \times \mathbb{T} \mapsto \mathbb{R}$. Traveling waves of the form $u(t, x) = \psi(x - ct)$ satisfy Eq. (1.1), where c is the wave speed and b is an integration constant. The fractional mKdV equation (1.2) admits formally the following conserved quantities:

$$E(u) = \frac{1}{2} \int_{-\pi}^{\pi} \left[(D^{\frac{\alpha}{2}} u)^2 - u^4 \right] dx, \tag{1.3}$$

$$F(u) = \frac{1}{2} \int_{-\pi}^{\pi} u^2 dx, \tag{1.4}$$

and

$$M(u) = \int_{-\pi}^{\pi} u \, dx, \tag{1.5}$$

which have meaning of energy, momentum, and mass, respectively. Eq. (1.1) is the Euler–Lagrange equation for the action functional,

$$G(u) = E(u) + cF(u) + bM(u), \tag{1.6}$$

so that $G'(\psi) = 0$. The Hessian operator from the action functional (1.6) yields the linearized operator around the wave ψ in the form:

$$\mathcal{L} := G''(\psi) = D^\alpha + c - 6\psi^2. \tag{1.7}$$

In addition to questions of the existence, we study the spectral stability of the periodic wave with the spatial profile ψ in the time evolution of the fractional mKdV equation (1.2) according to the following definition.

Definition 1.2 The periodic wave is said to be spectrally stable with respect to perturbations of the same period if $\sigma(\partial_x \mathcal{L}) \subset i\mathbb{R}$ in $L^2(\mathbb{T})$. Otherwise, it is said to be *spectrally unstable*.

The fractional mKdV equation (1.2) appears to be a generic model for one-dimensional long waves with weak dispersion (see recent review in [28]). For $\alpha = 2$, it is the completely integrable mKdV equation [1]. For $\alpha = 1$, it is referred to as the modified Benjamin–Ono equation [8].

Global well-posedness results for the initial data in $H^s(\mathbb{R})$ with $s > \frac{1}{4}$ and in $H^s(\mathbb{T})$ with $s \geq \frac{1}{2}$ were obtained for $\alpha = 2$ in [16]. Local well-posedness results for initial data in $H^s(\mathbb{R})$ with $s \geq \frac{1}{2}$ were obtained for $\alpha = 1$ in [27]. Energy and momentum are conserved in the time evolution of such solutions. Local solutions with sufficiently large initial data in $H^{\frac{1}{2}}(\mathbb{R})$ blow up in a finite time [26,31].

Periodic waves of Eq. (1.1) were only studied in the local case of $\alpha = 2$. There exist two families of periodic solutions for $b = 0$ with the single-lobe profile: sign-definite solutions are expressed by the dnoidal elliptic functions and sign-indefinite solutions are expressed by the cnoidal elliptic functions. For $b \neq 0$, all periodic waves can be expressed as a rational function of Jacobian elliptic functions [13]. Spectral and orbital stability of periodic waves in the local case of $\alpha = 2$ was also considered in the literature.

Employing the arguments in [9,35,36], orbital stability of sign-definite dnoidal waves with $b = 0$ was proven in [5]. Spectral stability of sign-indefinite cnoidal waves with $b = 0$ was studied in [17] by using the count of negative eigenvalues of the operator $\mathcal{L}$ restricted to the orthogonal complement of span$(1, \psi)$ (also also [22,34]). It was discovered in [17] that the cnoidal waves were spectrally stable for smaller speeds c and spectrally unstable for larger speeds c. Spectral and orbital stability and instability of the cnoidal waves was proven in [7] by adopting the arguments of [30] in the periodic context and employing the approach in [23] based on the existence of a sufficiently smooth data-to-solution-map. Orbital stability of a particular family of positive periodic waves of the dnoidal type with $b \neq 0$ was proven in [4] by adopting the arguments of [20].

In the limit of $c \to \infty$, periodic waves with the single-lobe profile on the fixed circle $\mathbb{T}$ concentrate near centers of symmetry and approach the solitary waves. With the scaling transformation,

$$\psi(x) = c^{\frac{1}{2}} Q(c^{\frac{1}{\alpha}} x), \quad x \in \mathbb{R}, \tag{1.8}$$

for $c > 0$, the solitary wave profile $Q(x) : \mathbb{R} \mapsto \mathbb{R}$ satisfies the c-independent problem

$$D^\alpha Q + Q = 2Q^3. \tag{1.9}$$

Existence and uniqueness (modulus translations) of solitary waves with the spatial profile Q satisfying Eq. (1.9) was shown in [18] based on their variational characterization as minimizers of the Gagliardo–Nirenberg-type inequality considered in [37,38]:

$$\|u\|_{L^4(\mathbb{R})}^4 \leq C_\alpha \|D^{\frac{\alpha}{2}} u\|_{L^2(\mathbb{R})}^{\frac{2}{\alpha}} \|u\|_{L^2(\mathbb{R})}^{4-\frac{2}{\alpha}}, \quad u \in H^{\frac{\alpha}{2}}(\mathbb{R}), \tag{1.10}$$

where $C_\alpha > 0$ is the u-independent constant. Solitary waves satisfying (1.9) give the best value for C_α which saturates the Gagliardo–Nirenberg-type inequality (1.10).

For $\alpha = 2$, the classical arguments in [2,9,20] can be used to prove the orbital stability of solitary waves in the energy space $H^1(\mathbb{R})$. Spectral and orbital stability of solitary waves in the general case $\alpha \in \left(\frac{1}{2}, 2\right)$ was considered in [6] based on the dependence of the momentum $F(\psi) = c^{1-\frac{1}{\alpha}} F(Q)$ on the wave speed $c > 0$. It was shown that the solitary waves were unstable if $\alpha \in \left(\frac{1}{2}, 1\right)$ and stable if $\alpha \in (1, 2)$ in agreement with increasing and decreasing dependence of $F(\psi)$ versus c respectively. The critical case $\alpha = 1$ was inconclusive because

$\frac{d}{dc} F(\psi) = 0$ in this case. For the critical case, existence of blow up solutions with minimal mass was proven in [31] by combining sharp energy estimates and a refined localized argument from [26]. Hence, the solitary waves are unstable for $\alpha = 1$.

The previous review of literature shows that Eq. (1.1) in the periodic domain $\mathbb{T}$ has been unexplored for $\alpha < 2$. Compared to the cubic case, a similar problem with the quadratic nonlinearity has been recently studied in many aspects, e.g. existence and stability of traveling periodic waves were analyzed by using perturbative [25], variational [10,12,24], and fixed-point [11,29] methods.

The standard approach to characterize the spectral and orbital stability of periodic waves with respect to perturbations of the same period is based on the minimization of energy $E(u)$ subject to the fixed momentum $F(u)$ and mass $M(u)$ [24]. Parameters c and b appear to be Lagrange multipliers of the action functional (1.6). It is important to realize that smoothness of the momentum and mass with respect to Lagrange multipliers cannot be taken for granted (compared to what was done in [24]).

In order to resolve problems of the variational characterization of the traveling periodic waves in the fractional KdV equation (with quadratic nonlinearity), two new approaches were recently developed.

In [21], the periodic waves with the single-lobe profile were constructed by minimizing the energy $E(u)$ subject to only one constraint of the fixed momentum $F(u)$. It was shown that such minimizers were degenerate only up to the translation symmetry and were spectrally stable.

From a different point of view, we characterized the traveling periodic waves in our previous work [32] by minimizing the quadratic part of the action functional $G(u)$ subject to the fixed cubic part of the energy $E(u)$ and the zero mean constraint $M(u) = 0$. This approach combined with the Galilean transformation allowed us to represent all possible periodic waves of the single-lobe profile ψ and to derive a simple stability criterion from the derivative of the momentum $F(\psi)$ with respect to the wave speed c.

When ideas of [32] are extended to the cubic nonlinearity in the framework of Eq. (1.1), we face the difficulty that the Galilean transformation generates a quadratic nonlinear term and connects solutions of the fractional mKdV equation to solutions of the fractional Gardner equation. As a result, we are not able yet to characterize all possible periodic waves of the single-lobe profile in Eq. (1.1). Instead, we shall study the two particular families of solutions which correspond to $b = 0$ and generalize the sign-definite dnoidal and sign-indefinite cnoidal elliptic solutions of the local case $\alpha = 2$. Both families are obtained as minimizers of the quadratic part of the action functional $G(u)$ subject to the fixed quartic part of the energy, but one family is obtained in the subspace of even functions and the other family is obtained in the subspace of odd functions. For the purpose of simplicity, we refer to the first family as *the even periodic waves* and to the second family as *the odd periodic waves*.

The following two theorems present the main results of this paper. In what follows, we write H_{per}^s instead of $H_{\text{per}}^s(\mathbb{T})$. The subspace of odd (even) functions in L^2 is denoted by L_{odd}^2 (L_{even}^2). Similarly, the subspace of odd (even) periodic functions in H_{per}^s is denoted by $H_{\text{per,odd}}^s$ ($H_{\text{per,even}}^s$).

Theorem 1.3 (Odd periodic wave) *Fix $\alpha \in \left(\frac{1}{2}, 2\right]$. For every $c_0 \in (-1, \infty)$, there exists a solution to Eq. (1.1) with $b = 0$ and the odd, single-lobe profile ψ_0, which is obtained from a constrained minimizer of the following variational problem:*

$$\inf_{u \in H^{\frac{\alpha}{2}}_{per,odd}} \left\{ \int_{-\pi}^{\pi} \left[(D^{\frac{\alpha}{2}}u)^2 + c_0 u^2 \right] dx : \int_{-\pi}^{\pi} u^4 dx = 1 \right\}. \tag{1.11}$$

There exists a C^1 mapping $c \mapsto \psi(\cdot, c) \in H^{\alpha}_{per,odd}$ in a local neighborhood of c_0 such that $\psi(\cdot, c_0) = \psi_0$. The spectrum of $\mathcal{L}$ in $L^2(\mathbb{T})$ has exactly two negative eigenvalues and if $1 \in$ Range($\mathcal{L}$), a simple zero eigenvalue. Assuming $1 \in$ Range($\mathcal{L}$) and setting $\sigma_0 := \langle \mathcal{L}^{-1} 1, 1 \rangle$, the periodic wave with the profile ψ_0 is spectrally stable if

$$\sigma_0 \le 0, \quad \frac{d}{dc} \|\psi\|_{L^2}^2 \ge 0 \tag{1.12}$$

and is spectrally unstable with exactly one real, positive eigenvalue of $\partial_x \mathcal{L}$ in $L^2(\mathbb{T})$ if

$$\text{either} \quad \sigma_0 \frac{d}{dc} \|\psi\|_{L^2}^2 > 0 \quad \text{or} \quad \sigma_0 = 0, \quad \frac{d}{dc} \|\psi\|_{L^2}^2 < 0, \quad \text{or} \quad \sigma_0 > 0, \quad \frac{d}{dc} \|\psi\|_{L^2}^2 = 0. \tag{1.13}$$

If $1 \notin$ Range($\mathcal{L}$), then the periodic wave is spectrally unstable with exactly one real positive eigenvalue of $\partial_x \mathcal{L}$ in $L^2(\mathbb{T})$ if

$$\frac{d}{dc} \|\psi\|_{L^2}^2 \ge 0. \tag{1.14}$$

Remark 1.4 If $\sigma_0 = 0$, the odd periodic wave of Theorem 1.3 undertakes the stability bifurcation, which also results in the bifurcation of new solutions in Eq. (1.1) with $b \ne 0$. The stability bifurcation was first discovered in [17] for $\alpha = 2$. We show numerically that this bifurcation is generic for every $\alpha \in \left(\frac{1}{2}, 2\right)$.

Remark 1.5 Based on numerical studies, we conjecture that the case $1 \notin$ Range($\mathcal{L}$) is impossible for the odd periodic wave in Theorem 1.3 for every $\alpha \in \left(\frac{1}{2}, 2\right]$ and every $c \in (-1, \infty)$. Nevertheless, the case $1 \notin$ Range($\mathcal{L}$) is observed for the new solutions bifurcating from the odd periodic wave in Theorem 1.3.

Remark 1.6 Although the solution ψ_0 is obtained as a global minimizer of the variational problem (1.11), the solution $\psi(\cdot, c) \in H^{\alpha}_{per,odd}$ in a local neighborhood of c_0 is continued from the Euler–Lagrange equation. Therefore, even if the solution $\psi(\cdot, c) \in H^{\alpha}_{per,odd}$ is C^1 with respect to c, it may not coincide with the global minimizer of (1.11) for $c \ne c_0$ because uniqueness of minimizers of the variational problem (1.11) is not proven. Nevertheless, the spectral stability conclusions of Theorem 1.3 apply to every global minimizer of the variational problem (1.11) for every $c_0 \in (-1, \infty)$.

Theorem 1.7 (Even periodic wave) *Fix $\alpha \in \left(\frac{1}{2}, 2\right]$. For every $c_0 \in \left(\frac{1}{2}, \infty\right)$, there exists a solution to Eq. (1.1) with $b = 0$ and the even, single-lobe profile ψ_0, which is obtained from a constrained minimizer of the following variational problem:*

$$\inf_{u \in H^{\frac{\alpha}{2}}_{per,even}} \left\{ \int_{-\pi}^{\pi} \left[(D^{\frac{\alpha}{2}}u)^2 + c_0 u^2 \right] dx : \int_{-\pi}^{\pi} u^4 dx = 1 \right\}. \tag{1.15}$$

The spectrum of $\mathcal{L}$ in $L^2(\mathbb{T})$ has exactly one simple negative eigenvalue and if $1 \in$ Range($\mathcal{L}$), a simple zero eigenvalue. With the transformation,

$$\psi_0(x) = a_0 + \phi_0(x), \quad a_0 := \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi_0(x) dx, \quad \omega_0 := c_0 - 6a_0^2, \tag{1.16}$$

assuming $\omega_0 \in (-1, \infty)$, there exists a C^1 mapping $(\omega, a) \mapsto \phi(\cdot, \omega, a) \in H^\alpha_{\text{per,even}}$ in a local neighborhood of (ω_0, a_0) such that $\phi(\cdot, \omega_0, a_0) = \phi_0$ and the mean value of ϕ is zero. The periodic wave ψ_0 is spectrally stable if

$$\frac{\partial}{\partial \omega} \|\phi\|^2_{L^2} \geq 0 \tag{1.17}$$

and is spectrally unstable with exactly one real, positive eigenvalue of $\partial_x \mathcal{L}$ in $L^2(\mathbb{T})$ if

$$\frac{\partial}{\partial \omega} \|\phi\|^2_{L^2} < 0. \tag{1.18}$$

Remark 1.8 We derive the criterion for $1 \notin \text{Range}(\mathcal{L})$, in which case $\mathcal{L}$ has the double zero eigenvalue and the even periodic wave of Theorem 1.7 undertakes the fold bifurcation. Two solutions of Eq. (1.1) with $b = 0$ coexist for the same value of c near the fold bifurcation. We show numerically that the fold bifurcation is generic for every $\alpha \in \left(\frac{1}{2}, \alpha_0\right)$, where

$$\alpha_0 := \frac{\log 8 - \log 5}{\log 2} \approx 0.6781.$$

If $\alpha \in \left(\frac{1}{2}, \alpha_0\right)$, two solutions with the even, single-lobe profile exist for the same value of $c \in \left(c_0, \frac{1}{2}\right)$ with $c_0 \in \left(0, \frac{1}{2}\right)$ beyond the admissible range of values of c in Theorem 1.7.

Remark 1.9 Based on numerical evidences, we conjecture that $\omega \in (-1, \infty)$ is always satisfied for the even periodic wave in Theorem 1.7.

Remark 1.10 Similarly to Remark 1.6, the smooth continuation of the solution $\psi(\cdot, \omega, a) = a + \phi(\cdot, \omega, a)$ with $\phi(\cdot, \omega, a) \in H^\alpha_{\text{per,even}}$ is obtained from the Euler–Lagrange equation and the solution $\psi(\cdot, \omega, a)$ may not coincide with the global minimizer of (1.15) for $c(\omega, a) \neq c_0$. Nevertheless, under the assumption $\omega_0 \in (-1, \infty)$, the spectral stability conclusions of Theorem 1.7 apply to every global minimizer of the variational problem (1.15) for every $c_0 \in (\frac{1}{2}, \infty)$.

The paper is organized as follows. The odd periodic wave of Theorem 1.3 is characterized in Sect. 2. Examples of the odd periodic waves are given in Sect. 3 with Stokes expansions, exact elliptic solutions in the local case $\alpha = 2$, and numerical approximations for $\alpha = 1$. The even periodic wave of Theorem 1.7 is characterized in Sect. 4. Similar examples of the even periodic waves are given in Sect. 5. Section 6 discusses an open problem to characterize all possible periodic waves with the single-lobe profile among solutions to Eq. (1.1) with $b \neq 0$.

2 Odd Periodic Waves

Here we consider *the odd periodic waves* and provide the proof of Theorem 1.3. First, we obtain the variational characterization of the odd periodic waves. Next, we consider a smooth continuation of the odd periodic waves with respect to the speed parameter c. Finally, we obtain the spectral stability conclusions.

2.1 Variational Characterization

If $\psi \in H^\alpha_{\text{per}}$ is a solution to Eq. (1.1) with $b = \frac{1}{\pi} \int_{-\pi}^{\pi} \psi^3 dx$, then ψ satisfies the zero-mean constraint and the boundary-value problem:

$$D^\alpha \psi + c\psi = 2\Pi_0 \psi^3, \tag{2.1}$$

where $\Pi_0 f := f - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$ is the projection operator reducing the mean value of 2π-periodic functions to zero.

Since we use variational methods, we consider weak solutions of the boundary-value problem (2.1) in $H_{\text{per}}^{\frac{\alpha}{2}}$. By the same bootstrapping argument as in Proposition 1 in [21] or Proposition 2.4 in [32], if $\psi \in H_{\text{per}}^{\frac{\alpha}{2}}$ is a weak solution of the boundary-value problem (2.1), then $\psi \in H_{\text{per}}^{\infty}$ and, in particular, it is a strong solution to the boundary-value problem (2.1) in H_{per}^{α}.

The following theorem and its corollary give the construction and properties of the periodic waves in a subspace of odd functions which satisfy the boundary-value problem (2.1).

Theorem 2.1 *Fix* $\alpha > \frac{1}{2}$. *For every* $c > -1$, *there exists the ground state (minimizer)* $\chi \in H_{\text{per,odd}}^{\frac{\alpha}{2}}$ *of the following constrained minimization problem:*

$$q_{c,\text{odd}} := \inf_{u \in H_{\text{per,odd}}^{\frac{\alpha}{2}}} \left\{ \mathcal{B}_c(u) : \int_{-\pi}^{\pi} u^4 dx = 1 \right\}, \tag{2.2}$$

where

$$\mathcal{B}_c(u) := \frac{1}{2}\int_{-\pi}^{\pi}\left[(D^{\frac{\alpha}{2}}u)^2 + cu^2 \right] dx. \tag{2.3}$$

If $\alpha \leq 2$, *the ground state has the single-lobe profile, which is even with respect to the points at* $x = \pm\pi/2$.

Proof It follows that $\mathcal{B}_c$ is a smooth functional bounded on $H_{\text{per,odd}}^{\frac{\alpha}{2}}$. Moreover, $\mathcal{B}_c$ is proportional to the quadratic form of the operator $c + D^{\alpha}$ with the spectrum in L_{odd}^2 given by $\{c + m^{\alpha}, \ m \in \mathbb{N}\}$. By Friedrichs's inequality, we have

$$\mathcal{B}_c(u) \geq \frac{1}{2}(c+1)\|u\|_{L^2}^2, \quad u \in H_{\text{per,odd}}^{\frac{\alpha}{2}}, \tag{2.4}$$

and by Gårding's inequality, for every $c > -1$ there exists $C > 0$ such that

$$\mathcal{B}_c(u) \geq C\|u\|_{H_{\text{per}}^{\frac{\alpha}{2}}}^2, \quad u \in H_{\text{per,odd}}^{\frac{\alpha}{2}}.$$

Hence $\mathcal{B}_c$ is equivalent to the squared norm in $H_{\text{per,odd}}^{\frac{\alpha}{2}}$ so that $q_{c,\text{odd}} \geq 0$.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for the constrained minimization problem (2.2), that is, a sequence satisfying

$$\mathcal{B}_c(u_n) \to q_{c,\text{odd}} \quad \text{as} \quad n \to \infty.$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H_{\text{per,odd}}^{\frac{\alpha}{2}}$, there exists $\chi \in H_{\text{per,odd}}^{\frac{\alpha}{2}}$ such that, up to a subsequence,

$$u_n \rightharpoonup \chi \quad \text{in } H_{\text{per,odd}}^{\frac{\alpha}{2}}, \quad \text{as} \quad n \to \infty.$$

For every $\alpha > \frac{1}{2}$, the energy space $H_{\text{per,odd}}^{\frac{\alpha}{2}}$ is compactly embedded in L_{odd}^4 (see, e.g., Theorem 4.2 in [3]). Thus, there is a positive constant C such that for every $\chi \in H_{\text{per,odd}}^{\frac{\alpha}{2}}$,

$$\|\chi\|_{L^4} \leq C\|\chi\|_{H^{\frac{\alpha}{2}}}, \tag{2.5}$$

and

$$u_n \to \chi \quad \text{in } L^4_{\text{odd}}, \quad \text{as} \quad n \to \infty.$$

Using the estimate

$$\left| \int_{-\pi}^{\pi} (u_n^4 - \chi^4) dx \right| \leq \int_{-\pi}^{\pi} |u_n^4 - \chi^4| dx$$
$$\leq \left(\|\chi\|_{L^4}^3 + \|\chi\|_{L^4}^2 \|u_n\|_{L^4} + \|\chi\|_{L^4} \|u_n\|_{L^4}^2 + \|u_n\|_{L^4}^3 \right) \|u_n - \chi\|_{L^4},$$

it follows that χ satisfies the constraint: $\int_{-\pi}^{\pi} \chi^4 dx = 1$. In view of (2.5), this implies that $q_{c,\text{odd}} > 0$.

Thanks to the weak lower semi-continuity of $\mathcal{B}_c$, we have

$$\mathcal{B}_c(\chi) \leq \liminf_{n \to \infty} \mathcal{B}(u_n) = q_{c,\text{odd}}.$$

Since χ satisfies the constraint, we also have $\mathcal{B}_c(\chi) \geq q_{c,\text{odd}}$, hence $\mathcal{B}_c(\chi) = q_{c,\text{odd}}$ and $\chi \in H^{\frac{\alpha}{2}}_{\text{per,odd}}$ is the ground state (global minimizer) of the variational problem (2.2).

If $\alpha \in (0, 2]$, the symmetric rearrangements of u do not increase $\mathcal{B}_c(u)$ while leaving the constraint on the L^4-norm invariant due to the fractional Polya–Szegö inequality, see Lemma A.1 in [15]. As a result, the minimizer $\chi \in H^{\frac{\alpha}{2}}_{\text{per,odd}}$ of $\mathcal{B}_c(u)$ must decrease symmetrically away from the maximum point. Since $\chi(x) = 0$ at $x = 0$ and $x = \pm\pi$, the symmetry points of χ are located at $x = \pm\pi/2$, so that the single-lobe profile is even with respect to the points at $x = \pm\pi/2$.   □

Corollary 2.2 *Let χ be the ground state of Theorem 2.1. There exists $C > 0$ such that $\psi(x) = C\chi(x)$ satisfies Eq. (1.1) with $b = 0$.*

Proof By Lagrange's Multiplier Theorem, the constrained minimizer $\chi \in H^{\frac{\alpha}{2}}_{\text{per,odd}}$ satisfies the stationary equation

$$D^\alpha \chi + c\chi = \mu \chi^3, \tag{2.6}$$

where $\mu = 2\mathcal{B}_c(\chi)$ is the Lagrange multiplier found from the constraint $\int_{-\pi}^{\pi} \chi^4 dx = 1$. Since $\mathcal{B}_c(\chi) > 0$, the scaling transformation $\psi = C\chi$ with $C := \sqrt{\mathcal{B}_c(\chi)}$ maps Eq. (2.6) to the form (1.1) with $b = 0$.   □

Lemma 2.3 *Let χ be the ground state of Theorem 2.1 and $q_{c,\text{odd}} = \mathcal{B}_c(\chi)$. Then $q_{c,\text{odd}}$ is continuous in c for $c > -1$ and $q_{c,\text{odd}} \to 0$ as $c \to -1$.*

Proof The proof of continuity of $q_{c,\text{odd}}$ follows the proof of Lemma 2.3 in [32] verbatim. In order to show that $q_{c,\text{odd}} \to 0$ as $c \to -1$, we consider the following function

$$u(x) = A \sin(x),$$

which satisfies the constraint in (2.2) for $A := \left(\frac{4}{3\pi}\right)^{1/4}$. Substituting u into $\mathcal{B}_c(u)$ yields

$$\mathcal{B}_c(u) = \frac{\pi}{2} A^2 (1 + c) = \frac{\sqrt{\pi}}{\sqrt{3}} (1 + c).$$

Since

$$0 \leq q_{c,\text{odd}} \leq \mathcal{B}_c(u),$$

it follows that $q_{c,\text{odd}} \to 0$ as $c \to -1$.   □

2.2 Smooth Continuation

Let $\psi \in H^\alpha_{\text{per,odd}}$ be a solution to the boundary-value problem (2.1) for some $c \in (-1, \infty)$ obtained by Theorem 2.1 and Corollary 2.2. Let $\mathcal{L}$ be the linearized operator around the wave ψ given by (1.7) and

$$\mathcal{L}: \quad H^\alpha_{\text{per}} \subset L^2(\mathbb{T}) \to L^2(\mathbb{T}). \tag{2.7}$$

In what follows, we determine the multiplicity of the zero eigenvalue of $\mathcal{L}$ denoted as $z(\mathcal{L})$ and the number of negative eigenvalues of $\mathcal{L}$ with the account of their multiplicities denoted as $n(\mathcal{L})$. It follows from Eq. (1.1) with $b = 0$ that

$$\mathcal{L}1 = c - 6\psi^2 \tag{2.8}$$

and

$$\mathcal{L}\psi = -4\psi^3. \tag{2.9}$$

By the translational symmetry, we always have $\mathcal{L}\partial_x \psi = 0$.

　Since ψ is the single-lobe profile of the periodic wave in the sense of Definition 1.1 and since ψ is even with respect to the points at $x = \pm\pi/2$, then we can place the unique maximum of ψ at $x = \pi/2$ and adopt several results of [24] with the same proof after translation $x \mapsto x - \pi/2$.

Proposition 2.4 [24] *Let $\alpha \in (\frac{1}{2}, 2]$ and $\psi \in H^\alpha_{\text{per,odd}}$ be a solution obtained in Theorem 2.1 and Corollary 2.2. An eigenfunction of $\mathcal{L}$ defined by (1.7) and (2.7) corresponding to the n-th eigenvalue of $\mathcal{L}$ for $n = 1, 2, 3$ changes its sign at most $2(n-1)$ times over $\mathbb{T}$. An eigenfunction of $\mathcal{L}|_{L^2_{\text{even}}}$ for the n-th eigenvalue of $\mathcal{L}|_{L^2_{\text{even}}}$ changes its sign at most $2(n-1)$ times over $\mathbb{T}$.*

Proposition 2.5 [24] *Assume $\alpha \in (\frac{1}{2}, 2]$ and $\psi \in H^\alpha_{\text{per,odd}}$ be a solution obtained in Theorem 2.1 and Corollary 2.2. If $\{1, \psi, \psi^2\} \in \text{Range}(\mathcal{L})$, then $\text{Ker}(\mathcal{L}) = \text{span}(\partial_x \psi)$.*

Proposition 2.6 [24] *Assume $\alpha \in (\frac{1}{2}, 2]$ and $\psi \in H^\alpha_{\text{per,odd}}$ be a solution obtained in Theorem 2.1 and Corollary 2.2. Then, $\partial_x \psi \in \text{Ker}(\mathcal{L})$ corresponds to the lowest eigenvalue of $\mathcal{L}$ in the space of odd functions with respect to $x = \pi/2$.*

Remark 2.7 Compared to [24], the potential of $\mathcal{L}$ is $\psi^2 \in H^\alpha_{\text{per}}$ rather than $\psi \in H^\alpha_{\text{per}}$, where we recall that $\alpha > \frac{1}{2}$. This implies that the spatial period of the potential of $\mathcal{L}$ is π rather than 2π. Nevertheless, we are interested in the properties of $\mathcal{L}$ on $L^2(\mathbb{T})$ rather than on $L^2(\frac{1}{2}\mathbb{T})$.

　By an elementary application of the implicit function theorem (similarly to Lemma 3.8 in [32]), we can also obtain the following result.

Lemma 2.8 *Assume $\alpha \in (\frac{1}{2}, 2]$ and $\psi_0 \in H^\alpha_{\text{per,odd}}$ be a solution obtained in Theorem 2.1 and Corollary 2.2 for $c = c_0$. Assume $\text{Ker}(\mathcal{L}|_{L^2_{\text{odd}}})$ is trivial. Then, there exists a C^1 mapping in an open subset of c_0 denoted by $\mathcal{I} \subset \mathbb{R}$:*

$$\mathcal{I} \ni c \mapsto \psi(\cdot, c) \in H^\alpha_{\text{per,odd}} \tag{2.10}$$

such that $\psi(\cdot, c) = \psi_0$ and $\mathcal{L}\partial_c \psi(\cdot, c_0) = -\psi_0$.

Proof Let $\Upsilon : (-1, \infty) \times H^{\alpha}_{\text{per,odd}} \to L^2_{\text{odd}}(\mathbb{T})$ be defined by $\Upsilon(c, f) := D^{\alpha} f + cf - 2f^3$. By hypothesis of the lemma, we have $\Upsilon(c_0, \psi_0) = 0$. Moreover, Υ is smooth and its Fréchet derivative with respect to f evaluated at (c_0, ψ_0) is given by $\mathcal{L}$ computed at ψ_0. Since $\text{Ker}(\mathcal{L}|_{L^2_{\text{odd}}})$ is empty by the assumption, we conclude that $\mathcal{L}$ is one-to-one. It is also onto since its spectrum consists of nonzero isolated eigenvalues with finite algebraic multiplicities because $H^{\alpha}_{\text{per,odd}}$ is compactly embedded in $L^2_{\text{odd}}(\mathbb{T})$ if $\alpha > 1/2$ and because $\mathcal{L}$ is a self-adjoint operator. Hence, $\mathcal{L}$ has a bounded inverse. Thus, since Υ and its derivative with respect to f are smooth maps on their domains, the result follows from the implicit function theorem. □

By using Propositions 2.4–2.6, and Lemma 2.8, we compute $n(\mathcal{L})$ and $z(\mathcal{L})$ in the following lemma.

Lemma 2.9 *Let $\alpha \in (\frac{1}{2}, 2]$ and $\psi \in H^{\alpha}_{\text{per,odd}}$ be a solution obtained in Theorem 2.1 and Corollary 2.2. Then, $n(\mathcal{L}) = 2$ and*

$$z(\mathcal{L}) = \begin{cases} 1, & \text{if } 1 \in \text{Range}(\mathcal{L}), \\ 2, & \text{if } 1 \notin \text{Range}(\mathcal{L}). \end{cases}$$

Proof Since $\psi \in H^{\alpha}_{\text{per,odd}}$ is a minimizer of the constrained variational problem (2.2) with only one constraint, we have $n(\mathcal{L}|_{L^2_{\text{odd}}}) \leq 1$. On the other hand, we have

$$\langle \mathcal{L}\psi, \psi \rangle_{L^2} = -4\|\psi\|^4_{L^4} < 0,$$

with odd ψ, hence $n(\mathcal{L}|_{L^2_{\text{odd}}}) \geq 1$, so that $n(\mathcal{L}|_{L^2_{\text{odd}}}) = 1$.

Since $\partial_x \psi \in \text{Ker}(\mathcal{L})$ and $\partial_x \psi$ is even with two nodes on $\mathbb{T}$, then by Proposition 2.4, 0 is not the first eigenvalue of $\mathcal{L}|_{L^2_{\text{even}}}$, so that $n(\mathcal{L}|_{L^2_{\text{even}}}) \geq 1$. However, another negative eigenvalue of $\mathcal{L}|_{L^2_{\text{even}}}$ is impossible since the eigenfunction for the second eigenvalue of $\mathcal{L}|_{L^2_{\text{even}}}$ must have two nodes by Proposition 2.4 and the nodes are located at the symmetry points $x = \pm \pi/2$, hence this eigenfunction is not orthogonal to $\partial_x \psi \in \text{Ker}(\mathcal{L})$. Therefore, 0 is the second eigenvalue of $\mathcal{L}|_{L^2_{\text{even}}}$, which yields $n(\mathcal{L}|_{L^2_{\text{even}}}) = 1$ and

$$n(\mathcal{L}) = n(\mathcal{L}|_{L^2_{\text{odd}}}) + n(\mathcal{L}|_{L^2_{\text{even}}}) = 2.$$

It remains to consider $z(\mathcal{L}) \geq 1$. For illustration purposes, we give the first five eigenfunctions of the operator $\mathcal{L}$ in $L^2(\mathbb{T})$ for $\alpha = 2$ on Fig. 1. By the symmetry of ψ and ψ^2, the operator $\mathcal{L}$ in (1.7) and (2.7) has a π-periodic potential, which is even with respect to both $x = 0$ and $x = \pi/2$. The negative eigenvalue of $\mathcal{L}$ in L^2_{even}, which is the lowest eigenvalue of $\mathcal{L}$ in $L^2(\mathbb{T})$, corresponds to the sign-definite π-periodic function, which is even with respect to both $x = 0$ and $x = \pi/2$. The negative eigenvalue in L^2_{odd}, which is the second eigenvalue of $\mathcal{L}$ in $L^2(\mathbb{T})$, corresponds to the eigenfunction with two nodes on $\mathbb{T}$, which is even with respect to $x = \pi/2$. The eigenfunction $\partial_x \psi$ for the zero eigenvalue in L^2_{even}, which is the third eigenvalue of $\mathcal{L}$ in $L^2(\mathbb{T})$, has two nodes and is odd with respect to $x = \pi/2$. By Proposition 2.6, the zero eigenvalue is the lowest eigenvalue for the eigenfunctions that are odd with respect to $x = \pi/2$.

Finally, we consider the eigenfunctions with four nodes on $\mathbb{T}$ since 0 is the third eigenvalue of $\mathcal{L}$. These eigenfunctions have the same parity with respect to $x = 0$ and $x = \pi/2$, hence odd functions in L^2_{odd} are also odd with respect to $x = \pi/2$. Since the zero eigenvalue is the lowest eigenvalue for the eigenfunctions that are odd with respect to $x = \pi/2$, the odd eigenfunction of $\mathcal{L}$ in L^2_{odd} with four nodes corresponds to the positive eigenvalue of $\mathcal{L}$. Therefore, $z(\mathcal{L}|_{L^2_{\text{odd}}}) = 0$ and by Lemma 2.8, the mapping $c \mapsto \psi(\cdot; c)$ is C^1 in c with $\mathcal{L}\partial_c \psi = -\psi$, so that $\psi \in \text{Range}(\mathcal{L})$.
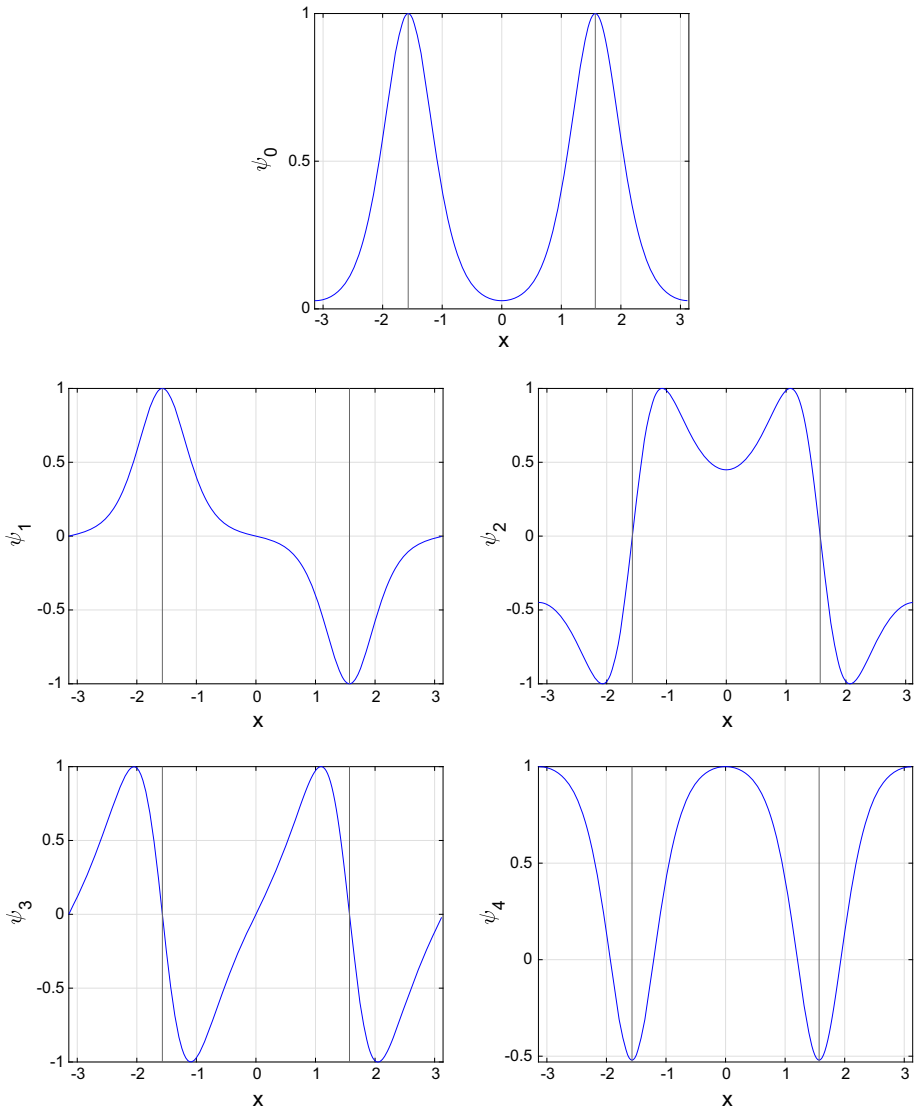
Fig. 1 Normalized eigenfunctions of $\mathcal{L}$ on $\mathbb{T}$ for $\alpha = 2$ computed from the exact expressions in Sect. 3.2

Assume that the even eigenfunction of $\mathcal{L}$ in L^2_{even} with four nodes (call it f) belongs to $\text{Ker}(\mathcal{L})$, hence $\text{Ker}(\mathcal{L}) = \text{span}(\partial_x \psi, f)$. Either $\langle 1, f \rangle_{L^2} = 0$ or $\langle 1, f \rangle_{L^2} \neq 0$. If $\langle 1, f \rangle_{L^2} = 0$, then $1 \in \text{Range}(\mathcal{L})$. It follows from (2.8) that if $1 \in \text{Range}(\mathcal{L})$, then $\psi^2 \in \text{Range}(\mathcal{L})$. Therefore, $\{1, \psi, \psi^2\} \in \text{Range}(\mathcal{L})$ and by Proposition 2.5, $\text{Ker}(\mathcal{L}) = \text{span}(\partial_x \psi)$, so that the existence of $f \in \text{Ker}(\mathcal{L})$ leads to a contradiction. Hence, $z(\mathcal{L}) = 1$ if $1 \in \text{Range}(\mathcal{L})$. If $\langle 1, f \rangle_{L^2} \neq 0$, then $1 \notin \text{Range}(\mathcal{L})$ because $\text{Range}(\mathcal{L})$ is orthogonal to $\text{Ker}(\mathcal{L})$. Hence, $z(\mathcal{L}) = 2$ if and only if $1 \notin \text{Range}(\mathcal{L})$. $\square$

Next, we introduce the subspace of L^2 with zero mean and denote it by X_0:

$$X_0 := \left\{ f \in L^2(\mathbb{T}) : \quad \int_{-\pi}^{\pi} f(x)dx = 0 \right\}. \tag{2.11}$$

Denote $\Pi_0 \mathcal{L} \Pi_0$ by $\mathcal{L}|_{X_0}$. By an explicit computation, it follows that if $f \in H_{\text{per}}^{\alpha} \cap X_0$, then

$$\mathcal{L}|_{X_0} f := \mathcal{L}f + \frac{3}{\pi} \langle f, \psi^2 \rangle. \tag{2.12}$$

The following result is similar to Lemma 3.5 in [32].

Lemma 2.10 *Let $\alpha \in (\frac{1}{2}, 2]$ and $\psi \in H_{\text{per,odd}}^{\alpha}$ be a solution obtained in Theorem 2.1 and Corollary 2.2. If there exists a nonzero $f \in \text{Ker}(\mathcal{L}|_{X_0})$ such that $\langle f, \partial_x \psi \rangle = 0$, then*

$$z(\mathcal{L}) = 1, \quad \text{and} \quad \langle f, \psi^2 \rangle \neq 0. \tag{2.13}$$

Proof Since $f \in \text{Ker}(\mathcal{L}|_{X_0})$, then $\langle 1, f \rangle = 0$ and f satisfies

$$\mathcal{L}f = -\frac{3}{\pi} \langle f, \psi^2 \rangle. \tag{2.14}$$

Either $\langle f, \psi^2 \rangle = 0$ or $\langle f, \psi^2 \rangle \neq 0$.

If $\langle f, \psi^2 \rangle = 0$, then $f \in \text{Ker}(\mathcal{L})$ so that $z(\mathcal{L}) = 2$ and $1 \notin \text{Range}(\mathcal{L})$ by Lemma 2.9. However, $1 \perp \text{span}(\partial_x \psi, f) = \text{Ker}(\mathcal{L})$ implies $1 \in \text{Range}(\mathcal{L})$, which is a contradiction.

If $\langle f, \psi^2 \rangle \neq 0$, then it follows from (2.14) that $1 \in \text{Range}(\mathcal{L})$ and hence $z(\mathcal{L}) = 1$ by Lemma 2.9. This yields (2.13). □

Remark 2.11 Assuming $1 \in \text{Range}(\mathcal{L})$, let us define $\sigma_0 := \langle \mathcal{L}^{-1}1, 1 \rangle$. Then, $z(\mathcal{L}|_{X_0}) = 2$ if and only if $\sigma_0 = 0$. On the other hand, $z(\mathcal{L}) = 2$ if and only if σ_0 is unbounded.

2.3 Spectral Stability

Next, we consider if the ground state of the variational problem (2.2) with a single constraint is a local minimizer of the following variational problem with two constraints:

$$r_c := \inf_{u \in H_{\text{per}}^{\frac{\alpha}{2}}} \left\{ \mathcal{B}_c(u) : \quad \int_{-\pi}^{\pi} u^4 dx = 1, \quad \int_{-\pi}^{\pi} u dx = 0 \right\}. \tag{2.15}$$

It is clear that $r_c \leq q_{c,\text{odd}}$ and therefore, minimizers of (2.2) could be saddle points of (2.15). The following lemma provides the relevant criterion. Its proof relies on the following proposition formulated as Theorem 4.1 in [33].

Proposition 2.12 *Let L be a self-adjoint operator in a Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ such that L has $n(L)$ negative eigenvalues (counting their multiplicities) and $z(L)$ multiplicity of the zero eigenvalue bounded away from the positive spectrum of L. Let $\{v_j\}_{j=1}^{N}$ be a linearly independent set in H and define*

$$H_c := \{ f \in H : \quad \{\langle f, v_j \rangle = 0\}_{j=1}^{N} \}.$$

Let $A(\lambda)$ be the matrix-valued function defined by its elements

$$A_{ij}(\lambda) := \langle (L - \lambda I)^{-1} v_i, v_j \rangle, \quad 1 \leq i, j \leq N, \quad \lambda \notin \sigma(L).$$

Then,

$$\begin{cases} n(L\big|_{H_c}) = n(L) - n_0 - z_0, \\ z(L\big|_{H_c}) = z(L) + z_0 - z_\infty, \end{cases} \tag{2.16}$$

where n_0, z_0, and p_0 are the numbers of negative, zero, and positive eigenvalues of $\lim_{\lambda \uparrow 0} A(\lambda)$ (counting their multiplicities) and $z_\infty = N - n_0 - z_0 - p_0$ is the number of eigenvalues of $A(\lambda)$ diverging in the limit $\lambda \uparrow 0$.

Lemma 2.13 *Let $\alpha \in (\frac{1}{2}, 2]$ and $\psi \in H^\alpha_{\mathrm{per,odd}}$ be a solution obtained in Theorem 2.1 and Corollary 2.2. If $1 \in \mathrm{Range}(\mathcal{L})$, then*

$$n(\mathcal{L}|_{\{1,\psi^3\}^\perp}) = \begin{cases} 0, & \sigma_0 \leq 0, \\ 1, & \sigma_0 > 0, \end{cases} \quad z(\mathcal{L}|_{\{1,\psi^3\}^\perp}) = \begin{cases} 1, & \sigma_0 \neq 0, \\ 2, & \sigma_0 = 0, \end{cases} \tag{2.17}$$

where $\sigma_0 := \langle \mathcal{L}^{-1} 1, 1 \rangle$. If $1 \notin \mathrm{Range}(\mathcal{L})$, then

$$n(\mathcal{L}|_{\{1,\psi^3\}^\perp}) = 1, \quad z(\mathcal{L}|_{\{1,\psi^3\}^\perp}) = 1. \tag{2.18}$$

Proof By using the result of Proposition 2.12, we construct the following symmetric 2-by-2 matrix related to the two constraints in (2.15):

$$P(\lambda) := \begin{bmatrix} \langle (\mathcal{L} - \lambda I)^{-1} \psi^3, \psi^3 \rangle & \langle (\mathcal{L} - \lambda I)^{-1} \psi^3, 1 \rangle \\ \langle (\mathcal{L} - \lambda I)^{-1} 1, \psi^3 \rangle & \langle (\mathcal{L} - \lambda I)^{-1} 1, 1 \rangle \end{bmatrix}.$$

If $1 \in \mathrm{Range}(\mathcal{L})$, then

$$\langle \mathcal{L}^{-1} 1, 1 \rangle = \sigma_0, \quad \langle \mathcal{L}^{-1} 1, \psi^3 \rangle = \langle \mathcal{L}^{-1} \psi^3, 1 \rangle = 0, \quad \langle \mathcal{L}^{-1} \psi^3, \psi^3 \rangle = s - \frac{1}{4} \int_{-\pi}^{\pi} \psi^4 dx, \tag{2.19}$$

thanks to Eq. (2.9). By Proposition 2.12, we have the following identities:

$$\begin{cases} n(\mathcal{L}|_{\{1,\psi^3\}^\perp}) = n(\mathcal{L}) - n_0 - z_0, \\ z(\mathcal{L}|_{\{1,\psi^3\}^\perp}) = z(\mathcal{L}) + z_0, \end{cases} \tag{2.20}$$

where n_0 and z_0 are the numbers of negative and zero eigenvalues of $P(0)$. Since $n(\mathcal{L}) = 2$ and $z(\mathcal{L}) = 1$ by Lemma 2.9, the count (2.20) yields (2.17) due to (2.19).

If $1 \notin \mathrm{Range}(\mathcal{L})$, then $z(\mathcal{L}) = 2$ but $z(\mathcal{L}|_{X_0}) = 1$ by Lemma 2.10. By Proposition 2.12, the count (2.20) must be replaced by

$$\begin{cases} n(\mathcal{L}|_{\{1,\psi^3\}^\perp}) = n(\mathcal{L}) - n_0 - z_0, \\ z(\mathcal{L}|_{\{1,\psi^3\}^\perp}) = z(\mathcal{L}) + z_0 - z_\infty, \end{cases} \tag{2.21}$$

where $z_\infty = 1$, $z_0 = 0$, and $n_0 = 1$. The count (2.21) yields (2.18).   □

It follows by Lemma 2.13 that the ground state of the variational problem (2.2) is a local minimizer of the variational problem (2.15) if $\sigma_0 \leq 0$, which is only degenerate by the translational symmetry if $\sigma_0 \neq 0$, whereas it is the saddle point of the variational problem (2.15) if $\sigma_0 > 0$ or if $1 \notin \mathrm{Range}(\mathcal{L})$, in which case σ_0 is unbounded.

Equipped with the variational characterization of Lemma 2.13, we can clarify the spectral stability of the odd periodic waves. The following theorem gives the relevant result.

Theorem 2.14 *Let $\alpha \in (\frac{1}{2}, 2]$ and $\psi \in H^\alpha_{\text{per,odd}}$ be a solution obtained in Theorem 2.1 and Corollary 2.2. If $1 \in \text{Range}(\mathcal{L})$, then the periodic wave is spectrally stable if*

$$\sigma_0 \leq 0, \quad \frac{d}{dc}\|\psi\|^2_{L^2} \geq 0, \tag{2.22}$$

and is spectrally unstable with exactly one real positive eigenvalue of $\partial_x \mathcal{L}$ in $L^2(\mathbb{T})$ if

$$\text{either } \sigma_0 \frac{d}{dc}\|\psi\|^2_{L^2} > 0 \quad \text{or } \sigma_0 = 0, \quad \frac{d}{dc}\|\psi\|^2_{L^2} < 0, \quad \text{or } \sigma_0 > 0, \quad \frac{d}{dc}\|\psi\|^2_{L^2} = 0, \tag{2.23}$$

where $\sigma_0 := \langle \mathcal{L}^{-1} 1, 1 \rangle$. If $1 \notin \text{Range}(\mathcal{L})$, then the periodic wave is spectrally unstable with exactly one real positive eigenvalue of $\partial_x \mathcal{L}$ in $L^2(\mathbb{T})$ if

$$\frac{d}{dc}\|\psi\|^2_{L^2} \geq 0. \tag{2.24}$$

Proof It is well-known [22] that the periodic wave ψ is spectrally stable if it is a constrained minimizer of energy (1.3) under fixed momentum (1.4) and mass (1.5). Since $\mathcal{L}$ is the Hessian operator for $G(u)$ in (1.7), the spectral stability holds if

$$\mathcal{L}|_{\{1,\psi\}^\perp} \geq 0. \tag{2.25}$$

On the other hand, the periodic wave ψ is spectrally unstable with exactly one real positive eigenvalue if $n\left(\mathcal{L}|_{\{1,\psi\}^\perp}\right) = 1$, whereas the case $n\left(\mathcal{L}|_{\{1,\psi\}^\perp}\right) = 2$ is inconclusive (see [34]).

Similarly to the proof of Lemma 2.13, we construct the following symmetric 2-by-2 matrix related to the two constraints in (2.25):

$$D(\lambda) := \begin{bmatrix} \langle (\mathcal{L} - \lambda I)^{-1}\psi, \psi \rangle & \langle (\mathcal{L} - \lambda I)^{-1}\psi, 1 \rangle \\ \langle (\mathcal{L} - \lambda I)^{-1}1, \psi \rangle & \langle (\mathcal{L} - \lambda I)^{-1}1, 1 \rangle \end{bmatrix}.$$

If $1 \in \text{Range}(\mathcal{L})$, then

$$\langle \mathcal{L}^{-1}1, 1 \rangle = \sigma_0, \quad \langle \mathcal{L}^{-1}1, \psi \rangle = \langle \mathcal{L}^{-1}\psi, 1 \rangle = 0, \quad \langle \mathcal{L}^{-1}\psi, \psi \rangle = -\frac{1}{2}\frac{d}{dc}\|\psi\|^2_{L^2}, \tag{2.26}$$

where we have used $\mathcal{L}\partial_c \psi = -\psi$ from Lemma 2.8, which can be applied since $z(\mathcal{L}|_{L^2_{\text{odd}}}) = 0$ follows from the proof of Lemma 2.9. By Proposition 2.12, we have the following identities:

$$\begin{cases} n(\mathcal{L}|_{\{1,\psi\}^\perp}) = n(\mathcal{L}) - n_0 - z_0, \\ z(\mathcal{L}|_{\{1,\psi\}^\perp}) = z(\mathcal{L}) + z_0, \end{cases} \tag{2.27}$$

where n_0 and z_0 are the numbers of negative and zero eigenvalues of $D(0)$. Since $n(\mathcal{L}) = 2$ and $z(\mathcal{L}) = 1$ by Lemma 2.9, the count (2.27) implies $n(\mathcal{L}|_{\{1,\psi\}^\perp}) = 0$ due to (2.26) if the conditions (2.22) are satisfied and $n(\mathcal{L}|_{\{1,\psi\}^\perp}) = 1$ if the condition (2.23) is satisfied.

If $1 \notin \text{Range}(\mathcal{L})$, then $z(\mathcal{L}) = 2$ but $z(\mathcal{L}|_{X_0}) = 1$ by Lemma 2.10. By Proposition 2.12, the count (2.27) must be replaced by

$$\begin{cases} n(\mathcal{L}|_{\{1,\psi\}^\perp}) = n(\mathcal{L}) - n_0 - z_0, \\ z(\mathcal{L}|_{\{1,\psi\}^\perp}) = z(\mathcal{L}) + z_0 - z_\infty, \end{cases} \tag{2.28}$$

where $z_\infty = 1$ and $n_0 + z_0 = 1$ if the condition (2.24) is satisfied. In this case, $n(\mathcal{L}|_{\{1,\psi\}^\perp}) = 1$ and the periodic wave is spectrally unstable. □

Remark 2.15 If $1 \in \text{Range}(\mathcal{L})$, the case $\sigma_0 > 0$ and $\frac{d}{dc}\|\psi\|_{L^2}^2 < 0$ is inconclusive because $n(\mathcal{L}|_{\{1,\psi\}^\perp}) = 2$. In this case, one needs to find if the spectral stability problem has eigenvalues $\lambda \in i\mathbb{R}$ with so-called negative Krein signature, see [34] for further details. The same is true if $1 \notin \text{Range}(\mathcal{L})$ and $\frac{d}{dc}\|\psi\|_{L^2}^2 < 0$.

3 Examples of Odd Periodic Waves

Here we present three examples of the odd periodic waves obtained in Theorem 2.1 and Corollary 2.2. Small-amplitude waves are studied with the explicit analytical computations of the perturbation expansions. Periodic waves in the local case $\alpha = 2$ are handled by explicit analytical computations of elliptic functions. Periodic waves for $\alpha = 1$ are considered by means of numerical approximations.

3.1 Stokes Expansion of Small-Amplitude Waves

Stokes expansions of small-amplitude periodic waves near the bifurcation point $c = -1$ are rather standard in getting precise results on the existence and stability of periodic waves [25, 29,32]. The following proposition describes the properties of the small-amplitude periodic waves.

Proposition 3.1 *For each $\alpha \in \left(\frac{1}{2}, 2\right]$, there exists $c_0 \in (-1, \infty)$ such that the odd periodic wave exists for $c \in (-1, c_0)$ with $n(\mathcal{L}) = 2$, $z(\mathcal{L}) = 1$ and is spectrally stable.*

Proof We solve Eq. (1.1) with $b = 0$ in the space of odd functions by using Stokes expansions in terms of small amplitude A:

$$\psi(x) = A\psi_1(x) + A^3\psi_3(x) + \mathcal{O}(A^5) \tag{3.1}$$

and

$$c = -1 + A^2 c_2 + \mathcal{O}(A^4). \tag{3.2}$$

We obtain recursively: $\psi_1(x) = \sin(x)$,

$$\psi_3(x) = \frac{1}{2(1 - 3^\alpha)}\sin(3x),$$

and $c_2 = \frac{3}{2}$ uniformly in α.

Since $\mathcal{L} = D^\alpha - 1 + \mathcal{O}(A^2)$, then $1 \in \text{Range}(\mathcal{L})$ for small A, so that $n(\mathcal{L}) = 2$ and $z(\mathcal{L}) = 1$ for any $\alpha \in \left(\frac{1}{2}, 2\right]$ by Lemma 2.9.

Furthermore, $\mathcal{L}^{-1}1 = -1 + \mathcal{O}(A^2)$, so that $\sigma_0 = \langle \mathcal{L}^{-1}1, 1 \rangle = -2\pi + \mathcal{O}(A^2) < 0$, which implies $n(\mathcal{L}|_{\{1,\psi^3\}^\perp}) = 0$ and $z(\mathcal{L}|_{\{1,\psi^3\}^\perp}) = 1$ by Lemma 2.13. Hence, the odd periodic wave for small amplitude A represents a local minimizer of the variational problem (2.15) with two constraints for any $\alpha \in \left(\frac{1}{2}, 2\right]$.

Finally, we obtain $\|\psi\|_{L^2}^2 = \pi A^2 + \mathcal{O}(A^4)$ so that

$$\frac{d}{dc}\|\psi\|_{L^2}^2 = \frac{2\pi}{3}\left[1 + \mathcal{O}(A^2)\right] > 0.$$

By Theorem 2.14, the Stokes wave (3.1) for small A is spectrally stable since the criterion (2.22) is satisfied. □

Remark 3.2 For c near -1, the small-amplitude wave of Proposition 3.1 coincides with the odd periodic wave obtained in Theorem 2.1 and Corollary 2.2. This follows from local uniqueness of the small-amplitude wave in the neighborhood of $(0, -1)$ in $H_{\text{per,odd}}^{\frac{\alpha}{2}} \times \mathbb{R} \ni (\psi, c)$ and the result of Lemma 2.3 which guarantees that $\psi \to 0$ in $H_{\text{per,odd}}^{\frac{\alpha}{2}}$ as $c \to -1$.

3.2 Local Case $\alpha = 2$

In the case of the modified KdV equation ($\alpha = 2$), Eq. (1.1) with $b = 0$ can be solved in the space of odd functions by using the Jacobian cnoidal function [7,17].

Let us recall the normalized second-order equation

$$\psi_0''(z) + (1 - 2k^2)\psi_0(z) + 2\psi_0(z)^3 = 0, \tag{3.3}$$

which admits the periodic solution $\psi_0(z) = k\,\text{cn}(z; k)$ with the period $4K(k)$, where $K(k)$ is the complete elliptic integral of the first kind and $k \in (0, 1)$. The periodic solution also satisfies the first-order invariant given by

$$(\psi_0')^2 + (1 - 2k^2)\psi_0^2 + \psi_0^4 = k^2(1 - k^2). \tag{3.4}$$

By adopting a scaling transformation and a translation of the even function $\text{cn}(z; k)$ by a quarter-period, we obtain the exact solution for the odd periodic wave in the form:

$$\psi(x) = \frac{2}{\pi}kK(k)\text{cn}\left[\frac{2}{\pi}K(k)x - K(k); k\right] = \frac{2}{\pi}k\sqrt{1-k^2}K(k)\frac{\text{sn}\left[\frac{2}{\pi}K(k)x; k\right]}{\text{dn}\left[\frac{2}{\pi}K(k)x; k\right]} \tag{3.5}$$

with

$$c = \frac{4}{\pi^2}K(k)^2(2k^2 - 1). \tag{3.6}$$

We recall some properties of complete elliptic integrals $K(k)$ and $E(k)$ of the first and second kinds, respectively:

(a) $E(0) = K(0) = \dfrac{\pi}{2}$,

(b) $E(k) \to 1, \quad K(k) \to \infty, \quad \text{as} \quad k \to 1,$

and

(c) $\dfrac{d}{dk}E(k) = \dfrac{E(k) - K(k)}{k} < 0, \quad \dfrac{d}{dk}K(k) = \dfrac{E(k)}{k(1 - k^2)} - \dfrac{K(k)}{k} > 0.$

It follows from (a) and (c) that

$$(1 - k^2)K(k) < E(k) < K(k), \quad k \in (0, 1). \tag{3.7}$$

The following proposition summarizes properties of the odd periodic waves for $\alpha = 2$. These properties were also studied in [7,17].

Proposition 3.3 *Fix $\alpha = 2$. The odd periodic wave (3.5) exists for every $c \in (-1, \infty)$ with $n(\mathcal{L}) = 2$ and $z(\mathcal{L}) = 1$. There exists $c_* \in (-1, \infty)$ such that the odd periodic wave is spectrally stable for $c \in (-1, c_*]$ and is spectrally unstable with one real positive eigenvalue for $c \in (c_*, \infty)$.*

Proof The mapping $(0, 1) \ni k \mapsto c(k) \in (-1, \infty)$ is one-to-one and onto. This follows from

$$\frac{\pi^2}{8} \frac{dc}{dk} = \frac{K(k)}{k(1-k^2)} \left[(1-k^2)[K(k) - E(k)] + k^2 E(k) \right] > 0,$$

where property (3.7) has been used. Hence, the odd periodic wave parameterized by $k \in (0, 1)$ in (3.5) and (3.6) exists for every $c \in (-1, \infty)$.

The first five eigenvalues and eigenfunctions of the normalized linearized operator

$$\mathcal{L}_0 = -\partial_z^2 + 2k^2 - 1 - 6k^2 \text{cn}(z; k)^2 \tag{3.8}$$

are known in space $L^2(-2K(k), 2K(k))$ in the explicit form [17]. The two negative eigenvalues and a simple zero eigenvalue with the corresponding eigenfunctions are given by

$$
\begin{aligned}
\lambda_0 &= 1 - 2k^2 - 2\sqrt{1 - k^2 + k^4}, &\varphi_0(z) &= 1 + k^2 + \sqrt{1 - k^2 + k^4} - 3k^2 \text{sn}(z; k)^2, \\
\lambda_1 &= -3k^2, &\varphi_1(z) &= \text{cn}(z; k)\text{dn}(z; k), \\
\lambda_2 &= 0, &\varphi_2(z) &= \text{sn}(z; k)\text{dn}(z; k).
\end{aligned}
$$

The next two positive eigenvalues with the corresponding eigenfunctions are given by

$$
\begin{aligned}
\lambda_3 &= 3(1 - k^2), &\varphi_3(z) &= \text{sn}(z; k)\text{cn}(z; k), \\
\lambda_4 &= 1 - 2k^2 + 2\sqrt{1 - k^2 + k^4}, &\varphi_4(z) &= 1 + k^2 - \sqrt{1 - k^2 + k^4} - 3k^2 \text{sn}(z; k)^2.
\end{aligned}
$$

Eigenvalues and eigenvectors of the linearized operator $\mathcal{L}$ are obtained after the same scaling and translational transformation as in (3.5). In agreement with Lemma 2.9, we have $n(\mathcal{L}) = 2$, $z(\mathcal{L}) = 1$, and $1 \in \text{Range}(\mathcal{L})$ for every $c \in (-1, \infty)$. Moreover, we compute

$$\frac{1}{2\sqrt{1 - k^2 + k^4}} \mathcal{L}_0 \left[\frac{\lambda_4 \varphi_0 - \lambda_0 \varphi_4}{\lambda_0 \lambda_4} \right] = 1 \quad \text{and} \quad \frac{1}{2\sqrt{1 - k^2 + k^4}} [\varphi_0 - \varphi_4] = 1,$$

from which it follows that

$$\langle \mathcal{L}_0^{-1} 1, 1 \rangle = \frac{\lambda_4 \langle \varphi_0, 1 \rangle - \lambda_0 \langle \varphi_4, 1 \rangle}{2\sqrt{1 - k^2 + k^4} \lambda_0 \lambda_4} = -4 [2E(k) - K(k)].$$

Since

$$\frac{d}{dk} k E(k) = 2E(k) - K(k) \quad \text{and} \quad \frac{d^2}{dk^2} k E(k) = \frac{(1-k^2)[E(k) - K(k)] - k^2 E(k)}{k(1-k^2)} < 0,$$

in addition to (b), there exists exactly one value of k, labeled as $k^* \approx 0.909$ in [17], such that $\langle \mathcal{L}_0^{-1} 1, 1 \rangle < 0$ for $k \in (0, k^*)$ and $\langle \mathcal{L}_0^{-1} 1, 1 \rangle > 0$ for $k \in (k^*, 1)$. Up to a positive scaling factor, $\langle \mathcal{L}_0^{-1} 1, 1 \rangle$ gives the value of $\sigma_0 = \langle \mathcal{L}^{-1} 1, 1 \rangle$. By Lemma 2.13, this implies that $n(\mathcal{L}|_{\{1, \psi^3\}^\perp}) = 0$ for $k \in (0, k^*]$ and $n(\mathcal{L}|_{\{1, \psi^3\}^\perp}) = 1$ for $k \in (k^*, 1)$. Therefore, there exists a bifurcation at $k = k^*$ such that the odd periodic wave (3.5) is a local minimizer of the variational problem (2.15) with two constraints for $k \in (0, k^*)$ and a saddle point for $k \in (k^*, 1)$. The value of k^* defines uniquely a value $c^* \approx 1.425$ by (3.6).

Finally, we obtain

$$\|\psi\|_{L^2}^2 = \frac{8}{\pi} K(k) \left[E(k) - (1 - k^2) K(k) \right] > 0$$

and

$$\frac{\pi}{9} \frac{d}{dk} \|\psi\|_{L^2}^2 = \frac{1}{k(1-k^2)} \left[(1-k^2) K(k)[K(k) - E(k)] + E(k)[E(k) - (1-k^2)K(k)] \right] > 0,$$

for every $k \in (0, 1)$, where the property (3.7) has been used. By Theorem 2.14 due to the stability and instability criteria (2.22) and (2.23), the odd periodic wave (3.5) with the speed (3.6) is spectrally stable for $c \in (-1, c^*]$ and is spectrally unstable with exactly one real positive eigenvalue if $c \in (c^*, \infty)$. □

Remark 3.4 The cnoidal wave of Proposition 3.3 coincides with the odd periodic wave obtained in Theorem 2.1 and Corollary 2.2 for $\alpha = 2$. This follows from uniqueness of smooth, odd, and 2π-periodic solutions of the differential equation

$$-\psi'' + c\psi = 2\psi^3, \tag{3.9}$$

with $c \in (-1, \infty)$, where Eq. (3.9) is the Euler–Lagrange equation for the variational problem in Theorem 2.1 and Corollary 2.2.

The claim in Remark 3.4 is based on the following proposition. Since the previous results in [14,19] are not sufficient for the proof of this proposition, we provide a simple proof based on explicit computations.

Proposition 3.5 *For every $c \in \mathbb{R}$, there exists a family of L-periodic, sign-indefinite solutions of Eq. (3.9), which can be parameterized by the value $\mathcal{I}$ of the first-order invariant*

$$\mathcal{I} = (\psi')^2 - c\psi^2 + \psi^4. \tag{3.10}$$

The mapping $\mathcal{I} \mapsto L$ is monotonically decreasing for every $c \in \mathbb{R}$ with $L \in (0, 2\pi|c|^{-1/2})$ for $c < 0$ and $L \in (0, \infty)$ for $c \geq 0$. Consequently, the odd, 2π-periodic solution of Eq. (3.9) for $c \in (-1, \infty)$ is unique.

Proof Elementary phase-plane analysis (see [14,19]) shows the existence of the L-periodic, sign-indefinite solutions of Eq. (3.9) integrable with the first-order invariant (3.10). By using the scaling transformation, the L-periodic sign-indefinite solution is obtained from the periodic solution $\psi_0(z) = k \operatorname{cn}(z; k)$ of the normalized equations (3.3) and (3.4) in the form:

$$\psi(x) = k\,\alpha\,\operatorname{cn}(\alpha x; k), \quad \alpha := \left(\frac{c}{2k^2 - 1}\right)^{1/2}, \tag{3.11}$$

where $k \in (0, \frac{1}{\sqrt{2}})$ if $c < 0$ and $k \in (\frac{1}{\sqrt{2}}, 1)$ if $c > 0$. For $c = 0$, the choice $k = \frac{1}{\sqrt{2}}$ is unique but parameter α is arbitrary.

It follows from (3.11) that period L and parameter $\mathcal{I}$ are expressed uniquely by

$$L = 4\alpha^{-1}K(k), \quad \mathcal{I} = \alpha^4 k^2 (1 - k^2), \tag{3.12}$$

where α depends on k if $c \neq 0$. Computing derivatives in k yields

$$\frac{d\mathcal{I}}{dk} = \frac{kc^2}{(1 - 2k^2)^3}, \quad \frac{dL}{dk} = -\frac{4}{\sqrt{c(2k^2 - 1)}}\left[(1 - 2k^2)\frac{d}{dk}K(k) - 2kK(k)\right].$$

If $c > 0$ and $k \in (\frac{1}{\sqrt{2}}, 1)$, then $\frac{d\mathcal{I}}{dk} < 0$ and $\frac{dL}{dk} > 0$ so that the mapping $\mathcal{I} \mapsto L$ is monotonically decreasing. As $k \to \frac{1}{\sqrt{2}}$, $\alpha \to \infty$ and $L \to 0$. As $k \to 1$, $K(k) \to \infty$ and $L \to \infty$.

If $c < 0$ and $k \in (0, \frac{1}{\sqrt{2}})$, then $\frac{d\mathcal{I}}{dk} > 0$ and $\frac{dL}{dk} < 0$ due to (3.7) and

$$\frac{dL}{dk} = \frac{4}{\sqrt{c(2k^2 - 1)}k(1 - k^2)}\left[(1 - k^2)(E(k) - K(k)) - k^2 E(k)\right] < 0,$$

so that the mapping $\mathcal{I} \mapsto L$ is also monotonically decreasing. As $k \to \frac{1}{\sqrt{2}}$, $\alpha \to \infty$ and $L \to 0$. As $k \to 0$, $K(k) \to \frac{\pi}{2}$ and $L \to 2\pi |c|^{-1/2}$.

If $c = 0$ and $k = \frac{1}{\sqrt{2}}$, then the parameter $\alpha \in \mathbb{R}$ is arbitrary and it follows from (3.11) that $\mathcal{I} = C/L^4$ for some $C > 0$ so that the mapping $\mathcal{I} \mapsto L$ is also monotonically decreasing and $L \in (0, \infty)$.

Since the period function $L = L(\mathcal{I})$ is monotonically decreasing in $\mathcal{I}$, there exists exactly one odd, 2π-periodic solution for every $c \in (-1, \infty)$ and by uniqueness of solutions to differential equations, this unique solution is given by the cnoidal wave (3.5) and (3.6). □

3.3 Numerical Approximations

Here we numerically compute solutions of Eq. (1.1) using Newton's method in the Fourier space, similar to our previous work [32]. For better performance, the odd periodic wave with profile ψ in Theorem 2.1 is translated by a quarter period $\pi/2$ to an even function of x. The starting iteration is generated from the Stokes expansion (3.1) after the translation and this solution is uniquely continued in c for all $c \in (-1, \infty)$. This family of solutions correspond to $b = 0$ in Eq. (1.1).

Additionally, we add a perturbation to the profile ψ to preserve the even symmetry but to break the odd symmetry after the translation. Numerical iterations converge back to the same family of solutions with $b = 0$ for $c < c_*$, where $c_* \in (-1, \infty)$ is the bifurcation point for which a nontrivial solution in Lemma 2.10 exists. The value of c_* exists for all $\alpha \in \left(\frac{1}{2}, 2\right]$. When $c > c_*$, numerical iterations converge to a new family of solutions to Eq. (1.1) with $b \neq 0$, which is then continued with respect to c. Convergence of numerical iterations is measured by the L^2 norm of the residual equation (1.1), with the tolerance equals to 10^{-10}.

Figure 2 presents the periodic wave solutions to Eq. (1.1) for $\alpha = 2$. The top left panel shows the profiles of ψ of the family with $b = 0$ for three different values of c: near the Stokes wave limit (blue curve), near the bifurcation point c_* (black curve) and when c is away from the bifurcation point c_* (red curve). The top right panel shows the profiles of ψ of the bifurcating family with $b \neq 0$ near the bifurcation point c_* (black curve) and increasingly away from the bifurcation point (blue and red curves). The vertical lines show the symmetry points at $x = \pm \pi/2$. The family with $b = 0$ has odd symmetry with respect to these points, whereas the family with $b \neq 0$ does not have this symmetry; both families are even at $x = 0$ and $x = \pm \pi$.

The middle left panel of Fig. 2 shows the dependence of b in Eq. (1.1) versus speed c. The pitchfork bifurcation point is located at $c_* \approx 1.425$. The two symmetric branches of solutions with $b > 0$ and $b < 0$ are obtained by using the positive and negative perturbations to the family of solutions with $b = 0$.

The middle right panel of Fig. 2 shows the momentum $F(\psi)$ versus c. The bottom left panel shows the dependence of σ_0 versus c. The bottom right panel displays the lowest eigenvalues of $\mathcal{L}$ versus c. The blue curve shows the family of solutions with $b = 0$, whereas the red curve shows the family of solutions with $b \neq 0$.

As shown on the middle right panel of Fig. 2, the momentum $F(\psi)$ is increasing function of c for both the families. In agreement with the theory, σ_0 for the family with $b = 0$ changes sign from negative to positive when c passes through the bifurcation point c_*, see the bottom left panel of Fig. 2. By Theorem 2.14, it follows that the family of solutions with $b = 0$ is spectrally stable for $c < c_*$ and spectrally unstable for $c > c_*$.
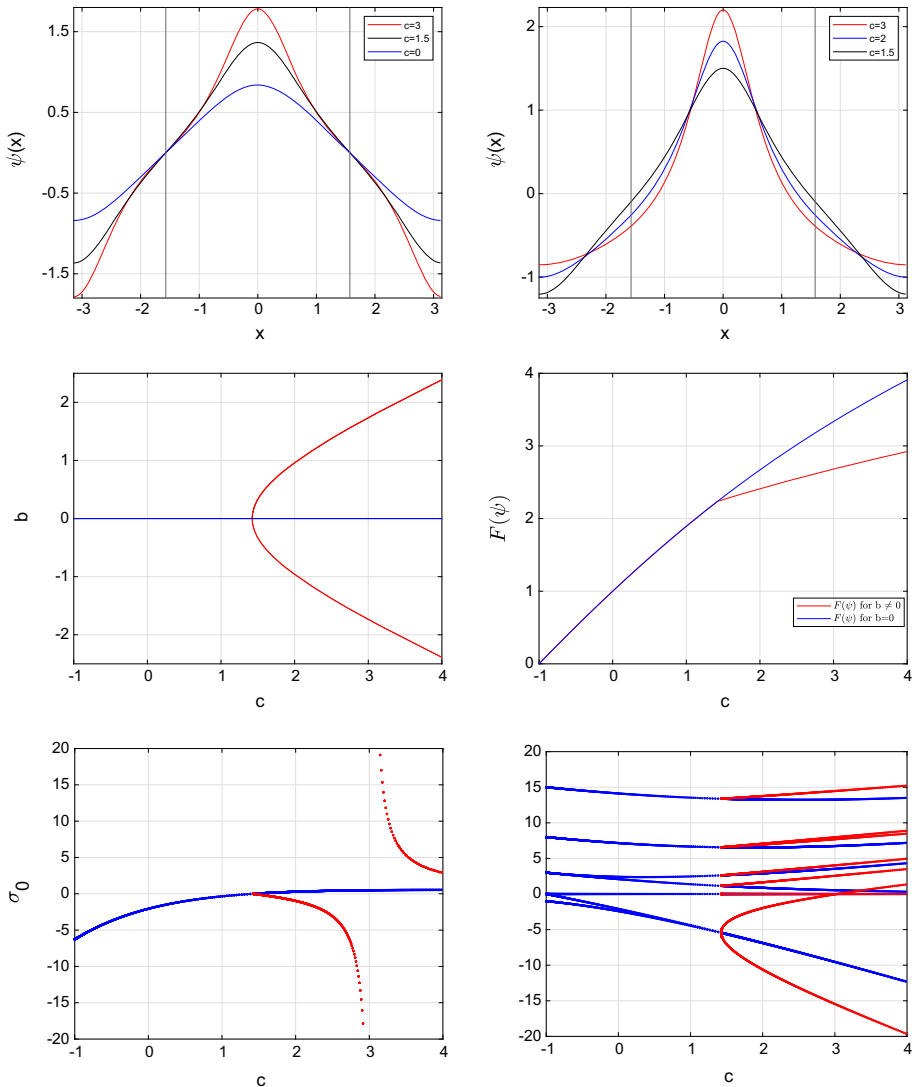
Fig. 2 Odd periodic waves for $\alpha = 2$. Top left: Profiles of ψ with $b = 0$ for three different values of c. Top right: Profiles of ψ with $b \neq 0$ for three values of c. Middle left: Dependence of b versus c showing the pitchfork bifurcation point c_*. Middle right: Dependence of the momentum $F(\psi)$ versus c. Bottom left: Dependence of σ_0 versus c. Bottom right: The lowest eigenvalues of $\mathcal{L}$ versus c. The blue (red) line corresponds to the family with $b = 0$ ($b \neq 0$)

On the other hand, the bifurcating family with $b \neq 0$ has $\sigma_0 < 0$ near the bifurcation point but there exists another point $\hat{c}_* > c_*$ such that σ_0 diverges at $c = \hat{c}_*$ and becomes positive for $c > \hat{c}_*$. This agrees with the behavior of the lowest eigenvalues of $\mathcal{L}$ shown on the bottom right panel of Fig. 2 since $z(\mathcal{L}) = 2$ at $c = \hat{c}_*$, $n(\mathcal{L}) = 2$ for $c < \hat{c}_*$ and $n(\mathcal{L}) = 1$ for $c > \hat{c}_*$. Lemma 2.13 and Theorem 2.14 are trivially extended to the family with $b \neq 0$ and they confirm that for both cases of $c < \hat{c}_*$ and $c > \hat{c}_*$, the periodic waves of the family
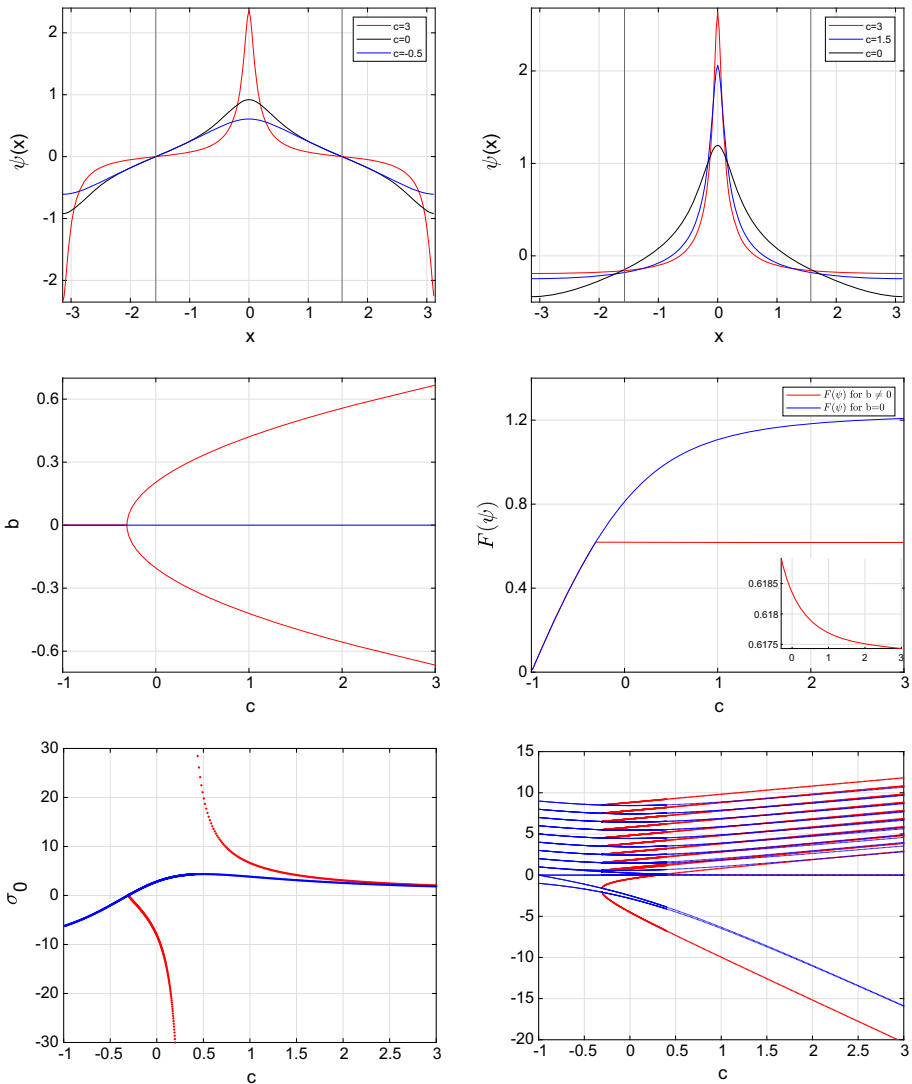
Fig. 3 The same as Fig. 2 but for $\alpha = 1$

with $b \neq 0$ correspond to minimizers of the constrained variational problem (2.15) and they are spectrally stable for $c > c_*$.

Figure 3 presents similar results for the periodic wave solutions to Eq. (1.1) for $\alpha = 1$. Note that bifurcation point c_* moves to the left and becomes $c_* \approx -0.310$. The existence and stability of the family of solutions with $b = 0$ is very similar with the only difference that the dependence of the momentum $F(\psi)$ versus speed c approaches the horizontal asymptote as $c \to \infty$ since $\alpha = 1$ is the L^2-critical modified Benjamin–Ono equation [8,26,31] and the periodic waves with the single-lobe profile converge to the solitary waves in the limit $c \to \infty$.

The stability of the family of solutions with $b \neq 0$ is however different. The momentum $F(\psi)$ is a decreasing function of the speed c, as the insert shows, hence the family of solutions is spectrally unstable for all $c > c_*$. It also approaches to the horizontal asymptote as $c \to \infty$. Profiles of both the families in the limit of large c approach the soliton profile, but the family with $b = 0$ contains two solitons on the period, whereas the family with $b \neq 0$ contains a single soliton on the period. Hence the momentum $F(\psi)$ of the family with $b = 0$ approaches the double horizontal asymptote as $c \to \infty$ compared to the momentum $F(\psi)$ of the family with $b \neq 0$.

We have checked again that σ_0 along the family with $b = 0$ changes sign from negative to positive at the bifurcation point $c = c_*$, whereas σ_0 along the family with $b \neq 0$ is negative for $c \in (c_*, \hat{c}_*)$ and positive for $c \in (\hat{c}_*, \infty)$, where $\hat{c}_*$ is the point where $z(\mathcal{L}) = 2$ along the family with $b \neq 0$.

4 Even Periodic Waves

Here we consider *the even periodic waves* and provide the proof of Theorem 1.7. The study follows the same structure as in the case of the odd periodic waves.

4.1 Variational Characterization

The odd periodic wave constructed in Theorem 2.1 and Corollary 2.2 is even after translation $x \mapsto x - \pi/2$. However, since $n(\mathcal{L}) = 2$ by Lemma 2.9, the odd periodic wave translated into an even function cannot be a solution of the constrained minimization problem with a single constraint. Therefore, the same constrained minimization problem (2.2) in a subspace of even functions yields a different branch of periodic waves.

The following theorem gives the construction and properties of the even periodic waves.

Theorem 4.1 *Let $\alpha > \frac{1}{2}$ be fixed. For every $c > 0$, there exists the ground state (minimizer) $\chi \in H_{\mathrm{per,even}}^{\frac{\alpha}{2}}$ of the following constrained minimization problem:*

$$q_{c,\mathrm{even}} := \inf_{u \in H_{\mathrm{per,even}}^{\frac{\alpha}{2}}} \left\{ \mathcal{B}_c(u) : \int_{-\pi}^{\pi} u^4 dx = 1 \right\}, \tag{4.1}$$

with the same $\mathcal{B}_c(u)$ as in (2.3). There exists $C > 0$ such that $\psi(x) = C\chi(x)$ satisfies Eq. (1.1) with $b = 0$. If $\alpha \leq 2$, the ground state is the constant solution for $c \in \left(0, \frac{1}{2}\right]$ and has the single-lobe profile for $c \in \left(\frac{1}{2}, \infty\right)$.

Proof It follows that $\mathcal{B}_c$ is a smooth, bounded, and coercive functional on $H_{\mathrm{per,even}}^{\frac{\alpha}{2}}$ if $c > 0$, hence $q_{c,\mathrm{even}} \geq 0$. It follows from the bound (2.5) and the constraint in (4.1) that $q_{c,\mathrm{even}} > 0$. The existence of the minimizer $\chi \in H_{\mathrm{per,even}}^{\frac{\alpha}{2}}$ of the constrained minimization problem (4.1) is proven exactly like in the proof of Theorem 2.1. Moreover, for $\alpha \in (0, 2]$, the symmetric rearrangements suggest that the minimizer $\chi \in H_{\mathrm{per,even}}^{\frac{\alpha}{2}}$ is either constant or it must decrease symmetrically away from the maximum point.

In order to ensure that the minimizer has the single-lobe profile, we need to eliminate the constant solution in $H_{\mathrm{per,even}}^{\frac{\alpha}{2}}$. By Lagrange's Multiplier Theorem, the constrained minimizer

$\chi \in H_{\mathrm{per,even}}^{\frac{\alpha}{2}}$ satisfies the stationary equation

$$D^\alpha \chi + c\chi = \mu\chi^3, \tag{4.2}$$

where $\mu = 2\mathcal{B}_c(\chi)$ due to the normalization in (4.1). Since $\mathcal{B}_c(\chi) > 0$, the scaling transformation $\psi = C\chi$ with $C := \sqrt{\mathcal{B}_c(\chi)}$ maps Eq. (4.2) to the form (1.1) with $b = 0$. The constant nonzero solution to Eq. (1.1) with $b = 0$ is given by $\psi(x) = \sqrt{c/2}$ up to a sign choice. The linearization operator $\mathcal{L}$ in (1.7) evaluated at the constant solution is given by

$$\mathcal{L} = D^\alpha + c - 6\psi^2 = D^\alpha - 2c.$$

Since $n(\mathcal{L}) = 1$ if and only if $c \in \left(0, \frac{1}{2}\right]$, the constant wave is a constrained minimizer of (4.1) for $c \in \left(0, \frac{1}{2}\right]$ and a saddle point of (4.1) for $c \in \left(\frac{1}{2}, \infty\right)$. By the symmetric rearrangements, the global minimizer is given by the constant solution in the former case and by a non-constant solution with the single-lobe profile in the latter case.   □

Remark 4.2 Periodic waves obtained in Theorem 4.1 do not satisfy the boundary-value problem (2.1) if the mean value of $\psi \in H_{\mathrm{per,even}}^\alpha$ is nonzero.

4.2 Smooth Continuation

Let $\psi \in H_{\mathrm{per,even}}^\alpha$ be a solution to Eq. (1.1) with $b = 0$ for $c \in \left(\frac{1}{2}, \infty\right)$ obtained by Theorem 4.1. We introduce again the linearized operator $\mathcal{L}$ by (1.7) and (2.7). Equalities (2.8) and (2.9) hold true for the even periodic wave and so do Propositions 2.4, 2.5, and 2.6 (no translation is needed for the last proposition).

The following lemma presents the count of $n(\mathcal{L})$ and $z(\mathcal{L})$ for the even periodic wave.

Lemma 4.3 *Let* $\alpha \in \left(\frac{1}{2}, 2\right]$ *and* $\psi \in H_{\mathrm{per,even}}^\alpha$ *be a solution obtained in Theorem* 4.1. *Then,* $n(\mathcal{L}) = 1$ *and*

$$z(\mathcal{L}) = \begin{cases} 1, & \text{if } 1 \in \mathrm{Range}(\mathcal{L}), \\ 2, & \text{if } 1 \notin \mathrm{Range}(\mathcal{L}). \end{cases}$$

Proof Since $\psi \in H_{\mathrm{per,even}}^\alpha$ is a minimizer of the constrained variational problem (4.1) with only one constraint, we have $n(\mathcal{L}|_{L_{\mathrm{even}}^2}) \leq 1$. On the other hand, we have

$$\langle \mathcal{L}\psi, \psi \rangle_{L^2} = -4\|\psi\|_{L^4}^4 < 0,$$

with even ψ, hence $n(\mathcal{L}|_{L_{\mathrm{even}}^2}) \geq 1$, so that $n(\mathcal{L}|_{L_{\mathrm{even}}^2}) = 1$. By Proposition 2.6 (without translation), $n(\mathcal{L}|_{L_{\mathrm{odd}}^2}) = 0$ and $z(\mathcal{L}|_{L_{\mathrm{odd}}^2}) = 1$. Hence, $n(\mathcal{L}) = 1$.

It remains to consider $z(\mathcal{L}) \geq 1$. Since 0 is the second eigenvalue of $\mathcal{L}$, Proposition 2.4 suggests that if $z(\mathcal{L}) = 2$, then the even eigenfunction of $\mathrm{Ker}(\mathcal{L})$ has at most two symmetric nodes on $\mathbb{T}$. If the periodic wave has the single-lobe profile ψ, then ψ^3 has also the single-lobe profile. By using the same argument as in the proof of Proposition 3.1 in [24], it follows that $z(\mathcal{L}) = 1$ if and only if $\{1, \psi^3\} \in \mathrm{Range}(\mathcal{L})$.

Indeed, if $h \in \mathrm{Ker}(\mathcal{L})$ is an even eigenfunction in the case $z(\mathcal{L}) = 2$ and $\{1, \psi^3\} \in \mathrm{Range}(\mathcal{L})$, then $\langle h, 1 \rangle = 0$ and $\langle h, \psi^3 \rangle = 0$. The first condition suggests that h is signindefinite with exactly two symmetric nodes at $\pm x_0$ with $x_0 \in (0, \pi)$, but then $\langle h, \psi^3 - \psi^3(x_0) \rangle$ is sign-definite and cannot be zero, so that no $h \in \mathrm{Ker}(\mathcal{L})$ exists.

Since $\psi^3 \in \mathrm{Range}(\mathcal{L})$ due to Eq. (2.9), it follows that $z(\mathcal{L}) = 1$ if and only if $1 \in \mathrm{Range}(\mathcal{L})$.   □

The definition of $\mathcal{L}|_{X_0}$, where $X_0 \subset L^2(\mathbb{T})$ is defined by (2.11), is the same as in (2.12). The result of Lemma 2.10 holds true for the even periodic wave $\psi \in H^\alpha_{\text{per,even}}$. In order to count the indices $n(\mathcal{L}|_{X_0})$ and $z(\mathcal{L}|_{X_0})$, we shall re-parameterize the even periodic wave to the zero-mean periodic wave.

Recall from Remark 4.2 that ψ has generally nonzero mean value and does not satisfy the boundary-value problem (2.1). Let us define $\psi(x) = a + \phi(x)$, where $a := \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(x)dx$. Then, $\phi \in H^\alpha_{\text{per,even}} \cap X_0$ is a solution of the stationary equation:

$$D^\alpha \phi + \omega \phi + \beta = 2\left(\phi^3 + 3a\phi^2\right), \tag{4.3}$$

where $\omega := c - 6a^2$ and $\beta := ca - 2a^3$. Since ϕ has zero mean, β can be equivalently written as

$$\beta := \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\phi^3 + 3a\phi^2\right)dx, \tag{4.4}$$

so that Eq. (4.3) can be rewritten as the boundary-value problem:

$$D^\alpha \phi + \omega \phi = 2\Pi_0 \left(\phi^3 + 3a\phi^2\right). \tag{4.5}$$

The following lemma presents the computation of $n(\mathcal{L}|_{X_0})$ and $z(\mathcal{L}|_{X_0})$.

Lemma 4.4 *Let $\alpha \in (\frac{1}{2}, 2]$ and $\psi \in H^\alpha_{\text{per,even}}$ be a solution obtained in Theorem 4.1. Assume that $\omega \in (-1, \infty)$ after the transformation to Eq. (4.3). Then, $n(\mathcal{L}|_{X_0}) = 1$ and $z(\mathcal{L}|_{X_0}) = 1$.*

Proof Transformation $\psi = a + \phi$ and $\omega = c - 6a^2$ changes $\mathcal{L}$ given by (1.7) into the equivalent form:

$$\mathcal{L} = D^\alpha + c - 6\psi^2 = D^\alpha + \omega - 6\phi^2 - 12a\phi =: \tilde{\mathcal{L}}. \tag{4.6}$$

Then, it follows directly that

$$\langle \mathcal{L}|_{X_0}\phi, \phi \rangle = -4 \int_{-\pi}^{\pi} \phi^4 dx - 6a \int_{-\pi}^{\pi} \phi^3 dx. \tag{4.7}$$

Taking an inner product of Eq. (4.3) with ϕ yields the Pohozhaev-type identity

$$B_\omega(\phi) = \int_{-\pi}^{\pi} \phi^4 dx + 3a \int_{-\pi}^{\pi} \phi^3 dx. \tag{4.8}$$

where $B_\omega(\phi)$ is defined by (2.3). Since $\omega \in (-1, \infty)$ and $\phi \in H^\alpha_{\text{per,even}} \cap X_0$, we have $B_\omega(\phi) \geq 0$, so that the equality (4.7) can be estimated by

$$\langle \mathcal{L}|_{X_0}\phi, \phi \rangle \leq -2 \int_{-\pi}^{\pi} \phi^4 dx < 0. \tag{4.9}$$

Hence $n(\mathcal{L}|_{X_0}) \geq 1$ and since $n(\mathcal{L}) = 1$ by Lemma 4.3, we have $n(\mathcal{L}|_{X_0}) = 1$. By Proposition 2.12, we have the following identities:

$$\begin{cases} n(\mathcal{L}|_{X_0}) = n(\mathcal{L}) - n_0 - z_0, \\ z(\mathcal{L}|_{X_0}) = z(\mathcal{L}) + z_0 - z_\infty, \end{cases} \tag{4.10}$$

where $z_\infty = 1$ if $1 \notin \text{Range}(\mathcal{L})$. It follows from the first equality in (4.10) that $n_0 = z_0 = 0$ since $n(\mathcal{L}) = n(\mathcal{L}|_{X_0}) = 1$. Then, the second equality yields $z(\mathcal{L}|_{X_0}) = z(\mathcal{L}) - z_\infty$. If $z(\mathcal{L}|_{X_0}) = 2$, then $z(\mathcal{L}) \geq 2$, which is in contradiction with Lemma 2.10 extended to the even periodic wave $\psi \in H^\alpha_{\text{per,even}}$. Hence, $z(\mathcal{L}|_{X_0}) = 1$, in which case $z(\mathcal{L}) = 1 + z_\infty$ in agreement with Lemma 4.3. $\qquad\square$

Remark 4.5 It follows from the proof of Lemma 4.4 that $\sigma_0 > 0$ if $1 \in \text{Range}(\mathcal{L})$, where $\sigma_0 := \langle \mathcal{L}^{-1} 1, 1 \rangle$.

In order to derive the spectral stability result, we shall now extend solutions to Eq. (4.3) with respect to two independent parameters (ω, a) and with β being a C^1 function of (ω, a). Since the periodic waves satisfy Eq. (1.1) with $b = 0$, where c is the only parameter, parameters ω, a, and β in Eq. (4.3) are parametrized by c, hence a is not independent of ω. The following lemma allows us to extend zero-mean solutions to the boundary-value problem (4.5) with respect to independent parameters (ω, a) near each uniquely defined point (ω_0, a_0).

Lemma 4.6 *Assume $\alpha \in (\frac{1}{2}, 2]$ and $\phi_0 \in H^\alpha_{\text{per,even}} \cap X_0$ be a solution to the boundary-value problem (4.5) with $\omega = \omega_0 \in (-1, \infty)$ and $a = a_0 \in \mathbb{R}$. Then, there exists a C^1 mapping in an open subset of (ω_0, a_0) denoted by $\mathcal{O} \subset \mathbb{R}^2$:*

$$\mathcal{O} \ni (\omega, a) \mapsto \phi(\cdot, \omega, a) \in H^\alpha_{\text{per,even}} \cap X_0, \tag{4.11}$$

such that $\phi(\cdot, \omega_0, a_0) = \phi_0$.

Proof The proof repeats the arguments in the proof of Lemma 2.8. Let $\Upsilon : (-1, \infty) \times \mathbb{R} \times H^\alpha_{\text{per,even}} \cap X_0 \to L^2_{\text{even}}(\mathbb{T}) \cap X_0$ be defined by

$$\Upsilon(\omega, a, g) := D^\alpha g + \omega g - 2\Pi_0 \left(g^3 + 3ag^2 \right). \tag{4.12}$$

By hypothesis we have $\Upsilon(\omega_0, a_0, \phi_0) = 0$. Moreover, since Υ is smooth, its Fréchet derivative with respect to g evaluated at (ω_0, a_0, ϕ_0) is given by

$$D_g \Upsilon(\omega_0, a_0, \phi_0) = D^\alpha + \omega_0 - 6\Pi_0 \left(\phi_0^2 + 2a_0 \phi_0 \right) = D^\alpha + c_0 - 6\Pi_0 \psi_0^2 = \mathcal{L}|_{X_0} \tag{4.13}$$

where we have unfolded the previous transformation $\psi_0 = a_0 + \phi_0$ and $\omega_0 = c_0 - 6a_0^2$ and used the same operator as in (2.12) computed at ψ_0.

Since $z(\mathcal{L}|_{X_0}) = 1$ by Lemma 4.4 and $\text{Ker}(\mathcal{L}|_{X_0}) = \text{span}\{\partial_x \phi_0\}$ with $\partial_x \phi_0 \notin H^\alpha_{\text{per,even}} \cap X_0$, we conclude that $D_g \Upsilon(\omega_0, a_0, \phi_0)$ is one-to-one. Next, we show that $D_g \Upsilon(\omega_0, a_0, \phi_0)$ is onto. Since $H^\alpha_{\text{per,even}} \cap X_0$ is compactly embedded in $L^2_{\text{even}}(\mathbb{T}) \cap X_0$ if $\alpha > 1/2$, the operator $\mathcal{L}|_{X_0}$ has compact resolvent. In addition, $\mathcal{L}|_{X_0}$ is a self-adjoint operator, hence its spectrum $\sigma(\mathcal{L}|_{X_0})$ consists of isolated eigenvalues with finite algebraic multiplicities. Since $D_g \Upsilon(\omega_0, a_0, \phi_0)$ is one-to-one, it follows that 0 is not in the spectrum of $D_g \Upsilon(\omega_0, a_0, \phi_0)$, so that it is onto. Hence, $D_g \Upsilon(\omega_0, a_0, \phi_0)$ is a bounded linear operator with a bounded inverse. Thus, since Υ and its derivative with respect to g are smooth maps on their domains, the result follows from the implicit function theorem. $\square$

Recall that $\mathcal{L} = \tilde{\mathcal{L}}$ in (4.6). Extension of relations (2.8) and (2.9) yields

$$\tilde{\mathcal{L}} 1 = \omega - 12a\phi - 6\phi^2 \tag{4.14}$$

and

$$\tilde{\mathcal{L}} \phi = -\beta - 6a\phi^2 - 4\phi^3, \tag{4.15}$$

where $\beta = \beta(\omega, a)$ is a C^1 function by Lemma 4.6 and the representation (4.4). Therefore, we also obtain two more relations:

$$\tilde{\mathcal{L}} \partial_\omega \phi = -\partial_\omega \beta - \phi, \tag{4.16}$$

and

$$\tilde{\mathcal{L}} \partial_a \phi = -\partial_a \beta + 6\phi^2. \tag{4.17}$$

These relations allow us to completely characterize $\text{Ker}(\mathcal{L})$, which can be two-dimensional if $1 \notin \text{Range}(\mathcal{L})$ by Lemma 4.3.

Lemma 4.7 *Assume $\alpha \in (\frac{1}{2}, 2]$ and $\phi \in H^{\alpha}_{\text{per,even}} \cap X_0$ be a single-lobe solution to the boundary-value problem (4.5) with $\omega \in (-1, \infty)$ and $a \in \mathbb{R}$. Then, $z(\mathcal{L}) = 1$ if and only if $s_0 := \omega - \partial_a \beta + 12a \partial_\omega \beta \neq 0$.*

Proof Eliminating ϕ and ϕ^2 from (4.14), (4.16), and (4.17) yields

$$\tilde{\mathcal{L}} (1 + \partial_a \phi - 12a \partial_\omega \phi) = \omega - \partial_a \beta + 12a \partial_\omega \beta =: s_0. \tag{4.18}$$

Recall that $\tilde{\mathcal{L}} = \mathcal{L}$ in (4.6). If $s_0 \neq 0$, then $1 \in \text{Range}(\mathcal{L})$, so that $z(\mathcal{L}) = 1$ holds by Lemma 4.3. If $s_0 = 0$, then $1 + \partial_a \phi - 12a \partial_\omega \phi \in \text{Ker}(\mathcal{L})$ in addition to $\partial_x \phi \in \text{Ker}(\mathcal{L})$. □

4.3 Spectral Stability

We are now ready to provide the criterion for spectral stability of the even periodic waves. This result is given by the following theorem.

Theorem 4.8 *Assume $\alpha \in (\frac{1}{2}, 2]$ and $\phi \in H^{\alpha}_{\text{per,even}} \cap X_0$ be a single-lobe solution to the boundary-value problem (4.5) with $\omega \in (-1, \infty)$ and $a \in \mathbb{R}$. The periodic wave is spectrally stable if and only if*

$$\frac{\partial}{\partial \omega} \|\phi\|^2_{L^2} \geq 0, \tag{4.19}$$

independently of either $z(\mathcal{L}) = 1$ or $z(\mathcal{L}) = 2$.

Proof We proceed similarly to the proof of Theorem 2.14. If $1 \in \text{Range}(\mathcal{L})$, we use (4.18) and compute

$$\sigma_0 := \langle \mathcal{L}^{-1} 1, 1 \rangle = \frac{2\pi}{s_0},$$

where $s_0 \neq 0$ by Lemma 4.7. For the even periodic wave, we have $\sigma_0 > 0$ (see Remark 4.5), so that $s_0 > 0$. Eliminating constant term from (4.16) and (4.18) yields

$$\tilde{\mathcal{L}} \left[\partial_\omega \phi + s_0^{-1} \partial_\omega \beta (1 + \partial_a \phi - 12a \partial_\omega \phi) \right] = -\phi, \tag{4.20}$$

By projecting (4.16) to $\partial_a \phi$ and (4.17) to $\partial_\omega \phi$, it is easy to verify that

$$6 \langle \phi^2, \partial_\omega \phi \rangle + \langle \phi, \partial_a \phi \rangle = 0. \tag{4.21}$$

Using (4.20) yields

$$\langle \mathcal{L}^{-1} 1, \phi \rangle = \langle \mathcal{L}^{-1} \phi, 1 \rangle = -\sigma_0 \partial_\omega \beta,$$

where we have used that

$$2\pi \partial_\omega \beta = 6 \langle \phi^2, \partial_\omega \phi \rangle + 12a \langle \phi, \partial_\omega \phi \rangle = -\langle \phi, \partial_a \phi \rangle + 12a \langle \phi, \partial_\omega \phi \rangle,$$

which follows from (4.4) and (4.21). Finally, we obtain from (4.20) and (4.21) that

$$\langle \mathcal{L}^{-1} \phi, \phi \rangle = -\langle \phi, \partial_\omega \phi \rangle + \sigma_0 (\partial_\omega \beta)^2.$$

By Proposition 2.12, we have the following identities:

$$\begin{cases} n(\mathcal{L}|_{\{1,\psi\}^\perp}) = n(\mathcal{L}) - n_0 - z_0, \\ z(\mathcal{L}|_{\{1,\psi\}^\perp}) = z(\mathcal{L}) + z_0, \end{cases} \tag{4.22}$$

where n_0 and z_0 are the numbers of negative and zero eigenvalues of $D(0)$ in the proof of Theorem 2.14. Since

$$\det D(0) = -\sigma_0 \langle \phi, \partial_\omega \phi \rangle.$$

and $\sigma_0 > 0$, we have $n_0 + z_0 = 1$ if the condition (4.19) is satisfied and $n_0 + z_0 = 0$ if it is not satisfied. Since $n(\mathcal{L}) = 1$ and $z(\mathcal{L}) = 1$ by Lemmas 4.3 and 4.7, the count (4.22) implies $n(\mathcal{L}|_{\{1,\psi\}^\perp}) = 0$ if the condition (4.19) is satisfied and $n(\mathcal{L}|_{\{1,\psi\}^\perp}) = 1$ if it is not satisfied. This gives the assertion of the theorem if $1 \in \text{Range}(\mathcal{L})$.

If $1 \notin \text{Range}(\mathcal{L})$, then $z(\mathcal{L}) = 2$ and $s_0 = 0$. In this case, the count (4.22) should be adjusted as

$$\begin{cases} n(\mathcal{L}|_{\{1,\psi\}^\perp}) = n(\mathcal{L}) - n_0 - z_0, \\ z(\mathcal{L}|_{\{1,\psi\}^\perp}) = z(\mathcal{L}) + z_0 - z_\infty, \end{cases} \tag{4.23}$$

where $z_\infty = 1$. At the same time, $n_0 + z_0 = 1$ if and only if the same condition (4.19) is satisfied and $n_0 + z_0 = 0$ if it is not satisfied. Hence the stability conclusion remains unchanged if $1 \notin \text{Range}(\mathcal{L})$.   □

Remark 4.9 The momentum (1.4) computed at the even periodic wave with the profile ψ and the decomposition $\psi = a + \phi$ is given by

$$F(\psi) = F(\phi) + \pi a^2.$$

If ω and a are independent parameters, it is true that

$$\frac{\partial}{\partial \omega} F(\phi) = \frac{\partial}{\partial \omega} F(\psi), \tag{4.24}$$

however, this quantity is not defined by the dependence of the momentum $F(\psi)$ on the original wave speed c. In addition, if ψ satisfies Eq. (1.1) with $b = 0$, then a depends on ω, therefore, the dependence of $F(\psi)$ versus ω does not generally provide information about the slope condition (4.24). See also numerical approximations in the next section.

5 Examples of Even Periodic Waves

Here we present three examples of the even periodic waves obtained in Theorem 4.1. These examples mimic the corresponding examples of the odd periodic waves.

5.1 Stokes Expansion of Small-Amplitude Waves

Stokes expansion gives again a direct way to illustrate small-amplitude periodic waves bifurcating from the constant solutions at $c = \frac{1}{2}$. In order to eliminate the constant wave, we set

$$\psi(x) = \frac{\sqrt{c}}{\sqrt{2}} + \varphi(x), \tag{5.1}$$

where φ is not required to satisfy the zero-mean property. Eq. (1.1) with $b = 0$ is written in the equivalent form:

$$D^\alpha \varphi - 2c\varphi = 2\varphi^3 + 3\sqrt{2c}\varphi^2. \tag{5.2}$$

By using the Stokes expansion in terms of small amplitude A:

$$\begin{cases} \varphi(x) = A\varphi_1(x) + A^2\varphi_2(x) + A^3\varphi_3(x) + \mathcal{O}(A^4), \\ 2c = 1 + A^2\gamma_2 + \mathcal{O}(A^4), \end{cases} \tag{5.3}$$

we obtain recursively: $\varphi_1(x) = \cos(x)$,

$$\varphi_2(x) = -\frac{3}{2} + \frac{3}{2(2^\alpha - 1)}\cos(2x),$$

$$\varphi_3(x) = \frac{1}{2(3^\alpha - 1)}\left[1 + \frac{9}{2^\alpha - 1}\right]\cos(3x),$$

and

$$\gamma_2 = \frac{15}{2} - \frac{9}{2(2^\alpha - 1)}.$$

It follows that $\gamma_2 = 0$ if and only if $2^\alpha = \frac{8}{5}$, which is true at

$$\alpha_0 := \frac{\log 8 - \log 5}{\log 2} \approx 0.6781. \tag{5.4}$$

The following proposition summarizes properties of the small-amplitude periodic waves.

Proposition 5.1 *Let α_0 be given by (5.4). For each $\alpha \in (\alpha_0, 2]$, there exists $c_0 > \frac{1}{2}$ such that the even periodic wave exists for $c \in \left(\frac{1}{2}, c_0\right)$ with $n(\mathcal{L}) = 1$, $z(\mathcal{L}) = 1$ and is spectrally stable. For each $\alpha \in \left(\frac{1}{2}, \alpha_0\right)$, there exists $c_0 < \frac{1}{2}$ such that the even periodic wave exists for $c \in \left(c_0, \frac{1}{2}\right)$ with $n(\mathcal{L}) = 2$, $z(\mathcal{L}) = 1$, and is spectrally stable.*

Proof The existence statement follows from the Stokes expansion (5.3) with small wave amplitude A since $\gamma_2 > 0$ for $\alpha > \alpha_0$ and $\gamma_2 < 0$ for $\alpha < \alpha_0$.

In order to compute $n(\mathcal{L})$ and $z(\mathcal{L})$, we substitute (5.1) and (5.3) into (1.7) and obtain

$$\mathcal{L} = D^\alpha - 1 - A\cos(x) - A^2\left[\gamma_2 + 6\varphi_2(x) - 6\cos^2(x)\right] + \mathcal{O}(A^3).$$

We solve the spectral problem $\mathcal{L}v = \lambda v$ perturbatively near the eigenvalue $\lambda = 0$ associated with the subspace of even functions in $L^2(\mathbb{T})$. Hence, we expand

$$u = \cos(x) + Au_1(x) + A^2u_2(x) + \mathcal{O}(A^3), \quad \lambda = A^2\lambda_2 + \mathcal{O}(A^4),$$

and obtain recursively: $u_1(x) = 2\varphi_2(x)$ and $\lambda_2 = 2\gamma_2$. Hence, $\lambda > 0$ if $\gamma_2 > 0$ and $\lambda < 0$ if $\gamma_2 < 0$. The zero eigenvalue associated with the subspace of odd functions in $L^2(\mathbb{T})$ is preserved at zero for every A due to $\partial_x\psi \in \text{Ker}(\mathcal{L})$. In addition, there exists a negative eigenvalue of $\mathcal{L}$ associated with the constant functions at $A = 0$. Hence, we confirm that $n(\mathcal{L}) = 1$ for $\alpha > \alpha_0$ and $n(\mathcal{L}) = 2$ for $\alpha < \alpha_0$, whereas $z(\mathcal{L}) = 1$ for both $\alpha > \alpha_0$ and $\alpha < \alpha_0$.

In order to deduce the spectral stability conclusion, we use transformation $\psi(x) = a + \phi(x)$, where the zero-mean function ϕ satisfies the boundary-value problem (4.5). Computing the mean value

$$a := \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(x)dx = \frac{1}{2} + \frac{3}{8}\left[1 - \frac{3}{2^\alpha - 1}\right]A^2 + \mathcal{O}(A^4)$$

we obtain

$$\omega := c - 6a^2 = -1 + \frac{3}{2}\left[1 + \frac{3}{2^\alpha - 1}\right]A^2 + \mathcal{O}(A^2)$$

and

$$\beta := ca - 2a^3 = \frac{3}{2}A^2 + \mathcal{O}(A^4).$$

No fold point occurs in the expansion of ω with respect to the Stokes amplitude A, in particular,

$$\frac{d\omega}{dA^2} = \frac{3}{2}\left[1 + \frac{3}{2^\alpha - 1}\right] + \mathcal{O}(A^2) > 0.$$

Since $\|\phi\|_{L^2}^2 = \pi A^2 + \mathcal{O}(A^4)$, we have

$$\frac{d}{d\omega}\|\phi\|_{L^2}^2 = \frac{2\pi}{3}\frac{2^\alpha - 1}{2^\alpha + 2} + \mathcal{O}(A^2) > 0. \tag{5.5}$$

In Appendix A, we show that $\sigma_0 > 0$ for $\alpha > \alpha_0$ and $\sigma_0 < 0$ for $\alpha < \alpha_0$. In addition, we show that

$$\frac{\partial}{\partial\omega}\|\phi\|_{L^2}^2 = \frac{d}{d\omega}\|\phi\|_{L^2}^2 + \mathcal{O}(A^2) > 0.$$

By Theorem 4.8, the periodic waves are spectrally stable for small A both for $\alpha > \alpha_0$ and $\alpha < \alpha_0$. $\hspace{2cm}\square$

Remark 5.2 For $\alpha > \alpha_0$, the small-amplitude periodic wave in Proposition 5.1 have the same properties $n(\mathcal{L}) = 1$, $z(\mathcal{L}) = 1$, and $\sigma_0 > 0$ as the even periodic wave in Theorem 4.1. However, for $\alpha < \alpha_0$, the small-amplitude periodic wave in Proposition 5.1 cannot be a minimizer of the constrained variational problem (4.1) in Theorem 4.1 because it exists for $c < \frac{1}{2}$ and has $n(\mathcal{L}) = 2$ and $\sigma_0 < 0$. Spectral stability of the periodic wave with $n(\mathcal{L}) = 2$, $\sigma_0 < 0$, and the slope condition (4.19) follows from the same computation as in the proof of Theorem 4.8.

5.2 Local Case with $\alpha = 2$

In the case of the modified KdV equation ($\alpha = 2$), Eq. (1.1) with $b = 0$ can be solved in the space of even functions by using the Jacobian dnoidal function [5,17].

Let us recall the normalized second-order equation

$$\psi_0''(z) + (k^2 - 2)\psi_0(z) + 2\psi_0(z)^3 = 0, \tag{5.6}$$

which admits the periodic solution $\psi_0(z) = \mathrm{dn}(z; k)$ with the period $2K(k)$, where $K(k)$ is the complete elliptic integral of the first kind. Adopting an elementary scaling transformation yields the exact solution in the form:

$$\psi(x) = \frac{1}{\pi}K(k)\mathrm{dn}\left[\frac{1}{\pi}K(k)x; k\right] \tag{5.7}$$

with

$$c = \frac{1}{\pi^2}K(k)^2(2 - k^2). \tag{5.8}$$

The following proposition summarizes properties of the even periodic waves for $\alpha = 2$. These properties were studied in [5,17].

Proposition 5.3 *Fix $\alpha = 2$. The even periodic wave (5.7) exists and is spectrally stable for every $c \in \left(\frac{1}{2}, \infty\right)$. Moreover, $n(\mathcal{L}) = 1$, $z(\mathcal{L}) = 1$, and $\sigma_0 > 0$ for every $c \in \left(\frac{1}{2}, \infty\right)$.*

Proof The mapping $(0, 1) \ni k \mapsto c(k) \in \left(\frac{1}{2}, \infty\right)$ is one-to-one and onto. This follows from

$$\frac{\pi^2}{2} \frac{dc}{dk} = \frac{K(k)}{k(1 - k^2)} \left[(2 - k^2)E(k) - 2(1 - k^2)K(k)\right] > 0, \tag{5.9}$$

where the latter inequality was proved in [5] (see also [17]). Indeed, if

$$f(k) := (2 - k^2)E(k) - 2(1 - k^2)K(k),$$

then $f(0) = 0$, whereas $f'(k) = 3k[K(k) - E(k)] > 0$ so that $f(k) > 0$ for $k \in (0, 1)$.

The mean value of the periodic wave in (5.7) is computed explicitly by

$$a := \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(x)dx = \frac{1}{\pi} \int_0^{K(k)} \mathrm{dn}(z; k)dz = \frac{1}{2}.$$

Hence, the zero-mean function $\phi(x) := \psi(x) - a$ is a solution to the boundary-value problem (4.5) with

$$\omega = c - \frac{3}{2}, \quad \beta = \frac{1}{2}\left(c - \frac{1}{2}\right).$$

This gives the straight line dependence $\beta = \frac{1}{2}(\omega + 1)$ for the periodic waves with the single-lobe profile. Furthermore, we can compute

$$\|\phi\|_{L^2}^2 = \frac{2}{\pi}K(k)E(k) - \frac{\pi}{2},$$

from which we verify that

$$\langle \phi, \partial_k \phi \rangle = -\frac{1}{\pi k(1 - k^2)} \left[K(k)^2(1 - k^2) - E(k)^2\right] > 0.$$

The latter inequality is also proven directly by setting

$$f(k) := K(k)^2(1 - k^2) - E(k)^2$$

such that $f(0) = 0$ and $f'(k) = -2k^{-1}[K(k) - E(k)]^2 < 0$ so that $f(k) < 0$ for $k \in (0, 1)$. By Theorem 4.8, the even periodic wave (5.7) with the speed (5.8) satisfying (5.9) is spectrally stable for $c \in \left(\frac{1}{2}, \infty\right)$.

Other properties such as $1 \in \mathrm{Range}(\mathcal{L})$, $\sigma_0 > 0$, and $n(\mathcal{L}) = 1$ for every $k \in (0, 1)$ can be confirmed by explicit computations. The normalized linearized operator is given by

$$\mathcal{L}_0 = -\partial_z^2 - 4 + 5k^2 - 6k^2\mathrm{cn}(z; k)^2. \tag{5.10}$$

Eigenvalues of $\mathcal{L}_0$ in (5.10) are given by subtracting $3(1 - k^2)$ from eigenvalues of $\mathcal{L}_0$ in (3.8). However, $\mathcal{L}_0$ in (5.10) is considered in space $L^2(-K(k), K(k))$ so that the eigenvalues λ_1 and λ_2 given below (5.10) with the eigenfunctions in $L^2(-2K(k), 2K(k))$ are not relevant. Hence, the first three eigenvalues of $\mathcal{L}_0$ in (5.10) are given by

$$
\begin{aligned}
\lambda_0 &= -2 + k^2 - 2\sqrt{1 - k^2 + k^4}, & \varphi_0(z) &= 1 + k^2 + \sqrt{1 - k^2 + k^4} - 3k^2\mathrm{sn}(z; k)^2, \\
\lambda_1 &= 0, & \varphi_1(z) &= \mathrm{sn}(z; k)\mathrm{cn}(z; k), \\
\lambda_2 &= -2 + k^2 + 2\sqrt{1 - k^2 + k^4}, & \varphi_2(z) &= 1 + k^2 - \sqrt{1 - k^2 + k^4} - 3k^2\mathrm{sn}(z; k)^2.
\end{aligned}
$$

Eigenvalues and eigenvectors of the linearized operator $\mathcal{L}$ are obtained after the same scaling transformation as in (5.7). In agreement with Lemma 4.3, we have $n(\mathcal{L}) = 1$ and $z(\mathcal{L}) = 1$. The property $1 \in \text{Range}(\mathcal{L})$ follows from the representation

$$\frac{1}{2\sqrt{1-k^2+k^4}}\mathcal{L}_0\left[\frac{\lambda_2\varphi_0 - \lambda_0\varphi_2}{\lambda_0\lambda_2}\right] = 1 \quad \text{and} \quad \frac{1}{2\sqrt{1-k^2+k^4}}\left[\varphi_0 - \varphi_2\right] = 1.$$

Direct computations yield

$$\langle \mathcal{L}_0^{-1}1, 1\rangle = \frac{\lambda_2\langle\varphi_0, 1\rangle - \lambda_0\langle\varphi_2, 1\rangle}{2\sqrt{1-k^2+k^4}\lambda_0\lambda_2} = \frac{2}{k^4}\left[(2-k^2)K(k) - 2E(k)\right] > 0,$$

where the latter inequality is justified by assigning

$$f(k) := (2-k^2)K(k) - 2E(k)$$

with $f(0) = 0$ and $f'(k) = k(1-k^2)^{-1}[E(k) - (1-k^2)K(k)] > 0$ so that $f(k) > 0$ for $k \in (0, 1)$. □

Remark 5.4 Explicit computations in the proof of Proposition 5.3 repeat computations in [17], however, the expression for $\langle \mathcal{L}_0^{-1}1, 1\rangle$ was typed incorrectly in [17].

Remark 5.5 The dnoidal wave of Proposition 5.3 coincides with the even periodic wave obtained in Theorem 4.1 for $\alpha = 2$. Similarly to Remark 3.4, this follows from uniqueness of smooth, positive, and 2π-periodic solutions of Eq. (3.9) with $c \in (\frac{1}{2}, \infty)$, where Eq. (3.9) is the Euler–Lagrange equation for the variational problem in Theorem 4.1. Compared to Proposition 3.5, it is well-known (see, e.g., [39]) that the mapping $\mathcal{I} \mapsto L$ is monotonically increasing for the L-periodic, positive solutions of Eq. (3.9) with the first-order invariant (3.10) for $c > 0$ such that $L \in (2\pi(2c)^{-1/2}, \infty)$. Therefore, there exists exactly one even, positive, 2π-periodic solution for every $c \in (\frac{1}{2}, \infty)$ and by uniqueness of solutions to differential equations, this unique solution is given by the dnoidal wave (5.7) and (5.8).

5.3 Numerical Approximations

Here we numerically compute solutions of Eq. (1.1) with $b = 0$ using Newton's method in the Fourier space. The starting iteration is generated from the Stokes expansion (5.3) and this solution is uniquely continued in c for all $c \in (\frac{1}{2}, \infty)$ if $\alpha > \alpha_0$.

Figure 4 presents the periodic wave solutions for $\alpha = 2$. The top panel shows the profiles of ψ for three different values of c. The bottom panels show the dependence of $F(\psi)$ versus c (left) and the dependence of $F(\phi)$ versus ω (right), where ϕ and ω was computed from the transformation $\phi(x) = \psi(x) - a$ and $\omega = c - 6a^2$ with $a := \frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(x)dx$. The even periodic wave with the single-lobe profile ψ (red line) bifurcates at $c = \frac{1}{2}$ from the constant wave (grey line) shown on the bottom left. Since $F(\phi)$ is increasing in ω and $a = \frac{1}{2}$ is independent of ω, the even periodic wave is stable by Theorem 4.8.

Figure 5 presents similar results but for $\alpha = 1$. The periodic wave (red line on the top right panel) still bifurcates from the constant wave (grey line on the top right panel) to the right of the bifurcation point at $c = \frac{1}{2}$. However, a depends on ω for the even periodic wave, hence

$$\frac{d}{d\omega}F(\phi) = \frac{\partial}{\partial\omega}F(\phi) + \frac{da}{d\omega}\frac{\partial}{\partial a}F(\phi)$$
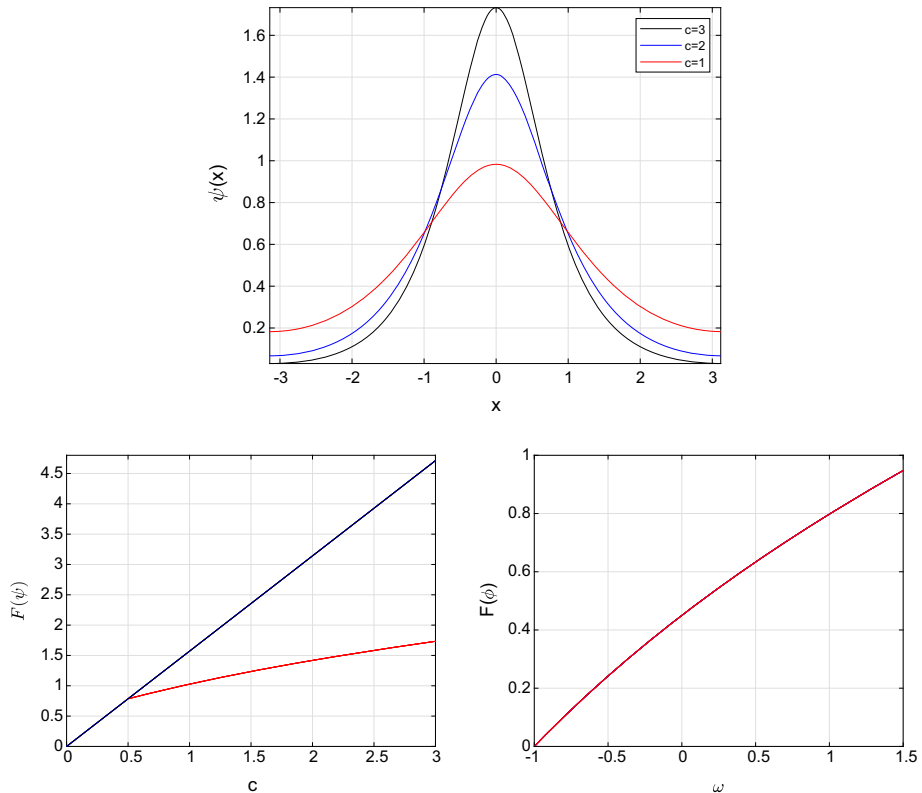
Fig. 4 Even periodic waves for $\alpha = 2$. Top: Profiles of ψ for three different values of c. Bottom: Dependence of the momentum $F(\psi)$ versus c (left) and $F(\phi)$ versus ω (right)

by the chain rule. In the Stokes limit, we have shown in Appendix A that $\frac{\partial}{\partial a} F(\phi) = \mathcal{O}(A^2)$ for small A so that

$$\frac{\partial}{\partial \omega} F(\phi) = \frac{d}{d\omega} F(\phi) + \mathcal{O}(A^2) > 0.$$

However, a discrepancy between partial and ordinary derivatives of $F(\phi)$ in ω exists away from the Stokes limit. The additional bottom right panel on Fig. 5 (compared to Fig. 4) shows the partial and ordinary derivatives on the same graph by the solid and dashed lines respectively. Since $\frac{\partial}{\partial \omega} F(\phi)$ remains positive, the even periodic wave is stable by Theorem 4.8. Since $F(\psi)$ for the even periodic wave is decreasing in c towards the horizontal asymptote as $c \to \infty$, it is clear that the stability conclusion does not follow from the dependence of the momentum $F(\psi)$ versus the wave speed c (see Remark 4.9).

Figure 6 presents similar results but for $\alpha = 0.6 < \alpha_0$. The periodic wave with the single-lobe profile ψ bifurcates to the left of the bifurcation point at $c = \frac{1}{2}$. There exists a fold point $c = c_0 \approx 0.4722$, where the branch turns and extends to all values of $c > c_0$. The upper branch (shown in red line on the top right panel) in $c \in (c_0, \frac{1}{2})$ has $n(\mathcal{L}) = 2$, whereas the lower branch (shown in blue line on the top right panel) has $n(\mathcal{L}) = 1$. The two branches were found iteratively from different initial approximations: the Stokes expansion was used for the upper branch and the periodic wave with larger $c > \frac{1}{2}$ was used for the lower branch,
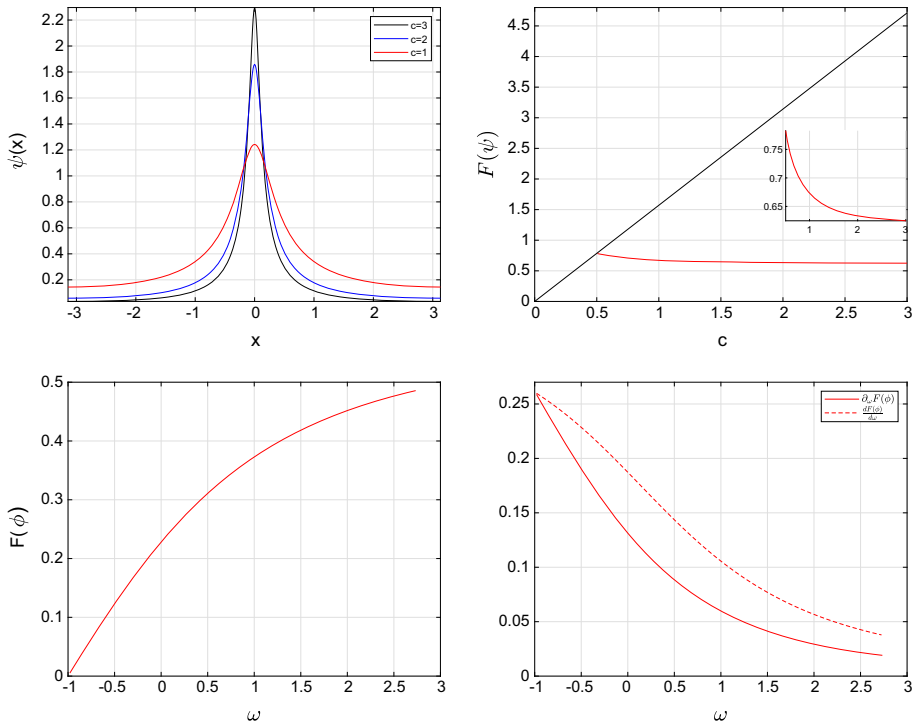
Fig. 5 Even periodic waves for $\alpha = 1$. Top: Profiles of ψ for three different values of c (left). Dependence of the momentum $F(\psi)$ versus c (right). Bottom: Dependence of $F(\phi)$ versus ω (left). and derivatives of $F(\phi)$ in ω (right). The solid (dashed) line shows the partial (ordinary) derivative in ω

then the two branches were continued in either direction. The grey line on the top right panel shows the momentum $F(\psi)$ of the constant solution.

It follows from the graph of $F(\phi)$ versus ω and its derivatives (bottom panels) that the periodic wave is stable near the bifurcation point before and after the fold point but there exists $c_* \approx 0.4774$ such that the even periodic wave is stable for $c < c_*$ and unstable for $c > c_*$. By comparing the partial and ordinary derivatives of $F(\phi)$ with respect to ω (solid and dashed lines, respectively), we can see that the partial derivative becomes zero for a smaller value of ω, which gives the correct transition from stability to instability at $c = c_*$ by Theorem 4.8.

6 Discussion

It remains an open problem to characterize the most general solution of Eq. (1.1) with arbitrary b. Here we generalize the two alternative parametrizations used in this work for the odd and even periodic waves.

Let $\psi \in H^\alpha_{\mathrm{per,even}}$ be a periodic wave solution to Eq. (1.1) with parameters (c, b) defined in an open region $\mathcal{I} \subset \mathbb{R}^2$. As in Sect. 5, we can define $\psi(x) = a + \phi(x)$, where $a := \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(x)dx$ so that $\langle 1, \phi \rangle = 0$. Then $\phi \in H^\alpha_{\mathrm{per,even}} \cap X_0$ is a solution of Eq. (4.3), where $\omega := c - 6a^2$ and $\beta := b + ca - 2a^3$. Parameters (ω, a) are defined in an open
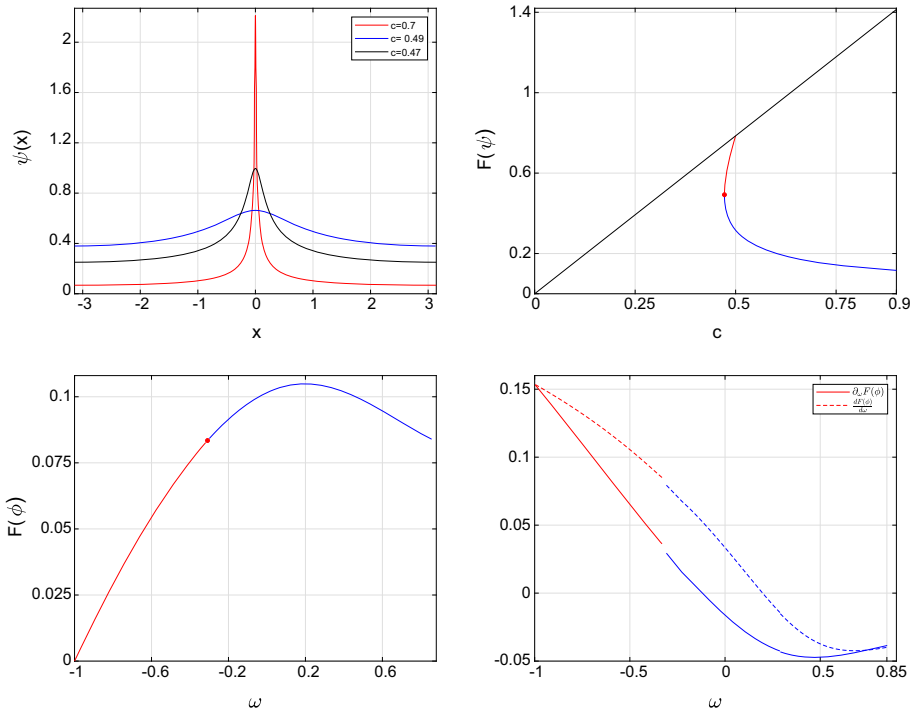
Fig. 6 The same as Fig. 5 but for $\alpha = 0.6$

region $\mathcal{O} \subset \mathbb{R}^2$, which is the image of the transformation $\omega = \omega(c, b)$ and $a = a(c, b)$, whereas $\beta = \beta(c, b)$ is uniquely determined by (4.4). Smoothness and invertibility of this transformation are described as follows.

Proposition 6.1 *Assume that $z(\mathcal{L}) = 1$ for a periodic wave with the single-lobe profile $\psi \in H^\alpha_{\mathrm{per,even}}$. Then, the mapping $\mathcal{I} \ni (c, b) \mapsto (\omega, a) \in \mathcal{O}$ is C^1. The transformation is invertible if and only if*

$$\frac{\partial a}{\partial b} \neq 0. \tag{6.1}$$

Proof In the case $z(\mathcal{L}) = 1$, the mapping $\mathcal{I} \ni (c, b) \mapsto \psi \in H^\alpha_{\mathrm{per,even}}$ is C^1 by the implicit function theorem (applied similarly to the proof of Lemma 2.8). Let $\omega := c - 6a^2$ and $a := \frac{1}{2\pi} \int_\mathbb{T} \psi(x) dx$. The mapping $\mathcal{I} \ni (c, b) \mapsto (\omega, a) \in \mathcal{O}$ is C^1 and the Jacobian of the transformation is given by

$$\begin{vmatrix} 1 - 12a\frac{\partial a}{\partial c} & -12a\frac{\partial a}{\partial b} \\ \frac{\partial a}{\partial c} & \frac{\partial a}{\partial b} \end{vmatrix} = \frac{\partial a}{\partial b},$$

so that the transformation is invertible if and only if the condition (6.1) is satisfied. □

Remark 6.2 Since the mapping $\mathcal{I} \ni (c, b) \mapsto \psi \in H^\alpha_{\mathrm{per,even}}$ is C^1 in Proposition 6.1, we have

$$\mathcal{L} \frac{\partial \psi}{\partial b} = -1 \quad \Rightarrow \quad \sigma_0 = \langle \mathcal{L}^{-1} 1, 1 \rangle = -\left\langle \frac{\partial \psi}{\partial b}, 1 \right\rangle = -2\pi \frac{\partial a}{\partial b}.$$

It follows that the criterion (6.1) is equivalent to the criterion $\sigma_0 \neq 0$ for $z(\mathcal{L}|_{X_0}) = 1$ (see Remark 4.5). If $\frac{\partial a}{\partial b} = 0$, then $z(\mathcal{L}|_{X_0}) = 2$ and $\frac{\partial \psi}{\partial b} \in \mathrm{Ker}(\mathcal{L}|_{X_0})$. In the latter case, we have bifurcation studied in Proposition 3.3 for $\alpha = 2$.

Similarly, but in the opposite direction, let $\phi \in H^\alpha_{\mathrm{per,even}} \cap X_0$ be a periodic wave solution to Eq. (4.3) with parameters (ω, a) defined in an open region $\mathcal{O} \subset \mathbb{R}^2$ and parameter $\beta = \beta(\omega, a)$ being uniquely defined by (4.4). Then, $\psi(x) = a + \phi(x) \in H^\alpha_{\mathrm{per,even}}$ is a solution of Eq. (1.1), where $c := \omega + 6a^2$ and $b := \beta(\omega, a) - \omega a - 4a^3$. Parameters (c, b) are defined in an open region $\mathcal{I} \subset \mathbb{R}^2$, which is the image of the transformation $c = c(\omega, a)$ and $b = b(\omega, a)$. Smoothness and invertibility of this transformation are described as follows.

Proposition 6.3 *Assume that $z(\mathcal{L}|_{X_0}) = 1$ for a zero-mean periodic wave with the single-lobe profile $\phi \in H^\alpha_{\mathrm{per,even}} \cap X_0$. Then, the mapping $\mathcal{O} \ni (\omega, a) \mapsto (c, b) \in \mathcal{I}$ is C^1. The transformation is invertible if and only if*

$$\omega - \frac{\partial \beta}{\partial a} + 12a \frac{\partial \beta}{\partial \omega} \neq 0. \tag{6.2}$$

Proof In the case $z(\mathcal{L}|_{X_0}) = 1$, the mapping $\mathcal{O} \ni (\omega, a) \mapsto \phi \in H^\alpha_{\mathrm{per,even}} \cap X_0$ is C^1 by the implicit function theorem (applied similarly to the proof of Lemma 4.6). Let $c := \omega + 6a^2$ and $b := \beta(\omega, a) - \omega a - 4a^3$. The mapping $\mathcal{O} \ni (\omega, a) \mapsto (c, b) \in \mathcal{I}$ is C^1 and the Jacobian of the transformation is given by

$$\begin{vmatrix} 1 & 12a \\ \frac{\partial \beta}{\partial \omega} - a & \frac{\partial \beta}{\partial a} - \omega - 12a^2 \end{vmatrix} = \frac{\partial \beta}{\partial a} - 12a \frac{\partial \beta}{\partial \omega} - \omega,$$

so that the transformation is invertible if and only if the condition (6.2) is satisfied. $\qquad \square$

Remark 6.4 The criterion (6.2) is equivalent to the criterion for $z(\mathcal{L}) = 1$ as in Lemma 4.7. If

$$\omega - \frac{\partial \beta}{\partial a} + 12a \frac{\partial \beta}{\partial \omega} = 0,$$

then $z(\mathcal{L}) = 2$ as follows from equality (4.18). In the latter case, we have bifurcation studied in Proposition 5.1 for small-amplitude periodic waves.

Both parameters c and b arise as Lagrange multipliers of the following variational problem with two constraints

$$r_{c,m} := \inf_{u \in H^{\frac{\alpha}{2}}_{\mathrm{per}}} \left\{ \mathcal{B}_c(u) : \int_{-\pi}^{\pi} u^4 dx = 1, \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} u \, dx = m \right\}. \tag{6.3}$$

When $m = 0$, the variational problem (6.3) reduces to the form (2.15) which was used in the context of the odd periodic waves. Without loss of generality, it suffices to consider (6.3) for $m \geq 0$ as solutions for $m \leq 0$ are mapped to solutions for $m \geq 0$ by the transformation $u \mapsto -u$. As is shown in Appendix B, the variational problem (6.3) defines constant solutions if $m = m_0 := (2\pi)^{-\frac{1}{4}}$ and periodic waves with the single-lobe profile if $m \in (-m_0, m_0)$.

Further studies are needed to investigate how the variational problem (6.3) with two constraints recovers the most general periodic solution to Eq. (1.1) with two parameters (c, b).

Appendix A. Stokes Expansion of General Small-Amplitude Waves

Here we generalize the Stokes expansion of Sect. 5.1 in order to prove that $\sigma_0 > 0$ for $\alpha > \alpha_0$ and $\sigma_0 < 0$ for $\alpha < \alpha_0$, where $\sigma_0 = \langle \mathcal{L}^{-1} 1, 1 \rangle$ is computed on the small-amplitude wave (5.1) and (5.3) in terms of the small amplitude A.

Let ψ satisfy Eq. (1.1) for (c, b) defined in an open neighborhood $\mathcal{I} \subset \mathbb{R}^2$ of the point $\left(\frac{1}{2}, 0 \right)$. We generalize the decomposition (5.1) by setting

$$\psi(x) = \psi_0 + \varphi(x), \tag{6.4}$$

where $\psi_0 = \psi(c, b)$ is a root of the cubic equation $b + c\psi_0 = 2\psi_0^3$ and φ is not required to satisfy the zero-mean property. Since three roots exist for ψ_0 at $b = 0$, we are picking uniquely the positive root by using the expansion

$$\psi_0(c, b) = \frac{1}{2} \sqrt{2c} + \frac{b}{2c} + \mathcal{O}(b^2). \tag{6.5}$$

Eq. (1.1) is written in the equivalent form:

$$D^\alpha \varphi + (c - 6\psi_0^2)\varphi = 2\varphi^3 + 6\psi_0\varphi^2, \tag{6.6}$$

which generalizes (5.2). By using the Stokes expansion in terms of small amplitude A:

$$\begin{cases} \varphi(x) = A\varphi_1(x) + A^2\varphi_2(x) + A^3\varphi_3(x) + \mathcal{O}(A^4), \\ c - 6\psi_0^2 = -1 + A^2\omega_2 + \mathcal{O}(A^4), \end{cases} \tag{6.7}$$

we obtain recursively: $\varphi_1(x) = \cos(x)$,

$$\varphi_2(x) = -3\psi_0 + \frac{3\psi_0}{2^\alpha - 1} \cos(2x),$$

$$\varphi_3(x) = \frac{1}{3^\alpha - 1} \left[\frac{1}{2} + \frac{18\psi_0^2}{2^\alpha - 1} \right] \cos(3x),$$

and

$$\omega_2 = \frac{3}{2} - 36\psi_0^2 + \frac{18\psi_0^2}{2^\alpha - 1}.$$

By substituting ω_2 to the expansion for c in (6.7) and using expansion (6.5), we obtain

$$\gamma_2 A^2 = 2c - 1 + 6b + \mathcal{O}((2c - 1)^2 + b^2), \tag{6.8}$$

where $\gamma_2 = -\omega_2|_{\psi_0 = \frac{1}{2}}$ is the same as in (5.3). By using (6.8), we obtain perturbatively:

$$a = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(x) dx = \psi_0 \left[1 - 3A^2 + \mathcal{O}(A^4) \right]$$

$$= \frac{1}{2} - a_1(2c - 1) - a_2 b + \mathcal{O}((2c - 1)^2 + b^2),$$

$$\omega = c - 6a^2$$
$$= -1 + \frac{1}{2}(1 + 12a_1)(2c - 1) + 6a_2 b + \mathcal{O}((2c - 1)^2 + b^2),$$

and

$$\beta = b + ca - 2a^3$$
$$= \frac{1}{4}(1 + 4a_1)(2c - 1) + (1 + a_2)b + \mathcal{O}((2c - 1)^2 + b^2),$$

where

$$a_1 := \frac{3}{8\gamma_2} \frac{4 - 2^\alpha}{2^\alpha - 1}, \quad a_2 := \frac{3}{2\gamma_2} \frac{2 + 2^\alpha}{2^\alpha - 1}.$$

If $\gamma_2 \neq 0$ for $\alpha \neq \alpha_0$ given by (5.4), the transformation $\mathcal{I} \ni (c, b) \mapsto (\omega, a) \in \mathcal{O}$ is C^1 and invertible with the inverse transformation

$$c = \frac{1}{2} + (\omega + 1) + 6(a - \frac{1}{2}) + \mathcal{O}((\omega + 1)^2 + (2a - 1)^2),$$
$$b = -\frac{1}{a_2} \left[2a_1(\omega + 1) + \frac{1}{2}(1 + 12a_1)(2a - 1) + \mathcal{O}((\omega + 1)^2 + (2a - 1)^2) \right],$$

from which we obtain $\beta = \beta(\omega, a)$:

$$\beta = \frac{a_2 - 4a_1}{2a_2}(\omega + 1) + \frac{2a_2 - 1 - 12a_1}{2a_2}(2a - 1) + \mathcal{O}((\omega + 1)^2 + (2a - 1)^2)$$

and

$$s_0 = \omega - \partial_a \beta + 12a \partial_\omega \beta = \frac{1}{a_2} + \mathcal{O}((\omega + 1)^2 + (2a - 1)^2).$$

Since $\sigma_0 = \frac{2\pi}{s_0}$, we have $\text{sign}(\sigma_0) = \text{sign}(a_2) = \text{sign}(\gamma_2)$, from which it follows that $\sigma_0 > 0$ for $\alpha > \alpha_0$ and $\sigma_0 < 0$ for $\alpha < \alpha_0$.

Furthermore, it follows from (6.8) that

$$\gamma_2 A^2 = \frac{2(a_2 - 6a_1)}{a_2}(\omega + 1) + \frac{3(2a_2 - 1 - 12a_1)}{a_2}(2a - 1) + \mathcal{O}((\omega + 1)^2 + (2a - 1)^2).$$

Explicit computation shows that $2a_2 - 1 - 12a_1 = 0$, hence $\|\phi\|_{L^2}^2 = \pi A^2 + \mathcal{O}(A^4)$ as a function of (ω, a) satisfies

$$\frac{\partial}{\partial \omega} \|\phi\|_{L^2}^2 = \frac{2\pi(a_2 - 6a_1)}{\gamma_2 a_2} + \mathcal{O}(A^2) = \frac{2\pi}{3} \frac{2^\alpha - 1}{2^\alpha + 2} + \mathcal{O}(A^2) = \frac{d}{d\omega} \|\phi\|_{L^2}^2 + \mathcal{O}(A^2),$$

in agreement with (5.5). In other words, although a is defined by ω at the periodic waves satisfying $b = 0$ by

$$2a - 1 = -\frac{4a_1}{1 + 12a_1}(\omega + 1) + \mathcal{O}((\omega + 1)^2),$$

this dependence does not result in the discrepancy between partial and ordinary derivatives of $\|\phi\|_{L^2}^2$ in ω along the family of even periodic waves in the limit $A \to 0$.

Appendix B. On the Variational Problem (6.3) with Two Constraints

We show that the periodic solutions to Eq. (1.1) with two parameters (c, b) can be recovered from the ground state of the variational problem (6.3).

Proposition 6.5 *Fix* $\alpha > \frac{1}{2}$ *and* $m_0 := (2\pi)^{-\frac{1}{4}}$. *For every* $m \in [-m_0, m_0]$ *and every* $c \in (-1, \infty)$, *there exists the ground state (minimizer)* $\chi \in H_{per}^{\frac{\alpha}{2}}$ *of the variational problem* (6.3). *If* $m \in (-m_0, m_0)$, *the ground state has the single-lobe profile and there exists* $C > 0$ *such that* $\psi(x) = C\chi(x)$ *satisfies Eq.* (1.1) *with some* b.

Proof The bound $|m| \le m_0$ follows by the Hölder's inequality

$$\left| \int_{\mathbb{T}} u \, dx \right| \le \left(\int_{\mathbb{T}} 1^{\frac{4}{3}} dx \right)^{\frac{3}{4}} \left(\int_{\mathbb{T}} u^4 dx \right)^{\frac{1}{4}} = (2\pi)^{\frac{3}{4}}.$$

The quadratic functional $B_c(u)$ in (2.3) is bounded from below by the Poincaré inequality:

$$B_c(u) \ge \frac{1}{2} \|u - m\|_{L^2}^2 + \frac{1}{2} c \|u\|_{L^2}^2 = \frac{1}{2}(1+c)\|u\|_{L^2}^2 - \pi m^2.$$

By the same analysis as in the proof of Theorem 2.1, for every $m \in [-m_0, m_0]$ and every $c \in (-1, \infty)$, there exists the ground state $\chi \in H_{per}^{\frac{\alpha}{2}}$ of the variational problem (6.3). Moreover, by the symmetric rearrangements, the ground state is either constant or has the single-lobe profile. The constant solution corresponds to $|m| = m_0$, hence the ground state has the single-lobe profile if $m \in (-m_0, m_0)$.

With two Lagrange multipliers μ and ν, the ground state $\chi \in H_{per}^{\frac{\alpha}{2}}$ satisfies the stationary equation

$$D^\alpha \chi + c\chi + \nu = \mu \chi^3. \tag{6.9}$$

Lagrange multipliers satisfy two relations due to the constraints in (6.3):

$$\mu = 2B_c(\chi) + 2\pi m\nu, \quad \mu \int_{\mathbb{T}} \chi^3 dx = 2\pi(cm + \nu). \tag{6.10}$$

Eliminating ν yields

$$\left[1 - m \int_{\mathbb{T}} \chi^3 dx \right] \mu = 2 \left[B_c(\chi) - \pi cm^2 \right]. \tag{6.11}$$

The left-hand side of (6.11) can be rewritten in the equivalent symmetrized form:

$$1 - m \int_{\mathbb{T}} \chi^3 dx = \frac{1}{2\pi} \left[\left(\int_{\mathbb{T}} dx \right) \left(\int_{\mathbb{T}} \chi^4 dx \right) - \left(\int_{\mathbb{T}} \chi dx \right) \left(\int_{\mathbb{T}} \chi^3 dx \right) \right]$$

$$= \frac{1}{16\pi} \int_{\mathbb{T}} \int_{\mathbb{T}} \left([\chi(x) - \chi(y)]^4 + 3 \left[\chi^2(x) - \chi^2(y) \right]^2 \right) dx dy,$$

from which it follows that it is strictly positive if $\chi(x)$ is not constant. Similarly, the right-hand side of (6.11) is strictly positive if $\chi(x)$ is not constant due to the following inequality

$$2B_c(\chi) - 2\pi cm^2 = \|D^{\frac{\alpha}{2}} \chi\|_{L^2}^2 + c\|u - m\|_{L^2}^2 \ge (1+c)\|u - m\|_{L^2}^2 > 0.$$

Since the ground state is non-constant if $m \in (-m_0, m_0)$, we obtain the unique $\mu > 0$ from (6.11) such that the transformation $\psi = C\chi$ with $C := \sqrt{\mu}/\sqrt{2}$ reduces (6.9) to Eq. (1.1) with $b = \nu \sqrt{\mu}/\sqrt{2}$, where ν is uniquely found from (6.10). $\square$

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