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Asymptotic stability of viscous shocks in the modular Burgers equation

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Abstract

Dynamics of viscous shocks is considered in the modular Burgers equation, where the time evolution becomes complicated due to singularities produced by the modular nonlinearity. We prove that the viscous shocks are asymptotically stable under odd and general perturbations. For the odd perturbations, the proof relies on the reduction of the modular Burgers equation to a linear diffusion equation on a half-line. For the general perturbations, the proof is developed by converting the time-evolution problem to a system of linear equations coupled with a nonlinear equation for the interface position. Exponential weights in space are imposed on the initial data of general perturbations in order to gain the asymptotic decay of perturbations in time. We give numerical illustrations of asymptotic stability of the viscous shocks under general perturbations.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Modular nonlinearity is commonly used for approximations of nonlinear interactions between particles by piecewise linear functions [14, 36]. Unidirectional propagation of waves in chains

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of particles is described by simplified nonlinear evolution equations with modular nonlinearity such as the modular Burgers [25, 26, 28, 30] and modular Korteweg–de Vries [21, 27, 29] equations.

Traveling solutions of modular evolution equations such as viscous shocks and solitary waves are found from differential equations by matching solutions of linear equations with suitable condition at the interface where the modular nonlinearity jumps. On the other hand, the time evolution of the modular equations is a more complicated problem because the transport term tends to break the solution along the characteristic lines whereas the diffusion or dispersion terms smoothen out the solution and affect propagation of waves near the interface. It is unclear without detailed analysis if the initial-value problem can be solved in a suitable function space due to singularities arising from the modular nonlinearity. Because of these reasons, stability of traveling waves remains open in the modular equations.

Similar questions arise in the context of granular chains and involve the logarithmic versions of the Burgers and Korteweg–de Vries equations [10, 11]. The logarithmic nonlinearity is more singular than the modular nonlinearity, hence questions of well-posedness and stability of nonlinear waves remain open for some time [4, 20, 24].

Stable viscous shocks are important for understanding nonlinear dynamics of the mathematical models and for matching with experimental data of real-world applications. *The purpose of this work is to clarify stability of viscous shocks in the modular Burgers equation.* We take the modular Burgers equation in the following normalized form:

$$\frac{\partial w}{\partial t} = \frac{\partial |w|}{\partial x} + \frac{\partial^2 w}{\partial x^2}, \tag{1.1}$$

where $w(t,x) : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$. Preliminary numerical approximations of time-dependent solutions to the modular Burgers equation (1.1) were constructed with the Fourier sine series in [25]. Traveling wave solutions were constructed analytically in [26, 28]. Collisions of compactly supported pulses were considered in [14] by using heuristic approximation methods. However, no rigorous analysis of well-posedness, stability of viscous shocks, or numerical approximations with the control of error terms has been developed so far.

In a similar context of the diffusion equation with the piecewisely defined nonlinearity, we mention the Kolmogorov–Petrovskii–Piskunov (KPP) model with the cutoff reaction rate proposed in [3]. Existence and asymptotic stability of traveling viscous shocks were analyzed in [6] and more recently in [34, 35], where the method of matched asymptotic expansions in the dynamically moving coordinate frame has been used.

Viscous shocks and metastable N-waves of the classical Burgers equation were studied in [17]. Stability arguments for viscous shocks and metastable N-waves have been developed in [2, 19] by using the linearization analysis and dynamical system methods. In particular, algebraic weights were used to study the spectrum of linearized operators at the metastable N-waves. Viscous shocks of the classical Burgers equation were also analyzed in [22, 23] in the context of the enstrophy growth in the limit of small dissipation.

General stability results of the shock waves in the scalar conservation laws were considered in [16]. Recent work in [7] deals with the asymptotic stability of the shocks under piecewise regular perturbations by using estimates from the linearized equations of motion. Complete classification of traveling waves of scalar conservation laws from the point of view of their spectral and nonlinear stability under (piecewise) smooth perturbations is given in [8].

Non-smoothness of the nonlinear term in the modular Burgers equation (1.1) restricts us from using the dynamical system methods in the analysis of asymptotic stability of viscous shocks. Nevertheless, we are able to use the linearized estimates due to the piecewise definition of the nonlinear term in this model.

The main novelty of this paper is the rigorous analysis of the modular nonlinearity. We keep the functional-analytic framework as simple as possible. If the perturbation has the odd spatial symmetry, the asymptotic stability result follows from analysis of the linear diffusion equation. For general perturbations, we impose the spatial exponential decay on the initial data in order to gain the asymptotic decay of perturbations in time. This technique is definitely not novel, see [9, 15, 31] for earlier studies in similar contexts. Further improvements of the asymptotic stability results in less restrictive function spaces are left for future work.

The paper is organized as follows. Main results are described in section 2. Properties of the heat kernel, convolution estimates, solutions to the linear advection–diffusion equations, and solutions to the Abel integral equations are reviewed in section 3. Asymptotic stability of viscous shocks in the space of odd and general functions is proven in sections 4 and 5 respectively. Numerical illustrations are given in section 6. The summary and open directions are described in section 7. Appendix describes the central-difference Crank–Nicholson method used for numerical simulations.

2. Main results

In what follows, we use the classical notations $H^k(\mathbb{R})$ for the Sobolev space of squared integrable distributions on $\mathbb{R}$ with squared integrable derivatives up to the integer order $k \in \mathbb{N}$. In particular, the norms in H^1 and H^2 are defined by

$$\|f\|_{H^1} := \left(\|f\|_{L^2}^2 + \|f'\|_{L^2}^2\right)^{1/2},$$
$$\|f\|_{H^2} := \left(\|f\|_{L^2}^2 + \|f'\|_{L^2}^2 + \|f''\|_{L^2}^2\right)^{1/2}.$$

Similarly, we consider $W^{1,\infty}$ and $W^{2,\infty}$ for bounded functions with bounded derivatives up the first and second order respectively. To simplify the notations, we use

$$\|f\|_{H^k\cap W^{k,\infty}} := \max\{\|f\|_{H^k}, \|f\|_{W^{k,\infty}}\}.$$

We start with the existence of *the traveling viscous shock* of the modular Burgers equation (1.1). Substituting $w(t,x) = W_c(x - ct)$ in (1.1) yields the differential equation

$$W_c''(x) + \mathrm{sign}(W_c)W_c'(x) + cW_c'(x) = 0. \tag{2.1}$$

Solutions of (2.1) are piecewise C^2 functions satisfying the interface condition

$$[W_c'']_-^+(x_0) = -2W_c'(x_0) \tag{2.2}$$

at each interface located at x_0, where $[f]_-^+(x_0) = f(x_0^+) - f(x_0^-)$ is the jump of a piecewise continuous function f across x_0. The following theorem gives the exact solution for the traveling viscous shock.

Theorem 2.1. *The only piecewise C^2 solution of the differential equation (2.1) satisfying the boundary conditions $W_c(x) \to W_\pm$ as $x \to \pm\infty$ with $W_- < 0 < W_+$ is given by*

$$W_c(x) = \begin{cases} W_+(1 - \mathrm{e}^{-(1+c)x}), & x > 0, \\ W_-(1 - \mathrm{e}^{(1-c)x}), & x < 0, \end{cases} \tag{2.3}$$

where the speed c is uniquely defined by

$$c = \frac{W_+ + W_-}{W_- - W_+}. \tag{2.4}$$

The solution W_c satisfies the jump condition (2.2) with $x_0 = 0$ and can be translated in x.

Proof. The second-order differential equation (2.1) is integrable with the first-order invariant

$$W_c'(x) + |W_c(x)| + cW_c(x) = d, \tag{2.5}$$

where d is constant. Since W_c is piecewise C^2, it follows that $W_c \in C^1(\mathbb{R})$ and the value of d is the same independently of the sign of $W_c'(x)$. From the boundary conditions $W_c(x) \to W_\pm$ as $x \to \pm\infty$, we find $d = (c+1)W_+ = (c-1)W_-$, which yields (2.4). Since $W_- < 0 < W_+$ and the equilibrium points W_- and W_+ are reached in infinite 'time' x, the value of $W_c'(x)$ is sign-definite (positive) both for $W_c(x) < 0$ and $W_c(x) > 0$. Therefore, there exists only one interface $x_0 \in \mathbb{R}$ where $W_c(x_0) = 0$ and $W_c'(x_0) = d$. Up to the translational invariance, one can chose $x_0 = 0$, after which the exact solution of the first-order differential equation (2.5) is found in the piecewise form (2.3). □

Remark 2.2. If $W_+ = -W_-$, then $c = 0$ and the viscous shock W_0 is time-independent.

The modular Burgers equation (1.1) on the line $\mathbb{R}$ can be closed on the half-line in the space of odd functions. In this case, the evolution equation with the normalized boundary conditions takes the form:

$$\begin{cases} w_t = w_x + w_{xx}, & x > 0, \\ w(t,0) = 0, & \\ w(t,x) \to 1 & \text{as } x \to +\infty, \end{cases} \tag{2.6}$$

subject to the positivity condition

$$w(t,x) > 0, \quad x > 0. \tag{2.7}$$

The classical solution of the boundary-value problem (2.6) satisfies the constraint

$$w_x(t,0^+) + w_{xx}(t,0^+) = 0. \tag{2.8}$$

If a classical solution $w(t,x) : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}$ to the boundary-value problem (2.6) is extended to the odd function $w_{\text{ext}}(t,x) : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$, then $w_{\text{ext}}(t,\cdot)$ is a piecewise C^2 function satisfying the interface condition

$$[w_{xx}]_-^+(t,0) = -2w_x(t,0), \tag{2.9}$$

where $w \equiv w_{\text{ext}}$ for simplicity of notations.

The following theorem states the asymptotic stability of the viscous shock (2.3) with $c = 0$ under the odd perturbations from the analysis of the boundary-value problem (2.6) subject to the positivity condition (2.7) and the boundary constraint (2.8). The proof of this theorem is presented in section 4.

Theorem 2.3. *For every $\epsilon > 0$ there is $\delta > 0$ such that for every odd w_0 satisfying*

$$\|w_0 - W_0\|_{H^2} < \delta, \tag{2.10}$$

there exists a unique odd solution $w(t,x)$ *to the modular Burgers equation* (1.1) *with* $w(0,x) = w_0(x)$ *satisfying*

$$\|w(t,\cdot) - W_0\|_{H^2} < \epsilon, \quad t > 0 \tag{2.11}$$

and

$$\|w(t,\cdot) - W_0\|_{W^{2,\infty}} \to 0 \quad \text{as } t \to +\infty. \tag{2.12}$$

The solution belongs to the class of functions such that $w - W_0 \in C(\mathbb{R}_+, H^2(\mathbb{R}))$.

Remark 2.4. Since $H^2(\mathbb{R})$ is continuously embedded into $C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ with functions and their first derivatives decaying to zero at infinity, whereas $W_0(0) = 0$, $W_c'(0) > 0$, and $W_0(x) \to 1$ as $x \to \infty$, the only interface of the solution $w(t,\cdot)$ in theorem 2.3 with small $\epsilon > 0$ is located at the origin. The positivity condition (2.7) is satisfied for all $t \in \mathbb{R}_+$.

Remark 2.5. The following transformation

$$w(t,x) = \begin{cases} W_+ v((1+c)^2 t, (1+c)(x-ct)), & x - ct > 0, \\ W_- v((1-c)^2 t, (1-c)(x-ct), & x - ct < 0, \end{cases} \tag{2.13}$$

where c is given by (2.4), relates solutions $w(t,x)$ with $W_+ \neq -W_-$ to solutions $v(t,x)$ with normalized boundary conditions $v(t,x) \to \pm 1$ as $x \to \pm\infty$. If $v(t,x)$ is odd in x, then it satisfies the same boundary-value problem (2.6) subject to the same constraints (2.7) and (2.8). Hence theorem 2.3 can be extended trivially to the traveling viscous shock W_c with $c \neq 0$ under the odd perturbation of $v(t,x)$ in (2.13).

For the general perturbations, we consider the solution $w(t,x)$ to the modular Burgers equation (1.1) with exactly one interface located dynamically at $x = \xi(t)$. Without loss of generality, we assume $\xi(0) = 0$. The evolution equation with the normalized boundary conditions takes the form:

$$\begin{cases} w_t = \pm w_x + w_{xx}, & \pm(x - \xi(t)) > 0, \\ w(t, \xi(t)) = 0, & \\ w(t,x) \to \pm 1 & \text{as } x \to \pm\infty, \end{cases} \tag{2.14}$$

subject to the positivity conditions

$$\pm w(t,x) > 0, \quad \pm(x - \xi(t)) > 0. \tag{2.15}$$

Piecewise C^2 solutions of the boundary-value problem (2.14) satisfy the interface condition

$$[w_{xx}]_-^+(t,\xi(t)) = -2w_x(t,\xi(t)), \tag{2.16}$$

whereas the boundary condition $w(t,\xi(t)) = 0$ implies

$$w_t(t,\xi(t)) + \xi'(t) w_x(t,\xi(t)) = 0, \tag{2.17}$$

for continuous w_t and w_x across the interface at $x = \xi(t)$.

The following theorem states the asymptotic stability of the viscous shock (2.3) with $c = 0$ under general perturbations from the analysis of the boundary-value problem (2.14) subject to the positivity conditions (2.15) and the interface conditions (2.16) and (2.17). The proof of this theorem is presented in section 5.

Theorem 2.6. *Fix* $\alpha \in \left(0, \frac{1}{2}\right)$. *For every* $\epsilon > 0$ *there is* $\delta > 0$ *such that for every* w_0 *satisfying*

$$\|w_0 - W_0\|_{H^2 \cap W^{2,\infty}} + \|e^{\alpha|\cdot|}(w_0 - W_0)\|_{W^{2,\infty}} < \delta \tag{2.18}$$

and $w_0(0) = 0$, *there exists a unique solution* $w(t, x)$ *to the modular Burgers equation* (1.1) *with* $w(0, x) = w_0(x)$ *satisfying*

$$\|w(t, \cdot + \xi(t)) - W_0\|_{H^2 \cap W^{2,\infty}} < \epsilon, \quad t > 0 \tag{2.19}$$

and

$$\|w(t, \cdot + \xi(t)) - W_0\|_{W^{2,\infty}} \to 0 \quad \text{as } t \to +\infty, \tag{2.20}$$

where $\xi(t)$ *is the uniquely determined interface position satisfying* $\xi(0) = 0$ *and* $\xi' \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. *The solution belongs to the class of functions such that*

$$w(t, \cdot + \xi(t)) - W_0 \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})) \tag{2.21}$$

and

$$e^{\alpha|\cdot + \xi(t)|}[w(t, \cdot + \xi(t)) - W_0] \in L^\infty(\mathbb{R}_+, W^{2,\infty}(\mathbb{R})). \tag{2.22}$$

Remark 2.7. The additional requirement $w_0 - W_0 \in H^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ for the initial data w_0 in theorem 2.6 compared to $w_0 - W_0 \in H^2(\mathbb{R})$ in theorem 2.3 is due to the necessity to control $\xi'(t)$ from the interface conditions (2.16) and (2.17). As we will show in lemma 5.1, this is possible if the solution stays in the class of functions satisfying (2.21).

Remark 2.8. We assume in (2.18) that $|w_0(x) - W_0(x)| \to 0$ as $|x| \to \infty$ at least exponentially with the decay rate $\alpha \in (0, \frac{1}{2})$. This gives the asymptotic stability resulting in

$$\xi'(t) \to 0 \quad \text{and} \quad \|w(t, \cdot + \xi(t)) - W_0\|_{W^{2,\infty}} \to 0 \quad \text{as } t \to +\infty.$$

The exponential decay in space is preserved in time as is shown in (2.22). It is opened for further studies to relax the exponential decay requirement on the general initial data w_0.

Remark 2.9. Thanks to the transformation (2.13), theorem 2.6 can be extended trivially to the traveling viscous shock W_c with $c \neq 0$ under a general perturbation of $v(t, x)$.

Remark 2.10. Since $\xi' \in L^1(\mathbb{R}_+)$, there exists $\xi_\infty := \lim_{t \to +\infty} \xi(t)$. The value of ξ_∞ depends on the asymmetry of w_0 and satisfies generally $\xi_\infty \neq 0$ even if $\xi(0) = 0$.

Numerical illustrations of the asymptotic stability of the viscous shock (2.3) with $c = 0$ for two examples of general perturbations are given in section 6, where the boundary-value problem (2.14) with (2.15)–(2.17) is approximated by using the central-difference Crank–Nicholson method. Error of the central-difference numerical approximation is controlled by the standard analysis. The two examples are constructed for perturbations with the Gaussian and exponential decay at infinity. Numerical simulations illustrate the asymptotic stability result of theorem 2.6.

3. Preliminary results

We review here properties of the heat kernel, convolution estimates, solutions to the linear advection–diffusion equations, and solutions to the Abel integral equations.

3.1. Heat kernel and linear diffusion equations

The heat kernel is defined by

$$G(t,x) := \frac{1}{\sqrt{4\pi t}}\, \mathrm{e}^{-\frac{x^2}{4t}}. \tag{3.1}$$

It follows from explicit computations of integrals that the heat kernel satisfies the properties:

$$\|G(t,\cdot)\|_{L^1(\mathbb{R})} = 1, \qquad \|G(t,\cdot)\|_{L^2(\mathbb{R})} = \frac{1}{(8\pi t)^{1/4}}, \qquad \|G(t,\cdot)\|_{L^\infty(\mathbb{R})} = \frac{1}{(4\pi t)^{1/2}}, \tag{3.2}$$

$$\|\partial_x G(t,\cdot)\|_{L^1(\mathbb{R})} = \frac{1}{(\pi t)^{1/2}}, \qquad \|\partial_x G(t,\cdot)\|_{L^2(\mathbb{R})} = \frac{1}{2(8\pi)^{1/4} t^{3/4}}, \tag{3.3}$$

and

$$\|\partial_x G(t,\cdot)\|_{L^\infty(\mathbb{R})} = \frac{1}{2(2\pi e)^{1/2} t}. \tag{3.4}$$

The heat kernel $G(t,x)$ is used to solve the following Dirichlet problem for the linear diffusion equation on the half-line:

$$\begin{cases} v_t = v_{xx}, & x > 0,\ t > 0, \\ v(t,0) = 0, & t > 0, \\ v(0,x) = v_0(x), & x > 0. \end{cases} \tag{3.5}$$

For a rather general class of functions $v_0(x) : \mathbb{R}_+ \mapsto \mathbb{R}$ (not necessarily decaying to zero at infinity), the Dirichlet problem (3.5) can be solved by the method of images:

$$v(t,x) = \int_0^\infty v_0(y)\,[G(t,x-y) - G(t,x+y)]\,\mathrm{d}y. \tag{3.6}$$

Convolution integrals in (3.6) are analyzed with Young's convolution inequality:

$$\|f * g\|_{L^r(\mathbb{R})} \leqslant \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}, \quad p,q,r \geqslant 1, \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \tag{3.7}$$

for every $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, where $(f * g)(x) := \int_{\mathbb{R}} f(y) g(x-y)\mathrm{d}y$ is the convolution integral. When integration is needed to be restricted on $\mathbb{R}_+$ as in (3.6), we can use the characteristic function $\chi_{\mathbb{R}_+}$ defined by $\chi_{\mathbb{R}_+}(x) = 1$ for $x > 0$ and $\chi_{\mathbb{R}_+}(x) = 0$ for $x < 0$.

For the inhomogeneous linear diffusion equation on the half-line:

$$\begin{cases} v_t = v_{xx} + f(t,x), & x > 0,\ t > 0, \\ v(t,0) = 0, & t > 0, \\ v(0,x) = v_0(x), & x > 0. \end{cases} \tag{3.8}$$

with given $v_0(x) : \mathbb{R}_+ \mapsto \mathbb{R}$ and $f(t, x) : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}$, the exact solution is written in the form

$$v(t, x) = \int_0^\infty v_0(y)\,[G(t, x - y) - G(t, x + y)]\,\mathrm{d}y + \int_0^t \int_0^\infty f(\tau, y)\,[G(t - \tau, x - y) - G(t - \tau, x + y)]\,\mathrm{d}y\,\mathrm{d}\tau. \quad (3.9)$$

The exact solutions to the linear diffusion equations (3.5) and (3.8) are used in the proof of lemmas 4.3 and 5.5.

3.2. Convolution estimates

Convolution integrals in time are analyzed with Young's convolution inequality:

$$\|\beta \star \gamma\|_{L^r(\mathbb{R}_+)} \leqslant \|\beta\|_{L^p(\mathbb{R}_+)} \|\gamma\|_{L^q(\mathbb{R}_+)}, \quad p, q, r \geqslant 1, \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad (3.10)$$

for every $\beta \in L^p(\mathbb{R}_+)$ and $\gamma \in L^q(\mathbb{R}_+)$, where $(\beta \star \gamma)(t) := \int_0^t \beta(t - \tau)\gamma(\tau)\mathrm{d}\tau$ is the convolution integral in time. The following two lemmas give bounds used in the proof of lemmas 3.3, 3.6, 5.7, 5.8, and 5.10.

Lemma 3.1. *For every $\gamma \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ and every $s \in [0, 1)$, there exists a positive constant C_s such that*

$$\int_0^t \frac{|\gamma(\tau)|}{(t - \tau)^s}\,\mathrm{d}\tau \leqslant C_s \|\gamma\|_{L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)}, \quad t > 0. \quad (3.11)$$

Proof. For every fixed $T > 0$, it is obvious that

$$\int_0^t \frac{|\gamma(\tau)|}{(t - \tau)^s}\,\mathrm{d}\tau \leqslant \frac{T^{1-s}}{1 - s} \|\gamma\|_{L^\infty(\mathbb{R}_+)}, \quad t \in [0, T].$$

Then, provided $T > 1$, we get the bounds

$$\int_0^t \frac{|\gamma(\tau)|}{(t - \tau)^s}\,\mathrm{d}\tau = \int_0^{t-1} \frac{|\gamma(\tau)|}{(t - \tau)^s}\,\mathrm{d}\tau + \int_{t-1}^t \frac{|\gamma(\tau)|}{(t - \tau)^s}\,\mathrm{d}\tau \leqslant \|\gamma\|_{L^1(\mathbb{R}_+)} + \frac{1}{1 - s} \|\gamma\|_{L^\infty(\mathbb{R}_+)}, \quad t > T,$$

and the bound (3.11) holds. □

Lemma 3.2. *For every $\gamma \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ satisfying $\gamma(t) \to 0$ as $t \to +\infty$ and every $\lambda > 0$ and $s \in [0, 1)$, we have*

$$\lim_{t \to +\infty} \int_0^t \frac{|\gamma(\tau)| \mathrm{e}^{-\lambda(t-\tau)}}{(t - \tau)^s}\,\mathrm{d}\tau = 0. \quad (3.12)$$

Proof. One can write for $t > 0$:

$$\int_0^t \frac{|\gamma(\tau)| \mathrm{e}^{-\lambda(t-\tau)}}{(t - \tau)^s}\,\mathrm{d}\tau = \int_0^{t/2} \frac{|\gamma(\tau)| \mathrm{e}^{-\lambda(t-\tau)}}{(t - \tau)^s}\,\mathrm{d}\tau + \int_{t/2}^t \frac{|\gamma(\tau)| \mathrm{e}^{-\lambda(t-\tau)}}{(t - \tau)^s}\,\mathrm{d}\tau \leqslant \|\gamma\|_{L^1(\mathbb{R}_+)} 2^s t^{-s}\,\mathrm{e}^{-\frac{\lambda t}{2}} + \sup_{\tau \in [t/2, t]} |\gamma(\tau)| \int_0^{t/2} \tau^{-s}\,\mathrm{e}^{-\lambda \tau}\,\mathrm{d}\tau,$$

from which the limit (3.12) follows from the assumptions of the lemma. □

3.3. Linear advection–diffusion equation

For a given function $\gamma \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, we analyze the following initial-value problem for the advection–diffusion equation:

$$\begin{cases} \nu_t = \nu_y + \nu_{yy} + 2\gamma(t)\delta(y), & y \in \mathbb{R},\ t > 0, \\ \nu(0, y) = 0, & y \in \mathbb{R}, \end{cases} \tag{3.13}$$

where δ is the Dirac distribution centered at zero. The advection–diffusion equation is used in the proof of lemmas 5.7 and 5.8.

In order to construct the exact solution to the initial-value problem (3.13), we use the Laplace transform in time t defined by

$$\hat{\gamma}(p) := \mathcal{L}(\gamma)(p) = \int_0^\infty \gamma(t)\mathrm{e}^{-pt}\,\mathrm{d}t, \quad p \geqslant 0. \tag{3.14}$$

We also use the following relations from the table of Laplace transforms for every $y \in \mathbb{R}$:

$$\mathcal{L}\left(\frac{1}{\sqrt{\pi t}}\,\mathrm{e}^{-\frac{y^2}{4t}}\right) = \frac{1}{\sqrt{p}}\,\mathrm{e}^{-\sqrt{p}|y|}, \quad p > 0 \tag{3.15}$$

and

$$\mathcal{L}\left(\frac{1}{\sqrt{\pi t}}\frac{y}{2t}\,\mathrm{e}^{-\frac{y^2}{4t}}\right) = \mathrm{sign}(y)\,\mathrm{e}^{-\sqrt{p}|y|}, \quad p > 0. \tag{3.16}$$

The following lemma gives the exact solution to the initial-value problem (3.13).

Lemma 3.3. *For every $\gamma \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}_+)$, there exists a unique solution to the initial-value problem (3.13) in the exact form:*

$$\nu(t, y) := 2\int_0^t \frac{\gamma(\tau)}{\sqrt{4\pi(t-\tau)}}\,\mathrm{e}^{-\frac{(y+t-\tau)^2}{4(t-\tau)}}\,\mathrm{d}\tau, \quad y \in \mathbb{R},\ t > 0. \tag{3.17}$$

Moreover, ν belongs to the class of functions in $L^\infty(\mathbb{R}_+, H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}_+))$ satisfying

$$\nu_y(t, 0^\pm) + \frac{1}{2}\nu(t, 0) = \mp\gamma(t), \quad t > 0. \tag{3.18}$$

Proof. By using (3.14) and (3.15), we compute from (3.17):

$$\hat{\nu}(p, y) = \mathcal{L}\left(\frac{1}{\sqrt{\pi t}}\,\mathrm{e}^{-\frac{(y+t)^2}{4t}}\right)(p) \times \mathcal{L}(\gamma)(p) = \mathrm{e}^{-\frac{y}{2}}\frac{\mathrm{e}^{-\sqrt{p+\frac{1}{4}}|y|}}{\sqrt{p+\frac{1}{4}}}\hat{\gamma}(p),$$

where we have used the following properties of the Laplace transform:

$$\mathcal{L}(f(t)\mathrm{e}^{-\frac{t}{4}})(p) = \hat{f}\left(p + \frac{1}{4}\right) \quad \text{and} \quad \mathcal{L}\left(\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau\right)(p) = \hat{f}(p)\hat{g}(p).$$

Differentiations of $\hat{\nu}(p, y)$ in y yield

$$\hat{\nu}_y = -\frac{1}{2}\hat{\nu} - \mathrm{sign}(y)\mathrm{e}^{-\frac{y}{2}}\,\mathrm{e}^{-\sqrt{p+\frac{1}{4}}|y|}\hat{\gamma}(p), \tag{3.19}$$

$$\hat{\nu}_{yy} = -\frac{1}{2}\hat{\nu}_y - 2\delta(y)\hat{\gamma}(p) + \frac{1}{2}\,\mathrm{sign}(y)\mathrm{e}^{-\frac{y}{2}}\,\mathrm{e}^{-\sqrt{p+\frac{1}{4}}|y|}\hat{\gamma}(p) + \left(p + \frac{1}{4}\right)\hat{\nu}. \tag{3.20}$$

Combining (3.19) and (3.20) yields

$$\hat{\nu}_{yy} = -\hat{\nu}_y - 2\delta(y)\hat{\gamma}(p) + p\hat{\nu}, \tag{3.21}$$

which becomes the initial-value problem (3.13) after the inverse Laplace transform. It follows from (3.19) for $p \geqslant 0$ that

$$\hat{\nu}_y(p, 0^{\pm}) = -\frac{1}{2}\hat{\nu}(p, 0) \mp \hat{\gamma}(p),$$

which yields (3.18) after the inverse Laplace transform. Uniqueness of the solution (3.17) is proven from uniqueness of the zero solution in the homogeneous version of the initial-value problem (3.13).

It remains to estimate the solution (3.17) in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}_+)$ provided that $\gamma \in L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R}_+)$. By using (3.2), we obtain

$$\|\nu(t,\cdot)\|_{L^2(\mathbb{R})} \leqslant \frac{2}{(8\pi)^{1/4}} \int_0^t \frac{|\gamma(\tau)|}{(t-\tau)^{1/4}}\,\mathrm{d}\tau \tag{3.22}$$

and

$$\|\nu(t,\cdot)\|_{L^{\infty}(\mathbb{R})} \leqslant \frac{1}{\sqrt{\pi}} \int_0^t \frac{|\gamma(\tau)|}{(t-\tau)^{1/2}}\,\mathrm{d}\tau. \tag{3.23}$$

The derivative $\nu(t, y)$ in y is given by

$$\nu_y(t, y) = -\int_0^t \frac{\gamma(\tau)(y + t - \tau)}{\sqrt{4\pi(t-\tau)^3}}\,\mathrm{e}^{-\frac{(y+t-\tau)^2}{4(t-\tau)}}\,\mathrm{d}\tau. \tag{3.24}$$

By using (3.3), we obtain

$$\|\nu_y(t,\cdot)\|_{L^2(\mathbb{R})} \leqslant \frac{1}{(8\pi)^{1/4}} \int_0^t \frac{|\gamma(\tau)|}{(t-\tau)^{3/4}}\,\mathrm{d}\tau. \tag{3.25}$$

It follows from (3.22) and (3.25) with lemma 3.1 that $\nu \in L^{\infty}(\mathbb{R}_+, H^1(\mathbb{R}))$ if $\gamma \in L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R}_+)$. By Sobolev embedding, $\nu \in L^{\infty}(\mathbb{R}_+, L^{\infty}(\mathbb{R}))$, which also follows from (3.23).

Finally, we show that $\nu_y \in L^{\infty}(\mathbb{R}_+, L^{\infty}(\mathbb{R}_+))$. Due to (3.4), direct estimates on $\|\nu_y(t,\cdot)\|_{L^{\infty}(\mathbb{R})}$ from (3.24) produce a non-integrable singularity in the convolution integral in time. Nevertheless, we show hereafter that $\|\nu_y(t,\cdot)\|_{L^{\infty}(\mathbb{R}_+)}$ can be estimated in terms of $|\gamma(t)|$.

The initial-value problem (3.13) can be rewritten in the piecewise form:

$$\begin{cases} \nu_t = \nu_y + \nu_{yy}, & \pm y > 0,\ t > 0, \\ \nu_y(t, 0^+) - \nu_y(t, 0^-) = -2\gamma(t), & t > 0, \\ \nu(0, y) = 0, & y \in \mathbb{R}. \end{cases} \tag{3.26}$$

With the transformation

$$\nu(t, y) = \mathrm{e}^{-\frac{y}{2} - \frac{t}{4}}\tilde{\nu}(t, y),$$

the initial-boundary-value problem (3.26) is equivalently written as

$$\begin{cases} \tilde{\nu}_t = \tilde{\nu}_{yy}, & \pm y > 0,\ t > 0, \\ \tilde{\nu}_y(t, 0^+) - \tilde{\nu}_y(t, 0^-) = -2\gamma(t)\mathrm{e}^{\frac{t}{4}}, & t > 0, \\ \tilde{\nu}(0, y) = 0, & y \in \mathbb{R}. \end{cases} \tag{3.27}$$

Due to the parity symmetry of the boundary and initial conditions in (3.27), $\tilde{\nu}$ is even in y, $\tilde{\nu}_y$ is odd in y, so that $\tilde{\nu}_y$ solves Dirichlet's problems for the diffusion equation on the quarter planes $\{y > 0, t > 0\}$ and $\{y < 0, t > 0\}$ subject to the boundary conditions $\tilde{\nu}_y(t, 0^+) = -\gamma(t)\mathrm{e}^{\frac{t}{4}}$ and $\tilde{\nu}_y(t, 0^-) = \gamma(t)\mathrm{e}^{\frac{t}{4}}$ respectively. It follows by the maximum principle for the diffusion equation that

$$\|\tilde{\nu}_y(t, \cdot)\|_{L^\infty(\mathbb{R})} \leqslant |\gamma(t)|\mathrm{e}^{\frac{t}{4}}, \quad t > 0, \tag{3.28}$$

which yields

$$\|\nu_y(t, \cdot)\|_{L^\infty(\mathbb{R}_+)} \leqslant \frac{1}{2}\|\nu(t, \cdot)\|_{L^\infty(\mathbb{R}_+)} + |\gamma(t)|, \quad t > 0, \tag{3.29}$$

since $\nu_y + \frac{1}{2}\nu = \mathrm{e}^{-\frac{y}{2} - \frac{t}{4}}\tilde{\nu}_y$ and $\mathrm{e}^{-\frac{y}{2}} \leqslant 1$ for $y \geqslant 0$. Hence, $\nu_y \in L^\infty(\mathbb{R}_+, L^\infty(\mathbb{R}_+))$. □

Remark 3.4. Since $\mathrm{e}^{-\frac{y}{2}}$ is unbounded for $y \in \mathbb{R}_-$, no bound on $\|\nu_y(t, \cdot)\|_{L^\infty(\mathbb{R}_-)}$ can be obtained from the estimate (3.28). However, we only need to use $\nu(t, y)$ for $t > 0$ and $y > 0$.

3.4. Abel's integral equations

For a given function $f \in W^{1,\infty}(\mathbb{R}_+)$, we solve the linear integral equation

$$\mathcal{M}(\gamma) = \frac{1}{\sqrt{4\pi t}}\int_0^\infty f(\eta)\mathrm{e}^{-\frac{\eta^2}{4t}}\,\mathrm{d}\eta, \quad t > 0, \tag{3.30}$$

where

$$\mathcal{M}(\gamma) := \int_0^t \frac{\gamma(\tau)}{\sqrt{\pi(t-\tau)}}\,\mathrm{d}\tau - \int_0^t \frac{\gamma(\tau)}{\sqrt{4\pi(t-\tau)}}\int_0^\infty \mathrm{e}^{-\frac{\eta}{2}}\,\mathrm{e}^{-\frac{\eta^2}{4(t-\tau)}}\,\mathrm{d}\eta\,\mathrm{d}\tau. \tag{3.31}$$

The linear equation (3.30) with the integral operator (3.31) is used in the proof of lemma 5.6.

The linear integral equations (3.30) and (3.31) is related to Abel's integral equation [32, 33]. We use again the Laplace transform in time t, as is defined in (3.14). The following lemma gives the exact solution to the integral equation (3.30) in the space of bounded functions.

Lemma 3.5. *For every $f \in W^{1,\infty}(\mathbb{R}_+)$ satisfying $f(0) = 0$, there exists a unique solution $\gamma \in L^\infty(\mathbb{R}_+)$ to the integral equation (3.30) in the exact form:*

$$\gamma(t) = \frac{1}{\sqrt{4\pi t}}\int_0^\infty f(\eta)\left(\frac{\eta + t}{2t}\right)\mathrm{e}^{-\frac{\eta^2}{4t}}\,\mathrm{d}\eta, \quad t > 0, \tag{3.32}$$

or, equivalently,

$$\gamma(t) = \frac{1}{\sqrt{4\pi t}}\int_0^\infty \left[f'(\eta) + \frac{1}{2}f(\eta)\right]\mathrm{e}^{-\frac{\eta^2}{4t}}\,\mathrm{d}\eta, \quad t > 0. \tag{3.33}$$

Proof. By using (3.14) and (3.15), we rewrite the integral equation (3.30) in the product form:

$$\frac{1}{\sqrt{p}}\hat{\gamma}(p) - \frac{1}{2\sqrt{p}}\hat{\gamma}(p)\int_0^\infty e^{-\frac{\eta}{2}}\, e^{-\sqrt{p}\eta}\, d\eta = \frac{1}{2\sqrt{p}}\int_0^\infty f(\eta) e^{-\sqrt{p}\eta}\, d\eta, \quad p > 0.$$

Evaluating the integral gives the solution in the Laplace transform space:

$$\hat{\gamma}(p) = \frac{1}{2}\int_0^\infty f(\eta)\left(1 + \frac{1}{2\sqrt{p}}\right) e^{-\sqrt{p}\eta}\, d\eta.$$

After the inverse Laplace transform, we obtain the exact solution (3.32) with the use of (3.16). The equivalent form (3.33) is obtained from (3.32) after integration by parts if $f \in W^{1,\infty}(\mathbb{R}_+)$ and $f(0) = 0$. It follows from (3.2) and (3.33) that

$$|\gamma(t)| \leqslant \frac{1}{2}\|f'\|_{L^\infty(\mathbb{R}_+)} + \frac{1}{4}\|f\|_{L^\infty(\mathbb{R}_+)}, \quad t > 0,$$

so that $\gamma \in L^\infty(\mathbb{R}_+)$. □

Similarly to lemma 3.5, we solve the linear integral equations

$$\mathcal{M}(\gamma) = \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}}\int_0^\infty g(\tau,\eta) e^{-\frac{\eta^2}{4(t-\tau)}}\, d\eta\, d\tau, \quad t > 0 \tag{3.34}$$

and

$$\mathcal{M}(\gamma) = \int_0^t \frac{h(\tau)d\tau}{\sqrt{4\pi(t-\tau)}}, \quad t > 0, \tag{3.35}$$

where $\mathcal{M}(\gamma)$ is given by (3.31), $g \in L^1(\mathbb{R}_+, L^\infty(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+, L^\infty(\mathbb{R}_+))$ and $h \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ are given functions. The linear equations (3.34) and (3.35) are also used in the proof of lemma 5.6. The following lemma gives the exact solutions of the integral equations (3.34) and (3.35) in the space of bounded functions.

Lemma 3.6. *For every $g \in L^1(\mathbb{R}_+, L^\infty(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+, L^\infty(\mathbb{R}_+))$, there exists a unique solution $\gamma \in L^\infty(\mathbb{R}_+)$ to the integral equation (3.34) in the exact form:*

$$\gamma(t) = \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}}\int_0^\infty g(\tau,\eta)\left(\frac{\eta + t - \tau}{2(t-\tau)}\right) e^{-\frac{\eta^2}{4(t-\tau)}}\, d\eta\, d\tau, \quad t > 0. \tag{3.36}$$

For every $h \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, there exists a unique solution $\gamma \in L^\infty(\mathbb{R}_+)$ to the integral equation (3.35) in the exact form:

$$\gamma(t) = \frac{1}{2}h(t) + \frac{1}{2}\int_0^t \frac{h(\tau)d\tau}{\sqrt{4\pi(t-\tau)}}, \quad t > 0. \tag{3.37}$$

Proof. By using (3.14) and (3.15), we solve the integral equation (3.34) for the Laplace transform:

$$\hat{\gamma}(p) = \frac{1}{2}\int_0^\infty \hat{g}(p,\eta)\left(1 + \frac{1}{2\sqrt{p}}\right) e^{-\sqrt{p}\eta}\, d\eta.$$

After the inverse Laplace transform, we obtain the exact solution (3.36) with the use of (3.16). By using the first integrals in (3.2) and (3.3), we obtain

$$|\gamma(t)| \leqslant \frac{1}{4}\int_0^t \|g(\tau,\cdot)\|_{L^\infty(\mathbb{R}_+)}\,\mathrm{d}\tau + \frac{1}{2}\int_0^t \frac{\|g(\tau,\cdot)\|_{L^\infty(\mathbb{R}_+)}\,\mathrm{d}\tau}{\sqrt{\pi(t-\tau)}}, \quad t>0,$$

where the upper bound is bounded if $\|g(t,\cdot)\|_{L^\infty(\mathbb{R}_+)}$ belongs to $L^1(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+)$ by lemma 3.1.

For the integral equation (3.35), we use the substitution $\gamma(t)=\frac{1}{2}h(t)+\upsilon(t)$, where $\upsilon(t)$ satisfies the integral equation

$$\mathcal{M}(\upsilon) = \frac{1}{2}\int_0^t \frac{h(\tau)}{\sqrt{4\pi(t-\tau)}}\int_0^\infty \mathrm{e}^{-\frac{\eta}{2}}\,\mathrm{e}^{-\frac{\eta^2}{4(t-\tau)}}\,\mathrm{d}\eta\,\mathrm{d}\tau, \quad t>0.$$

Since $g(\tau,\eta):=\frac{1}{2}h(\tau)\mathrm{e}^{-\frac{\eta}{2}}$ belongs to $L^1(\mathbb{R}_+,L^\infty(\mathbb{R}_+))\cap L^\infty(\mathbb{R}_+,L^\infty(\mathbb{R}_+))$, we can use the exact solution (3.36) and obtain

$$\upsilon(t) = \frac{1}{2}\int_0^t \frac{h(\tau)}{\sqrt{4\pi(t-\tau)}}\int_0^\infty \mathrm{e}^{-\frac{\eta}{2}}\left(\frac{\eta+t-\tau}{2(t-\tau)}\right)\mathrm{e}^{-\frac{\eta^2}{4(t-\tau)}}\,\mathrm{d}\eta\,\mathrm{d}\tau, \quad t>0.$$

Integrating the inner integral by parts gives

$$\upsilon(t) = \frac{1}{2}\int_0^t \frac{h(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}}, \quad t>0,$$

which recovers (3.37) for $\gamma(t)=\frac{1}{2}h(t)+\upsilon(t)$. Again, we have $\gamma\in L^\infty(\mathbb{R}_+)$ if $h\in L^1(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+)$ by lemma 3.1. □

Remark 3.7. Compared to the decomposition method $\gamma=\frac{1}{2}h+\upsilon$ in the proof of lemma 3.6, the exact solution (3.37) can be independently obtained by using the Laplace transform (3.14) in the linear equation (3.35).

4. Asymptotic stability under odd perturbations

Here we study the boundary-value problem (2.6) in order to prove theorem 2.3. The boundary-value problem (2.6) is solved by direct methods. First, we decompose

$$w(t,x) = W_0(x) + u(t,x), \quad x>0, \tag{4.1}$$

where $W_0(x)=1-\mathrm{e}^{-x}$ is the viscous shock given by (2.3) with $c=0$ and $W_+=-W_-=1$. The perturbation $u(t,x)$ satisfies the following boundary-value problem:

$$\begin{cases} u_t = u_x + u_{xx}, & x>0,\ t>0,\\ u(t,0)=0, & t>0,\\ u(t,x)\to 0 \quad \text{as } x\to+\infty, & t>0, \end{cases} \tag{4.2}$$

subject to the initial condition $u(0,x)=w(0,x)-W_0(x)=:u_0(x)$.

Remark 4.1. Although the boundary-value problem (4.2) can be solved with the unified transform method which involves complex analysis, implicit solutions, and numerical computations [1], we can solve this problem explicitly using the exact formula (3.6) from the method of images and the exponential transformation.

In order to prove theorem 2.3, we first derive *a priori* energy estimates (lemma 4.2) and then explore the exact formula (3.6) to study the solution in H^2 (lemma 4.3) and in $W^{2,\infty}$ (lemma 4.5).

The following lemma implies that the H^1-norm of a smooth solution $u(t,\cdot)$ is decreasing in time t. The result is obtained by using *a priori* energy estimates.

Lemma 4.2. *Assume existence of the solution $u \in C(\mathbb{R}_+, H^2(\mathbb{R}_+))$ to the boundary-value problem (4.2) with the initial condition $u(0,x) = u_0(x)$. Then, for every $t > 0$:*

$$\|u(t,\cdot)\|_{L^2} \leqslant \|u_0\|_{L^2}, \quad \|u(t,\cdot)\|_{H^1} \leqslant \|u_0\|_{H^1}.$$

Proof. Multiplying $u_t = u_x + u_{xx}$ by u and u_{xx} and integrating by parts yield for $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|_{L^2}^2 = -2\|u_x(t,\cdot)\|_{L^2}^2, \tag{4.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_x(t,\cdot)\|_{L^2}^2 = [u_x(t,0)]^2 - 2\|u_{xx}(t,\cdot)\|_{L^2}^2, \tag{4.4}$$

where $u_t(t,0) = 0$ has been used due to the boundary condition $u(t,0) = 0$ for $t > 0$. It follows from (4.3) that $\|u(t,\cdot)\|_{L^2} \leqslant \|u_0\|_{L^2}$ for every $t > 0$. By Sobolev embedding, it follows for every $f \in H^1(\mathbb{R}_+)$ that

$$[f(0)]^2 = -2\int_0^\infty f(x)f'(x)\mathrm{d}x \leqslant \|f\|_{H^1}^2, \tag{4.5}$$

so that we obtain by adding both equations (4.3) and (4.4) together and using (4.5) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|_{H^1}^2 = [u_x(t,0)]^2 - 2\|u_x(t,\cdot)\|_{H^1}^2 \leqslant -\|u_x(t,\cdot)\|_{H^1}^2,$$

hence $\|u(t,\cdot)\|_{H^1} \leqslant \|u_0\|_{H^1}$ for every $t > 0$. □

Lemma 4.2 implies uniqueness and continuous dependence of solutions to the boundary-value problem (4.2) with initial condition $u(0,x) = u_0(x)$. It remains to show existence of a solution $u \in C(\mathbb{R}_+, H^2(\mathbb{R}_+))$ for any given $u_0 \in H^2(\mathbb{R}_+)$. The following lemma explores an explicit formula for solutions to the boundary-value problem (4.2).

Lemma 4.3. *For any given $u_0 \in H^2(\mathbb{R}_+)$ satisfying $u_0(0) = 0$, there exists a solution $u(t,x)$ to the boundary-value problem (4.2) with the initial condition $u(0,x) = u_0(x)$ given explicitly by*

$$u(t,x) = \frac{1}{\sqrt{4\pi t}}\int_0^\infty u_0(y)\left[\mathrm{e}^{-\frac{(x-y+t)^2}{4t}} - \mathrm{e}^{-x}\,\mathrm{e}^{-\frac{(x+y-t)^2}{4t}}\right]\mathrm{d}y. \tag{4.6}$$

Moreover, $u \in C(\mathbb{R}_+, H^2(\mathbb{R}_+))$.

Proof. By using the transformation

$$u(t,x) = \mathrm{e}^{-\frac{x}{2}-\frac{t}{4}}v(t,x), \tag{4.7}$$

we can write the boundary-value problem (4.2) in the form (3.5) with the initial condition $v_0(x) = \mathrm{e}^{\frac{x}{2}}u_0(x)$. By substituting the transformation (4.7) to the exact solution (3.6) and completing squares for the heat kernel $G(t,x)$, we obtain the exact representation (4.6).

It remains to show that $u \in C(\mathbb{R}_+, H^2(\mathbb{R}_+))$ if $u_0 \in H^2(\mathbb{R}_+)$. The convolution integrals in (4.6) are analyzed by means of Young's inequality (3.7) with $p = r = 2$ and $q = 1$:

$$\|u_0\chi_{\mathbb{R}_+} * G(t, \cdot + t)\|_{L^2(\mathbb{R}_+)} \leqslant \|u_0\|_{L^2(\mathbb{R}_+)}\|G(t, \cdot + t)\|_{L^1(\mathbb{R})} \leqslant \|u_0\|_{L^2(\mathbb{R}_+)}$$

and

$$\|u_0\chi_{\mathbb{R}_+} * G(t, -\cdot + t)\|_{L^2(\mathbb{R}_+)} \leqslant \|u_0\|_{L^2(\mathbb{R}_+)}\|G(t, -\cdot + t)\|_{L^1(\mathbb{R})} \leqslant \|u_0\|_{L^2(\mathbb{R}_+)}.$$

At the same time, $\mathrm{e}^{-x} \leqslant 1$ for $x \geqslant 0$, so that

$$\|u(t, \cdot)\|_{L^2} \leqslant 2\|u_0\|_{L^2}, \tag{4.8}$$

where the L^2 norms are understood as $L^2(\mathbb{R}_+)$. In order to obtain similar estimates for u_x and u_{xx}, we differentiate (4.6) in x, use integration by parts under the consistency condition $u_0(0) = 0$, and obtain

$$\begin{aligned} u_x(t, x) = &\frac{1}{\sqrt{4\pi t}}\int_0^\infty u_0'(y)\left[\mathrm{e}^{-\frac{(x-y+t)^2}{4t}} + \mathrm{e}^{-x}\,\mathrm{e}^{-\frac{(x+y-t)^2}{4t}}\right]\mathrm{d}y \\ &+ \frac{1}{\sqrt{4\pi t}}\,\mathrm{e}^{-x}\int_0^\infty u_0(y)\mathrm{e}^{-\frac{(x+y-t)^2}{4t}}\,\mathrm{d}y \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} u_{xx}(t, x) = &\frac{1}{\sqrt{4\pi t}}\int_0^\infty u_0''(y)\left[\mathrm{e}^{-\frac{(x-y+t)^2}{4t}} - \mathrm{e}^{-x}\,\mathrm{e}^{-\frac{(x+y-t)^2}{4t}}\right]\mathrm{d}y \\ &- \frac{1}{\sqrt{\pi t}}\,\mathrm{e}^{-x}\int_0^\infty u_0'(y)\mathrm{e}^{-\frac{(x+y-t)^2}{4t}}\,\mathrm{d}y \\ &- \frac{1}{\sqrt{4\pi t}}\,\mathrm{e}^{-x}\int_0^\infty u_0(y)\mathrm{e}^{-\frac{(x+y-t)^2}{4t}}\,\mathrm{d}y. \end{aligned} \tag{4.10}$$

By the same estimates used in (4.8), we obtain:

$$\|u_x(t, \cdot)\|_{L^2} \leqslant 2\|u_0'\|_{L^2} + \|u_0\|_{L^2}, \tag{4.11}$$

$$\|u_{xx}(t, \cdot)\|_{L^2} \leqslant 2\|u_0''\|_{L^2} + 2\|u_0'\|_{L^2} + \|u_0\|_{L^2}. \tag{4.12}$$

This shows that $u(t, \cdot) \in H^2(\mathbb{R}_+)$ continuously in $t \in \mathbb{R}_+$. □

Remark 4.4. It follows from (4.9) and (4.10) as $x \to 0^+$ that the solution $u \in C(\mathbb{R}_+, H^2(\mathbb{R}_+))$ satisfies the interface condition

$$u_x(t, 0^+) + u_{xx}(t, 0^+) = 0, \quad t > 0. \tag{4.13}$$

The decay condition $u(t, x) \to 0$ as $x \to \infty$ is satisfied by the continuous embedding of $H^2(\mathbb{R}_+)$ into $C^1(\mathbb{R}_+) \cap W^{1,\infty}(\mathbb{R}_+)$ with functions and their first derivatives decaying to zero at infinity.

The following lemma establishes the decay of $\|u(t, \cdot)\|_{W^{2,\infty}}$ to zero as $t \to +\infty$.

Lemma 4.5. *Let $u \in C(\mathbb{R}_+, H^2(\mathbb{R}_+))$ be the solution to the boundary-value problem (4.2) given by lemma 4.3. Then, we have*

$$\|u(t, \cdot)\|_{W^{2,\infty}} \to 0 \quad \text{as} \quad t \to \infty. \tag{4.14}$$

Proof. For $t \geqslant 1$, we can estimate the convolution integrals in (4.6) by means of Young's inequality (3.7) with $p = q = 2$ and $r = \infty$:

$$\begin{aligned}\|u_0\chi_{\mathbb{R}_+} * G(t, \cdot + t)\|_{L^\infty(\mathbb{R}_+)} &\leqslant \|u_0\|_{L^2(\mathbb{R}_+)}\|G(t, \cdot + t)\|_{L^2(\mathbb{R})}\\ &\leqslant \frac{1}{(8\pi t)^{1/4}}\|u_0\|_{L^2(\mathbb{R}_+)}\end{aligned}$$

and

$$\begin{aligned}\|u_0\chi_{\mathbb{R}_+} * G(t, -\cdot + t)\|_{L^\infty(\mathbb{R}_+)} &\leqslant \|u_0\|_{L^2(\mathbb{R}_+)}\|G(t, -\cdot + t)\|_{L^2(\mathbb{R})}\\ &\leqslant \frac{1}{(8\pi t)^{1/4}}\|u_0\|_{L^2(\mathbb{R}_+)}.\end{aligned}$$

Using these estimates in (4.6), (4.9), and (4.10), we obtain

$$\begin{aligned}\|u(t, \cdot)\|_{L^\infty} &\leqslant \frac{2}{(8\pi t)^{1/4}}\|u_0\|_{L^2},\\ \|u_x(t, \cdot)\|_{L^\infty} &\leqslant \frac{1}{(8\pi t)^{1/4}}(2\|u_0'\|_{L^2} + \|u_0\|_{L^2}),\\ \|u_{xx}(t, \cdot)\|_{L^\infty} &\leqslant \frac{1}{(8\pi t)^{1/4}}(2\|u_0''\|_{L^2} + 2\|u_0'\|_{L^2} + \|u_0\|_{L^2}),\end{aligned}$$

which prove the decay (4.14). □

Proof of theorem 2.3. By lemma 4.3 and the bounds (4.8), (4.11), and (4.12), if $u_0 \in H^2(\mathbb{R}_+)$ satisfies $\|u_0\|_{H^2} < \delta$ as in (2.10), then

$$\|u(t, \cdot)\|_{H^2} \leqslant C\|u_0\|_{H^2} < C\delta$$

for a fixed δ-independent positive constant C. Hence, for every $\epsilon > 0$, there is $\delta := \epsilon/C$ such that the odd perturbation $u(t, \cdot)$ to the viscous shock W_0 in the decomposition (4.1) is bounded in $H^2(\mathbb{R})$ norm for every $t > 0$ according to the bound (2.11). The decay (2.12) follows from the decay (4.14) in lemma 4.5.

The constraint (2.8) is satisfied because both W_0 and u in the decomposition (4.1) satisfy this constraint. Under the constraint (2.8), the solutions $w(t, x) : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}$ to the boundary-value problem (2.6) are extended to the odd function $w_{\text{ext}}(t, x) : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ satisfying the interface condition (2.9).

It remains to verify that $w(t, x) = W_0(x) + u(t, x) > 0$ for every $x > 0$. The positivity condition (2.7) is necessary for reduction of the modular Burgers equation (1.1) with the odd functions to the boundary-value problem (2.6). By Sobolev embedding of $H^2(\mathbb{R}_+)$ into $C^1(\mathbb{R}_+) \cap W^{1,\infty}(\mathbb{R}_+)$, we obtain

$$\|u(t, \cdot)\|_{L^\infty} + \|u_x(t, \cdot)\|_{L^\infty} < \epsilon, \quad t > 0,$$

where ϵ is small. The symmetry point $x = 0$ is a simple root of $w(t, \cdot)$ for every $t > 0$ because $W_0(0) = 0$, $W_0'(0) = 1$, $u(t, 0) = 0$, and $|u_x(t, 0)| < \epsilon$ is small. Therefore, there exists an ϵ-independent $x_0 > 0$ such that $w(t, x) > 0$ for every $t > 0$ and $x \in (0, x_0)$. Now, $W_0(x) \geqslant W_0(x_0) > 0$ for every $x \geqslant x_0$ and since $|u(t, x)| < \epsilon$ for every $t > 0$ and $x > 0$, then $w(t, x) > 0$ for every $t > 0$ and $x \geqslant x_0$ if ϵ is sufficiently small. Combining these two estimates together yields $w(t, x) > 0$ for every $t > 0$ and $x > 0$. □

5. Asymptotic stability under general perturbations

Here we study the boundary-value problem (2.14) in order to prove theorem 2.6. The boundary-value problem (2.14) can be reformulated by using the decomposition

$$w(t,x) = W_0(x-\xi(t)) + u(t,x-\xi(t)), \quad x \in \mathbb{R}, \tag{5.1}$$

where W_0 is the viscous shock (2.3) with $c = 0$ and $W_+ = -W_- = 1$, $\xi(t)$ is the location of a single interface, and $u(t,y)$ with $y := x - \xi(t)$ is a perturbation satisfying

$$\begin{cases} u_t = (\xi'(t) \pm 1)u_y + u_{yy} + \xi'(t)W_0'(y), & \pm y > 0, \\ u(t,0) = 0, & \\ u(t,y) \to 0 & \text{as } y \to \pm\infty, \end{cases} \tag{5.2}$$

subject to the initial condition $u(0,x) = w(0,x) - W_0(x) =: u_0(x)$. We assume without loss of generality that $\xi(0) = 0$. The interface dynamics is defined by the following lemma.

Lemma 5.1. *Let $u(t,\cdot) \in C^1(\mathbb{R}) \cap C^2(\mathbb{R}\backslash\{0\})$ be a solution of the boundary-value problem (5.2) for $t \in \mathbb{R}_+$. Then, $\xi'(t)$ can be expressed in two equivalent ways by*

$$\xi'(t) = -\frac{u_y(t,0^+) + u_{yy}(t,0^+)}{1 + u_y(t,0^+)} = \frac{u_y(t,0^-) - u_{yy}(t,0^-)}{1 + u_y(t,0^-)}, \quad t \in \mathbb{R}_+. \tag{5.3}$$

Proof. It follows from (2.16) and (5.1) that piecewise C^2 solutions of the boundary-value problem (5.2) satisfy the interface condition

$$[u_{yy}]_-^+(t,0) = -2u_y(t,0). \tag{5.4}$$

On the other hand, it follows from (2.17) and (5.1) that $u_t(t,0) = 0$. Taking the limits $y \to 0^\pm$ in $u_t = (\xi'(t) \pm 1)u_y + u_{yy} + \xi'(t)W_0'(y)$ yields

$$(\xi'(t) \pm 1)u_y(t,0^\pm) + u_{yy}(t,0^\pm) + \xi'(t) = 0,$$

since $W_0'(0) = 1$. This balance yields the dynamical equation (5.3). The two equalities in (5.3) are consistent under the interface condition (5.4) since $u_y(t,0^+) = u_y(t,0^-)$. □

Remark 5.2. The system of equations (5.2)–(5.4) is derived under the conditions

$$\pm\,[W_0(y) + u(t,y)] > 0, \quad \pm y > 0 \tag{5.5}$$

which follow from (2.15) and (5.1). Since $W_0(0) = 0$, $W_0'(0) = 1$, $u(t,0) = 0$, and $u(t,\cdot) \in C^1(\mathbb{R})$, the positivity conditions (5.5) are attained near $y = 0$ if $1 + u_y(t,0) > 0$, which also ensures that the interface dynamics is well defined by the evolution equation (5.3).

Let us define

$$u^+(t,y) := u(t,y), \qquad u^-(t,y) := u(t,-y), \quad y > 0. \tag{5.6}$$

We also define $\gamma(t) := \xi'(t)$ and use $W_0'(y) = \mathrm{e}^{-|y|}$. The boundary-value problem (5.2) can be rewritten in the equivalent form

$$\begin{cases} u_t^\pm = (1 \pm \gamma)u_y^\pm + u_{yy}^\pm + \gamma\,\mathrm{e}^{-y}, & y > 0, \\ u^\pm(t,0) = 0, & \\ u^\pm(t,y) \to 0 & \text{as } y \to \infty, \end{cases} \tag{5.7}$$

subject to the continuity condition

$$u_y^+(t,0^+) = -u_y^-(t,0^+), \tag{5.8}$$

the interface condition

$$u_{yy}^+(t,0^+) - u_{yy}^-(t,0^+) = -2u_y^+(t,0^+), \tag{5.9}$$

and the dynamical condition

$$\gamma(t) = -\frac{u_y^+(t,0^+) + u_{yy}^+(t,0^+)}{1 + u_y^+(t,0^+)} = -\frac{u_y^-(t,0^+) + u_{yy}^-(t,0^+)}{1 - u_y^-(t,0^+)}. \tag{5.10}$$

The proof of theorem 2.6 is divided into two steps.

In the first step, for a given $\gamma \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, we show that the boundary-value problems (5.7) equipped with the initial conditions $u^\pm(0,y) = u_0^\pm(y)$ can be uniquely solved provided the norms of $\gamma \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ and $u_0^\pm \in H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$ are small. The unique global solutions $u^\pm \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$ satisfy the dynamical conditions (5.10) for any $t > 0$.

The two solutions for u^+ and u^- are uncoupled if γ is given. However, if the solutions u^+ and u^- are required to satisfy the continuity condition (5.8), then this constraint yields an integral equation on $\gamma \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. In the second step, we prove that the integral equation for $\gamma \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ can be uniquely solved provided $u_0^\pm \in H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$ are small and satisfy an additional requirement of the exponential decay in space.

Finally, the two conditions (5.8) and (5.10) imply the interface condition (5.9), which is thus redundant in the boundary-value problem.

The following lemma gives *a priori* energy estimates for the boundary-value problems (5.7) completed with the continuity condition (5.8). These energy estimates imply monotonicity of the H^1-norm of a smooth solution in time t.

Lemma 5.3. *Assume existence of the solutions $u^\pm \in C(\mathbb{R}_+, H^2(\mathbb{R}_+))$ to the boundary-value problem (5.7) completed with the continuity condition (5.8) for the initial conditions $u^\pm(0,y) = u_0^\pm(y)$ and for some $\gamma \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. Then, for every $t > 0$:*

$$\|u^+(t,\cdot)\|_{H^1}^2 + \|u^-(t,\cdot)\|_{H^1}^2 \leqslant \|u_0^+\|_{H^1}^2 + \|u_0^-\|_{H^1}^2. \tag{5.11}$$

Proof. Multiplying $u_t^\pm = (1 \pm \gamma)u_y^\pm + u_{yy}^\pm + \gamma\,\mathrm{e}^{-y}$ by $u^\pm$ and $u_{yy}^\pm$ and integrating by parts yield for $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^\pm(t,\cdot)\|_{L^2}^2 = -2\|u_y^\pm(t,\cdot)\|_{L^2}^2 + 2\gamma\int_0^\infty u^\pm(t,y)\mathrm{e}^{-y}\,\mathrm{d}y, \tag{5.12}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u_y^\pm(t,\cdot)\|_{L^2}^2 &= (1 \pm \gamma)\left[u_y^\pm(t,0^+)\right]^2 - 2\|u_{yy}^\pm(t,\cdot)\|_{L^2}^2 \\ &\quad + 2\gamma u_y^\pm(t,0^+) - 2\gamma\int_0^\infty u^\pm(t,y)\mathrm{e}^{-y}\,\mathrm{d}y,\end{aligned}$$

where $u_t^\pm(t,0) = 0$ has been used due to the boundary conditions $u^\pm(t,0) = 0$ for $t > 0$. Adding all equations and using the continuity condition (5.8) yield

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u^+(t,\cdot)\|_{H^1}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|u^-(t,\cdot)\|_{H^1}^2 &= \left[u_y^+(t,0^+)\right]^2 + \left[u_y^-(t,0^+)\right]^2 \\ &\quad - 2\|u_y^+(t,\cdot)\|_{H^1}^2 - 2\|u_y^-(t,\cdot)\|_{H^1}^2.\end{aligned}$$

By using the same inequality (4.5), we close the estimate and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|u^+(t,\cdot)\|_{H^1}^2 + \|u^-(t,\cdot)\|_{H^1}^2\right] \leqslant -\|u_y^+(t,\cdot)\|_{H^1}^2 - \|u_y^-(t,\cdot)\|_{H^1}^2 \leqslant 0,$$

from which the inequality (5.11) follows. □

Remark 5.4. Compared to lemma 4.2, we are not able to conclude on monotonicity of the L^2-norm of the solution. By using Cauchy–Schwarz inequality in (5.12), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^{\pm}(t,\cdot)\|_{L^2} \leqslant |\gamma| \|\mathrm{e}^{-y}\|_{L_y^2(\mathbb{R}_+)},$$

which yields the Stritcharz-type estimate

$$\sup_{t\in\mathbb{R}_+} \|u^{\pm}(t,\cdot)\|_{L^2} \leqslant \|u_0^{\pm}\|_{L^2} + \|\gamma\|_{L^1} \|\mathrm{e}^{-y}\|_{L_y^2(\mathbb{R}_+)},$$

where we write $\|\mathrm{e}^{-y}\|_{L_y^2(\mathbb{R}_+)}$ instead of $\|\mathrm{e}^{-\cdot}\|_{L^2}$ for better clarity.

We shall now consider the existence of solutions to the boundary-value problems (5.7) for a given $\gamma(t)$. Due to the condition (5.10) satisfied by smooth solutions $u^{\pm}(t,y)$, we need to require $u_{yy}^{\pm}$ be bounded in a one-sided neighborhood of $y=0$. This is achieved by using a sharper condition on the initial data $u_0^{\pm} \in H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$ compared to the requirement $u_0 \in H^2(\mathbb{R}_+)$ imposed in lemma 4.3. On the other hand, the L^∞ norm of the solution does not need to be continuous in time [18], hence we consider solutions to the boundary-value problems (5.7) in function space $L^\infty(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$.

The following lemma provides a convenient reformulation of the boundary-value problems (5.7) as systems of integral equations, where $u^{\pm}$ and γ are not required to satisfy the continuity condition (5.8), the interface condition (5.9), and the dynamical conditions (5.10).

Lemma 5.5. *There exist solutions $u^{\pm} \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$ to the boundary-value problems (5.7) with the initial conditions $u^{\pm}(0,y) = u_0^{\pm}(y)$ and the given function $\gamma(t)$ if there exist solutions $u^{\pm} \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$ to the following integral equations for $(t,y) \in \mathbb{R}_+ \times \mathbb{R}_+$:*

$$\begin{aligned} u^{\pm}(t,y) &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty u_0^{\pm}(\eta) \left[\mathrm{e}^{-\frac{(y-\eta+t)^2}{4t}} - \mathrm{e}^{-y}\,\mathrm{e}^{-\frac{(y+\eta-t)^2}{4t}}\right] \mathrm{d}\eta \\ &\quad + \int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}} \int_0^\infty e^{-\eta} \left[\mathrm{e}^{-\frac{(y-\eta+t-\tau)^2}{4(t-\tau)}} - \mathrm{e}^{-y}\,\mathrm{e}^{-\frac{(y+\eta-t+\tau)^2}{4(t-\tau)}}\right] \mathrm{d}\eta \\ &\quad \pm \int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}} \int_0^\infty u_\eta^{\pm}(\tau,\eta) \left[\mathrm{e}^{-\frac{(y-\eta+t-\tau)^2}{4(t-\tau)}} - \mathrm{e}^{-y}\,\mathrm{e}^{-\frac{(y+\eta-t+\tau)^2}{4(t-\tau)}}\right] \mathrm{d}\eta. \end{aligned}$$

Proof. Similar to the transformation formula (4.7) in the proof of lemma 4.3, the system of equation (5.7) can be simplified by using the transformation formulas:

$$u^{\pm}(t,y) = \mathrm{e}^{-\frac{y}{2}-\frac{t}{4}} v^{\pm}(t,y), \qquad \gamma(t) = \mathrm{e}^{-\frac{t}{4}} \tilde{\gamma}(t). \tag{5.13}$$

The boundary-value problems (5.7) can be rewritten in the form (3.8) with $v = v^{\pm}$,

$$f(t,y) = \tilde{\gamma}\,\mathrm{e}^{-\frac{y}{2}} \pm \tilde{\gamma}\,\mathrm{e}^{-\frac{t}{4}} \left(v_y^{\pm} - \frac{1}{2} v^{\pm}\right),$$

and $v_0(y) := u_0^{\pm}(y)\mathrm{e}^{\frac{y}{2}}$. The exact solution (3.9) yields the integral equations for $v^{\pm}$:

$$
\begin{aligned}
v^{\pm}(t,y) = {} & \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} v_0^{\pm}(\eta) \left[\mathrm{e}^{-\frac{(y-\eta)^2}{4t}} - \mathrm{e}^{-\frac{(y+\eta)^2}{4t}} \right] \mathrm{d}\eta \\
& + \int_0^t \frac{\tilde{\gamma}(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}} \int_0^{\infty} \mathrm{e}^{-\frac{\eta}{2}} \left[\mathrm{e}^{-\frac{(y-\eta)^2}{4(t-\tau)}} - \mathrm{e}^{-\frac{(y+\eta)^2}{4(t-\tau)}} \right] \mathrm{d}\eta \\
& \pm \int_0^t \frac{\tilde{\gamma}(\tau)\mathrm{e}^{-\frac{\tau}{4}}\,\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}} \int_0^{\infty} \left(v_y^{\pm}(\tau,\eta) - \frac{1}{2} v^{\pm}(\tau,\eta) \right) \left[\mathrm{e}^{-\frac{(y-\eta)^2}{4(t-\tau)}} - \mathrm{e}^{-\frac{(y+\eta)^2}{4(t-\tau)}} \right] \mathrm{d}\eta.
\end{aligned}
$$

Substituting the transformation (5.13) yields the integral equations for $u^{\pm}(t,y)$. □

If solutions $u^{\pm} \in L^{\infty}(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$ of the integral equations in lemma 5.5 are required to satisfy the continuity condition (5.8), then $\gamma \in L^1(\mathbb{R}_+) \cap L^{\infty}(\mathbb{R}_+)$ satisfies a certain constraint. Computing partial derivatives of $u^{\pm}(t,y)$ in y, taking the limit $y \to 0^+$, and substituting $u_y^{\pm}(t,0^+)$ into (5.8) yields the constraint in the form

$$
\begin{aligned}
& \frac{1}{\sqrt{4\pi t^3}} \int_0^{\infty} \left[u_0^+(\eta) + u_0^-(\eta) \right] \eta\, \mathrm{e}^{-\frac{(\eta-t)^2}{4t}}, \mathrm{d}\eta \\
& \quad + 2 \int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)^3}} \int_0^{\infty} \mathrm{e}^{-\eta} \eta\, \mathrm{e}^{-\frac{(\eta-t+\tau)^2}{4(t-\tau)}} \mathrm{d}\eta \\
& \quad + \int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)^3}} \int_0^{\infty} \left[u_\eta^+(\tau,\eta) - u_\eta^-(\tau,\eta) \right] \eta\, \mathrm{e}^{-\frac{(\eta-t+\tau)^2}{4(t-\tau)}} \mathrm{d}\eta = 0. \qquad (5.14)
\end{aligned}
$$

The following lemma rewrites the constraint (5.14) as the integral equation for $\gamma(t)$.

Lemma 5.6. *Assume that $u^{\pm} \in L^{\infty}(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$ are solutions to the boundary-value problems (5.7) with the initial conditions $u^{\pm}(0,y) = u_0^{\pm}(y)$ satisfying $u_0^{\pm}(0) = 0$ and $u_0^{+\prime}(0) + u_0^{-\prime}(0) = 0$. There exists a solution $\gamma \in L^1(\mathbb{R}_+) \cap L^{\infty}(\mathbb{R}_+)$ to the integral equation (5.14) if there exists a solution $\gamma \in L^1(\mathbb{R}_+) \cap L^{\infty}(\mathbb{R}_+)$ to the following integral equation*

$$
\begin{aligned}
\gamma(t) = {} & -\frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left[u_0^{+\prime}(\eta) + u_0^{-\prime}(\eta) + \frac{1}{2} u_0^+(\eta) + \frac{1}{2} u_0^-(\eta) \right] \left(\frac{\eta + t}{2t} \right) \mathrm{e}^{-\frac{(\eta-t)^2}{4t}} \mathrm{d}\eta \\
& - \frac{1}{2} \gamma(t) \left[u_y^+(t,0^+) - u_y^-(t,0^+) \right] \\
& - \frac{1}{2} \int_0^t \frac{\gamma(\tau)\mathrm{e}^{-\frac{t-\tau}{4}}}{\sqrt{4\pi(t-\tau)}} \left[u_y^+(\tau,0^+) - u_y^-(\tau,0^+) \right] \mathrm{d}\tau \\
& - \int_0^t \frac{\gamma(\tau)}{\sqrt{4\pi(t-\tau)}} \int_0^{\infty} \left[u_{yy}^+ - u_{yy}^- + \frac{1}{2} u_y^+ - \frac{1}{2} u_y^- \right] (\tau,\eta) \left(\frac{\eta + t - \tau}{2(t-\tau)} \right) \mathrm{e}^{-\frac{(\eta-t+\tau)^2}{4(t-\tau)}} \mathrm{d}\eta\, \mathrm{d}\tau.
\end{aligned}
$$

Proof. First, we integrate by parts in (5.14) with the use of the boundary conditions $u^{\pm}(t,0) = 0$ and $u_0^{\pm}(0) = 0$ in order to obtain the integral equations in time variable with a weakly singular kernel. A short computation yields the following integral equation:

$$\int_0^t \frac{\gamma(\tau)}{\sqrt{\pi(t-\tau)}}\,\mathrm{e}^{-\frac{t-\tau}{4}}\,\mathrm{d}\tau - \int_0^t \frac{\gamma(\tau)}{\sqrt{4\pi(t-\tau)}}\int_0^\infty \mathrm{e}^{-\frac{(\eta+t-\tau)^2}{4(t-\tau)}}\,\mathrm{d}\eta\,\mathrm{d}\tau$$
$$+\frac{1}{\sqrt{4\pi t}}\int_0^\infty\left[u_0^{+\prime}(\eta)+u_0^{-\prime}(\eta)+\frac{1}{2}u_0^+(\eta)+\frac{1}{2}u_0^-(\eta)\right]\mathrm{e}^{-\frac{(\eta-t)^2}{4t}}\,\mathrm{d}\eta$$
$$+\int_0^t \frac{\gamma(\tau)}{\sqrt{4\pi(t-\tau)}}\int_0^\infty\left[u_y^+-u_y^-+\frac{1}{2}u^+-\frac{1}{2}u^-\right](\tau,\eta)\left(\frac{\eta-t+\tau}{2(t-\tau)}\right)\mathrm{e}^{-\frac{(\eta-t+\tau)^2}{4(t-\tau)}}\,\mathrm{d}\eta\,\mathrm{d}\tau$$
$$=0.$$

By using the transformation (5.13), we rewrite the integral equation in the equivalent form:

$$\mathcal{M}(\tilde{\gamma})+\frac{1}{\sqrt{4\pi t}}\int_0^\infty\left[v_0^{+\prime}(\eta)+v_0^{-\prime}(\eta)\right]\mathrm{e}^{-\frac{\eta^2}{4t}}\,\mathrm{d}\eta$$
$$+\int_0^t\frac{\tilde{\gamma}(\tau)\mathrm{e}^{-\frac{\tau}{4}}}{\sqrt{4\pi(t-\tau)}}\int_0^\infty\left[v_y^+-v_y^-\right](\tau,\eta)\left(\frac{\eta-t+\tau}{2(t-\tau)}\right)\mathrm{e}^{-\frac{\eta^2}{4(t-\tau)}}\,\mathrm{d}\eta\,\mathrm{d}\tau=0, \qquad (5.15)$$

where the linear operator $\mathcal{M}$ is given by (3.31) and $v_0^\pm(y)=u_0^\pm(y)\mathrm{e}^{\frac{y}{2}}$. By using lemma 3.5 with $f\in W^{1,\infty}(\mathbb{R}_+)$ given by

$$f(\eta):=v_0^{+\prime}(\eta)+v_0^{-\prime}(\eta),$$

the linear operator $\mathcal{M}$ can be inverted on the second term of the integral equation (5.15). Note that $f(0)=0$ in lemma 3.5 is satisfied due to the consistency conditions $u_0^\pm(0)=0$ and $u_0^{+\prime}(0)+u_0^{-\prime}(0)=0$.

In order to invert the linear operator $\mathcal{M}$ on the third term of the integral equation (5.15), we integrate it by parts and obtain

$$\int_0^t\frac{\tilde{\gamma}(\tau)\mathrm{e}^{-\frac{\tau}{4}}}{\sqrt{4\pi(t-\tau)}}\int_0^\infty\left[v_y^+-v_y^-\right](\tau,\eta)\left(\frac{\eta-t+\tau}{2(t-\tau)}\right)\mathrm{e}^{-\frac{\eta^2}{4(t-\tau)}}\,\mathrm{d}\eta\,\mathrm{d}\tau$$
$$=\int_0^t\frac{\tilde{\gamma}(\tau)\mathrm{e}^{-\frac{\tau}{4}}}{\sqrt{4\pi(t-\tau)}}\left[v_y^+(\tau,0^+)-v_y^-(\tau,0^+)\right]\mathrm{d}\tau$$
$$+\int_0^t\frac{\tilde{\gamma}(\tau)\mathrm{e}^{-\frac{\tau}{4}}}{\sqrt{4\pi(t-\tau)}}\int_0^\infty\left[v_{yy}^+-v_{yy}^--\frac{1}{2}v_y^++\frac{1}{2}v_y^-\right](\tau,\eta)\mathrm{e}^{-\frac{\eta^2}{4(t-\tau)}}\,\mathrm{d}\eta\,\mathrm{d}\tau.$$

We are now in position to use lemma 3.6 with $g\in L^1(\mathbb{R}_+,L^\infty(\mathbb{R}_+))\cap L^\infty(\mathbb{R}_+,L^\infty(\mathbb{R}_+))$ given by

$$g(\tau,\eta)=\tilde{\gamma}(\tau)\mathrm{e}^{-\frac{\tau}{4}}\left[v_{yy}^+(\tau,\eta)-v_{yy}^-(\tau,\eta)-\frac{1}{2}v_y^+(\tau,\eta)+\frac{1}{2}v_y^-(\tau,\eta)\right]$$

and $h\in L^1(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+)$ given by

$$h(\tau)=\tilde{\gamma}(\tau)\mathrm{e}^{-\frac{\tau}{4}}\left[v_y^+(\tau,0^+)-v_y^-(\tau,0^+)\right].$$

By using lemmas 3.5 and 3.6 as described above, we obtain the following integral equation:

$$\begin{aligned}\tilde{\gamma}(t) = & -\frac{1}{\sqrt{4\pi t}}\int_0^{\infty}\left[v_0^{+\prime}(\eta)+v_0^{-\prime}(\eta)\right]\left(\frac{\eta+t}{2t}\right)\mathrm{e}^{-\frac{\eta^2}{4t}}\,\mathrm{d}\eta \\ & -\frac{1}{2}\tilde{\gamma}(t)\mathrm{e}^{-\frac{t}{4}}\left[v_y^{+}(t,0^{+})-v_y^{-}(t,0^{+})\right] \\ & -\frac{1}{2}\int_0^t \frac{\tilde{\gamma}(\tau)\mathrm{e}^{-\frac{\tau}{4}}}{\sqrt{4\pi(t-\tau)}}\left[v_y^{+}(\tau,0^{+})-v_y^{-}(\tau,0^{+})\right]\mathrm{d}\tau \\ & -\int_0^t \frac{\tilde{\gamma}(\tau)\mathrm{e}^{-\frac{\tau}{4}}}{\sqrt{4\pi(t-\tau)}}\int_0^{\infty}\left[v_{yy}^{+}-v_{yy}^{-}-\frac{1}{2}v_y^{+}+\frac{1}{2}v_y^{-}\right](\tau,\eta)\left(\frac{\eta+t-\tau}{2(t-\tau)}\right)\mathrm{e}^{-\frac{\eta^2}{4(t-\tau)}}\,\mathrm{d}\eta\,\mathrm{d}\tau.\end{aligned}$$

Substituting the transformation (5.13) yields the integral equation for $\gamma(t)$. □

Next, we solve the integral equations in lemmas 5.5 and 5.6.

The following lemma guarantees existence of the global solutions $u^{\pm} \in L^{\infty}(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$ to the boundary-value problems (5.7) for small initial data $u_0^{\pm} \in H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$ and small function $\gamma \in L^1(\mathbb{R}_+) \cap L^{\infty}(\mathbb{R}_+)$. The global solutions satisfy the dynamical conditions (5.10) for $t > 0$ but do not generally satisfy the additional conditions (5.8) and (5.9).

Lemma 5.7. *For every $\epsilon > 0$ (small enough), there is $\delta > 0$ such that for every $u_0^{\pm} \in H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$ and for every $\gamma \in L^1(\mathbb{R}_+) \cap L^{\infty}(\mathbb{R}_+)$ satisfying*

$$\|u_0^{+}\|_{H^2\cap W^{2,\infty}} + \|u_0^{-}\|_{H^2\cap W^{2,\infty}} + \|\gamma\|_{L^1\cap L^{\infty}} < \delta \tag{5.16}$$

and $u_0^{\pm}(0) = 0$, there exist unique solutions $u^{\pm} \in L^{\infty}(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$ to the integral equations in lemma 5.5. Moreover, the solutions satisfy

$$\|u^{+}(t,\cdot)\|_{H^2\cap W^{2,\infty}} + \|u^{-}(t,\cdot)\|_{H^2\cap W^{2,\infty}} < \epsilon \quad t > 0 \tag{5.17}$$

and the dynamical conditions (5.10) for $t > 0$.

Proof. We rewrite the integral equations in lemma 5.5 as the fixed-point equations associated with the following integral operators:

$$u^{\pm} = A^{\pm}(u^{\pm}) := u_1^{\pm} + u_2^{\pm} \pm u_3^{\pm}, \tag{5.18}$$

where

$$u_1^{\pm}(t,y) = \frac{1}{\sqrt{4\pi t}}\int_0^{\infty} u_0^{\pm}(\eta)\left[\mathrm{e}^{-\frac{(y-\eta+t)^2}{4t}} - \mathrm{e}^{-y}\,\mathrm{e}^{-\frac{(y+\eta-t)^2}{4t}}\right]\mathrm{d}\eta, \tag{5.19}$$

$$u_2^{\pm}(t,y) = \int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}}\int_0^{\infty}\mathrm{e}^{-\eta}\left[\mathrm{e}^{-\frac{(y-\eta+t-\tau)^2}{4(t-\tau)}} - \mathrm{e}^{-y}\,\mathrm{e}^{-\frac{(y+\eta-t+\tau)^2}{4(t-\tau)}}\right]\mathrm{d}\eta, \tag{5.20}$$

$$u_3^{\pm}(t,y) = \int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}}\int_0^{\infty} u_{\eta}^{\pm}(\tau,\eta)\left[\mathrm{e}^{-\frac{(y-\eta+t-\tau)^2}{4(t-\tau)}} - \mathrm{e}^{-y}\,\mathrm{e}^{-\frac{(y+\eta-t+\tau)^2}{4(t-\tau)}}\right]\mathrm{d}\eta. \tag{5.21}$$

The fixed-point equations (5.18) are considered in a small ball $B_{\epsilon} \subset X$ of radius $\epsilon > 0$ in Banach space

$$X := L^{\infty}(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)),$$

where $u_0^\pm \in H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$ and $\gamma \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ are given and satisfy the initial bound (5.16). We analyze hereafter each term in the definition of $A^\pm(u^\pm)$ in X.

The explicit expressions for $u_1^\pm$ in (5.19) coincide with (4.6) after the change of the initial data u_0 to $u_0^\pm$. By using the same analysis as in the proof of lemma 4.3, we obtain the same bounds (4.8), (4.11), and (4.12) for $u_1^\pm(t, \cdot)$ and their first and second y-derivatives in the $L^2(\mathbb{R}_+)$ norm. Similarly, the same bounds can be rederived in the $L^\infty(\mathbb{R}_+)$ norm. Combining them together, we deduce that there exists $C > 0$ such that

$$\|u_1^\pm(t, \cdot)\|_{H^2\cap W^{2,\infty}} \leqslant C\|u_0^\pm\|_{H^2\cap W^{2,\infty}}, \quad t > 0. \tag{5.22}$$

It follows from (4.13) in remark 4.4 that

$$\partial_y u_1^\pm(t, 0^+) + \partial_y^2 u_1^\pm(t, 0^+) = 0, \quad t > 0. \tag{5.23}$$

Let us now consider the explicit expressions for $u_2^\pm$ in (5.20). Recall from the proof of lemma 4.3 that the estimates for $\mathrm{e}^{-y}G(t, -y + t)$ in $L_y^2(\mathbb{R}_+) \cap L_y^\infty(\mathbb{R}_+)$ are identical to those for $G(t, y + t)$ and result in the double factors in the bounds (4.8), (4.11), and (4.12). In what follows, we only show the explicit estimates for the first term $G(t, y + t)$. By Young's inequality (3.7) with either $p = 1$ and $q = r = 2$ or $p = q = 2$ and $r = \infty$, we obtain

$$\begin{aligned}
\|u_2^\pm(t, \cdot)\|_{L^2\cap L^\infty} &\leqslant 2\int_0^t |\gamma(\tau)|\|\mathrm{e}^{-y} * G(t-\tau, y+t-\tau)\|_{L_y^2(\mathbb{R}_+)\cap L_y^\infty(\mathbb{R}_+)}\,\mathrm{d}\tau \\
&\leqslant 2\int_0^t |\gamma(\tau)|\|\mathrm{e}^{-y}\|_{L_y^1(\mathbb{R}_+)\cap L_y^2(\mathbb{R}_+)}\|G(t-\tau, y+t-\tau)\|_{L_y^2(\mathbb{R})}\,\mathrm{d}\tau \\
&\leqslant \frac{2}{(8\pi)^{1/4}}\int_0^t \frac{|\gamma(\tau)|}{(t-\tau)^{1/4}}\,\mathrm{d}\tau,
\end{aligned}$$

where the second equality in (3.2) has been used together with $\|\mathrm{e}^{-y}\|_{L_y^1(\mathbb{R}_+)} = 1$ and $\|\mathrm{e}^{-y}\|_{L_y^2(\mathbb{R}_+)} = \frac{1}{\sqrt{2}} < 1$. Computing derivatives in y and integrating by parts yield

$$\partial_y u_2^\pm(t, y) = -\int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}}\int_0^\infty \mathrm{e}^{-\eta}\,\mathrm{e}^{-\frac{(y-\eta+t-\tau)^2}{4(t-\tau)}}\,\mathrm{d}\eta + \nu(t, y) \tag{5.24}$$

and

$$\partial_y^2 u_2^\pm(t, y) = \int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}}\int_0^\infty \mathrm{e}^{-\eta}\,\mathrm{e}^{-\frac{(y-\eta+t-\tau)^2}{4(t-\tau)}}\,\mathrm{d}\eta - \frac{1}{2}\nu(t, y) + \nu_y(t, y), \tag{5.25}$$

where $\nu(t, y)$ is given by (3.17). By using estimates (3.22), (3.23), (3.25), and (3.29) in the proof of lemma 3.3, we obtain

$$\|\partial_y u_2^\pm(t, \cdot)\|_{L^2} \leqslant \frac{3}{(8\pi)^{1/4}}\int_0^t \frac{|\gamma(\tau)|}{(t-\tau)^{1/4}}\,\mathrm{d}\tau,$$

$$\|\partial_y u_2^\pm(t, \cdot)\|_{L^\infty} \leqslant \frac{3}{\sqrt{4\pi}}\int_0^t \frac{|\gamma(\tau)|}{(t-\tau)^{1/2}}\,\mathrm{d}\tau,$$

$$\|\partial_y^2 u_2^\pm(t, \cdot)\|_{L^2} \leqslant \frac{2}{(8\pi)^{1/4}}\int_0^t \frac{|\gamma(\tau)|}{(t-\tau)^{1/4}}\,\mathrm{d}\tau + \frac{1}{(8\pi)^{1/4}}\int_0^t \frac{|\gamma(\tau)|}{(t-\tau)^{3/4}}\,\mathrm{d}\tau,$$

and

$$\|\partial_y^2 u_2^\pm(t,\cdot)\|_{L^\infty} \leqslant \frac{3}{\sqrt{4\pi}} \int_0^t \frac{|\gamma(\tau)|}{(t-\tau)^{1/2}}\,\mathrm{d}\tau + |\gamma(t)|.$$

Combining all estimates together, we deduce that there exists $C > 0$ such that

$$\|u_2^\pm(t,\cdot)\|_{H^2\cap W^{2,\infty}} \leqslant C\left(\int_0^t \frac{|\gamma(\tau)|}{(t-\tau)^{1/4}}\,\mathrm{d}\tau + \int_0^t \frac{|\gamma(\tau)|}{(t-\tau)^{1/2}}\,\mathrm{d}\tau + \int_0^t \frac{|\gamma(\tau)|}{(t-\tau)^{3/4}}\,\mathrm{d}\tau + |\gamma(t)|\right). \tag{5.26}$$

By taking the limit $y \to 0^+$ in (5.24) and (5.25) and using (3.18) in lemma 3.3, we obtain

$$\partial_y u_2^\pm(t,0^+) + \partial_y^2 u_2^\pm(t,0^+) = \frac{1}{2}\nu(t,0^+) + \nu_y(t,0^+) = -\gamma(t), \quad t > 0. \tag{5.27}$$

Let us now consider the explicit expressions for $u_3^\pm$ in (5.21). Integrating by parts with the boundary conditions $u^\pm(t,0^+) = 0$, we obtain

$$\begin{aligned} u_3^\pm(t,y) = &-\int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}} \int_0^\infty u^\pm(\tau,\eta)\left(\frac{y-\eta+t-\tau}{2(t-\tau)}\right)\mathrm{e}^{-\frac{(y-\eta+t-\tau)^2}{4(t-\tau)}}\,\mathrm{d}\eta \\ &-\int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}} \int_0^\infty u^\pm(\tau,\eta)\left(\frac{y+\eta-t+\tau}{2(t-\tau)}\right)\mathrm{e}^{-y}\,\mathrm{e}^{-\frac{(y+\eta-t+\tau)^2}{4(t-\tau)}}\,\mathrm{d}\eta. \end{aligned} \tag{5.28}$$

The second term again enjoys the same estimates as the first term and give a double factor in the resulting bounds. By Young's inequality (3.7) with $p = 1$ and either $q = r = 2$ or $q = r = \infty$, we obtain

$$\begin{aligned} \|u_3^\pm(t,\cdot)\|_{L^2\cap L^\infty} &\leqslant 2\int_0^t |\gamma(\tau)|\|u^\pm(\tau,y) * \partial_y G(t-\tau, y+t-\tau)\|_{L_y^2(\mathbb{R}_+)\cap L_y^\infty(\mathbb{R}_+)}\,\mathrm{d}\tau \\ &\leqslant 2\int_0^t |\gamma(\tau)|\|u^\pm(\tau,\cdot)\|_{L^2\cap L^\infty}\|\partial_y G(t-\tau,y+t-\tau)\|_{L_y^1(\mathbb{R})}\,\mathrm{d}\tau \\ &\leqslant 2\int_0^t \frac{|\gamma(\tau)|}{\sqrt{\pi(t-\tau)}}\|u^\pm(\tau,\cdot)\|_{L^2\cap L^\infty}\,\mathrm{d}\tau, \end{aligned}$$

where the first equality in (3.3) has been used. Computing derivative in y and integrating by parts yield

$$\begin{aligned} \partial_y u_3^\pm(t,y) = &-\int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}} \int_0^\infty u_y^\pm(\tau,\eta)\left(\frac{y-\eta+t-\tau}{2(t-\tau)}\right)\mathrm{e}^{-\frac{(y-\eta+t-\tau)^2}{4(t-\tau)}}\,\mathrm{d}\eta \\ &+\int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}} \int_0^\infty u_y^\pm(\tau,\eta)\left(\frac{y+\eta-t+\tau}{2(t-\tau)}\right)\mathrm{e}^{-y}\,\mathrm{e}^{-\frac{(y+\eta-t+\tau)^2}{4(t-\tau)}}\,\mathrm{d}\eta \\ &+\int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}} \int_0^\infty u^\pm(\tau,\eta)\left(\frac{y+\eta-t+\tau}{2(t-\tau)}\right)\mathrm{e}^{-y}\,\mathrm{e}^{-\frac{(y+\eta-t+\tau)^2}{4(t-\tau)}}\,\mathrm{d}\eta. \end{aligned}$$

With similar estimates as above, we obtain

$$\|\partial_y u_3^\pm(t,\cdot)\|_{L^2\cap L^\infty} \leqslant \int_0^t \frac{|\gamma(\tau)|}{\sqrt{\pi(t-\tau)}}(2\|\partial_y u^\pm(\tau,\cdot)\|_{L^2\cap L^\infty} + \|u^\pm(\tau,\cdot)\|_{L^2\cap L^\infty})\,\mathrm{d}\tau.$$

Computing another derivative in y and integrating by parts yield

$$\begin{aligned}\partial_y^2 u_3^{\pm}(t,y) = &-\int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}}\int_0^\infty u_{yy}^{\pm}(\tau,\eta)\left(\frac{y-\eta+t-\tau}{2(t-\tau)}\right)\mathrm{e}^{-\frac{(y-\eta+t-\tau)^2}{4(t-\tau)}}\,\mathrm{d}\eta\\ &-\int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}}\int_0^\infty u_{yy}^{\pm}(\tau,\eta)\left(\frac{y+\eta-t+\tau}{2(t-\tau)}\right)\mathrm{e}^{-y}\,\mathrm{e}^{-\frac{(y+\eta-t+\tau)^2}{4(t-\tau)}}\,\mathrm{d}\eta\\ &-2\int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}}\int_0^\infty u_{y}^{\pm}(\tau,\eta)\left(\frac{y+\eta-t+\tau}{2(t-\tau)}\right)\mathrm{e}^{-y}\,\mathrm{e}^{-\frac{(y+\eta-t+\tau)^2}{4(t-\tau)}}\,\mathrm{d}\eta\\ &-\int_0^t \frac{\gamma(\tau)\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}}\int_0^\infty u^{\pm}(\tau,\eta)\left(\frac{y+\eta-t+\tau}{2(t-\tau)}\right)\mathrm{e}^{-y}\,\mathrm{e}^{-\frac{(y+\eta-t+\tau)^2}{4(t-\tau)}}\,\mathrm{d}\eta\\ &-\int_0^t \frac{\gamma(\tau)}{\sqrt{4\pi(t-\tau)}}\partial_y u^{\pm}(\tau,0^+)\left(\frac{y}{t-\tau}\right)\mathrm{e}^{-\frac{(y+t-\tau)^2}{4(t-\tau)}}\,\mathrm{d}\tau,\end{aligned}$$

where the last term can be written as $\tilde{\nu}_y(t,y)+\frac{1}{2}\tilde{\nu}(t,y)$ with

$$\tilde{\nu}(t,y) := 2\int_0^t \frac{\gamma(\tau)\partial_y u^{\pm}(\tau,0^+)}{\sqrt{4\pi(t-\tau)}}\,\mathrm{e}^{-\frac{(y+t-\tau)^2}{4(t-\tau)}}\,\mathrm{d}\tau.$$

All terms in $\partial_y^2 u_3^{\pm}$ including the last one are estimated similarly to what was done above. As a result, we obtain

$$\begin{aligned}\|\partial_y^2 u_3^{\pm}(t,\cdot)\|_{L^2\cap L^\infty} \leqslant{}& 2\int_0^t \frac{|\gamma(\tau)|\mathrm{d}\tau}{\sqrt{\pi(t-\tau)}}\left(\|\partial_y^2 u^{\pm}(\tau,\cdot)\|_{L^2\cap L^\infty}+\|\partial_y u^{\pm}(\tau,\cdot)\|_{L^2\cap L^\infty}\right)\mathrm{d}\tau\\ &+\int_0^t \frac{|\gamma(\tau)|\mathrm{d}\tau}{\sqrt{\pi(t-\tau)}}\|u^{\pm}(\tau,\cdot)\|_{L^2\cap L^\infty}\,\mathrm{d}\tau+\|\tilde{\nu}_y(t,\cdot)+\frac{1}{2}\tilde{\nu}(t,\cdot)\|_{L^2\cap L^\infty},\end{aligned}$$

where the following estimates from the proof of lemma 3.3 can be used:

$$\begin{aligned}\|\tilde{\nu}_y(t,\cdot)+\frac{1}{2}\tilde{\nu}(t,\cdot)\|_{L^2} \leqslant{}& \frac{1}{(8\pi)^{1/4}}\int_0^t \frac{|\gamma(\tau)||\partial_y u^{\pm}(\tau,0^+)|}{(t-\tau)^{1/4}}\,\mathrm{d}\tau\\ &+\frac{1}{(8\pi)^{1/4}}\int_0^t \frac{|\gamma(\tau)||\partial_y u^{\pm}(\tau,0^+)|}{(t-\tau)^{3/4}}\,\mathrm{d}\tau\end{aligned}$$

and

$$\|\tilde{\nu}_y(t,\cdot)+\frac{1}{2}\tilde{\nu}(t,\cdot)\|_{L^\infty} \leqslant |\gamma(t)||\partial_y u^{\pm}(t,0^+)|.$$

Combining all estimates together, we deduce that there exists $C>0$ such that

$$\begin{aligned}\|u_3^{\pm}(t,\cdot)\|_{H^2\cap W^{2,\infty}} \leqslant C\Bigg(&\int_0^t \frac{|\gamma(\tau)|}{(t-\tau)^{1/2}}\|u^{\pm}(\tau,\cdot)\|_{H^2\cap W^{2,\infty}}\,\mathrm{d}\tau\\ &+\int_0^t \frac{|\gamma(\tau)||\partial_y u^{\pm}(\tau,0^+)|}{(t-\tau)^{1/4}}\,\mathrm{d}\tau+\int_0^t \frac{|\gamma(\tau)||\partial_y u^{\pm}(\tau,0^+)|}{(t-\tau)^{3/4}}\,\mathrm{d}\tau\\ &+|\gamma(t)||\partial_y u^{\pm}(t,0^+)|\Bigg),\quad t>0.\end{aligned}\qquad(5.29)$$

By taking the limit $y \to 0^+$ in $\partial_y u_3^\pm(t,y)$ and $\partial_y^2 u_3^\pm(t,y)$ and using (3.18) in lemma 3.3, we obtain

$$\partial_y u_3^\pm(t,0^+) + \partial_y^2 u_3^\pm(t,0^+) = \tilde{\nu}_y(t,0^+) + \frac{1}{2}\tilde{\nu}(t,0^+) = -\gamma(t)\partial_y u^\pm(t,0^+), \quad t > 0. \quad (5.30)$$

Summing (5.23), (5.27), and (5.30) with the decomposition $u^\pm = u_1^\pm + u_2^\pm \pm u_3^\pm$ recovers the dynamical conditions (5.10) for $t > 0$.

Next, we run the fixed-point arguments for the fixed-point equations (5.18) in $B_\epsilon \subset X$. If $u_0^\pm$ and γ satisfy the initial bound (5.16), then there exists $C > 0$ such that

$$\|A^\pm(0)\|_X \leqslant C\delta$$

due to bounds (5.22) and (5.26), where we have also used the bound (3.11) in lemma 3.1. Furthermore, for every small $\epsilon > 0$, there is sufficiently small $\delta > 0$ such that if $u^\pm \in B_\epsilon \subset X$, then $A^\pm(u^\pm) \in B_\epsilon \subset X$; moreover $A^\pm$ are contractions on $B_\epsilon \subset X$ due to bounds (5.29), where the bound (3.11) is used again. Existence and uniqueness of the fixed points $u^\pm \in B_\epsilon \subset X$ to the fixed-point equations (5.18) follow by the Banach fixed-point theorem. Hence, the bound (5.17) is proven and the proof of the lemma is complete. □

When $u^\pm \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$ are substituted from lemma 5.7 into the integral equation (5.14), we are looking for a small solution $\gamma \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ in response to small initial data $u_0^\pm \in H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$. However, we were not able to close the fixed-point iterations unless we added the additional requirement of the spatial exponential decay of the initial data $u_0^\pm$.

The following lemma shows that the spatial exponential decay of the initial data $u_0^\pm$ is preserved in time.

Lemma 5.8. *In addition to (5.16), we assume that $u_0^\pm \in H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$ satisfy*

$$\|e^{\alpha\cdot} u_0^+\|_{W^{2,\infty}} + \|e^{\alpha\cdot} u_0^-\|_{W^{2,\infty}} < \delta, \quad (5.31)$$

for a fixed $\alpha \in (0, \frac{1}{2}]$ and that $|\gamma(t)| \to 0$ as $t \to +\infty$. The unique solutions $u^\pm \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$ of lemma 5.7 satisfy

$$\|e^{\alpha\cdot} u^+(t,\cdot)\|_{W^{2,\infty}} + \|e^{\alpha\cdot} u^-(t,\cdot)\|_{W^{2,\infty}} < \epsilon, \quad t > 0 \quad (5.32)$$

and

$$\|u^\pm(t,\cdot)\|_{W^{2,\infty}} \to 0 \quad \text{as } t \to +\infty. \quad (5.33)$$

Proof. By rearranging the heat kernels, we can rewrite (5.19), (5.20), and (5.28) as

$$e^{\alpha y} u_1^\pm(t,y) = \frac{e^{-\alpha(1-\alpha)t}}{\sqrt{4\pi t}} \int_0^\infty e^{\alpha\eta} u_0^\pm(\eta) \left[e^{-\frac{(y-\eta+(1-2\alpha)t)^2}{4t}} - e^{-(1-2\alpha)y} e^{-\frac{(y+\eta-(1-2\alpha)t)^2}{4t}} \right] d\eta, \quad (5.34)$$

$$e^{\alpha y} u_2^\pm(t,y) = \int_0^t \frac{\gamma(\tau) e^{-\alpha(1-\alpha)(t-\tau)}\, d\tau}{\sqrt{4\pi(t-\tau)}}$$
$$\times \int_0^\infty e^{-(1-\alpha)\eta} \left[e^{-\frac{(y-\eta+(1-2\alpha)(t-\tau))^2}{4(t-\tau)}} - e^{-(1-2\alpha)y} e^{-\frac{(y+\eta-(1-2\alpha)(t-\tau))^2}{4(t-\tau)}} \right] d\eta, \quad (5.35)$$

and

$$\mathrm{e}^{\alpha y}u_3^{\pm}(t,y) = -\int_0^t \frac{\gamma(\tau)\mathrm{e}^{-\alpha(1-\alpha)(t-\tau)}\,\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}}\int_0^\infty \mathrm{e}^{\alpha\eta}u^{\pm}(\tau,\eta)\left(\frac{y-\eta+t-\tau}{2(t-\tau)}\right)\mathrm{e}^{-\frac{(y-\eta+(1-2\alpha)(t-\tau))^2}{4(t-\tau)}}\,\mathrm{d}\eta$$
$$-\int_0^t \frac{\gamma(\tau)\mathrm{e}^{-\alpha(1-\alpha)(t-\tau)}\,\mathrm{d}\tau}{\sqrt{4\pi(t-\tau)}}\int_0^\infty \mathrm{e}^{\alpha\eta}u^{\pm}(\tau,\eta)\left(\frac{y+\eta-t+\tau}{2(t-\tau)}\right)\mathrm{e}^{-(1-2\alpha)y}\,\mathrm{e}^{-\frac{(y+\eta-(1-2\alpha)(t-\tau))^2}{4(t-\tau)}}\,\mathrm{d}\eta. \tag{5.36}$$

If $\alpha \in (0, \frac{1}{2}]$, the exponential function $\mathrm{e}^{-(1-2\alpha)y}$ is still bounded on $\mathbb{R}_+$, whereas $\mathrm{e}^{\alpha y}u_0^{\pm}(y)$ belongs to $W^{2,\infty}(\mathbb{R}_+)$ and satisfies the initial bound (5.31). All convolution estimates of lemma 5.7 hold true with some α-dependent constants and give the unique solution in $W^{2,\infty}(\mathbb{R}_+)$ satisfying the bound (5.32).

It remains to prove the asymptotic decay (5.33). Since

$$\|u\|_{L^\infty(\mathbb{R}_+)} \leqslant \|\mathrm{e}^{\alpha\cdot}u\|_{L^\infty(\mathbb{R}_+)},$$
$$\|u_y\|_{L^\infty(\mathbb{R}_+)} \leqslant \|(\mathrm{e}^{\alpha\cdot}u)_y\|_{L^\infty(\mathbb{R}_+)} + \alpha\|\mathrm{e}^{\alpha\cdot}u\|_{L^\infty(\mathbb{R}_+)},$$
$$\|u_{yy}\|_{L^\infty(\mathbb{R}_+)} \leqslant \|(\mathrm{e}^{\alpha\cdot}u)_{yy}\|_{L^\infty(\mathbb{R}_+)} + 2\alpha\|(\mathrm{e}^{\alpha\cdot}u)_y\|_{L^\infty(\mathbb{R}_+)} + \alpha^2\|\mathrm{e}^{\alpha\cdot}u\|_{L^\infty(\mathbb{R}_+)},$$

it is sufficient to prove the decay to zero as $t \to +\infty$ for $\mathrm{e}^{\alpha y}u^{\pm}(t,y)$ in $W^{2,\infty}(\mathbb{R}_+)$.

Similarly to the expression (5.34), one can write the integral representations for $\mathrm{e}^{\alpha y}\partial_y u_1^{\pm}$ and $\mathrm{e}^{\alpha y}\partial_y^2 u_1^{\pm}$ obtained from (4.9) and (4.10) with the exponential weights. It follows from these representations that if $\alpha \in (0, \frac{1}{2}]$, there exists an α-independent $C > 0$ such that

$$\|\mathrm{e}^{\alpha\cdot}u_1^{\pm}(t,\cdot)\|_{W^{2,\infty}} \leqslant C\,\mathrm{e}^{-\alpha(1-\alpha)t}\|\mathrm{e}^{\alpha\cdot}u_0^{\pm}\|_{W^{2,\infty}}, \tag{5.37}$$

where the upper bound decays to zero as $t \to +\infty$ exponentially fast.

Similarly, it follows from (5.35) and the integral representations for $\mathrm{e}^{\alpha y}\partial_y u_2^{\pm}$ and $\mathrm{e}^{\alpha y}\partial_y^2 u_2^{\pm}$ obtained from (5.24) and (5.25) that there exists an α-independent $C > 0$ such that

$$\|\mathrm{e}^{\alpha\cdot}u_2^{\pm}(t,\cdot)\|_{W^{2,\infty}} \leqslant C\left(\int_0^t |\gamma(\tau)|\mathrm{e}^{-\alpha(1-\alpha)(t-\tau)}\,\mathrm{d}\tau + \int_0^t \frac{|\gamma(\tau)|\mathrm{e}^{-\alpha(1-\alpha)(t-\tau)}}{\sqrt{4\pi(t-\tau)}}\,\mathrm{d}\tau + |\gamma(t)|\right), \tag{5.38}$$

where we have used the weighted representation for $\nu(t,y)$ in (3.17):

$$\mathrm{e}^{\alpha y}\nu(t,y) = 2\int_0^t \frac{\gamma(\tau)\mathrm{e}^{-\alpha(1-\alpha)(t-\tau)}}{\sqrt{4\pi(t-\tau)}}\,\mathrm{e}^{-\frac{(y+(1-2\alpha)(t-\tau))^2}{4(t-\tau)}}\,\mathrm{d}\tau$$

and the representation

$$\mathrm{e}^{\alpha y}\nu_y(t,y) + \frac{1}{2}\mathrm{e}^{\alpha y}\nu(t,y) = \mathrm{e}^{-\frac{1}{2}(1-2\alpha)y-\frac{t}{4}}\tilde{\nu}_y(t,y)$$

with $\tilde{\nu}_y$ satisfying the bound (3.28). By lemma 3.2, the upper bound in (5.38) decays to zero as $t \to +\infty$ since $\gamma \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $|\gamma(t)| \to 0$ as $t \to +\infty$.

Finally, it follows from (5.36) with (3.3) and Young's inequality (3.7) with $p = r = \infty$ and $q = 1$ that

$$\|e^{\alpha\cdot} u_3^{\pm}(t,\cdot)\|_{L^\infty} \leqslant 2 \sup_{t\in\mathbb{R}_+} \|e^{\alpha\cdot} u^{\pm}(t,\cdot)\|_{L^\infty} \left[\alpha \int_0^t |\gamma(\tau)| e^{-\alpha(1-\alpha)(t-\tau)}\, d\tau + \int_0^t \frac{|\gamma(\tau)| e^{-\alpha(1-\alpha)(t-\tau)}}{\sqrt{\pi(t-\tau)}}\, d\tau\right].$$

The bounds for $\|e^{\alpha\cdot}\partial_y u_3^{\pm}(t,\cdot)\|_{L^\infty}$ and $\|e^{\alpha\cdot}\partial_y^2 u_3^{\pm}(t,\cdot)\|_{L^\infty}$ are similar and follow from the integral representations for $e^{\alpha y}\partial_y u_3^{\pm}$ and $e^{\alpha y}\partial_y^2 u_3^{\pm}$ obtained in the proof of lemma 5.7 after adding exponential weights and using (3.3) and (3.7). As a result, we conclude again that there exists an α-independent $C > 0$ such that

$$\|e^{\alpha\cdot} u_3^{\pm}(t,\cdot)\|_{W^{2,\infty}} \leqslant C \sup_{t\in\mathbb{R}_+} \|e^{\alpha\cdot} u^{\pm}(t,\cdot)\|_{W^{2,\infty}} \left(\int_0^t |\gamma(\tau)| e^{-\alpha(1-\alpha)(t-\tau)}\, d\tau + \int_0^t \frac{|\gamma(\tau)| e^{-\alpha(1-\alpha)(t-\tau)}}{\sqrt{\pi(t-\tau)}}\, d\tau + |\gamma(t)|\right), \tag{5.39}$$

for every $u^{\pm} \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$ of lemma 5.7. By lemma 3.2, the upper bound in (5.39) decays to zero as $t \to +\infty$ since $\gamma \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $|\gamma(t)| \to 0$ as $t \to +\infty$.

Combining all estimates together, we obtain the asymptotic decay (5.33). □

Remark 5.9. Due to the exponential decay with $\alpha \in (0, \frac{1}{2}]$, we also have the bound

$$\|u\|_{H^2(\mathbb{R}_+)} \leqslant C_\alpha \|e^{\alpha\cdot} u\|_{W^{2,\infty}(\mathbb{R}_+)},$$

which implies that $\|u^{\pm}(t,\cdot)\|_{H^2} \to 0$ as $t \to \infty$.

The final lemma gives the existence of a unique solution to the integral equation (5.14) for $\gamma \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, where $u^{\pm} \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$ are substituted from lemmas 5.7 and 5.8 into the integral equation (5.14) and the initial data $u_0^{\pm} \in H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$ satisfy the bounds (5.16) and (5.31).

Lemma 5.10. *Fix $\alpha \in (0, \frac{1}{2}]$ and consider the integral equation (5.14) with the unique solutions $u^{\pm} \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$ defined in lemmas 5.7 and 5.8 that depend on (small) $\gamma \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. For every $\tilde{\epsilon} > 0$ (small enough), there is $\tilde{\delta} > 0$ such that for every $u_0^{\pm} \in H^2(\mathbb{R}_+ \cap W^{2,\infty}(\mathbb{R}_+))$ satisfying*

$$\|u_0^+\|_{H^2\cap W^{2,\infty}} + \|u_0^-\|_{H^2\cap W^{2,\infty}} + \|e^{\alpha\cdot} u_0^+\|_{W^{2,\infty}} + \|e^{\alpha\cdot} u_0^-\|_{W^{2,\infty}} \leqslant \tilde{\delta} \tag{5.40}$$

and the boundary conditions $u_0^{\pm}(0) = 0$ and $u_0^{+\prime}(0) + u_0^{-\prime}(0) = 0$, there exists a unique solution $\gamma \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ of the integral equation (5.14) satisfying

$$\|\gamma\|_{L^\infty\cap L^1} \leqslant \tilde{\epsilon} \tag{5.41}$$

and $|\gamma(t)| \to 0$ as $t \to +\infty$.

Proof. We rewrite the integral equation in lemma 5.6 as the fixed-point equation associated with the following integral operator:

$$\gamma = \mathcal{A}(\gamma) := \gamma_1 + \gamma_2 + \gamma_3, \tag{5.42}$$

where

$$\gamma_1(t) = -\frac{1}{\sqrt{4\pi t}}\int_0^{\infty}\left[u_0^{+\prime}(\eta) + u_0^{-\prime}(\eta) + \frac{1}{2}u_0^{+}(\eta) + \frac{1}{2}u_0^{-}(\eta)\right]\left(\frac{\eta+t}{2t}\right)\mathrm{e}^{-\frac{(\eta-t)^2}{4t}}\,\mathrm{d}\eta,$$

$$\gamma_2(t) = -\frac{1}{2}\gamma(t)\left[u_y^{+}(t,0^+) - u_y^{-}(t,0^+)\right]$$

$$-\frac{1}{2}\int_0^t \frac{\gamma(\tau)\mathrm{e}^{-\frac{t-\tau}{4}}}{\sqrt{4\pi(t-\tau)}}\left[u_y^{+}(\tau,0^+) - u_y^{-}(\tau,0^+)\right]\mathrm{d}\tau,$$

$$\gamma_3(t) = -\int_0^t \frac{\gamma(\tau)}{\sqrt{4\pi(t-\tau)}}\int_0^{\infty}\left[u_{yy}^{+} - u_{yy}^{-} + \frac{1}{2}u_y^{+} - \frac{1}{2}u_y^{-}\right](\tau,\eta)\left(\frac{\eta+t-\tau}{2(t-\tau)}\right)\mathrm{e}^{-\frac{(\eta-t+\tau)^2}{4(t-\tau)}}\,\mathrm{d}\eta\,\mathrm{d}\tau.$$

The fixed-point equation (5.42) is considered in a small ball $B_{\tilde{\epsilon}} \subset L^1(\mathbb{R}_+) \cap L^{\infty}(\mathbb{R}_+)$ of radius $\tilde{\epsilon} > 0$, where $u_0^{\pm} \in H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$ are given and satisfy (5.40) and $u^{\pm} \in L^{\infty}(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$ are defined in lemmas 5.7 and 5.8 such that $\tilde{\delta}$ and $\tilde{\epsilon}$ in (5.40) and (5.41) are smaller than δ in (5.16). We analyze hereafter each term in the definition of $\mathcal{A}(\gamma)$ in $L^1(\mathbb{R}_+) \cap L^{\infty}(\mathbb{R}_+)$.

Since the boundary conditions $u_0^{\pm}(0) = 0$ and $u_0^{+\prime}(0) + u_0^{-\prime}(0) = 0$ are satisfied by the initial condition $u_0^{\pm} \in W^{2,\infty}(\mathbb{R}_+)$, we can use the equivalent form (3.33) in lemma 3.5 and express $\gamma_1(t)$ in the form:

$$\gamma_1(t) = -\frac{1}{\sqrt{4\pi t}}\int_0^{\infty} f(\eta)\mathrm{e}^{-\frac{(\eta-t)^2}{4t}}\,\mathrm{d}\eta, \tag{5.43}$$

where

$$f(\eta) = u_0^{+\prime\prime}(\eta) + u_0^{-\prime\prime}(\eta) + \frac{3}{2}u_0^{+\prime}(\eta) + \frac{3}{2}u_0^{-\prime}(\eta) + \frac{1}{2}u_0^{+}(\eta) + \frac{1}{2}u_0^{-}(\eta).$$

It follows from the first identity in (3.2) that there exists $C > 0$ such that

$$|\gamma_1(t)| \leqslant C\left(\|u_0^{+}\|_{W^{2,\infty}} + \|u_0^{-}\|_{W^{2,\infty}}\right), \quad t > 0. \tag{5.44}$$

However, there is no bound on $\|\gamma_1\|_{L^1}$ unless we add some weights (e.g. the exponential weight in lemma 5.8) on the initial conditions $u_0^{\pm}$ and rewrite (5.43) in the form:

$$\gamma_1(t) = -\frac{\mathrm{e}^{-\alpha(1-\alpha)t}}{\sqrt{4\pi t}}\int_0^{\infty}\mathrm{e}^{\alpha\eta}f(\eta)\mathrm{e}^{-\frac{(\eta-(1-2\alpha)t)^2}{4t}}\,\mathrm{d}\eta.$$

Now, thanks to the exponential factor $\mathrm{e}^{-\alpha(1-\alpha)t}$ decaying to zero as $t \to +\infty$, we obtain

$$\|\gamma_1\|_{L^1} \leqslant \frac{1}{\alpha(1-\alpha)}\|\mathrm{e}^{\alpha\cdot}f\|_{L^{\infty}},$$

so there exists a positive α-dependent constant C_{α} such that

$$\|\gamma_1\|_{L^1} \leqslant C_{\alpha}\left(\|\mathrm{e}^{\alpha\cdot}u_0^{+}\|_{W^{2,\infty}} + \|\mathrm{e}^{\alpha\cdot}u_0^{-}\|_{W^{2,\infty}}\right). \tag{5.45}$$

Moreover, $|\gamma_1(t)| \leqslant e^{-\alpha(1-\alpha)t}\|e^{\alpha\cdot} f\|_{L^\infty}$, hence $|\gamma_1(t)| \to 0$ as $t \to +\infty$.

For $\gamma_2(t)$, we obtain

$$\|\gamma_2\|_{L^1\cap L^\infty} \leqslant \frac{1}{2}\left(1 + \int_0^\infty \frac{e^{-\frac{t}{4}}}{\sqrt{4\pi t}}\,dt\right)\|\gamma\|_{L^1\cap L^\infty} \times \sup_{t\in\mathbb{R}_+}\left(\|u^+(t,\cdot)\|_{W^{1,\infty}} + \|u^-(t,\cdot)\|_{W^{1,\infty}}\right), \tag{5.46}$$

where the expression in brackets is a finite constant. No weights are needed to estimate γ_2 in $L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. Moreover, by lemma 3.2, if $|\gamma(t)| \to 0$ as $t \to +\infty$ and $u^\pm \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$, then $|\gamma_2(t)| \to 0$ as $t \to +\infty$.

For $\gamma_3(t)$, we use (3.2) and (3.3), and Young's inequality (3.7) with $p = q = 2$ and $r = \infty$ and obtain

$$|\gamma_3(t)| \leqslant C\left(\int_0^t \frac{|\gamma(\tau)|d\tau}{(t-\tau)^{1/4}} + \int_0^t \frac{|\gamma(\tau)|d\tau}{(t-\tau)^{3/4}}\right)\sup_{t\in\mathbb{R}_+}\left(\|u^+(t,\cdot)\|_{H^2} + \|u^-(t,\cdot)\|_{H^2}\right), \quad t > 0,$$

which is bounded by lemma 3.1 if $\gamma \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. To get the bound on $\|\gamma_3\|_{L^1}$, we add the exponential weight and rewrite $\gamma_3(t)$ in the equivalent form:

$$\gamma_3(t) = -\int_0^t \frac{\gamma(\tau)e^{-\alpha(1-\alpha)(t-\tau)}}{\sqrt{4\pi(t-\tau)}}\int_0^\infty e^{\alpha\eta} g(\tau,\eta)\left(\frac{\eta+t-\tau}{2(t-\tau)}\right)e^{-\frac{(\eta-(1-2\alpha)(t-\tau))^2}{4(t-\tau)}}\,d\eta\,d\tau,$$

where

$$g(\tau,\eta) = u^+_{yy}(\tau,\eta) - u^-_{yy}(\tau,\eta) + \frac{1}{2}u^+_y(\tau,\eta) - \frac{1}{2}u^-_y(\tau,\eta).$$

By using Young's inequality (3.7) with $p = r = \infty$ and $q = 1$ and Young's inequality (3.10) with either $p = r = 1$ or $p = r = \infty$ and $q = 1$, we now obtain

$$\|\gamma_3\|_{L^1\cap L^\infty} \leqslant \left(\frac{1}{\alpha} + \int_0^\infty \frac{e^{-\alpha(1-\alpha)t}}{\sqrt{\pi t}}\,dt\right)\|\gamma\|_{L^1\cap L^\infty}\sup_{t\in\mathbb{R}_+}\|e^{\alpha\cdot}g(t,\cdot)\|_{L^\infty},$$

so there exists a positive α-dependent constant C_α such that

$$\|\gamma_3\|_{L^1\cap L^\infty} \leqslant C_\alpha\|\gamma\|_{L^1\cap L^\infty}\sup_{t\in\mathbb{R}_+}\left(\|e^{\alpha\cdot}u^+(t,\cdot)\|_{W^{2,\infty}} + \|e^{\alpha\cdot}u^-(t,\cdot)\|_{W^{2,\infty}}\right). \tag{5.47}$$

Moreover, by lemma 3.2, if $|\gamma(t)| \to 0$ as $t \to +\infty$ and $u^\pm \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$, then $|\gamma_3(t)| \to 0$ as $t \to +\infty$.

Next, we run the fixed-point arguments for the fixed-point equation (5.42) in $B_{\tilde{\epsilon}} \subset L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. If $u_0^\pm$ satisfy the initial bound (5.40) and $\gamma \in B_{\tilde{\epsilon}}$, then the solutions $u^\pm \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$ in lemmas 5.7 and 5.8 satisfy the bounds (5.17) and (5.32) if $\tilde{\delta} \leqslant \delta$ and $\tilde{\epsilon} \leqslant \delta$. The bounds (5.44)–(5.47) imply that $\mathcal{A}(\gamma) \in B_{\tilde{\epsilon}}$ for sufficiently small $\tilde{\delta}$ and given small $\tilde{\epsilon}$. Moreover, $\mathcal{A}$ is a contraction on $B_{\tilde{\epsilon}} \subset L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ due to the same bounds (5.46), and (5.47) and the smallness of the solutions $u^\pm \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+))$.

Existence and uniqueness of the fixed point $\gamma \in B_{\tilde{\epsilon}} \subset L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ to the fixed-point equation (5.42) follows from the Banach fixed-point theorem. Hence, the bound (5.41) is proven. Moreover, the decay $|\gamma(t)| \to 0$ as $t \to +\infty$ is preserved by the fixed-point iterations. The proof of the lemma is complete. □

Proof of theorem 2.6. The existence, uniqueness, and continuous dependence of the solutions $u^{\pm}$ to the boundary-value problems (5.7) with (5.8) and (5.10) is obtained from lemmas 5.7, 5.8, and 5.10 as follows. For a fixed ϵ in (5.17) and (5.32), there exists a small δ in (5.16) and (5.31), for which we select $\tilde{\epsilon}$ in (5.41) such that $\tilde{\epsilon} \leqslant \delta$. By lemma 5.10, there exists $\tilde{\delta}$ in (5.40) and, if necessary, we reduce $\tilde{\delta}$ so that $\tilde{\delta} \leqslant \delta$. Then, the results of lemmas 5.7, 5.8, and 5.10 hold simultaneously for the initial conditions satisfying (5.40), which is obtained from (2.18) by the transformations (5.1) and (5.6). The bound (2.19) follows from $u^{\pm} \in B_{\epsilon} \subset X$ in the proof of lemma 5.7 and the transformations (5.1) and (5.6). The decay (2.20) follows from the decay (5.33). By lemmas 5.7 and 5.8, the solutions belong to the spaces (2.21) and (2.22).

The interface condition (5.9) follows from (5.8) and (5.10). The interface condition (5.3) of lemma 5.1 follows from the transformation (5.6) and the dynamical condition (5.10). The positivity conditions (5.5) follow from the decomposition (5.1) and smallness of u in $W^{1,\infty}(\mathbb{R})$ similarly to the proof of theorem 2.3. □

6. Numerical simulations

Here we simulate numerically the boundary-value problem (5.2) completed with the dynamical equation (5.3) and the interface condition (5.4). The interface location $\xi(t)$ satisfies $\xi(0) = 0$. By using $\gamma(t) = \xi'(t)$, $W_0'(y) = \mathrm{e}^{-|y|}$, and the new variables $u^{\pm}(t, y)$ in (5.6) the boundary-value problem is rewritten as the system (5.7)–(5.10). Although this system can also be simulated numerically, we found more convenient to reformulate the boundary condition (5.8) as a Neumann condition for a single function. By introducing new variables

$$v^{\pm}(t, y) = u(t, y) \mp u(t, -y), \quad y > 0 \tag{6.1}$$

we rewrite the boundary-value problem as the following system of two coupled equations:

$$\begin{cases} v_t^+ = v_y^+ + v_{yy}^+ + \gamma v_y^-, & y > 0, \\ v_t^- = v_y^- + v_{yy}^- + \gamma v_y^+ + 2\gamma\,\mathrm{e}^{-y}, & y > 0, \end{cases} \tag{6.2}$$

subject to the boundary conditions

$$\begin{cases} v^{\pm}(t, 0) = 0, \\ v_y^-(t, 0) = 0, \\ v^{\pm}(t, y) \to 0 \quad \text{as } y \to \infty, \end{cases} \tag{6.3}$$

the interface condition

$$v_y^+(t, 0) + v_{yy}^+(t, 0) = 0, \tag{6.4}$$

and the dynamical condition

$$\gamma(t) = -\frac{v_{yy}^-(t, 0)}{2 + v_y^+(t, 0)}. \tag{6.5}$$

If $v^-(0, y) = 0$ initially, then $\gamma(t) = 0$ and $v^-(t, y) = 0$ are preserved in the time evolution of (6.2), (6.3), and (6.5). In this case, the variable $v^+(t, y)$ satisfies the boundary-value problem

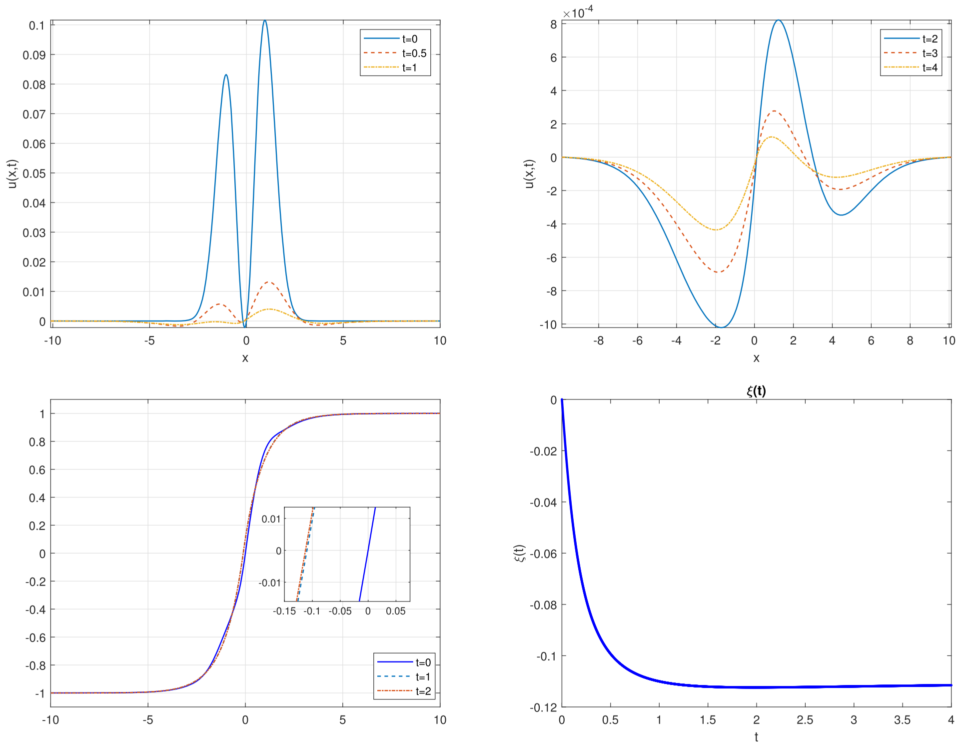

Figure 1. Numerical simulations for the initial conditions (6.6). Top: plot of $u(t, x)$ versus x for $t = 0, 0.5, 1$ (left) and $t = 2, 3, 4$ (right). Bottom: plot of $w(t, x)$ versus x for $t = 0, 1, 2$ (left) and plot of $\gamma(t)$ versus t (right).

(4.2), which is analyzed in theorem 2.3 for the odd perturbations to the viscous shock. In what follows, we consider the general case of $v^-(0, y) \neq 0$ which is analyzed in theorem 2.6.

We use the central-difference Crank–Nicholson scheme for numerical simulations of system (6.2)–(6.5). The numerical scheme is described in appendix. The spatial domain is chosen at $[0, L]$ with $L = 10$ and the equally spaced grid has the spacing $h = 0.05$. Simulations were performed with the time step $\tau = 0.001$.

Figure 1 reports the results of numerical simulations for the initial condition with the Gaussian decay:

$$\begin{cases} v^+(0, y) = 0.1(y - 0.5y^2)\mathrm{e}^{-y^2}, \\ v^-(0, y) = 0.5y^2\,\mathrm{e}^{-y^2}, \end{cases} \tag{6.6}$$

where the coefficients are carefully selected to satisfy the boundary conditions in (6.3) and the interface condition (6.4) at $t = 0$. After the solution $v^{\pm}(t, y)$ to the evolution problem (6.2) is approximated numerically, the function $u(t, y)$ is recovered from (6.1) and then plotted versus $x := \xi(t) + y$, where $\xi(t)$ is found from numerical integration of $\xi'(t) = \gamma(t)$ with $\gamma(t)$ approximated from (6.5).

Snapshots of $u(t, x)$ versus x for different values of t (top panels of figure 1) show that the solution quickly decays to zero in the supremum norm. Although the perturbation u is sign-indefinite, the values of u are smaller compared to the values of W_0 in the viscous shock, hence

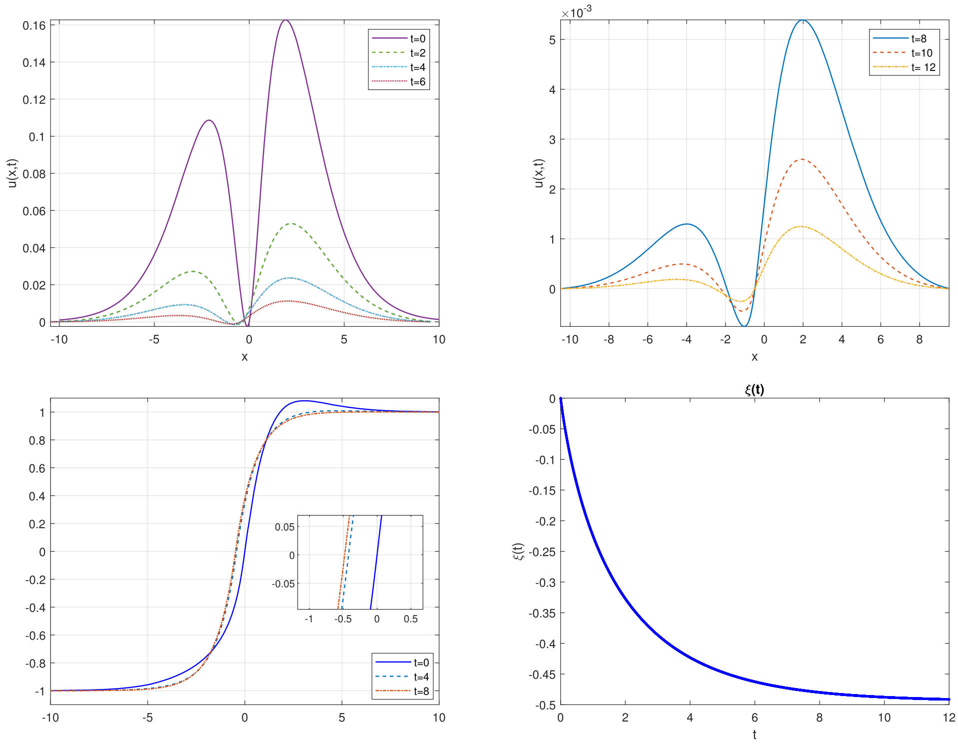

Figure 2. The same as in figure 1 but for the initial condition (6.7).

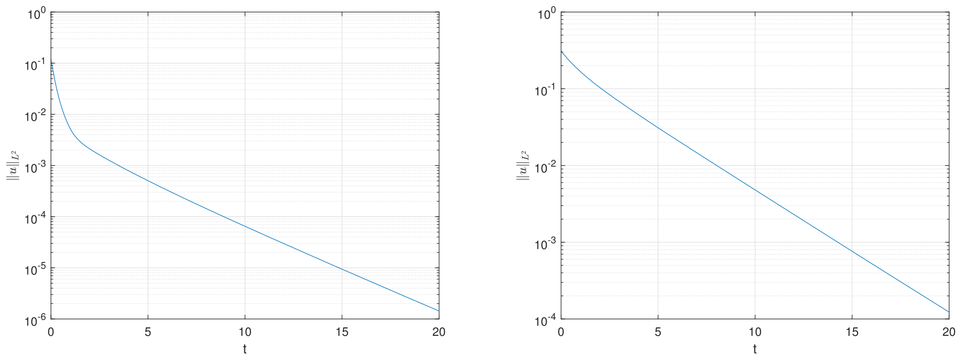

Figure 3. L^2 norm of the solution $u(t, \cdot)$ over time t for the numerical simulations on figure 1 (left) and figure 2 (right).

$w = W_0 + u$ remains positive (negative) to the right (left) of the interface located at $x = \xi(t)$. The snapshots of w are shown on the bottom left panel for $t = 0, 1, 2$ with the insert showing the profile of w near the interface. The bottom right panel shows the position of the interface ξ versus t. It quickly relaxes to the equilibrium position at $\xi_\infty \approx -0.11$. This behavior agrees with the asymptotic stability result of theorem 2.6 suggesting existence of $\xi_\infty \in \mathbb{R}$ such that $\xi(t) \to \xi_\infty$ as $t \to +\infty$ (remark 2.10). The value of ξ_∞ depends on the initial conditions. We

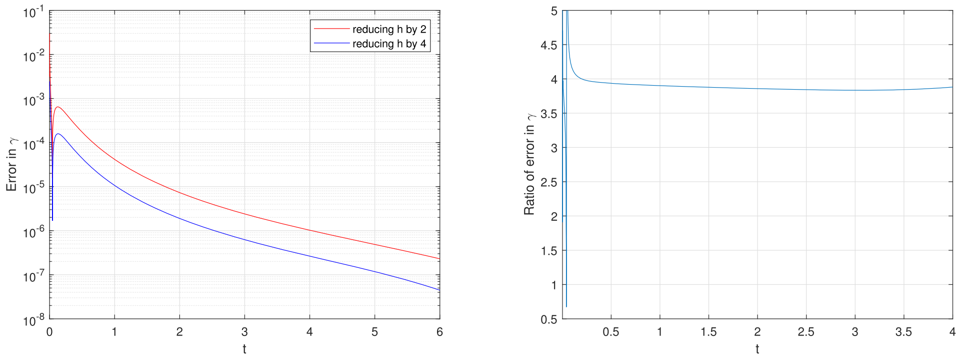

Figure 4. Distance between two numerical approximations of $\gamma(t_k)$ with double number of grid points (left) and their ratio showing the quadruple reduction (right).

have checked that if the sign of $v^-(0, y)$ in (6.6) is changed to the opposite, then the sign of ξ_∞ changes to the opposite.

Figure 2 reports similar results for the initial conditions with the exponential decay:

$$\begin{cases} v^+(0, y) = 0.1(y + 0.5y^2)e^{-y}, \\ v^-(0, y) = 0.5y^2\, e^{-y}, \end{cases} \tag{6.7}$$

which again satisfies the boundary conditions in (6.3) and the interface condition (6.4) at $t = 0$. Dynamics of the perturbation u in time t for the initial data (6.7) resembles the same dynamics as for the initial condition (6.6). However, the relaxation time is slower for the exponentially decaying perturbations, hence the time window is extended from $T = 4$ on figure 1 to $T = 12$ on figure 2. Nevertheless, the interface $\xi(t)$ moves to the left and relaxes to some equilibrium position $\xi_\infty \approx -0.49$.

Comparison between figures 1 and 2 illustrates the role of weights in the asymptotic stability of viscous shocks. The greater is the spatial decay of the initial perturbations, the quicker is the relaxation dynamics of perturbed solutions to the traveling viscous shocks. The proof of theorem 2.6 is obtained under the exponential decay but similar results are likely to be available for other (e.g., algebraic) decay conditions. Figure 3 shows the decay of $\|u(t, \cdot)\|_{L^2}$ over the extended time interval for the same initial conditions (6.6) and (6.7). The rate of decay is slower in the case of exponential weights (right) compared to the case of Gaussian weights (left).

In order to check the convergence of the numerical method, we have reduced the grid spacing h by the factor of 2 and the time step τ by the factor of 4. Figure 4 shows the errors of the numerical simulations (left) computed as a difference between $\gamma(t_k)$ obtained between two consequent approximations with double number of grid points. As we can see from their ratio (right), the errors reduce by the factor of 4, which is in agreement with the second-order accuracy of the central-difference Crank–Nicholson method.

7. Conclusion

We have considered the modular Burgers equation, where the advective nonlinearity produces singularities related to the modular functions. For the class of viscous shocks with a single interface at the zero value of the modular function, we have proven their asymptotic stability under a general perturbation with the spatial exponential decay at infinity. This work may open up new directions of research.

First, it is interesting to consider the existence and nonlinear dynamics of the viscous shocks with multiple interfaces. It is expected that the perturbations at the tails will behave similarly but the dynamics will be complicated by the internal interactions among the interfaces. The periodic waves with an infinite number of interfaces located at the equal distance is another interesting case for further studies, e.g., see [12, 13].

Second, one can wonder if the exponential weight requirement on the initial perturbations can be relaxed or completely removed. It may be relatively easy to replace the exponential weights with the algebraic weights as done in [2]. However, we are not able to close the fixed-point arguments for the perturbations to the viscous shocks in $H^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$, hence new ideas for analysis are needed to remove the weights.

Finally, the Burgers equation with more singular nonlinearity, e.g. given by the logarithmic functions, arises in the applications of granular chains [11]. It is definitely interesting if the asymptotic stability of viscous shocks can be proven for the logarithmic Burgers equations. Unfortunately, our methods rely on the reductions provided by the modular nonlinearity and cannot be extended to the case of logarithmic or other singular nonlinearities.

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Appendix Central-difference Crank–Nicholson method for system (6.2)

The spatial domain of system (6.2) is discretized at the points $y_n = nh$ with equal step size h for $n = 1, \dots, N$. It follows from the boundary conditions (6.3) that $v^{\pm}(t, y_0) = 0$ at $y_0 = 0$. Although the problem is unbounded in one direction, one can truncate the half-line on the finite interval $[0, L]$ with sufficiently large L and $y_{N+1} = L = (N+1)h$ and apply the Dirichlet condition $v^{\pm}(t, y_{N+1}) = 0$ at the end point. This approach of truncation is commonly adopted for the numerical approximation of evanescent waves in engineering [5] as the Dirichlet condition does not provide large errors due to reflections if the waves have fast spatial decay.

At each time level $t_k = k\tau$ with the time step τ, we approximate the spatial derivatives with the second-order central differences as follows:

$$v_y^{\pm}(t_k, y_n) = \frac{v_{n+1,k}^{\pm} - v_{n-1,k}^{\pm}}{2h}, \tag{A.1}$$

$$v_{yy}^{\pm}(t_k, y_n) = \frac{v_{n+1,k}^{\pm} - 2v_{n,k}^{\pm} + v_{n-1,k}^{\pm}}{h^2}, \tag{A.2}$$

where $v_{n,k}$ is a numerical approximation of $v(t_k, x_n)$. The Neumann condition $v_y^-(t, 0) = 0$ is modeled with the virtual grid point $y_{-1} = -h$ so that $v_{-1,k}^- = v_{1,k}^-$. By using the virtual grid point y_{-1} and the interface condition (6.4), we also express

$$v^{+}_{-1,k} = -\frac{2+h}{2-h} v^{+}_{1,k}, \tag{A.3}$$

after which the approximation of $\gamma(t_k)$ is obtained from (6.5) as follows:

$$\gamma(t_k) = -\frac{(2-h)v^{-}_{1,k}}{hv^{+}_{1,k} + h^2(2-h)}. \tag{A.4}$$

We use the Crank–Nicholson method in order to perform steps in time for the evolution system (6.2). For each equation of the form $\frac{\mathrm{d}v}{\mathrm{d}t} = f(v)$, the Crank–Nicholson method yields:

$$v_{k+1} - \frac{\tau}{2} f(v_{k+1}) = v_k + \frac{\tau}{2} f(v_k), \tag{A.5}$$

where f for the first and second equations of system (6.2) take the form:

$$[f^+]_{n,k} = \frac{v^{+}_{n+1,k} - v^{+}_{n-1,k}}{2h} + \frac{v^{+}_{n+1,k} - 2v^{+}_{n,k} + v^{+}_{n-1,k}}{h^2} + \gamma_k \frac{v^{-}_{n+1,k} - v^{-}_{n-1,k}}{2h}$$

and

$$[f^-]_{n,k} = \frac{v^{-}_{n+1,k} - v^{-}_{n-1,k}}{2h} + \frac{v^{-}_{n+1,k} - 2v^{-}_{n,k} + v^{-}_{n-1,k}}{h^2} + \gamma_k \frac{v^{+}_{n+1,k} - v^{+}_{n-1,k}}{2h} + 2\gamma_k\, \mathrm{e}^{-y_n}.$$

For simplicity, we use γ_k at the time level k on both sides of equation (A.5). Thus, in order to advance the solution of (6.2) to the next time level $k+1$, we have to solve the following algebraic system:

$$L(-\tau)\mathbf{v}_{k+1} = L(\tau)\mathbf{v}_k + \mathbf{c}_k, \tag{A.6}$$

where $\mathbf{v}_k$ and $\mathbf{c}_k$ are the $2N$ vectors with the elements

$$v_{n,k} = v^{+}_{n,k}, \quad 1 \leqslant n \leqslant N, \quad \text{and} \quad v_{n,k} = v^{-}_{n,k}, \quad N+1 \leqslant n \leqslant 2N, \tag{A.7}$$

and

$$c_{n,k} = 0, \quad 1 \leqslant n \leqslant N, \quad \text{and} \quad c_{n,k} = 2\tau\gamma_k\, \mathrm{e}^{-y_n}, \quad N+1 \leqslant n \leqslant 2N, \tag{A.8}$$

and $L(\tau)$ is the $(2N \times 2N)$ matrix defined in the block form:

$$L = \left[\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right], \tag{A.9}$$

with A and B are $(N \times N)$ three-diagonal matrices with the elements:

$$a_{j,j} = 1 - \frac{\tau}{h^2}, \qquad a_{j,j+1} = \frac{\tau}{2}\left(\frac{1}{2h} + \frac{1}{h^2}\right), \qquad a_{j,j-1} = \frac{\tau}{2}\left(-\frac{1}{2h} + \frac{1}{h^2}\right)$$

and

$$b_{j,j} = 0, \qquad b_{j,j+1} = \frac{\tau}{4h}\gamma_k, \qquad b_{j,j-1} = -\frac{\tau}{4h}\gamma_k.$$

The solution $u(t, y)$ to the boundary-value problem (5.2) for $y \in \mathbb{R}$ is recovered from solution $v^{\pm}(t, y)$ to system (6.2) for $y \in \mathbb{R}_+$ by using the transformation (6.1). Finally, we use $y = x - \xi(t)$ with $\xi(t) := \int_0^t \gamma(t')\mathrm{d}t'$ in order to display $u(t, x)$ versus x on $\mathbb{R}$.

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