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Moving gap solitons in periodic potentials

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SUMMARY

We address the existence of moving gap solitons (traveling localized solutions) in the Gross–Pitaevskii equation with a small periodic potential. Moving gap solitons are approximated by the explicit solutions of the coupled-mode system. We show, however, that exponentially decaying traveling solutions of the Gross–Pitaevskii equation do not generally exist in the presence of a periodic potential due to bounded oscillatory tails ahead and behind the moving solitary waves. The oscillatory tails are not accounted in the coupled-mode formalism and are estimated by using techniques of spatial dynamics and local center-stable manifold reductions. Existence of bounded traveling solutions of the Gross–Pitaevskii equation with a single bump surrounded by oscillatory tails on a large interval of the spatial scale is proven by using these techniques.

KEY WORDS: spatial dynamics; homoclinic orbits; moving gap solitons; Gross–Pitaevskii equation; periodic potentials

1. INTRODUCTION

Moving gap solitons are thought to be steadily traveling localized solutions of nonlinear partial differential equations (PDEs) with spatially periodic coefficients. The name of gap solitons comes from the fact that parameters of *stationary* localized solutions reside in the spectral gap of the

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associated Schrödinger operator. Existence of stationary solutions can be studied by separation of the time and space variables and reduction of the problem to an elliptic semi-linear equation. Since the variables are not separable for *traveling* solutions, little is known about the existence of moving gap solitons.

We address stationary and traveling localized solutions in the context of the Gross–Pitaevskii equation with an external periodic potential:

$$\mathrm{i}E_t = -E_{xx} + V(x)E + \sigma|E|^2 E \tag{1}$$

where $E(x,t): \mathbb{R}\times\mathbb{R}_+ \mapsto \mathbb{C}$, $V(x): \mathbb{R}\mapsto\mathbb{R}$, and $\sigma=\pm 1$. The Gross–Pitaevskii equation (1) is derived for the mean-field amplitude of the Bose–Einstein condensate placed in the optical lattice trap $V(x)$, where σ is the normalized scattering length [1]. Stationary solutions of the Gross–Pitaevskii equation are found from the solutions of the nonlinear differential equation:

$$-\phi''(x) + V(x)\phi(x) + \sigma|\phi(x)|^2\phi(x) = \omega\phi(x) \tag{2}$$

where $\phi(x): \mathbb{R}\mapsto\mathbb{C}$, $\omega\in\mathbb{R}$, and the exact reduction $E(x,t)=\phi(x)\mathrm{e}^{-\mathrm{i}\omega t}$ is used. Localized stationary solutions of (2) were proved to exist in Theorem 9.3 of [2] for $\sigma=-1$ and in Theorem 1 of [3] for any σ using a variational technique.

Theorem 1 (*Stuart* [2], *Pankov* [3])
Let $V(x)$ be a real-valued and periodic potential in $L^\infty(\mathbb{R})$. Let ω be in a finite gap of the purely continuous spectrum of $L=-\partial_x^2+V(x)$ in $L^2(\mathbb{R})$. There exists a non-trivial solution $\phi(x)$ of the differential equation (2) in $H^1(\mathbb{R})$.

In [4], we have obtained a more precise information on properties of the stationary solution $\phi(x)$ by working with a small potential $V(x)$, when the spectrum of L exhibits a sequence of narrow gaps bifurcating near *resonance* points $\omega=\omega_n=n^2/4$, $n\in\mathbb{N}$. We have justified the use of the stationary coupled-mode equations which have been used in the physics literature [5] for explicit approximations of stationary gap solitons. In this paper, we shall investigate whether the time-dependent coupled-mode equations can be used for approximation of moving gap solitons in the framework of the Gross–Pitaevskii equation (1).

Assumption 1
Let $V=\varepsilon W(x)$, where ε is a small parameter and $W(x)$ is a smooth, real-valued, ε-independent, and 2π-periodic function with zero mean and symmetry $W(-x)=W(x)$ on $\mathbb{R}$.

According to Assumption 1, the function $W(x)$ can be represented by the Fourier series:

$$W(x) = \sum_{m\in\mathbb{Z}} w_{2m}\mathrm{e}^{\mathrm{i}mx} \text{ such that } \sum_{m\in\mathbb{Z}} (1+m^2)^s |w_{2m}|^2 < \infty \quad \forall s\in\mathbb{R} \tag{3}$$

where $w_0=0$ and $w_{2m}=w_{-2m}=\bar{w}_{2m}$, $\forall m\in\mathbb{N}$. A formal asymptotic solution of the Gross–Pitaevskii equation (1) for small values of ε is represented in the form [6, 7]:

$$E(x,t) = \varepsilon^{1/2}[a(\varepsilon x, \varepsilon t)\mathrm{e}^{\mathrm{i}nx/2} + b(\varepsilon x, \varepsilon t)\mathrm{e}^{-\mathrm{i}nx/2}]\mathrm{e}^{-\mathrm{i}n^2 t/4} + O(\varepsilon^{3/2}), \quad n\in\mathbb{N} \tag{4}$$

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where the vector function $(a,b):\mathbb{R}\times\mathbb{R}_+\mapsto\mathbb{C}^2$ satisfies the coupled-mode system

$$\begin{cases} \mathrm{i}(a_T+na_X)=w_{2n}b+\sigma(|a|^2+2|b|^2)a \\ \mathrm{i}(b_T-nb_X)=w_{2n}a+\sigma(2|a|^2+|b|^2)b \end{cases} \tag{5}$$

in slow variables $X=\varepsilon x$ and $T=\varepsilon t$. System (5) admits a separation of variables [5]:

$$a=\left(\frac{n+c}{n-c}\right)^{1/4}A(\xi)\mathrm{e}^{-\mathrm{i}\mu n\tau},\quad b=\left(\frac{n-c}{n+c}\right)^{1/4}B(\xi)\mathrm{e}^{-\mathrm{i}\mu n\tau} \tag{6}$$

where $|c|<n$, the new independent coordinates (ξ,τ) are given by the Lorentz transformation

$$\xi=\frac{X-cT}{\sqrt{n^2-c^2}},\quad \tau=\frac{T-cX}{\sqrt{n^2-c^2}} \tag{7}$$

and the new functions $A(\xi)$ and $B(\xi)$ satisfy the ODE system

$$\begin{cases} (n-c)(\mathrm{i}A'-w_{2n}B)+\mu n(1-cn)A=\sigma[(n+c)|A|^2+2(n-c)|B|^2]A \\ -(n+c)(\mathrm{i}B'+w_{2n}A)+\mu n(1+cn)B=\sigma[2(n+c)|A|^2+(n-c)|B|^2]B \end{cases} \tag{8}$$

Since $|A(\xi)|^2-|B(\xi)|^2$ is constant in $\xi\in\mathbb{R}$ and the constant is zero for localized solutions as $|\xi|\to\infty$, we can further represent the localized solution in the form

$$A=\phi(\xi)\mathrm{e}^{\mathrm{i}\varphi(\xi)},\quad B=\bar{\phi}(\xi)\mathrm{e}^{\mathrm{i}\varphi(\xi)} \tag{9}$$

where the functions $\phi:\mathbb{R}\mapsto\mathbb{C}$ and $\varphi:\mathbb{R}\mapsto\mathbb{R}$ are solutions of the first-order equations

$$\begin{cases} \varphi'=\dfrac{nc(\mu(1-n^2)-2\sigma|\phi|^2)}{(n^2-c^2)} \\ \mathrm{i}\phi'=w_{2n}\bar{\phi}-\dfrac{\mu n^2(1-c^2)}{(n^2-c^2)}\phi+\sigma\dfrac{(3n^2-c^2)}{(n^2-c^2)}|\phi|^2\phi \end{cases} \tag{10}$$

The second equation of system (10) is closed on $\phi(\xi)$ and the explicit localized solution for $c\neq 0$ can be found from the corresponding solution for $c=0$ [4]. For instance, if $\sigma=-1$ and $w_{2n}>0$, the function $\phi(\xi)$ is found in the explicit form

$$\phi=\sqrt{\frac{2(n^2-c^2)}{(3n^2-c^2)}}\frac{\sqrt{w_{2n}^2-\mu_0^2}}{\sqrt{w_{2n}+\mu_0}\cosh(\sqrt{w_{2n}^2-\mu_0^2}\xi)-\mathrm{i}\sqrt{w_{2n}-\mu_0}\sinh(\sqrt{w_{2n}^2-\mu_0^2}\xi)} \tag{11}$$

where $\mu_0=\mu n^2(1-c^2)/(n^2-c^2)$ and $|\mu_0|<w_{2n}$. If $c=0$, then $\mu_0=\mu$ and the condition $|\mu|<w_{2n}$ indicates that the frequency parameter $\omega=\omega_n+\varepsilon\mu$ of the stationary gap soliton is chosen inside the newly formed gap of the continuous spectrum near the bifurcation point $\omega_n=n^2/4$ [4] given by

$$\omega_n-\varepsilon w_{2n}<\omega<\omega_n+\varepsilon w_{2n} \tag{12}$$

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The exact solution (11) can be easily extended to values $\sigma=1$ and $w_{2n}<0$. Given a localized solution for $\phi(\xi)$, we can integrate the first equation of system (10) and obtain a linearly growing solution for $\varphi(\xi)$:

$$\varphi=\frac{nc}{(n^2-c^2)}\left(\mu(1-n^2)\xi-2\sigma\int_0^{\xi}|\phi(\xi')|^2\,\mathrm{d}\xi'\right) \tag{13}$$

The trivial parameters of translations of solutions in ξ and φ are set to zero in the explicit expressions (11) and (13), such that the functions $A(\xi)$ and $B(\xi)$ given by the parametrization (9) satisfy the constraints $A(\xi)=\bar{A}(-\xi)$ and $B(\xi)=\bar{B}(-\xi)$.

Definition 1

The traveling solution of the coupled-mode system (5) in the form (6)–(7) is said to be a *reversible homoclinic orbit* if it decays to zero at infinity and satisfies the constraints $A(\xi)=\bar{A}(-\xi)$ and $B(\xi)=\bar{B}(-\xi)$.

We study persistence of the traveling solution of the coupled-mode system (5) in the Gross–Pitaevskii equation (1). We show that the moving gap solitons have bounded oscillatory tails in the far-field profile of the scale $\varepsilon^{-(N+1)}$, which are small in amplitude of order $\varepsilon^{N+1/2}$ for any $N\geqslant 1$. These small oscillatory tails are not accounted in the coupled-mode system (5). The main result is formulated as follows.

Theorem 2

Let Assumption 1 be satisfied. Fix $n\in\mathbb{N}$, such that $w_{2n}\neq 0$. Let $\omega=n^2/4+\varepsilon\Omega$, such that $|\Omega|<\Omega_0=|w_{2n}|\sqrt{(n^2-c^2)}/n$. Let $0<c<n$, such that $(n^2+c^2)/2c\notin\mathbb{Z}'$, where $\mathbb{Z}'$ is a set of odd (even) numbers for odd (even) n. Fix $N\in\mathbb{N}$. For sufficiently small ε, there are ε-independent constants $L>0$ and $C>0$, such that the Gross–Pitaevskii equation (1) admits an infinite-dimensional, continuous family of traveling solutions in the form $E(x,t)=\mathrm{e}^{-\mathrm{i}\omega t}\psi(x,y)$, where $y=x-ct$ and the function $\psi(x,y)$ is a periodic (anti-periodic) function of x for even (odd) n, satisfying the reversibility constraint $\psi(x,y)=\bar{\psi}(x,-y)$, and the bound

$$|\psi(x,y)-\varepsilon^{1/2}[a_\varepsilon(\varepsilon y)\mathrm{e}^{\mathrm{i}nx/2}+b_\varepsilon(\varepsilon y)\mathrm{e}^{-\mathrm{i}nx/2}]|\leqslant C_0\varepsilon^{N+1/2}$$
$$\forall x\in\mathbb{R}\ \ \forall y\in[-L/\varepsilon^{N+1},L/\varepsilon^{N+1}] \tag{14}$$

Here $a_\varepsilon(Y)=a(Y)+O(\varepsilon)$ and $b_\varepsilon(Y)=b(Y)+O(\varepsilon)$ on $Y=\varepsilon y\in\mathbb{R}$ are exponentially decaying reversible solutions as $|Y|\to\infty$ in the sense of Definition 1, while $a(Y)\mathrm{e}^{-\mathrm{i}\Omega T}$ and $b(Y)\mathrm{e}^{-\mathrm{i}\Omega T}$ are solutions of the coupled-mode system (5) with $Y=X-cT$.

Remark 1

(a) The solution $\psi(x,y)$ is a bounded non-decaying function on a large finite interval

$$y\in[-L/\varepsilon^{N+1},L/\varepsilon^{N+1}]\subset\mathbb{R}$$

but we do not claim that the solution $\psi(x,y)$ can be extended to a global bounded function on $y\in\mathbb{R}$.

(b) Since the homoclinic orbit (a,b) of the coupled-mode system (5) is single-bumped, the traveling solution $\psi(x,y)$ is represented by a single bump surrounded by bounded oscillatory tails.

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(c) The solution $(a_\varepsilon, b_\varepsilon)$ is defined up to the terms of $O(\varepsilon^N)$ and it satisfies an extended coupled-mode system, which is a perturbation of the coupled-mode system (5).

(d) We believe that the result of Theorem 2 can be extended to claim that the tails are exponentially small in ε on an exponentially large scale of the y-axis, following the analysis of [8].

Our work can be compared with three groups of papers. The first group covers rigorous justification of the validity of the coupled-mode system (5) for the system of cubic Maxwell equations [6] and for the Klein–Fock equation with quadratic nonlinearity [7]. The bound on the error terms was proved for a finite time interval, which depends on ε. By using this bound, one can see that the solution of the Gross–Pitaevskii equation (1) behaves as a moving gap soliton of the coupled-mode system (5) during the initial time evolution in $H^1(\mathbb{R})$ [6] or in $C_b^0(\mathbb{R})$ [7]. However, the error is not controlled on the entire time interval $t \in \mathbb{R}$ since other effects, such as radiation caused by interactions of the moving gap soliton with the stationary periodic potential, can destroy steady propagation of gap solitons.

The second group of articles covers analysis of persistence of small-amplitude localized modulated pulses in nonlinear dispersive systems such as the Maxwell equations with periodic coefficients [9], the nonlinear wave equation [10], and the quasi-linear wave equation [11]. Methods of spatial dynamics were applied in these works to show that a local center manifold spanned by oscillatory modes destroys exponential localization of the modulated pulses along the directions of the slow stable and unstable manifolds. As a result, the spatial field of the modulating pulse solutions decays to small-amplitude oscillatory disturbances in the far-field regions of the spatial scale.

The third group of papers addresses propagation of a moving solitary wave in a periodic potential $V(x)$ of a large period (see review in [12]). An effective particle equation is derived from the focusing Gross–Pitaevskii equation (1) with $\sigma = -1$ by a heuristic asymptotic expansion. The particle equation describes a steady propagation of the moving solitary wave with $\omega < 0$, which corresponds to the semi-infinite gap of the periodic potential. Radiation effects appear beyond all orders of the asymptotic expansion. They have been incorporated in the asymptotic formalism by using perturbation theory based on the inverse scattering transform [13]. The same methods were applied to the finite-period and small-period potentials [12]. Unfortunately, this group of articles does not connect individual results in a complete rigorous theory of the time evolution of a solitary wave in a periodic potential, although it does give a good intuition on what to expect from the time evolution. Recent numerical approximations of dynamics of moving gap solitons in periodic potentials are reported in [14].

The main result of this paper is a proof that a moving gap soliton of the Gross–Pitaevskii equation (1) with the periodic potential $V(x)$ is surrounded by the oscillatory tails which are bounded on *finite* intervals of the spatial scale. Unfortunately, we are not able to exclude the polynomial growth of the oscillatory tails in the far-field regions. This construction of traveling solutions on a finite spatial scale is related with the *finite-time* applicability of the coupled-mode equations (5) for the Cauchy problem associated with the Gross–Pitaevskii equation (1) [7]. It would be a drastic improvement to the constructed theory if we could extend the analysis of oscillatory tails to the *infinite* spatial scale by proving existence of *global* solutions with a single bump and *bounded* oscillatory tails.

Our article is structured as follows. Section 2 reformulates the existence problem for moving gap solitons as the spatial dynamical system. Section 3 presents the Hamiltonian structure for the spatial dynamical system and normal coordinates of the Hamiltonian system. Section 4 describes a transformation of the Hamiltonian system to the normal form and gives a proof of persistence of a

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reversible homoclinic orbit in the extended coupled-mode system. Section 5 presents a construction of a local center-saddle manifold that concludes the proof of Theorem 2.

2. SPATIAL DYNAMICS FORMULATION

We look for traveling solutions of the Gross–Pitaevskii equation (1) in the form

$$E(x,t)=\mathrm{e}^{-\mathrm{i}\omega t}\psi(x,y), \quad y=x-ct \tag{15}$$

where ω is a parameter of gap solitons and the coordinates (x,y) are linearly independent if $c\neq 0$. For simplicity, we consider only the case $c>0$. The envelope function $\psi(x,y)$ satisfies the PDE

$$(\omega-\mathrm{i}c\partial_y+\partial_x^2+2\partial_x\partial_y+\partial_y^2)\psi(x,y)=\varepsilon W(x)\psi(x,y)+\sigma|\psi(x,y)|^2\psi(x,y) \tag{16}$$

Equation (16) is *equivalent* to the original equation (1) if $c\neq 0$. We shall, however, specify the class of functions $\psi(x,y)$ to accommodate the moving gap solitons according to their leading-order representation given by (4), (6), and (7). In particular, we consider either periodic (for even n) or anti-periodic (for odd n) functions $\psi(x,y)$ in variable x on $[0,2\pi]$ and look for decaying or bounded solutions $\psi(x,y)$ in variable y on $\mathbb{R}$. Such solutions can be described by using the formalism of spatial dynamical systems [10, 11]. We make use of the periodic or anti-periodic conditions in variable x and represent the solution $\psi(x,y)$ in the form

$$\psi(x,y)=\sqrt{\varepsilon}\sum_{m\in\mathbb{Z}'}\psi_m(y)\mathrm{e}^{\mathrm{i}mx/2}, \quad \psi_m(y)=\frac{1}{2\pi\sqrt{\varepsilon}}\int_0^{2\pi}\psi(x,y)\mathrm{e}^{-\mathrm{i}mx/2}\,\mathrm{d}x, \quad m\in\mathbb{Z}' \tag{17}$$

where the factor $\sqrt{\varepsilon}$ is used for convenience and the set $\mathbb{Z}'$ contains even numbers if $\psi(x,y)$ is periodic in x and odd numbers if $\psi(x,y)$ is anti-periodic in x. The series representation (17) transforms the PDE system (16) to the nonlinear system of coupled ODEs

$$\psi_m''(y)+\mathrm{i}(m-c)\psi_m'(y)+\left(\omega-\frac{m^2}{4}\right)\psi_m(y)$$
$$=\varepsilon\sum_{m_1\in\mathbb{Z}'}w_{m-m_1}\psi_{m_1}(y)+\varepsilon\sigma\sum_{m_1\in\mathbb{Z}'}\sum_{m_2\in\mathbb{Z}'}\psi_{m_1}(y)\bar{\psi}_{-m_2}(y)\psi_{m-m_1-m_2}(y) \quad \forall m\in\mathbb{Z}' \tag{18}$$

The left-hand side of system (18) represents a linearized system at the zero solution for $\varepsilon=0$. Since the linearized system at $\varepsilon=0$ has a diagonal structure on $m\in\mathbb{Z}'$, its solutions are given by the eigenmodes $\psi_m(y)=\mathrm{e}^{\kappa y}\delta_{m,m_0}$ for all $m\in\mathbb{Z}'$ and a fixed $m_0\in\mathbb{Z}'$, where the values of κ are determined by the roots of quadratic equations:

$$\kappa=\kappa_m: \kappa^2+\mathrm{i}(m-c)\kappa+\omega-\frac{m^2}{4}=0 \quad \forall m\in\mathbb{Z}' \tag{19}$$

The zero root $\kappa=0$ exists if and only if $\omega=\omega_n=n^2/4$ for any fixed $n\in\mathbb{Z}$. The zero root has multiplicity two if $n\in\mathbb{N}$ and $c\neq\pm n$ and it appears in the equations for $m=n$ and $m=-n$. The special value $\omega=\omega_n$ corresponds to the bifurcation of periodic or anti-periodic stationary solutions as well as of the stationary gap solitons of the nonlinear ODE (2) [4]. We note that the values of n determine the choice for the set $\mathbb{Z}'$: it includes even (odd) numbers if n is even (odd). We shall hence focus on the bifurcation case $\omega=\omega_n$, when the two roots of the quadratic equations (19) are represented explicitly as follows:

$$\omega=\omega_n:\kappa=\kappa_m^{\pm}=\frac{\mathrm{i}(c-m)\pm\sqrt{2cm-n^2-c^2}}{2}\quad\forall m\in\mathbb{Z}' \tag{20}$$

When $m>m_0=[(n^2+c^2)/2c]'$, where the notation $[a]'$ denotes the integer part of the number a in the set $\mathbb{Z}'$, all roots are complex valued with $\mathrm{Re}(\kappa_m^{\pm})=\pm\sqrt{2cm-n^2-c^2}/2\neq 0$ and $\mathrm{Im}(\kappa_m^{\pm})=(c-m)/2$. When $m\leqslant m_0$, all roots κ are purely imaginary with $\kappa_m^{\pm}=\mathrm{i}k_m^{\pm}$ and $k_m^{\pm}=[(c-m)\pm\sqrt{(n^2+c^2-2cm)}]/2$.

Lemma 1
Let $\omega=\omega_n$ for a fixed $n\in\mathbb{N}$ and let $c>0$, such that $(n^2+c^2)/2c\notin\mathbb{Z}'$. Then,

(i) the phase space of the linearized system (18) at the zero solution for $\varepsilon=0$ decomposes into a direct sum of infinite-dimensional subspaces $E^{\mathrm{s}}\oplus E^{\mathrm{u}}\oplus E^{c^+}\oplus E^{c^-}$, where

$$E^{\mathrm{s}}=\oplus_{m>m_0}E_m^{+},\quad E^{\mathrm{u}}=\oplus_{m>m_0}E_m^{-},\quad E^{c^+}=\oplus_{m\leqslant m_0}E_m^{+},\quad E^{c^-}=\oplus_{m\leqslant m_0}E_m^{-} \tag{21}$$

The subspace $E_m^{\pm}$ consists of the eigenspace associated with the eigenmode $\psi_{m'}(y)=\mathrm{e}^{\kappa y}\delta_{m,m'}$, $\forall m'\in\mathbb{Z}'$ which corresponds to the root $\kappa=\kappa_m^{\pm}$ given by (20).

(ii) the double zero root $\kappa=0$ is associated with the subspaces E_n^{+} and E_{-n}^{-}. The purely imaginary roots $\kappa\in\mathrm{i}\mathbb{R}$ are semi-simple with a multiplicity of at most three. All other roots $\kappa\in\mathbb{C}$ are simple.

Proof
It follows from the quadratic equation (19) that a root $\kappa=\kappa_0$ is double if $\kappa_0=\mathrm{i}(c-m)/2$, which implies that $m=(n^2+c^2)/2c$. Under the non-degeneracy constraint $(n^2+c^2)/2c\notin\mathbb{Z}'$, all roots κ are semi-simple. When $m>m_0=[(n^2+c^2)/2c]'$, all roots are complex valued and simple, such that E^{s} and E^{u} are stable and unstable subspaces of the linearized system at the zero solution for $\varepsilon=0$.

When $m\leqslant m_0$, all roots κ are purely imaginary, such that $E^{c^+}\oplus E^{c^-}$ is a center subspace of the linearized system for $\varepsilon=0$. It is obvious that $\mathrm{Im}\,\kappa_m^{+}$ increases as m decreases, while $\mathrm{Im}\,\kappa_m^{-}$ decreases for $m_1<m\leqslant m_0$ and increases for $m\leqslant m_1$ as m decreases, where $m_1=[n^2/2c]'$. Therefore, the purely imaginary roots may have multiplicity of at most three. The zero eigenvalue, however, has multiplicity two since the two modes $m=n$ and $m=-n$ have simple zero eigenvalues and other modes have no zero eigenvalues. □

Lemma 2
Let $\omega=\omega_n$ for a fixed $n\in\mathbb{N}$ and let $0<c<n$. If c is irrational, all non-zero roots κ of the quadratic equations (19) are simple.

Proof
By Lemma 1, only imaginary roots κ can be semi-simple. Let two roots κ coincide for $m\leqslant m_0$ and $l\leqslant m_0$. Then, $\kappa=-\mathrm{i}(m+l)/4$ and a pair (m,l) satisfies the following equation:

$$(m-l)^2+4c(m+l)-4n^2=0,\quad m\leqslant m_0,\ \ l\leqslant m_0 \tag{22}$$

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If c is irrational, no solution exists for integers m and l. Therefore, all non-zero roots κ are simple. □

Example 1

Figure 1(a) illustrates the distribution of imaginary roots $\kappa_m^{\pm} = \mathrm{i}k_m^{\pm}$ *versus* $m \leqslant m_0$ for $c = \frac{1}{\sqrt{2}}$ and $n = 1$. Although all imaginary roots are simple for this (irrational) value of c, the purely imaginary roots κ can approach to each other arbitrarily close. Figure 1(b) shows a similar distribution for $c = \frac{1}{2}$ and $n = 1$. It follows from Equation (22) that an infinite sequence of semi-simple roots exists for this (rational) value of c.

Remark 2

The formalism of spatial dynamical systems fails if $c = 0$ since the second-order ODE (2) cannot be replaced by the PDE (16) if $y = x$ for $c = 0$ and hence it cannot be expressed as a system of infinitely many second-order ODEs (18).

Lemma 3

Consider a linear inhomogeneous equation

$$-\left(\partial_y^2 + \mathrm{i}(m-c)\partial_y + \frac{n^2 - m^2}{4}\right)\psi_m(y) = F_m(y) \quad \forall m > m_0 \tag{23}$$

where $F_m \in C_b^0(\mathbb{R})$. There exists a unique solution $\psi_m \in C_b^0(\mathbb{R})$, such that $\|\psi_m\|_{C_b^0(\mathbb{R})} \leqslant C\|F_m\|_{C_b^0(\mathbb{R})}$ for some $C > 0$.

Proof

Let $\psi_m = \mathrm{e}^{-\mathrm{i}\alpha y}\varphi_m(y)$ with $\alpha = (m-c)/2$. The function $\varphi_m(y)$ solves the ODE in the form

$$(\beta^2 - \partial_y^2)\varphi_m(y) = F_m(y)\mathrm{e}^{\mathrm{i}\alpha y} \quad \text{where } \beta^2 = \frac{2cm - n^2 - c^2}{4} > 0 \quad \text{for } m > m_0$$

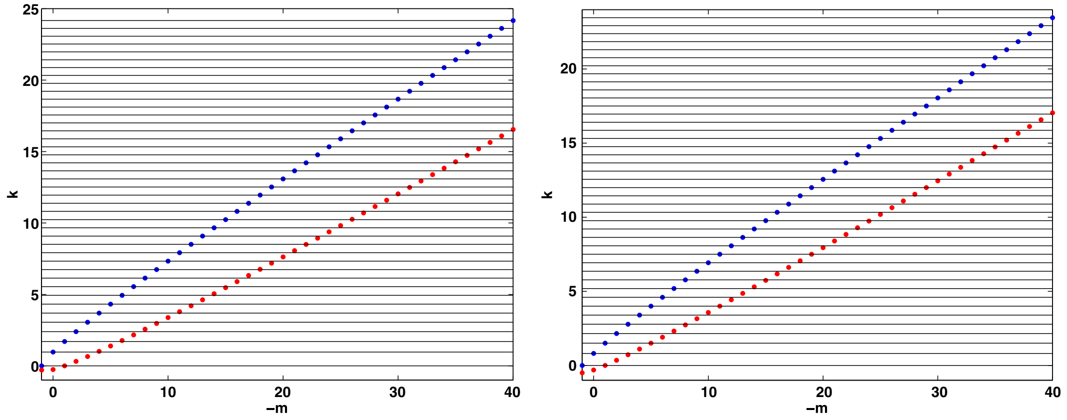

Figure 1. Purely imaginary roots $\kappa_m^{\pm} = \mathrm{i}k_m^{\pm}$ *versus* $m \leqslant m_0$ for $n = 1$, $c = \frac{1}{\sqrt{2}}$ (left) and for $n = 1$, $c = \frac{1}{2}$ (right). Larger values correspond to k_m^+ and smaller values correspond to k_m^-.

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Since the solutions of the homogeneous equation are exponentially decaying and growing as $\varphi_m \sim e^{\pm\beta y}$, there exists a unique bounded solution of the inhomogeneous equation in the integral form

$$\varphi_m(y) = \frac{1}{2\beta}\int_{-\infty}^{\infty} e^{-\beta|y-y'|} F_m(y') e^{i\alpha y'}\, dy'$$

such that $\|\varphi_m\|_{C_b^0(\mathbb{R})} \leqslant (1/\beta^2)\|F_m\|_{C_b^0(\mathbb{R})}$. □

Remark 3

By using Lemma 3 and the Implicit Function Theorem, one can solve the equations of system (18) for all $m>m_0$ and parameterize components $\psi_m(y)$ for all $m>m_0$ in terms of bounded components $\psi_m(y)$ for all $m\leqslant m_0$ if ε is sufficiently small. However, we do not perform this parametrization at this stage, since we will rewrite system (18) as a Hamiltonian dynamical system with a local symplectic structure and use a formalism of near-identity transformations and normal forms.

3. HAMILTONIAN FORMALISM AND NORMAL COORDINATES

We rewrite the system of second-order equations (18) as the system of first-order equations that admits a symplectic Hamiltonian structure. Let $\omega = n^2/4 + \varepsilon\Omega$, where $n \in \mathbb{N}$ and Ω is a free parameter. Let

$$\phi_m(y) = \psi_m'(y) - \frac{i}{2}(c-m)\psi_m(y) \quad \forall m \in \mathbb{Z}' \tag{24}$$

System (18) is equivalent to the first-order system

$$\begin{cases} \dfrac{d\psi_m}{dy} = \phi_m + \dfrac{i}{2}(c-m)\psi_m \\ \dfrac{d\phi_m}{dy} = -\dfrac{1}{4}(n^2+c^2-2cm)\psi_m + \dfrac{i}{2}(c-m)\phi_m - \varepsilon\Omega\psi_m + \varepsilon \displaystyle\sum_{m_1\in\mathbb{Z}'} w_{m-m_1}\psi_{m_1} \\ \qquad\qquad + \varepsilon\sigma \displaystyle\sum_{m_1\in\mathbb{Z}'}\sum_{m_2\in\mathbb{Z}'} \psi_{m_1}\bar{\psi}_{-m_2}\psi_{m-m_1-m_2} \end{cases} \tag{25}$$

Let the bolded symbol $\boldsymbol{\psi}$ denote a vector consisting of elements of the set $\{\psi_m\}_{m\in\mathbb{Z}'}$. The variables $\{\boldsymbol{\psi}, \boldsymbol{\phi}, \bar{\boldsymbol{\psi}}, \bar{\boldsymbol{\phi}}\}$ are canonical and system (25) is equivalent to the Hamilton equations of motion:

$$\frac{d\psi_m}{dy} = \frac{\partial H}{\partial \bar{\phi}_m}, \quad \frac{d\phi_m}{dy} = -\frac{\partial H}{\partial \bar{\psi}_m} \quad \forall m \in \mathbb{Z}' \tag{26}$$

where $H = H(\boldsymbol{\psi}, \boldsymbol{\phi}, \bar{\boldsymbol{\psi}}, \bar{\boldsymbol{\phi}})$ is the Hamiltonian function given by

$$\begin{aligned} H = \sum_{m\in\mathbb{Z}'} &\left[|\phi_m|^2 + \frac{1}{4}(n^2+c^2-2cm)|\psi_m|^2 + \frac{i}{2}(c-m)(\psi_m\bar{\phi}_m - \bar{\psi}_m\phi_m) + \varepsilon\Omega|\psi_m|^2 \right] \\ &- \varepsilon \sum_{m\in\mathbb{Z}'}\sum_{m_1\in\mathbb{Z}'} w_{m-m_1}\psi_{m_1}\bar{\psi}_m - \frac{\varepsilon\sigma}{2}\sum_{m\in\mathbb{Z}'}\sum_{m_1\in\mathbb{Z}'}\sum_{m_2\in\mathbb{Z}} \psi_{m_1}\bar{\psi}_{-m_2}\psi_{m-m_1-m_2}\bar{\psi}_m \end{aligned} \tag{27}$$

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Let us define the discrete weighted l^2-space by its norm

$$\forall \mathbf{u} \in l_s^2(\mathbb{Z}') : \|\mathbf{u}\|_{l_s^2}^2 = \sum_{m \in \mathbb{Z}'} (1+m^2)^s |u_m|^2 < \infty \tag{28}$$

Since $l_s^2(\mathbb{Z}')$ is a Banach algebra for any $s > \frac{1}{2}$, the convolution sums in the nonlinear system (25) map an element of $l_s^2(\mathbb{Z}')$ to an element of $l_s^2(\mathbb{Z}')$ for ψ if $\mathbf{W} \in l_s^2(\mathbb{Z})$ for any $s > \frac{1}{2}$ (Assumption 1). Owing to the unbounded linear part, the vector field of system (25) maps a domain in $D = \{(\psi, \phi, \bar{\psi}, \bar{\phi}) \in l_{s+1}^2(\mathbb{Z}', \mathbb{C}^4)\}$ to a range in $X = \{(\psi, \phi, \bar{\psi}, \bar{\phi}) \in l_s^2(\mathbb{Z}', \mathbb{C}^4)\}$ for any $s > \frac{1}{2}$. We note that $D \subset X$ and that X can be chosen as the phase space of the Hamiltonian system (26).

If $W(-x) = W(x)$, $\forall x \in \mathbb{R}$ (Assumption 1), then $w_{2m} = w_{-2m}$, $\forall m \in \mathbb{Z}$. In this case, the Hamiltonian system (25) is reversible and its solutions are invariant under the transformation

$$\psi(y) \mapsto \bar{\psi}(-y), \quad \phi(y) \mapsto -\bar{\phi}(-y) \ \ \forall y \in \mathbb{R} \tag{29}$$

In addition, the Hamiltonian function (27) is invariant with respect to the gauge transformation

$$\psi(y) \mapsto e^{i\alpha}\psi(y), \quad \phi(y) \mapsto e^{i\alpha}\phi(y) \ \ \forall \alpha \in \mathbb{R} \tag{30}$$

Let us define the domain for reversible solutions by

$$D_r = \{(\psi, \phi, \bar{\psi}, \bar{\phi}) \in D : \psi(-y) = \bar{\psi}(y),\ \phi(-y) = -\bar{\phi}(y),\ \forall y \in \mathbb{R}\} \tag{31}$$

If a local solution of system (25) is constructed on $y \in \mathbb{R}_+$ and it intersects at $y = 0$ with the reversibility constraint

$$\Sigma_r = \{(\psi, \phi, \bar{\psi}, \bar{\phi}) \in D : \mathrm{Im}\,\psi = 0, \mathrm{Re}\,\phi = 0\} \tag{32}$$

then the solution is extended to a global reversible solution in D_r on $y \in \mathbb{R}$ by using the reversibility transformation (29). The global reversible solution defines uniquely parameter α of the gauge transformation (30).

In order to construct non-trivial bounded solutions of the Hamiltonian system (25), we shall introduce normal coordinates for the infinite-dimensional stable, unstable and center subspaces of the linearized Hamiltonian system (25) at the zero solution for $\varepsilon = 0$ (Lemma 1). Let $\mathbb{Z}'_- = \{m \in \mathbb{Z}' : m \leqslant m_0\}$ and $\mathbb{Z}'_+ = \{m \in \mathbb{Z}' : m > m_0\}$. For the center subspace of the linearized Hamiltonian system, we set

$$\forall m \in \mathbb{Z}'_- : \psi_m = \frac{c_m^+(y) + c_m^-(y)}{\sqrt[4]{n^2 + c^2 - 2cm}}, \quad \phi_m = \frac{i}{2}\sqrt[4]{n^2 + c^2 - 2cm}[c_m^+(y) - c_m^-(y)] \tag{33}$$

For the stable and unstable subspaces of the linearized Hamiltonian system, we set

$$\forall m \in \mathbb{Z}'_+ : \psi_m = \frac{c_m^+(y) + c_m^-(y)}{\sqrt[4]{2cm - n^2 - c^2}}, \quad \phi_m = \frac{1}{2}\sqrt[4]{2cm - n^2 - c^2}[c_m^+(y) - c_m^-(y)] \tag{34}$$

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By using the normal coordinates (33)–(34), the Hamiltonian function (27) transforms to the new form

$$H=\sum_{m\in\mathbb{Z}'_-}(k_m^+|c_m^+|^2-k_m^-|c_m^-|^2)+\sum_{m\in\mathbb{Z}'_+}(\kappa_m^- c_m^-\bar{c}_m^+-\kappa_m^+c_m^+\bar{c}_m^-)+\varepsilon\Omega\sum_{m\in\mathbb{Z}'}\frac{|c_m^++c_m^-|^2}{\sqrt{|n^2+c^2-2cm|}}$$

$$-\varepsilon\sum_{m\in\mathbb{Z}'}\sum_{m_1\in\mathbb{Z}'}w_{m,m_1}(c_{m_1}^++c_{m_1}^-)(\bar{c}_m^++\bar{c}_m^-)-\frac{\varepsilon\sigma}{2}\sum_{m\in\mathbb{Z}'}\sum_{m_1\in\mathbb{Z}'}\sum_{m_2\in\mathbb{Z}}g_{m,m_1,m_2}(c_{m_1}^++c_{m_1}^-)$$

$$\times(\bar{c}_{-m_2}^++\bar{c}_{-m_2}^-)(c_{m-m_1-m_2}^++c_{m-m_1-m_2}^-)(\bar{c}_m^++\bar{c}_m^-) \tag{35}$$

where

$$k_m^\pm=\frac{c-m\pm\sqrt{n^2+c^2-2cm}}{2}\quad \forall m\in\mathbb{Z}'_-$$

$$\kappa_m^\pm=\frac{\mathrm{i}(c-m)\pm\sqrt{2cm-n^2-c^2}}{2}\quad \forall m\in\mathbb{Z}'_+$$

$$w_{m,m_1}=\frac{w_{m-m_1}}{\sqrt[4]{|n^2+c^2-2cm||n^2+c^2-2cm_1|}}\quad \forall m,m_1\in\mathbb{Z}'$$

and

$$g_{m,m_1,m_2}=\frac{1}{\sqrt[4]{|n^2+c^2-2cm_1||n^2+c^2+2cm_2||n^2+c^2-2c(m-m_1-m_2)||n^2+c^2-2cm|}}$$

for all $m,m_1,m_2\in\mathbb{Z}'$. The quadratic part of the Hamiltonian function in (35) for $\varepsilon=0$ is diagonal in normal coordinates for all $m\in\mathbb{Z}'_-$ and it is block-diagonal for all $m\in\mathbb{Z}'_+$. The Hamilton equations of motion (26) transform into new canonical variables of the form

$$\frac{\mathrm{d}c_m^+}{\mathrm{d}y}=\mathrm{i}\frac{\partial H}{\partial\bar{c}_m^+},\quad \frac{\mathrm{d}c_m^-}{\mathrm{d}y}=-\mathrm{i}\frac{\partial H}{\partial\bar{c}_m^-}\quad \forall m\in\mathbb{Z}'_- \tag{36}$$

and

$$\frac{\mathrm{d}c_m^+}{\mathrm{d}y}=-\frac{\partial H}{\partial\bar{c}_m^-},\quad \frac{\mathrm{d}c_m^-}{\mathrm{d}y}=\frac{\partial H}{\partial\bar{c}_m^+}\quad \forall m\in\mathbb{Z}'_+ \tag{37}$$

Since the convolution sums on $\{c_m^\pm\}_{m\in\mathbb{Z}'}$ involve decaying weights as $|m|\to\infty$ and the linear unbounded part on $\{c_m\}_{m\in\mathbb{Z}'}$ is linear in $|m|$ ($k_m^\pm$ and $\kappa_m^\pm$ grow linearly as $|m|\to\infty$), the vector field of system (36)–(37) has now a domain in $D'=\{(\mathbf{c}^+,\mathbf{c}^-,\bar{\mathbf{c}}^+,\bar{\mathbf{c}}^-)\in l^2_{s'+1}(\mathbb{Z}',\mathbb{C}^4)\}$ and range in $X'=\{(\mathbf{c}^+,\mathbf{c}^-,\bar{\mathbf{c}}^+,\bar{\mathbf{c}}^-)\in l^2_{s'}(\mathbb{Z}',\mathbb{C}^4)\}$ for $s'=s-\frac{1}{4}$. (Here we have used the fact that the convolution sum acts on $\{\psi_m\}_{m\in\mathbb{Z}}$, where ψ_m is given by $c_m^\pm$ in the transformation (33)–(34).) If the space X is a Banach algebra for $s>\frac{1}{2}$, the space X' is a Banach algebra with respect to the decaying weights for $s'>\frac{1}{4}$. The domain for reversible solutions becomes now

$$D'_r=\{(\mathbf{c}^+,\mathbf{c}^-,\bar{\mathbf{c}}^+,\bar{\mathbf{c}}^-)\in D':c_m^\pm(-y)=\bar{c}_m^\pm(y),\ \forall m\in\mathbb{Z}'_-,\ c_m^\pm(-y)=\bar{c}_m^\mp(y),\ \forall m\in\mathbb{Z}'_+\} \tag{38}$$

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and the reversibility constraint becomes

$$\Sigma_r' = \{(\mathbf{c}^+, \mathbf{c}^-, \bar{\mathbf{c}}^+, \bar{\mathbf{c}}^-) \in D' : \mathrm{Im}\, c_m^{\pm} = 0,\ \forall m \in \mathbb{Z}_-',\ \mathrm{Re}\, c_m^+ = \mathrm{Re}\, c_m^-,\ \mathrm{Im}\, c_m^+ = -\mathrm{Im} c_m^-,\ \forall m \in \mathbb{Z}_+'\} \quad (39)$$

In what follows, we are concerned with the reversible homoclinic orbit of the Hamilton equations (36)–(37) in normal coordinates (33)–(34). We will abbreviate $H(\mathbf{c}^+, \mathbf{c}^-, \bar{\mathbf{c}}^+, \bar{\mathbf{c}}^-)$ to $H(\mathbf{c}^+, \mathbf{c}^-)$ for notational simplicity.

Remark 4

Since $k_m^{\pm} > 0$ for sufficiently large negative $m \in \mathbb{Z}_-'$, the center subspace $E_c^+ \oplus E_c^-$ is spanned by an infinite set of modes with positive and negative energies. Therefore, we cannot use the technique of [10], which relies on the fact that the quadratic part of the Hamiltonian function is positive definite for the non-bifurcating modes of the center subspace. We can, however, use the technique of [11], which relies on the separation of slow motion for the modes, which correspond to the zero eigenvalue (modes c_n^+ and c_{-n}^-), and the fast motion of the other modes, which correspond to non-zero purely imaginary or complex eigenvalues (modes $c_m^{\pm}$ for all other m). Moreover, we can simplify the technique of [11] by incorporating the Hamiltonian structure (36)–(37) with the Hamiltonian function (35). Unlike [10], we use the Hamiltonian formalism not to control the solution at larger distances y but to perform persistence analysis of the reversible homoclinic orbits in the coupled-mode system. The crucial role of the Hamiltonian formalism is to ensure that the four-dimensional dynamical system in combination with two continuous symmetries has a two-dimensional invariant manifold, in which homoclinic orbits lie.

4. NORMAL FORM AND PERSISTENCE OF HOMOCLINIC ORBITS

We first show that the formal truncation of the Hamiltonian function (35) at the two bifurcating modes c_n^+ and c_{-n}^-, which correspond to the double zero eigenvalue of the linearized system (Lemma 1) leads to the coupled-mode system (5). Then, we derive an extended coupled-mode system for the modes c_n^+ and c_{-n}^- by using near-identity transformations of the Hamiltonian function and prove persistence of a reversible homoclinic orbit in the extended coupled-mode system. We will assume in what follows that $0 < c < n$.

Let us consider the subspace of the phase space of the Hamiltonian system (36)–(37):

$$S = \{c_m^+ = 0,\ \forall m \in \mathbb{Z}' \backslash \{n\},\ c_m^- = 0,\ \forall m \in \mathbb{Z}' \backslash \{-n\}\} \quad (40)$$

If the Hamiltonian function H is formally constrained on the subspace S, expression (35) takes the form

$$H|_S = \varepsilon \left[\frac{\Omega |c_n^+|^2}{n-c} + \frac{\Omega |c_{-n}^-|^2}{n+c} - \frac{w_{2n}(\bar{c}_n^+ c_{-n}^- + c_n^+ \bar{c}_{-n}^-)}{\sqrt{n^2-c^2}} - \frac{\sigma}{2} \left(\frac{|c_n^+|^4}{(n-c)^2} + \frac{4|c_n^+|^2 |c_{-n}^-|^2}{n^2-c^2} + \frac{|c_{-n}^-|^4}{(n+c)^2} \right) \right]$$

Since $(n, -n) \in \mathbb{Z}_-'$, we use the symplectic structure (36) to generate a system of first-order ODEs for normal coordinates c_n^+ and c_{-n}^-. By using the new independent variable $Y = \varepsilon y$ and the new dependent variables $a = c_n^+ / \sqrt{(n-c)}$ and $b = c_{-n}^- / \sqrt{(n+c)}$, we recover the ODE system

$$\begin{cases} \mathrm{i}(n-c)a' + \Omega a = w_{2n} b + \sigma(|a|^2 + 2|b|^2)a \\ -\mathrm{i}(n+c)b' + \Omega b = w_{2n} a + \sigma(2|a|^2 + |b|^2)b \end{cases} \quad (41)$$

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where the derivatives are taken with respect to $Y=\varepsilon y$. System (41) is nothing but the coupled-mode system (5) for solutions $a(Y)e^{-i\Omega T}$ and $b(Y)e^{-i\Omega T}$, where $Y=X-cT$. Solutions in the form (6)–(7) correspond to solutions of system (41) with the correspondence $Y=\sqrt{n^2-c^2}\xi$ and $\Omega=\mu n(1-c^2)/\sqrt{(n^2-c^2)}$. System (41) has a localized solution (a homoclinic orbit) for $w_{2n}\neq 0$ and $|\Omega|<\Omega_0$, where $\Omega_0=|w_{2n}|\sqrt{(n^2-c^2)}/n$ (Section 1).

The ODE system (41) is invariant with respect to translation $a(Y)\to a(Y-Y_0)$, $b(Y)\to b(Y-Y_0)$ for any $Y_0\in\mathbb{R}$ and gauge transformation $a(Y)\to e^{i\alpha}a(Y-Y_0)$, $b(Y)\to e^{i\alpha}b(Y-Y_0)$ for any $\alpha\in\mathbb{R}$. Therefore, any solution of the system is continued with a two-parameter group of symmetry transformations. However, these parameters are set uniquely in the reversible homoclinic orbit of Definition 1. In addition, we note that although the ODE system (41) is formulated in the four-dimensional phase space, it has two conserved quantities related to the translational and gauge symmetries. Indeed, the Hamiltonian $H|_S$ and the quadratic function

$$Q=|c_n^+(Y)|^2-|c_{-n}^-(Y)|^2=(n-c)|a(Y)|^2-(n+c)|b(Y)|^2 \tag{42}$$

are constant on $Y\in\mathbb{R}$. As a result, localized solutions of system (41) correspond to $H|_S=0$ and $Q=0$. After an appropriate parametrization (6) and (9), localized solutions of system (41) are obtained from a planar Hamiltonian system given by the second equation of system (10) in variables $(\phi,\bar{\phi})$. We recall that reversible homoclinic orbits of planar Hamiltonian systems are structurally stable with respect to parameter continuations.

To incorporate the ideas of integrability of the coupled-mode system on the subspace S and persistence of reversible homoclinic orbits in planar Hamiltonian systems, we extend the coupled-mode system by using near-identity transformations and the normal form theory.

Lemma 4
Let $0<c<n$, such that $(n^2+c^2)/2c\notin\mathbb{Z}'$. For each $N\in\mathbb{N}$ and a sufficiently small ε, there is a near-identity, analytic, symplectic change of coordinates in a neighborhood of the origin in X', such that the Hamiltonian function H in (35) transforms to the normal form up to the order of $O(\varepsilon^{N+1})$,

$$\begin{aligned} H=&\sum_{m\in\mathbb{Z}'_-}(k_m^+|c_m^+|^2-k_m^-|c_m^-|^2)+\sum_{m\in\mathbb{Z}'_+}(\kappa_m^-c_m^-\bar{c}_m^+-\kappa_m^+c_m^+\bar{c}_m^-)\\ &+\varepsilon H_S(c_n^+,c_{-n}^-)+\varepsilon H_T(c_n^+,c_{-n}^-,\mathbf{c}^+,\mathbf{c}^-)+\varepsilon^{N+1}H_R(c_n^+,c_{-n}^-,\mathbf{c}^+,\mathbf{c}^-) \end{aligned} \tag{43}$$

where H_S is a polynomial of degree $2N+2$ in (c_n^+,c_{-n}^-), H_T is a polynomial of degree $2N$ in (c_n^+,c_{-n}^-) and of degree 4 with no linear terms in all other variables $(\mathbf{c}^+,\mathbf{c}^-)$, and H_R is a polynomial of degree $8N+4$ in (c_n^+,c_{-n}^-) and of degree 4 in all other variables $(\mathbf{c}^+,\mathbf{c}^-)$. All components H_S, H_T and H_R depend on ε, such that H_S and H_T are polynomials in ε of degree $N-1$ and H_R is a polynomial in ε of degree $3N-1$. The reversibility (29) and the gauge transformation (30) are preserved by the change of the variables.

Proof
The existence of a near-identity symplectic transformation that maps H to the form (43) follows from the fact that the non-resonance conditions $l\kappa_0-\kappa_m^\pm\neq 0$, $\forall l\in\mathbb{Z}$, $\forall m\in\mathbb{Z}'$ are satisfied since $\kappa_0=0$ and all eigenvalues are semi-simple for $\varepsilon=0$. See [10] for an iterative sequence of symplectic transformations. The transformation is analytic in a local neighborhood of the origin in X' as

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the vector field of the Hamiltonian system (36)–(37) is analytic (given by a cubic polynomial). The reversibility (29) and gauge transformation (30) are preserved by the symplectic change of variables [10]. The count of the degree of polynomials H_S, H_T and H_R follows from the fact that the vector field of the Hamiltonian system (36)–(37) contains only linear and cubic terms in normal coordinates, while the near-identity transformation of $(\mathbf{c}^+, \mathbf{c}^-)$ up to the order $O(\varepsilon^{N+1})$ involves a polynomial in ε of degree N and in (c_n^+, c_{-n}^-) of degree $2N+1$. □

Remark 5

For each $N \in \mathbb{N}$, the subspace S defined by (40) is an invariant subspace of the Hamiltonian system (36)–(37) with the Hamiltonian function (43) truncated at $H_R \equiv 0$. The dynamics on S is given by the four-dimensional Hamiltonian system:

$$\frac{\mathrm{d}c_n^+}{\mathrm{d}Y} = \mathrm{i}\frac{\partial H_S}{\partial \bar{c}_n^+}, \quad \frac{\mathrm{d}c_{-n}^-}{\mathrm{d}Y} = -\mathrm{i}\frac{\partial H_S}{\partial \bar{c}_{-n}^+} \tag{44}$$

where $Y = \varepsilon y$. If $N = 1$, system (44) transforms to the coupled-mode system (41) in variables $a = c_n^+/\sqrt{(n-c)}$ and $b = c_{-n}^-/\sqrt{(n+c)}$. If $N > 1$, this system is referred to as the *extended* coupled-mode system.

Lemma 5

Let $w_{2n} \neq 0$ for a given $n \in \mathbb{N}$. For each $N \in \mathbb{N}$ and a sufficiently small ε, there exists a reversible homoclinic orbit of system (44) for $|\Omega| < \Omega_0 = |w_{2n}|\sqrt{(n^2 - c^2)}/n$ in the sense of Definition 1. Moreover, the solution for the homoclinic orbit satisfies the global bound

$$|c_n^+(y)| \leqslant C_+ \mathrm{e}^{-\varepsilon\gamma|y|}, \quad |c_{-n}^-(y)| \leqslant C_- \mathrm{e}^{-\varepsilon\gamma|y|} \quad \forall y \in \mathbb{R} \tag{45}$$

for some ε-independent constants $\gamma > 0$ and $C_\pm > 0$.

Proof

Owing to the gauge invariance of the polynomial Hamiltonian function $H_S = H_S(c_n^+, c_{-n}^-)$, H_S satisfies the PDE [15]:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha} H_S(\mathrm{e}^{\mathrm{i}\alpha} c_n^+, \mathrm{e}^{\mathrm{i}\alpha} c_{-n}^-)\right|_{\alpha=0} \simeq c_n^+ \frac{\partial H_S}{\partial c_n^+} - \bar{c}_n^+ \frac{\partial H_S}{\partial \bar{c}_n^+} + c_{-n}^- \frac{\partial H_S}{\partial c_{-n}^-} - \bar{c}_{-n}^- \frac{\partial H_S}{\partial c_{-n}^-} = 0 \tag{46}$$

It follows from system (44) and relation (46) that $Q = |c_n^+|^2 - |c_{-n}^-|^2$ is constant in $Y \in \mathbb{R}$ [15]. If localized solutions exist, then $Q = 0$. Let us represent solutions of the extended system (44) in the general form

$$c_n^+ = \sqrt{\rho + q}\,\mathrm{e}^{\mathrm{i}\varphi + \mathrm{i}\theta}, \quad c_{-n}^- = \sqrt{\rho - q}\,\mathrm{e}^{\mathrm{i}\varphi - \mathrm{i}\theta} \tag{47}$$

where $(\rho, q, \theta, \varphi)$ are new real-valued variables. Using the chain rule for $H_S \equiv H_S(\rho, q, \theta, \varphi)$, we find that if it satisfies (46), then H_s does not depend on φ, that is $H_s \equiv H_s(\rho, q, \theta)$. In addition, q is conserved and we set $q = 0$ for localized solutions. As a result, we find that (ρ, θ) satisfy a planar Hamiltonian system, while φ is found from a linear inhomogeneous equation:

$$2\frac{\mathrm{d}\rho}{\mathrm{d}Y} = -\frac{\partial}{\partial \theta} H_S(\rho, 0, \theta), \quad 2\frac{\mathrm{d}\theta}{\mathrm{d}Y} = \frac{\partial}{\partial \rho} H_S(\rho, 0, \theta), \quad 2\frac{\mathrm{d}\varphi}{\mathrm{d}Y} = \frac{\partial}{\partial q} H_S(\rho, 0, \theta) \tag{48}$$

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If $\varepsilon = 0$, system (48) reduces to the ODE system (10) rewritten in new coordinates for $\phi = \sqrt{\rho}e^{i\theta}$. The vector field of the extended coupled-mode system (44) is given by polynomials in c_n^+ and c_{-n}^- of degree $2N+1$ and in ε of degree $N-1$. Recall that the coupled-mode system (41) admits a reversible homoclinic orbit for $w_{2n} \neq 0$ and $|\Omega| < \Omega_0$, where $\Omega_0 = |w_{2n}|\sqrt{(n^2 - c^2)}/n$. Since a reversible homoclinic orbit is structurally stable in a planar Hamiltonian system with an analytic vector field, the homoclinic orbit persists in the extended coupled-mode system (44) for small ε. □

5. CONSTRUCTION OF A LOCAL CENTER-STABLE MANIFOLD

We study solutions of the Hamiltonian system of Equations (36)–(37) after the normal-form transformation of Lemma 4. We construct a local solution on $y \in [0, L/\varepsilon^{N+1}]$ for some ε-independent constant $L > 0$, which is close in X' to the homoclinic orbit of Lemma 5 by the distance $C\varepsilon^N$ for some ε-independent constant $C > 0$. This solution represents an infinite-dimensional local center-stable manifold and it is spanned by the small oscillatory and small exponentially decaying solutions near the finite exponentially decaying homoclinic solution of Lemma 5. Parameters of the local center-stable manifold are chosen to ensure that the manifold intersects at $y = 0$ with the symmetric section Σ'_r. This construction completes the proof of Theorem 2.

By using Lemma 4 and the explicit representation of the Hamiltonian function (43), we rewrite the Hamiltonian system of equations in the separated form

$$\frac{dc_n^+}{dy} = \varepsilon[\mu_\varepsilon^+ c_n^+ + v_\varepsilon^+ c_{-n}^- + F_S^+(c_n^+, c_{-n}^-) + F_T^+(c_n^+, c_{-n}^-, \mathbf{c})] + \varepsilon^{N+1} F_R^+(c_n^+, c_n^-, \mathbf{c}) \tag{49}$$

$$\frac{dc_{-n}^-}{dy} = \varepsilon[\mu_\varepsilon^- c_n^+ + v_\varepsilon^- c_{-n}^- + F_S^-(c_n^+, c_{-n}^-) + F_T^-(c_n^+, c_{-n}^-, \mathbf{c})] + \varepsilon^{N+1} F_R^-(c_n^+, c_n^-, \mathbf{c}) \tag{50}$$

$$\frac{d\mathbf{c}}{dy} = \Lambda_\varepsilon \mathbf{c} + \varepsilon \mathbf{F}_T(c_n^+, c_{-n}^-, \mathbf{c}) + \varepsilon^{N+1} \mathbf{F}_R(c_n^+, c_n^-, \mathbf{c}) \tag{51}$$

where $\mathbf{c}$ denotes all components of $(\mathbf{c}^+, \mathbf{c}^-, \bar{\mathbf{c}}^+, \bar{\mathbf{c}}^-)$ excluding (c_n^+, c_{-n}^-), $(\mu_\varepsilon^\pm, v_\varepsilon^\pm, \Lambda_\varepsilon)$ denotes the coefficient matrix for the linear part of the system and $(F_S^\pm, F_T^\pm, F_R^\pm, \mathbf{F}_T, \mathbf{F}_R)$ denotes the nonlinear (polynomial) part of the system. For briefness of notations, we do not rewrite system (49)–(50) for variables $(\bar{c}_n^+, \bar{c}_{-n}^-)$ and we do not display dependence of the nonlinear functions from these variables.

Lemma 6

Let $W(x)$ satisfy Assumption 1 and $w_{2n} \neq 0$ for a given $n \in \mathbb{N}$. Let $|\Omega| < \Omega_0 = |w_{2n}|\sqrt{(n^2 - c^2)}/n$ and $0 < c < n$. For sufficiently small ε, the linearized system (49)–(51) at the zero solution is topologically equivalent to the one for $\varepsilon = 0$, except that the double zero eigenvalue of the subsystem (49)–(50) splits into a pair of complex eigenvalues to the left and right half-planes.

Proof

The matrix operator Λ_ε is a polynomial in ε and Λ_0 is a diagonal unbounded matrix operator that consists of $(ik_m^+, -ik_m^-, -ik_m^+, ik_m^-)$ for all $m \in \mathbb{Z}'_-$ and of $(\kappa_m^+, \kappa_m^-, \bar{\kappa}_m^+, \bar{\kappa}_m^-)$ for all $m \in \mathbb{Z}'_+$. The matrix operator with elements $\varepsilon w_{m,m'}$ represents a small perturbation to Λ_0 if the vector of Fourier

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coefficients $\mathbf{W}$ is in $l_s^2(\mathbb{Z})\subset l^1(\mathbb{Z})$ for any $s>\frac{1}{2}$ according to Assumption 1. Since all eigenvalues of Λ_0 are semi-simple, they are structurally stable with respect to perturbations for sufficiently small $\varepsilon\neq 0$.

The coefficients $\mu_\varepsilon^\pm$ and $v_\varepsilon^\pm$ are polynomials in ε and

$$\mu_0^+=\frac{\mathrm{i}\Omega}{n-c},\quad \mu_0^-=\frac{\mathrm{i}w_{2n}}{\sqrt{n^2-c^2}},\quad v_0^+=\frac{-\mathrm{i}w_{2n}}{\sqrt{n^2-c^2}},\quad v_0^-=\frac{-\mathrm{i}\Omega}{n+c}$$

The characteristic equation of subsystem (49)–(50) at $\varepsilon=0$ is given by

$$(n^2-c^2)\kappa^2-2\mathrm{i}\varepsilon c\Omega\kappa+\varepsilon^2(\Omega^2-w_{2n}^2)=0$$

with two roots

$$\kappa=\kappa_\pm=\varepsilon\frac{\mathrm{i}\Omega c\pm\sqrt{(n^2-c^2)w_{2n}^2-n^2\Omega^2}}{n^2-c^2}$$

The two roots have $\mathrm{Re}\,\kappa_\pm\gtrless 0$ if $|\Omega|<\Omega_0$, where $\Omega_0=|w_{2n}|\sqrt{(n^2-c^2)}/n$. Under the same assumption on $\mathbf{W}$, perturbation terms in $\mu_\varepsilon^\pm$ and $v_\varepsilon^\pm$ are small compared with the leading-order terms $\mu_0^\pm$ and $v_0^\pm$, such that κ_+ and κ_- persist in the left and right half-planes, respectively. □

Corollary 1
For sufficiently small ε, a local neighborhood of the zero point in the phase space X' can be decomposed into the subspaces determined by the spectrum of the linearized system at the zero solution

$$X'=X_h\oplus X_c\oplus X_u\oplus X_s\equiv X_h\oplus X_h^\perp \tag{52}$$

where X_s and X_u are associated with the subspaces E^{s} and E^{u} of Lemma 1, while X_h and X_c are associated with the subspaces $E_n^+\oplus E_{-n}^-$ and $E^{c^+}\oplus E^{c^-}\backslash E_n^+\oplus E_{-n}^-$, respectively.

By Remark 5, the truncated system (49)–(51) with $F_R^\pm\equiv 0$ and $\mathbf{F}_R\equiv\mathbf{0}$ admits an invariant reduction on S. By Lemma 5, the extended coupled-mode system (44) has a reversible homoclinic orbit which satisfies the decay bound (45). This construction enables us to represent solutions of system (49)–(50) and its complex conjugate in the form $[c_n^+,c_{-n}^-,\bar{c}_n^+,\bar{c}_{-n}^-]^{\mathrm{T}}=\mathbf{c}_0(\varepsilon y)+\mathbf{c}_h(y)$, where $\mathbf{c}_0(\varepsilon y)$ is the homoclinic orbit of Lemma 5 and $\mathbf{c}_h(y)$ is a perturbation term. By using the decomposition, we rewrite system (49)–(51) in the equivalent form

$$\frac{\mathrm{d}\mathbf{c}_h}{\mathrm{d}y}=\varepsilon\Lambda_h(\mathbf{c}_0(\varepsilon y))\mathbf{c}_h+\varepsilon\mathbf{G}_T(\mathbf{c}_0(\varepsilon y))(\mathbf{c}_h,\mathbf{c})+\varepsilon^{N+1}\mathbf{G}_R(\mathbf{c}_0(\varepsilon y)+\mathbf{c}_h,\mathbf{c}) \tag{53}$$

$$\frac{\mathrm{d}\mathbf{c}}{\mathrm{d}y}=\Lambda_\varepsilon\mathbf{c}+\varepsilon\mathbf{F}_T(\mathbf{c}_0(\varepsilon y)+\mathbf{c}_h,\mathbf{c})+\varepsilon^{N+1}\mathbf{F}_R(\mathbf{c}_0(\varepsilon y)+\mathbf{c}_h,\mathbf{c}) \tag{54}$$

where $\Lambda_h(\mathbf{c}_0(\varepsilon y))$ is a 4×4 linearization matrix of the extended coupled-mode system (44) and its complex conjugate at the solution $\mathbf{c}_0(\varepsilon y)$ and $(\mathbf{G}_T,\mathbf{G}_R)$ is a nonlinear vector field obtained from system (49)–(50) and its conjugate. We note that the function $\mathbf{G}_T$ combines nonlinear terms in $\mathbf{c}_h$ from the functions $F_S^\pm$ and the nonlinear terms in $\mathbf{c}$ from the functions $F_T^\pm$, while the function $\mathbf{G}_R$ is obtained from the functions $F_R^\pm$ by direct substitution.

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Example 2
At the lowest order ($N=1$), the linearization matrix $\Lambda_h(\mathbf{c}_0(\varepsilon y))$ takes the explicit form

$$\mathrm{i}\begin{bmatrix} \frac{\Omega}{n-c}-\frac{2\sigma|c_n^+|^2}{(n-c)^2}-\frac{2\sigma|c_{-n}^-|^2}{n^2-c^2} & -\frac{w_{2n}}{\sqrt{n^2-c^2}}-\frac{2\sigma c_n^+\bar{c}_{-n}^-}{n^2-c^2} & -\frac{\sigma c_n^{+2}}{(n-c)^2} & -\frac{2\sigma c_n^+ c_{-n}^-}{n^2-c^2} \\ \frac{w_{2n}}{\sqrt{n^2-c^2}}+\frac{2\sigma\bar{c}_n^+ c_{-n}^-}{n^2-c^2} & -\frac{\Omega}{n+c}+\frac{2\sigma|c_n^+|^2}{n^2-c^2}+\frac{2\sigma|c_{-n}^-|^2}{(n+c)^2} & \frac{2\sigma c_n^+ c_{-n}^-}{n^2-c^2} & \frac{\sigma c_{-n}^{-2}}{(n+c)^2} \\ \frac{\sigma\bar{c}_n^{+2}}{(n-c)^2} & \frac{2\sigma\bar{c}_n^+\bar{c}_{-n}^-}{n^2-c^2} & -\frac{\Omega}{n-c}+\frac{2\sigma|c_n^+|^2}{(n-c)^2}+\frac{2\sigma|c_{-n}^-|^2}{n^2-c^2} & \frac{w_{2n}}{\sqrt{n^2-c^2}}+\frac{2\sigma\bar{c}_n^+ c_{-n}^-}{n^2-c^2} \\ -\frac{2\sigma\bar{c}_n^+\bar{c}_{-n}^-}{n^2-c^2} & -\frac{\sigma\bar{c}_{-n}^{-2}}{(n+c)^2} & -\frac{w_{2n}}{\sqrt{n^2-c^2}}-\frac{2\sigma c_n^+\bar{c}_{-n}^-}{n^2-c^2} & \frac{\Omega}{n+c}-\frac{2\sigma|c_n^+|^2}{n^2-c^2}-\frac{2\sigma|c_{-n}^-|^2}{(n+c)^2} \end{bmatrix}$$

where components of $[c_n^+, c_{-n}^-, \bar{c}_n^+, \bar{c}_{-n}^-]^{\mathrm{T}}$ are evaluated at the solution $\mathbf{c}_0$ of the coupled-mode system. As $|y|\to\infty$, matrix $\Lambda_h(\mathbf{c}_0(\varepsilon y))$ converges to the form

$$\Lambda_h(\mathbf{0})=\mathrm{i}\begin{bmatrix} \frac{\Omega}{n-c} & -\frac{w_{2n}}{\sqrt{n^2-c^2}} & 0 & 0 \\ \frac{w_{2n}}{\sqrt{n^2-c^2}} & -\frac{\Omega}{n+c} & 0 & 0 \\ 0 & 0 & -\frac{\Omega}{n-c} & \frac{w_{2n}}{\sqrt{n^2-c^2}} \\ 0 & 0 & -\frac{w_{2n}}{\sqrt{n^2-c^2}} & \frac{\Omega}{n+c} \end{bmatrix}$$

such that $\|\Lambda_h(\mathbf{c}_0(\varepsilon y))-\Lambda_h(\mathbf{0})\|_{X_h\mapsto X_h}\leqslant C\mathrm{e}^{-\varepsilon\gamma|y|}$ for some $C>0$ and $\gamma>0$ according to the decay bound (45).

Lemma 7
Let $w_{2n}\neq 0$ for a given $n\in\mathbb{N}$. Let $|\Omega|<\Omega_0=|w_{2n}|\sqrt{(n^2-c^2)}/n$ and $0<c<n$. Consider the linear inhomogeneous equation

$$\frac{\mathrm{d}\mathbf{c}_h}{\mathrm{d}y}-\varepsilon\Lambda_h(\mathbf{c}_0(\varepsilon y))\mathbf{c}_h=\mathbf{F}_h(y) \tag{55}$$

where $\mathbf{F}_h\in C_b^0(\mathbb{R})$. The homogeneous equation has a two-dimensional stable manifold spanned by the two fundamental solutions

$$\mathbf{s}_1=\mathbf{c}_0'(\varepsilon y), \quad \mathbf{s}_2=\sigma_1\mathbf{c}_0(\varepsilon y) \tag{56}$$

where σ_1 is a diagonal matrix of $(1,1,-1,-1)$. If components of $\mathbf{F}_h(y)$ satisfy the constraints

$$\begin{aligned} (\mathbf{F}_h)_1(y) &= (\bar{\mathbf{F}}_h)_1(-y), \quad (\mathbf{F}_h)_2(y)=(\bar{\mathbf{F}}_h)_2(-y) \\ (\mathbf{F}_h)_3(y) &= -(\bar{\mathbf{F}}_h)_1(y), \quad (\mathbf{F}_h)_4(y)=-(\bar{\mathbf{F}}_h)_4(y) \end{aligned} \tag{57}$$

for all $y\in\mathbb{R}$, then there exists a two-parameter family of solutions $\mathbf{c}_h\in C_b^0(\mathbb{R})$ in the form $\mathbf{c}_h=\alpha_1\mathbf{s}_1(y)+\alpha_2\mathbf{s}_2(y)+\tilde{\mathbf{c}}_h(y)$, where (α_1,α_2) are parameters and $\tilde{\mathbf{c}}_h(y)$ is a particular solution of

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the inhomogeneous equation (55), such that $\|\tilde{\mathbf{c}}_h\|_{C_b^0(\mathbb{R})}\leqslant(C/\varepsilon)\|\mathbf{F}_h\|_{C_b^0(\mathbb{R})}$ for an ε-independent constant C.

Proof

The existence of the two-dimensional kernel (56) follows from symmetries of the extended coupled-mode system (44) with respect to translation and gauge transformation. Since the subspace X_h associated with $\Lambda_h(\mathbf{0})$ is invariant under $\Lambda_h(\mathbf{c}_0(\varepsilon y))$, the kernel is exactly two-dimensional and the other two fundamental solutions of the homogeneous equation are exponentially growing. The adjoint homogeneous equation $\mathrm{d}\mathbf{u}/\mathrm{d}y=-\varepsilon\Lambda_h^*(\mathbf{c}_0(\varepsilon y))\mathbf{u}$ has also a two-dimensional stable manifold spanned by the two fundamental solutions:

$$\mathbf{s}_1^*=\sigma_2\mathbf{c}_0'(\varepsilon y), \quad \mathbf{s}_2^*=\sigma_3\mathbf{c}_0(\varepsilon y) \tag{58}$$

where σ_2 and σ_3 are diagonal matrices of $(1,-1,-1,1)$ and $(1,-1,1,-1)$, respectively. By the Fredholm Alternative Theorem, a solution of the linear inhomogeneous equation (55) is bounded on $y\in\mathbb{R}$ if and only if the function $\mathbf{F}_h$ is orthogonal to $\{\mathbf{s}_1^*,\mathbf{s}_2^*\}$. The orthogonality conditions $(\mathbf{s}_1^*,\mathbf{F}_h)=(\mathbf{s}_2^*,\mathbf{F}_h)=0$ are satisfied if the constraints (57) are met. Since the norm in $C_b^0(\mathbb{R})$ is invariant with respect to the transformation $y\mapsto Y=\varepsilon y$, if the constraints (57) are satisfied, there exists a solution $\tilde{\mathbf{c}}_h\in C_b^0(\mathbb{R})$ of the inhomogeneous equation (55) such that $\|\tilde{\mathbf{c}}_h\|_{C_b^0(\mathbb{R})}\leqslant(C/\varepsilon)\|\mathbf{F}_h\|_{C_b^0(\mathbb{R})}$ for an ε-independent constant C. A general solution has the form $\mathbf{c}_h=\alpha_1\mathbf{s}_1(y)+\alpha_2\mathbf{s}_2(y)+\tilde{\mathbf{c}}_h(y)$, where (α_1,α_2) are parameters. □

Lemma 8

The nonlinear part of the vector field of system (53)–(54) is bounded in a local neighborhood of the zero point in $X'=X_h\oplus X_h^{\perp}$ by

$$\|\mathbf{G}_R\|_{X_h}\leqslant N_R(\|\mathbf{c}_0+\mathbf{c}_h\|_{X_h}+\|\mathbf{c}\|_{X_h^{\perp}}), \quad \|\mathbf{F}_R\|_{X_h^{\perp}}\leqslant M_R(\|\mathbf{c}_0+\mathbf{c}_h\|_{X_h}+\|\mathbf{c}\|_{X_h^{\perp}}) \tag{59}$$

$$\|\mathbf{G}_T\|_{X_h}\leqslant N_T(\|\mathbf{c}_h\|_{X_h}^2+\|\mathbf{c}\|_{X_h^{\perp}}^2), \quad \|\mathbf{F}_T\|_{X'}\leqslant M_T(\|\mathbf{c}_0+\mathbf{c}_h\|_{X_h}+\|\mathbf{c}\|_{X_h^{\perp}})\|\mathbf{c}\|_{X_h^{\perp}} \tag{60}$$

for some $N_R,M_R,N_T,M_T>0$.

Proof

Equations of system (53)–(54) are semi-linear and the vector field for a finite $N\in\mathbb{N}$ is a polynomial acting on $D'\subset X'$. If X is the Banach algebra for $s>\frac{1}{2}$, then X' is the Banach algebra for $s'>\frac{1}{4}$. The derivation of estimates (59)–(60) follows similar to Lemmas 2 and 3 in [10]. The characterization of $\mathbf{G}_T$ and $\mathbf{F}_T$ is based on the fact that the Hamiltonian function H_T is quadratic with respect to $\mathbf{c}$ by Lemma 4. □

Remark 6

Using Corollary 1, we denote $\Lambda_c=\Lambda_\varepsilon|_{X_c}$, $\Lambda_u=\Lambda_\varepsilon|_{X_u}$, $\Lambda_s=\Lambda_\varepsilon|_{X_s}$ for a block-diagonal decomposition of Λ_ε on the invariant subspaces X_c, X_u and X_s, respectively. We also denote the coordinates of the decomposition by $\mathbf{c}=[\mathbf{c}_c,\mathbf{c}_u,\mathbf{c}_s]$ and the projection operators by P_c, P_u, P_s, respectively.

Theorem 3 (*Local center-stable manifold*)

Let $\mathbf{a}\in X_c$, $\mathbf{b}\in X_s$ and $(\alpha_1,\alpha_2)\in\mathbb{C}^2$ be small such that

$$\|\mathbf{a}\|_{X_c}\leqslant C_a\varepsilon^N, \quad \|\mathbf{b}\|_{X_s}\leqslant C_b\varepsilon^N, \quad |\alpha_1|+|\alpha_2|\leqslant C_\alpha\varepsilon^N \tag{61}$$

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for some ε-independent constants $C_a, C_b, C_\alpha > 0$. Under the conditions of Lemma 6, there exists a family of local solutions $\mathbf{c}_h = \mathbf{c}_h(y; \mathbf{a}, \mathbf{b}, \alpha_1, \alpha_2)$ and $\mathbf{c} = \mathbf{c}(y; \mathbf{a}, \mathbf{b}, \alpha_1, \alpha_2)$ of system (53)–(54) such that

$$\mathbf{c}_c(0) = \mathbf{a}, \quad \mathbf{c}_s = \mathrm{e}^{y\Lambda_s}\mathbf{b} + \tilde{\mathbf{c}}_s(y), \quad \mathbf{c}_h = \alpha_1 \mathbf{s}_1(y) + \alpha_2 \mathbf{s}_2(y) + \tilde{\mathbf{c}}_h(y) \tag{62}$$

where $\tilde{\mathbf{c}}_s(y)$ and $\tilde{\mathbf{c}}_h(y)$ are uniquely defined and the family of local solutions satisfies the bound

$$\sup_{\forall y \in [0, L/\varepsilon^{N+1}]} \|\mathbf{c}_h(y)\|_{X_h} \leqslant C_h \varepsilon^N, \qquad \sup_{\forall y \in [0, L/\varepsilon^{N+1}]} \|\mathbf{c}(y)\|_{X_h^\perp} \leqslant C \varepsilon^N \tag{63}$$

for some ε-independent constants $L > 0$ and $C_h, C > 0$.

Proof

We modify system (53)–(54) by the following trick. We multiply the nonlinear vector field of system (54) by the cut-off function $\chi_{[0,y_0]}(y)$, such that

$$\frac{\mathrm{d}\mathbf{c}}{\mathrm{d}y} - \Lambda_\varepsilon \mathbf{c} = \varepsilon \chi_{[0,y_0]}(y) \mathbf{F}_T(\mathbf{c}_0(\varepsilon y) + \mathbf{c}_h, \mathbf{c}) + \varepsilon^{N+1} \chi_{[0,y_0]}(y) \mathbf{F}_R(\mathbf{c}_0(\varepsilon y) + \mathbf{c}_h, \mathbf{c}) \tag{64}$$

where $\chi_{[0,y_0]}(y) = 1$ for all $y \in [0, y_0]$ and $\chi_{[0,y_0]}(y) = 0$ for all $y \in \mathbb{R} \backslash [0, y_0]$. Similarly, we multiply the nonlinear vector field of system (53) by the cut-off function $\chi_{[0,y_0]}(y)$ and add a symmetrically reflected vector field multiplied by the cut-off function $\chi_{[-y_0,0]}(y)$, such that

$$\begin{aligned} &\frac{\mathrm{d}\mathbf{c}_h}{\mathrm{d}y} - \varepsilon \Lambda_h(\mathbf{c}_0(\varepsilon y)) \mathbf{c}_h \\ &\quad = \varepsilon \chi_{[0,y_0]}(y) \mathbf{G}_T(\mathbf{c}_0(\varepsilon y))(\mathbf{c}_h, \mathbf{c}) + \varepsilon^{N+1} \chi_{[0,y_0]}(y) \mathbf{G}_R(\mathbf{c}_0(\varepsilon y) + \mathbf{c}_h, \mathbf{c}) \\ &\qquad + \varepsilon \chi_{[-y_0,0]}(y) \mathbf{G}_T^*(\mathbf{c}_0(\varepsilon y))(\mathbf{c}_h, \mathbf{c}) + \varepsilon^{N+1} \chi_{[-y_0,0]}(y) \mathbf{G}_R^*(\mathbf{c}_0(\varepsilon y) + \mathbf{c}_h, \mathbf{c}) \end{aligned} \tag{65}$$

where $(\mathbf{G}_{T,R}^*)_{1,2}(y) = (\bar{\mathbf{G}}_{T,R})_{1,2}(-y)$ and $(\mathbf{G}_{T,R}^*)_{3,4}(y) = -(\bar{\mathbf{G}}_{T,R}^*)_{1,2}(y)$ for all $y \in [-y_0, 0)$. We are looking for a global solution of system (64)–(65) in the space of bounded continuous functions $C_b^0(\mathbb{R})$. By uniqueness of solutions of semi-linear ODEs, this global solution on $y \in \mathbb{R}$ coincides with a local solution of system (53)–(54) on the interval $y \in [0, y_0] \subset \mathbb{R}$.

Let $\mathbf{c}_s(y) = \mathrm{e}^{y\Lambda_s}\mathbf{b} + \tilde{\mathbf{c}}_s(y)$ and look for a solution $\tilde{\mathbf{c}}_s(y)$ and $\mathbf{c}_u(y)$ of system (64) projected to X_s and X_u with operators P_s and P_u. By Lemmas 3, 8, and the Implicit Function Theorem, there exists a unique map from $C_b^0(\mathbb{R}, X_h \oplus X_c)$ to $C_b^0(\mathbb{R}, X_u \oplus X_s)$ such that

$$\begin{aligned} \sup_{\forall y \in [0,y_0]} [\|\mathbf{c}_u(y)\|_{X_u} + \|\mathbf{c}_s(y)\|_{X_s}] \leqslant \|\mathbf{b}\|_{X_s} &+ \varepsilon M_1 \sup_{\forall y \in [0,y_0]} [(1 + \|\mathbf{c}_h(y)\|_{X_h} + \|\mathbf{c}_c(y)\|_{X_c}) \|\mathbf{c}_c(y)\|_{X_c}] \\ &+ \varepsilon^{N+1} M_2 \sup_{\forall y \in [0,y_0]} [1 + \|\mathbf{c}_h(y)\|_{X_h} + \|\mathbf{c}_c(y)\|_{X_c}] \end{aligned} \tag{66}$$

for some $M_1, M_2 > 0$. To use Lemma 3, we note that the linear part of the system for $\mathbf{c}_s$ and $\mathbf{c}_u$ is not affected by the near-identity transformations of Lemma 4 and it can be converted back to the scalar second-order equation (23) by using transformations (24) and (34).

Let $\mathbf{c}_h(y) = \alpha_1 \mathbf{s}_1(y) + \alpha_2 \mathbf{s}_2(y) + \tilde{\mathbf{c}}_h(y)$ and look for a solution $\tilde{\mathbf{c}}_h(y)$ of system (65). Since system (53) is obtained from the Hamilton system of equations (36), the vector field of system (53) satisfies the constraints $(\mathbf{G}_{T,R})_{3,4}(y) = -(\bar{\mathbf{G}}_{T,R})_{1,2}(y)$ for all $y \in [0, y_0]$. By the construction of the

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modified vector field in system (65), the modified vector field satisfies the constraints (57) for all $y\in\mathbb{R}$. By Lemmas 7, 8, the bound (66), and the Implicit Function Theorem, there exists a unique map from $C_b^0(\mathbb{R}, X_c)$ to $C_b^0(\mathbb{R}, X_h)$ such that

$$\sup_{\forall y\in[0,y_0]}\|\mathbf{c}_h(y)\|_{X_h}\leqslant|\alpha_1|+|\alpha_2|+M_3\sup_{\forall y\in[0,y_0]}\|\mathbf{c}_c(y)\|^2_{X_c}+\varepsilon^N M_4\sup_{\forall y\in[0,y_0]}[1+\|\mathbf{c}_c(y)\|_{X_c}] \quad (67)$$

for some $M_3, M_4>0$.

Since the spectrum of Λ_c consists of pairs of semi-simple purely imaginary eigenvalues, the operator Λ_c generates a strongly continuous group $\mathrm{e}^{y\Lambda_c}$ for all $y\in\mathbb{R}$ on X_c such that

$$\sup_{\forall y\in\mathbb{R}}\|\mathrm{e}^{y\Lambda_c}\|_{X_c\mapsto X_c}\leqslant K \quad (68)$$

for some $K>0$. By variation of constant formula, the solution of system (64) projected to X_c can be rewritten in the integral form

$$\mathbf{c}_c(y)=\mathrm{e}^{y\Lambda_c}\mathbf{a}+\varepsilon\int_0^y \mathrm{e}^{(y-y')\Lambda_c}P_c[\mathbf{F}_T(\mathbf{c}_0(\varepsilon y')+\mathbf{c}_h(y'),\mathbf{c}(y'))$$
$$+\varepsilon^N\mathbf{F}_R(\mathbf{c}_0(\varepsilon y')+\mathbf{c}_h(y'),\mathbf{c}(y'))]\,\mathrm{d}y' \quad (69)$$

where $\mathbf{a}=\mathbf{c}_c(0)$. By using the bound (45) for $\mathbf{c}_0(\varepsilon y)$ and the bounds (66) and (67) on the components $\mathbf{c}_s$, $\mathbf{c}_u$, and $\mathbf{c}_h$, the integral equation (69) results in the bound

$$\sup_{\forall y\in[0,y_0]}\|\mathbf{c}_c(y)\|_{X_c}\leqslant K\Bigg(\|\mathbf{a}\|_{X_c}+\|\mathbf{b}\|_{X_s}+|\alpha_1|+|\alpha_2|+\varepsilon M_5\int_0^{y_0}\|\mathbf{c}_0(\varepsilon y)\|_{X_h}\|\mathbf{c}_c(y)\|_{X_c}\,\mathrm{d}y$$
$$+\varepsilon y_0 M_6\sup_{\forall y\in[0,y_0]}\|\mathbf{c}_c(y)\|^2_{X_c}+\varepsilon^{N+1}M_7$$
$$\times\int_0^{y_0}\|\mathbf{c}_0(\varepsilon y)\|_{X_h}\,\mathrm{d}y+\varepsilon^{N+1}y_0M_8\sup_{\forall y\in[0,y_0]}\|\mathbf{c}_c(y)\|_{X_c}\Bigg)$$

for some $M_5, M_6, M_7, M_8>0$. By the Gronwall inequality, we have thus obtained that

$$\sup_{\forall y\in[0,y_0]}\|\mathbf{c}_c(y)\|_{X_c}\leqslant K\mathrm{e}^{\varepsilon K M_5\int_0^{y_0}\|\mathbf{c}_0(\varepsilon y)\|_{X_h}\,\mathrm{d}y}\Bigg(\|\mathbf{a}\|_{X_c}+\|\mathbf{b}\|_{X_s}+|\alpha_1|+|\alpha_2|$$
$$+\varepsilon^N M_9+\varepsilon y_0 M_6\sup_{\forall y\in[0,y_0]}\|\mathbf{c}_c(y)\|^2_{X_c}+\varepsilon^{N+1}y_0M_8\sup_{\forall y\in[0,y_0]}\|\mathbf{c}_c(y)\|_{X_c}\Bigg)$$

for some $M_9>0$. It follows by the decay bound (45) that $\varepsilon\int_0^{y_0}\|\mathbf{c}_0(\varepsilon y)\|_{X_h}\,\mathrm{d}y\leqslant C$ for some ε-independent $C>0$. If we let $y_0=L/\varepsilon^M$, then the bound is consistent for $M\leqslant N+1$, where the value $M=N+1$ gives the balance of all terms in the upper bound above. If arbitrary vectors **a**, **b**, and (α_1,α_2) satisfy the bound (61), then the local solution $\mathbf{c}_c(y)$ satisfies the bound

$$\sup_{\forall y\in[0,L/\varepsilon^{N+1}]}\|\mathbf{c}_c(y)\|_{X_c}\leqslant\tilde{C}_c\varepsilon^N \quad (70)$$

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for some $\tilde{C}_c > 0$. By using the bounds (61), (66), (67), and (70), we have proved the bound (63) for some ε-independent constants $C_h, C > 0$. □

Remark 7
One can prove Theorem 3 by using the contraction mapping principle and the integral formulation for the local center manifold of system (53)–(54). This approach was undertaken in Section 4 of [10] (see Theorem 4 in [10]). We have avoided this unnecessary complication with the explicit decomposition (52) and analysis of system (53)–(54) decomposed into subsystems. Similar direct methods of analysis have been applied to problems without Hamiltonian structure such as the quasi-linear wave equation in [11], where an iteration scheme was employed to prove the bound on small local solutions along the local center-stable manifold.

Proof of Theorem 2
By Theorem 3, we have constructed an infinite-dimensional continuous family of local bounded solutions of system (53)–(54) on $y \in [0, L/\varepsilon^{N+1}]$ for some ε-independent constant $L > 0$. The solutions are close to the reversible homoclinic orbit of the extended coupled-mode system (44) in the sense of the bound (14) on $y \in [0, L/\varepsilon^{N+1}]$. It remains to extend the local solution to the symmetric interval $y \in [-L/\varepsilon^{N+1}, L/\varepsilon^{N+1}]$ under the reversibility constraints (38). To do so, we shall consider the intersections of the local invariant manifold of system (53)–(54) with the symmetric section Σ'_r defined by (39).

Since the initial data $\mathbf{c}_c(0) = \mathbf{a}$ in the local center-stable manifold of Theorem 3 are arbitrary, the components of $\mathbf{a}$ can be chosen to lie in the symmetric section Σ'_r, such that

$$\mathrm{Im}(\mathbf{a})_m^+ = 0 \;\; \forall m \in \mathbb{Z}'_- \backslash \{n\}, \quad \mathrm{Im}(\mathbf{a})_m^- = 0 \;\; \forall m \in \mathbb{Z}'_- \backslash \{-n\} \tag{71}$$

This construction still leaves infinitely many arbitrary parameters for

$$\mathrm{Re}(\mathbf{a})_m^+ \;\; \forall m \in \mathbb{Z}'_- \backslash \{n\}, \quad \mathrm{Re}(\mathbf{a})_m^- \;\; \forall m \in \mathbb{Z}'_- \backslash \{-n\} \tag{72}$$

to be chosen in the bound (61). The initial data $\mathbf{c}_h(0)$ and $\mathbf{c}_{s,u}(0)$ are not arbitrary since we have used the Implicit Function Theorem for the mappings (66) and (67). Therefore, we have to show that the components of $\mathbf{b}$ and (α_1, α_2) can be chosen uniquely so that the local center-stable manifold intersects at $y = 0$ with the symmetric section Σ'_r.

There are as many arbitrary parameters $\mathbf{b}$ and (α_1, α_2) in the local center-stable manifold as there are remaining constraints in the set Σ'_r. First, let us consider constraints in the set Σ'_r for all $m \in \mathbb{Z}'_+$, namely

$$\mathrm{Re}\, c_m^+(0) = \mathrm{Re}\, c_m^-(0), \quad \mathrm{Im}\, c_m^+(0) = -\mathrm{Im}\, c_m^-(0) \;\; \forall m \in \mathbb{Z}'_+ \tag{73}$$

Let $\mathbf{c}_s = \mathrm{e}^{y\Lambda_s}\mathbf{b} + \tilde{\mathbf{c}}_s(y)$ and rewrite the constraints in the form

$$\mathrm{Re}\, b_m + \mathrm{Re}(\tilde{\mathbf{c}}_s)_m(0) = \mathrm{Re}(\mathbf{c}_u)_m(0), \quad \mathrm{Im}\, b_m + \mathrm{Im}(\tilde{\mathbf{c}}_s)_m(0) = -\mathrm{Im}(\mathbf{c}_u)_m(0) \;\; \forall m \in \mathbb{Z}'_+ \tag{74}$$

where $\tilde{\mathbf{c}}_s(0)$ and $\mathbf{c}_u(0)$ are of order $O(\varepsilon^N)$ and depend on $\mathbf{b}$ in higher orders of $O(\varepsilon^k)$ with $k > N$. By the Implicit Function Theorem, there exists a unique solution $\mathbf{b}$ of the constraints (74) satisfying the bound (61).

Finally, let us consider constraints in the set Σ'_r for components of $\mathbf{c}_h$, namely

$$\mathrm{Im}\, c_n^+(0) = 0, \quad \mathrm{Im}\, c_{-n}^-(0) = 0 \tag{75}$$

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Let $\mathbf{c}_h = \alpha_1 \mathbf{s}_1 + \alpha_2 \mathbf{s}_2 + \tilde{\mathbf{c}}_s(y)$ and note that $\mathbf{s}_1$ and $\mathbf{s}_2$ violate the constraints (75). Let $\mathbf{a}$ and $\mathbf{b}$ be chosen so that $\mathbf{c}(0)$ belongs to the set Σ_r'. Owing to a construction of the modified vector field in system (65), if $\mathbf{c}(y)$ lies in the domain D_r' and $\alpha_1 = \alpha_2 = 0$, then $\mathbf{G}^*_{T,R} = \mathbf{G}_{T,R}$ for all $y \in [-y_0, y_0]$. In this case, the global solution $\tilde{\mathbf{c}}_s(y)$ constructed in Theorem 3 intersects the set Σ_r' at $y = 0$. Therefore, the choice $\alpha_1 = \alpha_2 = 0$ satisfies the constraints (75) identically. Uniqueness of $\alpha_1 = \alpha_2 = 0$ among solutions of the constraints (75) satisfying the bound (61) follows again by the Implicit Function Theorem for system (75).

We have thus constructed a family of reversible solutions in the symmetric interval $[-L/\varepsilon^{N+1}, L/\varepsilon^{N+1}]$ in variable y while preserving the bound (14). Tracing the coordinate transformations used in our analysis back to the original variable $\psi(x, y)$, we have thus completed the proof of Theorem 2. □

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