INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 6129–6140 doi:10.1088/0305-4470/38/27/003

# Spectral stability and time evolution of *N*-solitons in the KdV hierarchy

**Yuji Kodama**[1] **and Dmitry Pelinovsky**[2]

[1] Department of Mathematics, Ohio State University, Columbus, OH 43210, USA
[2] Department of Mathematics, McMaster University, Hamilton, ON L8S 4K1, Canada

Received 8 April 2005, in final form 10 April 2005
Published 22 June 2005
Online at stacks.iop.org/JPhysA/38/6129

**Abstract**

This paper concerns spectral stability and time evolution of $N$-solitons in the Korteweg–de Vries (KdV) hierarchy with mixed commuting time flows. Spectral stability problem is analysed by using a pair of self-adjoint operators with finite numbers of negative eigenvalues. We show that the absence of unstable eigenvalues in the stability problem is related to the absence of negative eigenvalues of these operators in the constrained function spaces. Time evolution of $N$-solitons is uniquely characterized from the inverse scattering transform technique.

PACS numbers: 02.30.Jr, 05.45.Yv

## 1. Introduction

We address the stability problem for solitary waves in the KdV-type evolution equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial}{\partial x}\frac{\delta H}{\delta u}, \tag{1.1}$$

where $u \in \mathbb{R}$, $x \in \mathbb{R}$, $t \in \mathbb{R}_+$ and $H(u)$ is the Hamiltonian. Besides the canonical Korteweg–de Vries (KdV) equation [N85], the time-evolution problem (1.1) includes higher order KdV equations of the integrable KdV hierarchy (e.g. see [KN78, KT78]). The integrable KdV hierarchy can be derived with the asymptotic multi-scale expansion technique for modelling of solitary waves in physical non-integrable problems, e.g. in ferromagnets [L02, L03] (see also [HK02] for the normal form of perturbed KdV equation).

Our work originates from analysis of spectral stability of solitary waves in the KdV-type evolution equations [BSS87, W87, SS90, PW92]. Let us assume that the evolution problem (1.1) has a solitary wave solution $u = u_0(x, t)$ that decays exponentially in space $x \in \mathbb{R}$ and changes in time $t \in \mathbb{R}_+$ according to symmetries of (1.1). The spectral stability problem for KdV solitary waves takes the general form

$$\partial_x \mathcal{L} v = \lambda v, \tag{1.2}$$

where $v \in \mathbb{C}$, $\lambda \in \mathbb{C}$ and the self-adjoint linearized operator $\mathcal{L}$ is computed at the solution $u_0(x,t)$ after separation of variables $(x,t)$, i.e. $\mathcal{L} = D^2 H(u_0)$. The solitary wave solution $u_0(x,t)$ is spectrally unstable if there exists an eigenvalue $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ and $v \in L^2(\mathbb{R})$. It is weakly spectrally stable if no eigenvalues $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ and $v \in L^2(\mathbb{R})$ exist.

We shall characterize unstable eigenvalues of the stability problem (1.2) from the study of the self-adjoint operator $\mathcal{L}$. The operator $\mathcal{L}$ defines the energy quadratic form

$$h(v) = (v, \mathcal{L}v) := \int_{\mathbb{R}} v(x)(\mathcal{L}v)(x)\,\mathrm{d}x. \tag{1.3}$$

The relation between eigenvalues of the stability problem and those of the linearized energy was recently studied in the context of spectral stability of solitary waves in the nonlinear Schrödinger equations [CPV05, P05]. The number of unstable eigenvalues in the spectral stability problem was found to be related to the number of negative eigenvalues of the energy quadratic form in a constrained function space. The same relation was derived independently in [KKS04] by extending earlier results [G90] to an abstract Hamiltonian dynamical system.

The methods of [KKS04, P05] are not applicable to the KdV-type evolution equations, since the symplectic operator $\partial_x$ is not invertible. Nevertheless, we will show with explicit analysis of quadratic forms that the spectral stability problem (1.2) can be embedded into a larger problem, which has the same structure as that considered in [KKS04, P05]. Since the operator $\mathcal{L}$ maps $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, the eigenfunction $v(x) \in L^2(\mathbb{R})$ for $\lambda \neq 0$ must satisfy the constraint

$$(1, v) = \int_{\mathbb{R}} v(x)\,\mathrm{d}x = 0. \tag{1.4}$$

If $v \in L^2(\mathbb{R})$ decays exponentially as $|x| \to \infty$ and satisfies the constraint (1.4), the eigenfunction of the spectral problem (1.2) can be represented as $v = -w'(x)$, where $w \in L^2(\mathbb{R})$, i.e. $v \in \mathrm{im}(\partial_x)$. We shall hence replace the stability problem (1.2) by the coupled system

$$\mathcal{L}v = -\lambda w, \qquad \mathcal{M}w = \lambda v, \tag{1.5}$$

where $\mathcal{M} = -\partial_x \mathcal{L} \partial_x$. Eliminating $w$ from the system (1.5), we reduce the coupled problem (1.5) to the scalar problem

$$\partial_x \mathcal{L} \partial_x \mathcal{L} v = \lambda^2 v, \tag{1.6}$$

which has both eigenvalues $\lambda$ and $-\lambda$ of the original problem (1.2). Moreover, if $\lambda$ is a simple eigenvalue of $\partial_x \mathcal{L}$, then the system (1.5) has the solution $v = -w'(x)$, while if $\lambda$ is a simple eigenvalue of $-\partial_x \mathcal{L}$, then the system (1.5) has the solution $v = w'(x)$.

The coupled system (1.5) is defined in a constrained subspace of $L^2(\mathbb{R}, \mathbb{C}^2)$. Assuming that the linear operator $\mathcal{L}$ has an isolated kernel and a positive continuous spectrum and that the eigenfunctions of $\ker(\mathcal{L})$ satisfy the constraint (1.4), we introduce the constrained subspaces

$$X_c(\mathbb{R}) = \{v \in L^2(\mathbb{R}) : (v, \ker(\mathcal{L}\partial_x)) = 0\}, \tag{1.7}$$

$$X_c'(\mathbb{R}) = \{w \in L^2(\mathbb{R}) : (w, \ker(\mathcal{L})) = 0\}. \tag{1.8}$$

If $(v, w) \in L^2(\mathbb{R}, \mathbb{C}^2)$ is the eigenvector of the system (1.5) for $\lambda \neq 0$, then $v \in X_c(\mathbb{R})$ and $w \in X_c'(\mathbb{R})$.

Within this general formalism, we shall study $N$-solitons of the KdV hierarchy. Time evolution of the $N$-solitons is defined by the mixed commuting flows of the higher order KdV equations. Using earlier results of [MS93], we shall prove that the operators $\mathcal{L}$ and $\mathcal{M}$ have no negative eigenvalues in the constrained spaces $X_c(\mathbb{R})$ and $X_c'(\mathbb{R})$, respectively. This fact has a consequence that the stability problem (1.2) and the coupled system (1.5) have no unstable eigenvalues with $\mathrm{Re}(\lambda) > 0$. We also use the inverse scattering transform technique [AK82] to characterize the time evolution of $N$-solitons in the mixed commuting flows of the KdV hierarchy and to prove uniqueness of $N$-solitons in a constrained variational problem. Both main results (spectral stability and time evolution) are not covered by the previous publications on $N$-solitons in the KdV hierarchy [MS93, MMT02].

The paper is organized as follows. In section 2, we briefly summarize the background of the integrable KdV hierarchy and give an explicit definition of the spaces $X_c(\mathbb{R})$ and $X_c'(\mathbb{R})$ for the $N$-solitons. Spectral stability of $N$-solitons in mixed commuting time flows is studied in section 3. Time evolution of $N$-solitons is characterized in section 4. Section 5 concludes the paper.

## 2. Review of $N$-solitons in the KdV hierarchy

The integrable KdV hierarchy (see the review in [N85]) is defined by the set of KdV-type evolution equations

$$\frac{\partial u}{\partial t_n} = \frac{\partial}{\partial x}\frac{\delta H_n}{\delta u}, \qquad n \in \mathbb{N}_+, \tag{2.1}$$

where $H_n(u)$ are the Hamiltonian in energy space $H^{n-1}(\mathbb{R})$. The Hamiltonians are constructed recursively as

$$\mathcal{J}\frac{\delta H_{n+1}}{\delta u} = \mathcal{K}\frac{\delta H_n}{\delta u}, \qquad n \in \mathbb{N}_+, \tag{2.2}$$

where the linear operators $\mathcal{J}$ and $\mathcal{K}$ take the form

$$\mathcal{J} = \frac{\partial}{\partial x}, \qquad \mathcal{K} = \frac{\partial^3}{\partial x^3} + 2\left(\frac{\partial}{\partial x}u + u\frac{\partial}{\partial x}\right) \tag{2.3}$$

and the lowest three Hamiltonians are

$$H_1 = \frac{1}{2}\int_{\mathbb{R}} u^2\,\mathrm{d}x, \tag{2.4}$$

$$H_2 = \frac{1}{2}\int_{\mathbb{R}} \left(u_x^2 - 2u^3\right)\mathrm{d}x, \tag{2.5}$$

$$H_3 = \frac{1}{2}\int_{\mathbb{R}} \left(u_{xx}^2 - 10uu_x^2 + 5u^4\right)\mathrm{d}x. \tag{2.6}$$

The Hamiltonians $H_1$, $H_2$ and $H_3$ generate the first three members of the KdV hierarchy, which are given by the transport equation $u_{t_1} = u_x$, the KdV equation,

$$u_{t_2} = -u_{xxx} - 6uu_x, \tag{2.7}$$

and the integrable fifth-order KdV equation,

$$u_{t_3} = u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x. \tag{2.8}$$

All members of the KdV hierarchy (2.1)–(2.6) have families of $2N$-parameter solutions called $N$-solitons which are expressed by the $\tau$ function [N85],

$$u(x; t_1, t_2, \ldots) = U_N(x - \theta_1, \ldots, x - \theta_N) = 2\frac{\partial^2}{\partial x^2}\log\tau(x - \theta_1, \ldots, x - \theta_N), \tag{2.9}$$

where

$$\theta_n(t_1, t_2, \ldots) = \delta_n + \sum_{k=1}^{\infty} (-1)^k c_n^{k-1} t_k = \delta_n - t_1 + c_n t_2 - c_n^2 t_3 + \cdots, \tag{2.10}$$

and $n = 1, 2, \ldots, N$. Parameters $\delta_n$ are arbitrary, while parameters $c_n$ satisfy the constraints $c_n > 0$ for all $n = 1, 2, \ldots, N$.

We give explicit formulae for the first two solitons (2.9) and (2.10). The 1-soliton is given by

$$U_1 = 2\frac{\partial^2}{\partial x^2} \log[1 + \mathrm{e}^{\sqrt{c_1}(x-\theta_1)}], \qquad c_1 > 0. \tag{2.11}$$

The function $U_1(x)$ is a decaying solution of the second-order ODE:

$$H_2'(u) + \nu_1 H_1'(u) = -u_{xx} - 3u^2 + c_1 u = 0, \tag{2.12}$$

where $\nu_1 = c_1$. The 2-solitons are given by

$$U_2 = 2\frac{\partial^2}{\partial x^2} \log\left[1 + \mathrm{e}^{\sqrt{c_1}(x-\theta_1)} + \mathrm{e}^{\sqrt{c_2}(x-\theta_2)} + \left(\frac{\sqrt{c_1} - \sqrt{c_2}}{\sqrt{c_1} + \sqrt{c_2}}\right)^2 \mathrm{e}^{\sqrt{c_1}(x-\theta_1)+\sqrt{c_2}(x-\theta_2)}\right], \tag{2.13}$$

where $c_1, c_2 > 0$ and $c_1 \neq c_2$. The function $U_2(x)$ is a decaying solution of the fourth-order ODE:

$$\begin{aligned} H_3'(u) + \nu_2 H_2'(u) + \nu_1 H_1'(u) &= u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3 \\ &\quad - (c_1 + c_2)(u_{xx} + 3u^2) + c_1 c_2 u = 0, \end{aligned} \tag{2.14}$$

where $\nu_1 = c_1 c_2$ and $\nu_2 = c_1 + c_2$.

In general, the functions $U_N(x)$ are critical points of the Lyapunov functional in $H^N(\mathbb{R})$:

$$\Lambda_N(u) = H_{N+1}(u) + \sum_{n=1}^{N} \nu_n H_n(u), \tag{2.15}$$

such that

$$\Lambda_N'(U_N) = H_{N+1}'(U_N) + \sum_{n=1}^{N} \nu_n H_n'(U_N) = 0, \tag{2.16}$$

where the Lagrange multipliers $\nu_1, \ldots, \nu_N$ are elementary symmetric functions of $c_1, \ldots, c_N$ due to normalization of $H_n$ (see [MS93]). Lyapunov stability of $N$-solitons as critical points of the constrained Hamiltonian $H_{N+1}(u)$ was proved in [MS93]. Specifically, $N$-solitons $U_N(x)$ are minimal points of $H_{N+1}(u)$ subject to $N$ constraints on the lower order Hamiltonians:

$$H_n(u) = \text{constant}, \qquad n = 1, \ldots, N, \tag{2.17}$$

so that the second variation of $\Lambda_N(u)$ is positive definite,

$$\frac{1}{2}(v, \mathcal{L}_N v) = \lim_{\epsilon \to 0} \frac{\Lambda_N(U_N + \epsilon v) - \Lambda_N(U_N)}{\epsilon^2} > 0, \qquad v \in X_{\mathrm{c}}(\mathbb{R}) \cap X_{\mathrm{c}}'(\mathbb{R}). \tag{2.18}$$

Here, $\mathcal{L}_N$ is the self-adjoint linearized operator of $2N$ order with finite number of negative eigenvalues, finite-dimensional kernel and positive continuous spectrum, bounded away from zero (see [MS93]), $X_{\mathrm{c}}(\mathbb{R})$ is the closed orthogonal complement of the kernel of $\mathcal{L}_N \partial_x$:

$$X_{\mathrm{c}}(\mathbb{R}) = \left\{ v \in L^2(\mathbb{R}) : \left( v, \frac{\delta H_n}{\delta U_N} \right) = 0, n = 1, \ldots, N \right\} \tag{2.19}$$

and $X_{\mathrm{c}}'(\mathbb{R})$ is the closed orthogonal complement of the kernel of $\mathcal{L}_N$:

$$X_{\mathrm{c}}'(\mathbb{R}) = \left\{ w \in L^2(\mathbb{R}) : \left( w, \frac{\partial}{\partial x} \frac{\delta H_n}{\delta U_N} \right) = 0, n = 1, \ldots, N \right\}. \tag{2.20}$$

The embedding $v \in X_c(\mathbb{R})$ follows from the constraints (2.17), while the embedding $v \in X'_c(\mathbb{R})$ is set artificially, in order to remove the zero eigenvalues of $\mathcal{L}_N$ and to ensure positivity of the energy quadratic form (2.18). Due to positivity (2.18), the functional $\Lambda(u)$ is convex at the point $u = U_N(x)$, such that Lyapunov stability of $N$-solitons in energy space $H^N(\mathbb{R})$ holds [MS93].

The linearized operators $\mathcal{L}_1$ and $\mathcal{L}_2$ for 1-soliton and 2-solitons are given explicitly as

$$\mathcal{L}_1 = -\partial_x^2 - 6U_1(x) + c_1 \tag{2.21}$$

and

$$\begin{aligned}\mathcal{L}_2 = \partial_x^4 &+ 10U_2(x)\partial_x^2 + 10U_2'(x)\partial_x + 10U_2''(x) + 30U_2^2(x)\\ &- (c_1 + c_2)\big(\partial_x^2 + 6U_2(x)\big) + c_1c_2.\end{aligned} \tag{2.22}$$

The first two eigenfunctions of the kernel of $\mathcal{L}_N$ are

$$v_1 = U_N'(x), \qquad v_2 = -U_N'''(x) - 6U_N(x)U_N'(x). \tag{2.23}$$

The first two eigenfunctions of the kernel of $\mathcal{L}_N\partial_x$ are

$$w_1 = U_N(x), \qquad w_2 = -U_N''(x) - 3U_N^2(x). \tag{2.24}$$

These explicit expressions illustrate the general construction of the linearized operator $\mathcal{L}_N$ and the kernels of $\mathcal{L}_N$ and $\mathcal{L}_N\partial_x$.

## 3. Spectral stability of $N$-solitons

Time evolution of $N$-solitons (2.9) and (2.10) is defined in the mixed commuting flows of the KdV hierarchy, which are generated by the Lyapunov functional (2.15):

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t_{N+1}} + \sum_{n=1}^{N} \nu_n \frac{\partial u}{\partial t_n} = \frac{\partial}{\partial x}\frac{\delta \Lambda_N}{\delta u}. \tag{3.1}$$

The family of $N$-solitons $u = U_N(x; c_1, \ldots, c_N; \delta_1, \ldots, \delta_N)$ is a time-independent space-decaying solution of the KdV-type evolution equation (3.1), where $\nu_1, \ldots, \nu_N$ are elementary symmetric functions of $c_1, \ldots, c_N$. Linearization of the evolution equation (3.1) as $u(x;t) = U_N(x) + V_N(x;t)$ and separation of variables as $V_N(x;t) = v(x)\mathrm{e}^{\lambda t}$ results in the spectral stability problem

$$\partial_x \mathcal{L}_N v = \lambda v, \tag{3.2}$$

where $\mathcal{L}_N$ is the same as in the energy quadratic form (2.18). We assume that the family of $N$-solitons is non-degenerate [MS93] and characterize the kernel of $\mathcal{L}_N$ in terms of the symmetries of the KdV hierarchy [KT78].

**Assumption 3.1.** *$N$-solitons $U_N(x; c_1, \ldots, c_N; \delta_1, \ldots, \delta_N)$ have distinct positive parameters $c_1, \ldots, c_N$.*

**Lemma 3.2.** *The kernel of $\mathcal{L}_N$ has a basis of $N$ linearly independent eigenfunctions $\{v_n(x)\}_{n=1}^N$ in $L^2(\mathbb{R})$, where*

$$v_n(x) = \frac{\partial}{\partial x}\frac{\delta H_n}{\delta U_N}, \qquad n = 1, \ldots, N. \tag{3.3}$$

**Proof.** Derivatives of $U_N(x; c_1, \ldots, c_N; \delta_1, \ldots, \delta_N)$ with respect to arbitrary parameters $\delta_1, \ldots, \delta_N$ form a basis of the kernel of $\mathcal{L}_N$, since they are linearly independent and $\mathcal{L}_N$ is a self-adjoint operator of $2N$ order. It follows from (2.10) that

$$\frac{\partial U_N}{\partial t_n} = \sum_{k=1}^{N} (-1)^{n-1} c_k^{n-1} \frac{\partial U_N}{\partial \theta_k} = \sum_{k=1}^{N} (-1)^{n-1} c_k^{n-1} \frac{\partial U_N}{\partial \delta_k}, \qquad n = 1, \ldots, N. \tag{3.4}$$

Since the Vandermonde determinant of $c_1, \ldots, c_N$ is non-singular under assumption 3.1, the basis of $\left\{\frac{\partial U_N}{\partial \delta_n}\right\}_{n=1}^N$ is equivalent to the basis $\left\{\frac{\partial U_N}{\partial t_n}\right\}_{n=1}^N$. It follows from the KdV hierarchy (2.1) that

$$\frac{\partial U_N}{\partial t_n} = \frac{\partial}{\partial x}\frac{\delta H_n}{\delta U_N}, \qquad n = 1, \ldots, N. \tag{3.5}$$

See also [KT78] and [MS93, lemma 3.4] for alternative proofs. □

**Lemma 3.3.** *The generalized kernel of* $\partial_x \mathcal{L}_N$ *has a basis of* $N$ *linearly independent eigenfunctions* $\left\{v_n^{(1)}(x)\right\}_{n=1}^N$ *in* $L^2(\mathbb{R})$, *where*

$$v_n^{(1)}(x) = -\frac{\partial U_N}{\partial \nu_n}, \qquad n = 1, \ldots, N. \tag{3.6}$$

**Proof.** The generalized kernel of $\partial_x \mathcal{L}_N$ is generated by solutions of the inhomogeneous problem

$$\partial_x \mathcal{L}_N v_n^{(1)} = v_n(x). \tag{3.7}$$

Integrating (3.7) in $x$ for $v_n^{(1)} \in L^2(\mathbb{R})$, we find that eigenfunctions $v_n^{(1)}(x)$ satisfy the inhomogeneous equations

$$\mathcal{L}_N v_n^{(1)} = \frac{\delta H_n}{\delta U_N}, \qquad n = 1, \ldots, N. \tag{3.8}$$

It follows from the derivative of the variation equations (2.16) in $\nu_n$ that

$$\mathcal{L}_N \frac{\partial U_N}{\partial \nu_n} = -\frac{\delta H_n}{\delta U_N}, \qquad n = 1, \ldots, N. \tag{3.9}$$

As a result, relations (3.6) hold. □

**Remark 3.4.** The symmetries $v_n(x)$ and $v_n^{(1)}(x)$ are shown to be expressed by the squared eigenfunctions of the inverse spectral method [KN78, KT78]. The squared eigenfunctions form a basis for a class of functions $u(x)$ satisfying the bounds:

$$\int_{-\infty}^{\infty} (1 + x^2)|u(x)|\,\mathrm{d}x < \infty. \tag{3.10}$$

Define the solution surface as $\Lambda_s = \Lambda_N(U_N) = \Lambda_s(\nu_1, \ldots, \nu_N)$. The Hessian matrix $\mathcal{H}$ of principal curvatures of the solution surface $\Lambda_s(\nu_1, \ldots, \nu_N)$ has the elements

$$\mathcal{H}_{n,m} = \frac{\partial^2 \Lambda_s}{\partial \nu_n \partial \nu_m} = \frac{\partial H_n(U_N)}{\partial \nu_m} = \frac{\partial H_m(U_N)}{\partial \nu_n}, \qquad n, m = 1, \ldots, N, \tag{3.11}$$

where

$$\frac{\partial H_n(U_N)}{\partial \nu_m} = \left(\frac{\partial U_N}{\partial \nu_m}, \frac{\delta H_n}{\delta U_N}\right) = -\left(\frac{\partial U_N}{\partial \nu_m}, \mathcal{L}_N \frac{\partial U_N}{\partial \nu_n}\right), \qquad n, m = 1, \ldots, N. \tag{3.12}$$

Let $z(\mathcal{H})$ be the number of zero eigenvalues of $\mathcal{H}$.

**Lemma 3.5.** *The second generalized kernel of* $\partial_x \mathcal{L}_N$ *is empty in* $L^2(\mathbb{R})$ *if* $z(\mathcal{H}) = 0$.

**Proof.** The second generalized kernel is generated by solutions of the inhomogeneous problem

$$\partial_x \mathcal{L}_N v^{(2)} = \sum_{n=1}^N a_n v_n^{(1)}(x), \tag{3.13}$$

where $\{a_n\}_{n=1}^N$ is a set of coefficients. Computing the inner products of the inhomogeneous problem (3.13) with the set of eigenfunctions $\left\{\frac{\delta H_n}{\delta U_N}\right\}_{n=1}^N$, we find that the solutions $v^{(2)} \in L^2(\mathbb{R})$ may exist only if the Hessian matrix $\mathcal{H}$ in (3.11) and (3.12) has a zero eigenvalue. They do not exist if $z(\mathcal{H}) = 0$. □

**Lemma 3.6.** *The kernel of $\mathcal{M}_N = -\partial_x \mathcal{L}_N \partial_x$ has a basis of $N$ linearly independent eigenfunctions $\{w_n(x)\}_{n=1}^N$ in $L^2(\mathbb{R})$, where*

$$w_n(x) = \frac{\delta H_n}{\delta U_N}, \qquad n = 1, \ldots, N, \tag{3.14}$$

**Proof.** Integrating $\partial_x \mathcal{L}_N \partial_x w = 0$ in $x$, we have $\mathcal{L}_N w'(x) = 0$ for $w'(x) \in L^2(\mathbb{R})$. Therefore, $w' \in \ker(\mathcal{L}_N)$, such that the eigenfunctions (3.14) follow from integration of eigenfunctions (3.3). □

We now work with the coupled problem

$$\mathcal{L}_N v = -\lambda w, \qquad \mathcal{M}_N w = \lambda v, \tag{3.15}$$

which is equivalent to the spectral problem (3.2) when $v = -w'(x)$, subject to the constraint (1.4). By lemmas 3.2 and 3.6, definitions (1.7) and (1.8) of the constrained spaces $X_c(\mathbb{R})$ and $X'_c(\mathbb{R})$ coincide with the definitions (2.19) and (2.20). It is therefore clear that $v \in X_c(\mathbb{R})$ and $w \in X'_c(\mathbb{R})$ for any solution $(v, w) \in L^2(\mathbb{R}, \mathbb{C}^2)$ of the coupled problem (3.15) with $\lambda \neq 0$.

Analysis of unstable eigenvalues in the coupled problem (3.15) is based on the fact that both operators $\mathcal{L}_N$ and $\mathcal{M}_N$ have finitely many negative eigenvalues in $L^2(\mathbb{R})$. The continuous spectrum of $\mathcal{L}_N$ is positive and bounded away from zero by $c_0 = \prod_{n=1}^N c_n > 0$, due to the exponential decay of potential functions and the factorization relation [MS93]

$$U_N(x) \equiv 0: \quad \mathcal{L}_N = \left(-\partial_x^2 + c_1\right) \cdots \left(-\partial_x^2 + c_N\right). \tag{3.16}$$

The continuous spectrum of $\mathcal{M}_N$ is non-negative, due to the explicit relation $\mathcal{M}_N = -\partial_x \mathcal{L}_N \partial_x$. The kernel of $\mathcal{L}_N$ belongs to $X_c(\mathbb{R})$, due to commutativity of the Hamiltonians $H_n(u)$ of the KdV hierarchy (2.1)–(2.6):

$$\left(\frac{\delta H_n}{\delta U_N}, \frac{\partial}{\partial x} \frac{\delta H_m}{\delta U_N}\right) = 0, \qquad n, m = 1, \ldots, N. \tag{3.17}$$

Equivalently, the kernel of $\mathcal{M}_N$ belongs to $X'_c(\mathbb{R})$. These facts enable us to consider the number of negative eigenvalues of $\mathcal{L}_N$ and $\mathcal{M}_N$ in $X_c(\mathbb{R})$ and $X'_c(\mathbb{R})$, respectively, and to relate the absence of negative eigenvalues to spectral stability of $N$-solitons in the spectral problem (3.2).

**Proposition 3.7.** *Let $n(\mathcal{L}_N)$ be the number of negative eigenvalues of $\mathcal{L}_N$ in $L^2(\mathbb{R})$ and $p(\mathcal{H})$ be the number of positive eigenvalues of $\mathcal{H}$. The numbers $\#_{<0}(\mathcal{L}_N)$ and $\#_{<0}(\mathcal{M}_N)$ of negative eigenvalues of $\mathcal{L}_N$ in $X_c(\mathbb{R})$ and of $\mathcal{M}_N$ in $X'_c(\mathbb{R})$, respectively, are given by*

$$\#_{<0}(\mathcal{L}_N) = \#_{<0}(\mathcal{M}_N) = n(\mathcal{L}_N) - p(\mathcal{H}).$$

**Proof.** It follows from [GSS90, theorem 3.1] and [MS93, lemma 2.1] that $\#_{<0}(\mathcal{L}_N) = n(\mathcal{L}_N) - p(\mathcal{H})$. Equivalently, the same result can be proved by minimization of quadratic forms with Lagrange multipliers, see [P05, CPV05]. In order to prove the relation $\#_{<0}(\mathcal{M}_N) = \#_{<0}(\mathcal{L}_N)$, we integrate the quadratic form $(w, \mathcal{M}_N w)$, $w \in X'_c(\mathbb{R})$ by parts:

$$(w, \mathcal{M}_N w) = -(w, \partial_x \mathcal{L}_N \partial_x w) = (w', \mathcal{L}_N w'). \tag{3.18}$$

By integrating the constraints in the inner products (2.20) by parts, we confirm that if $w \in X_c'(\mathbb{R})$, then $w' \in X_c(\mathbb{R})$, such that the quadratic form $(w', \mathcal{L}_N w')$ is defined in $w' \in X_c(\mathbb{R})$. Finally, it follows from (3.3) and (3.14) after integration by parts that

$$\left(v_n, \mathcal{M}_N^{-1} v_m\right) = \left(w_n, \mathcal{L}_N^{-1} w_m\right), \qquad n, m = 1, \ldots, N, \tag{3.19}$$

where the functions $\mathcal{M}_N^{-1} v_m$ are bounded due to the fact that $v_m(x)$ satisfy the constraint (1.4) for any $m$. Applying lemma 3.4 from [CPV05], we have the relation $\#_{<0}(\mathcal{M}_N) = n(\mathcal{L}_N) - p(\mathcal{H}) = \#_{<0}(\mathcal{L}_N)$. □

**Proposition 3.8.** *Let $p(\mathcal{H}) = n(\mathcal{L}_N)$. The coupled problem (1.5) has no eigenvalues $\lambda \in \mathbb{C}$ with* $\mathrm{Re}(\lambda) > 0$ *and* $(v, w) \in L^2(\mathbb{R}, \mathbb{C}^2)$.

**Proof.** When $p(\mathcal{H}) = n(\mathcal{L}_N)$, we have by proposition 3.7:

$$\forall v \in X_c(\mathbb{R}) : (v, \mathcal{L}_N v) \geqslant 0, \tag{3.20}$$

$$\forall w \in X_c'(\mathbb{R}) : (w, \mathcal{M}_N w) \geqslant 0, \tag{3.21}$$

where the zero values are achieved if and only if $v \in \ker(\mathcal{L}_N)$ in $X_c(\mathbb{R})$ and $w \in \ker(\mathcal{M}_N)$ in $X_c'(\mathbb{R})$. We assume that there exists an eigenvalue $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}(\lambda_0) > 0$ and $(v_0, w_0) \in L^2(\mathbb{R}, \mathbb{C}^2)$ and show the contradiction. If the eigenvalue $\lambda_0$ existed, then the coupled system (3.15) would result in the identity

$$\lambda_0(\bar{w}_0, \mathcal{M}_N w_0) + \bar{\lambda}_0(\bar{v}_0, \mathcal{L}_N v_0) = 0, \tag{3.22}$$

where the real-valued inner products are used for clarity of notation. Since $\mathrm{Re}(\lambda_0) > 0$ and quadratic forms associated with $\mathcal{L}_N$ and $\mathcal{M}_N$ are real valued, we have

$$(\bar{w}_0, \mathcal{M}_N w_0) = -(\bar{v}_0, \mathcal{L}_N v_0),$$

which is the contradiction, since both quadratic forms are non-negative for $v_0 \in X_c(\mathbb{R})$ and $w_0 \in X_c'(\mathbb{R})$ and non-zero for $v_0 \notin \ker(\mathcal{L}_N)$ and $w_0 \notin \ker(\mathcal{M}_N)$ (when $\lambda_0 \neq 0$). □

**Spectral stability theorem.** *Let N-solitons $U_N(x)$ of the KdV hierarchy (2.1)–(2.6) satisfy assumption 3.1. Then, N-solitons are weakly spectrally stable such that the spectral problem (3.2) has no eigenvalues $\lambda \in \mathbb{C}$ with* $\mathrm{Re}(\lambda) > 0$ *and* $v \in L^2(\mathbb{R}, \mathbb{C})$.

**Proof.** We show that conditions of proposition 3.8 are satisfied for $N$-solitons of the KdV hierarchy (2.1)–(2.6). It was proved in [MS93, lemma 3.5] and [MS93, lemma 3.6] that

$$n(\mathcal{L}_N) = \left[\frac{N+1}{2}\right], \qquad p(\mathcal{H}) = \left[\frac{N+1}{2}\right], \tag{3.23}$$

where $[z]$ is the integer part of $z$. Therefore, the result holds by proposition 3.8. □

**Remark 3.9.** It was also proved in [MS93, lemma 3.6] that $z(\mathcal{H}) = 0$. Therefore, the second generalized kernel of $\partial_x \mathcal{L}_N$ is empty by lemma 3.5, such that the zero eigenvalue $\lambda = 0$ in the spectral problem (1.2) is controlled by the symmetries of the KdV hierarchy.

## 4. Time evolution of $N$-solitons in the KdV hierarchy

According to (2.18), the Lyapunov functional $\Lambda_N(u)$ defined in (2.15) is convex at the point $u = U_N(x)$, where $U_N(x)$ is $N$-solitons of the KdV hierarchy (2.1)–(2.6). We give a direct proof of convexity based on the inverse spectral method [N85]. Moreover, we prove uniqueness of $N$-solitons as minimizers of the constrained variational problem (2.16)–(2.17). Using the asymptotic results from [AK82], we formulate the time evolution theorem for $N$-solitons in the energy space $H^N(\mathbb{R})$, which follows from Lyapunov stability of $N$-solitons.

These results improve the Lyapunov stability theorem of [MS93], where the time evolution of parameters $\theta_n(t)$ in the solution $U_N(x)$ remains undefined and uniqueness of minimizers in the constrained variational problem (2.16)–(2.17) remains open. The unique characterization of the time evolution of parameters of $N$-solitons is equivalent to the asymptotic stability of $N$-solitons in the mixed commuting time flows (3.1). We note that asymptotic stability of 1-soliton in the KdV equation was proved in [PW94] with exponentially weighted spaces (see also [B72, B75] for pioneer papers), while asymptotic stability of $N$-solitons in the energy space $H^1(\mathbb{R})$ was recently proved in [MMT02, corollary 1] with different analysis of energy functionals.

Let us recall some necessary formulae obtained in the inverse spectral method [N85]. The method is based on the isomorphism between the potential $u(x,t)$ of the Schrödinger equation and the scattering data $S(t)$ (see, e.g., [N85]). Consider the scattering problem

$$\left(\partial_x^2 + u(x,t) + k^2\right)\psi = 0, \tag{4.1}$$

with the boundary conditions

$$\psi(x,t;k) \longrightarrow \begin{cases} \mathrm{e}^{-\mathrm{i}kx} & \text{as} \quad x \to -\infty, \\ a(k)\,\mathrm{e}^{-\mathrm{i}kx} + b(t;k)\,\mathrm{e}^{\mathrm{i}kx} & \text{as} \quad x \to \infty. \end{cases} \tag{4.2}$$

Assuming that the potential $u(x,t)$ is exponentially decaying in $x$, it is proved that the function $a(k)$ is analytic on the upper-half plane of $k \in \mathbb{C}$, and the bound states are defined by the zeros of $a(k)$ at $k = i\kappa_j$, $\kappa_j > 0$, such that

$$a(i\kappa_j) = 0, \qquad j = 1, \ldots, N. \tag{4.3}$$

The scattering data is then defined by

$$S(t) = \left[\{\kappa_j, C_j(t)\}_{j=1}^N, \{r(t;k)\}_{k\in\mathbb{R}}\right], \tag{4.4}$$

where $r(t;k) = b(t;k)/a(k)$ is the reflection coefficient and $C_j(t)$ is the normalization constant (see [N85] for details). The parameter $c_j$ in the $N$-solitons (2.9) and (2.10) is given by $c_j = 4\kappa_j^2$ for all $j$. We assume that the parameters $\{\kappa_j\}_{j=1}^N$ are ordered as

$$\kappa_1 > \kappa_2 > \cdots > \kappa_N > 0. \tag{4.5}$$

By inverse scattering, the potential $u(x,t)$ can be expressed as

$$u(x,t) = 4\sum_{j=1}^N \kappa_j C_j(t)\psi^2(x,t;\mathrm{i}\kappa_j) + \frac{2\mathrm{i}}{\pi}\int_{-\infty}^{\infty} kr(t;k)\psi^2(x,t;k)\,\mathrm{d}k, \tag{4.6}$$

which shows that $u(x,t)$ consists of $N$-solitons and the radiation. With the formula (4.6), the Hamiltonian $H_n$ can be expressed in terms of the scattering data [N85],

$$H_n(u) = (-1)^{n+1}2^{2n+1}\left(\frac{1}{2n+1}\sum_{j=1}^N \kappa_j^{2n+1} - (-1)^n R_n\right), \tag{4.7}$$

where $R_n$ is the radiation part, given by

$$R_n = -\frac{1}{2\pi}\int_0^\infty k^{2n}\ln(1-|r(t;k)|^2)\,\mathrm{d}k \geqslant 0, \qquad n = 1, 2, \ldots. \tag{4.8}$$

We show that the alternate sign $(-1)^n$ in front of $R_n$ in the representation (4.7) plays a crucial role for the convexity.

**Proposition 4.1.** *The N-soliton $U_N(x)$ is uniquely determined in the variational problem (2.16) by the constraints (2.17) except the phase parameters $\delta_1, \ldots, \delta_N$. The Lyapunov functional $\Lambda_N(u)$ is convex at $u = U_N(x)$.*

**Proof.** Let us consider the variation of $H_n(u)$ at $u = U_N(x)$, which we denote as $\Delta H_n(U_N)$. The constraints (2.17) imply that $\Delta H_n(U_N) = 0$, such that

$$\sum_{j=1}^N \kappa_j^{2n}\Delta\kappa_j - (-1)^n\Delta R_n = 0, \qquad n = 1, \ldots, N, \tag{4.9}$$

where $\Delta\kappa_j$ is the variation of $\kappa_j$ and $\Delta R_n$ is the variation of $R_n$. Since $R_n = 0$ at $u = U_N(x)$, we understand from the explicit formula (4.8) that all $\Delta R_n \geqslant 0$. It is clear from the system (4.9) that the variations of $\kappa_j$ are balanced with the variations of $R_n$, so that the Hamiltonian $H_n$ remains constant. The variations $\Delta U_N$ belong then to the constrained space $X_c(\mathbb{R})$, defined in (2.19). We also note that the number of solitons may change under the variation of $H_n(u)$, but this does not appear in the constraints (4.9), since $\kappa_j = 0$ for $j > N$. We need to prove the convexity of the functional $\Lambda_N(u)$ at $u = U_N(x)$, which implies that

$$\Delta H_{N+1}(U_N) > 0. \tag{4.10}$$

From the system (4.9), the variation $\Delta H_{N+1}(U_N)$ can be expressed as

$$\begin{aligned}\Delta H_{N+1}(U_N) &= (-1)^N 2^{2N+3}\left(\sum_{j=1}^N \kappa_j^{2N+2}\Delta\kappa_j + (-1)^N\Delta R_{N+1}\right)\\ &= \frac{(-1)^N 2^{2N+3}}{D}\left(\sum_{l=1}^N (-1)^l D_l\Delta R_l + (-1)^N\Delta R_{N+1}\right),\end{aligned} \tag{4.11}$$

where $D$ is the determinant of the $N\times N$ matrix $K_N := \left\{\left(\kappa_i^{2j}\right) : 1 \leqslant i, j \leqslant N\right\}$ and $D_l$ is the determinant of the matrix $K_N$ after replacing the $l$th row by $\left\{\left(\kappa_i^{2N+2}\right) : 1 \leqslant i \leqslant N\right\}$. It is clear that $D$ is the Vandermonde determinant, computed as

$$D = \prod_{1\leqslant i<j\leqslant N}\left(\kappa_i^2 - \kappa_j^2\right), \tag{4.12}$$

so that $D > 0$ from the ordering (4.5). We further notice that $D_l$ can be written as

$$D_l = (-1)^{N-l}D_l^{N+1}, \tag{4.13}$$

where $D_l^{N+1}$ is the determinant of the $(N+1)\times(N+1)$ matrix $K_{N+1} = \left\{\left(\kappa_i^{2j}\right) : 1 \leqslant i, j \leqslant N+1\right\}$ after removing the $l$th row and the $(N+1)$th column. Thus, we have

$$\Delta H_{N+1}(U_N) = \frac{2^{2N+2}}{D}\left(\sum_{l=1}^N D_l^{N+1}\Delta R_l + \Delta R_{N+1}\right). \tag{4.14}$$

Since all $\Delta R_n \geqslant 0$, we only need to show that $D_l^{N+1} > 0$ for the convexity. This follows from the expression

$$D_l^{N+1} = \sigma_N\sigma_{N-l+1}D, \tag{4.15}$$

where $\sigma_k$ is the symmetric polynomial of $(\kappa_1^2, \ldots, \kappa_N^2)$ given by

$$\sigma_k = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant N} \kappa_{i_1}^2 \cdots \kappa_{i_k}^2 > 0. \tag{4.16}$$

As a result, $D_l^{N+1} > 0$ and the convexity (4.10) holds if there exists at least one $\Delta R_n \neq 0$. (Note that if $\Delta R_n \neq 0$ for some $n$, then it is not zero for any $n$.) Otherwise, i.e. when all $\Delta R_n = 0$, all $\Delta\kappa_n = 0$ from the system (4.9), and the variations of $U_N(x)$ simply translate the solution $U_N(x)$ along its phase parameters $\delta_1, \ldots, \delta_N$. When the variation $\Delta U_N \in X_c(\mathbb{R})$ belongs also to the constrained space $X_c'(\mathbb{R})$, defined in (2.20), the latter case is excluded. Thus, the minimal point of $H_{N+1}$ is given by the $N$-soliton $u = U_N(x)$, that is, all $R_n = 0$. Uniqueness of the minimizer $U_N(x)$ follows from the geometry. Each constraint $H_n(u) =$ constant gives a hypersurface in $(\kappa_1, \kappa_2, \ldots, \kappa_N) \in \mathbb{R}_+^N$. If all those hypersurfaces intersect transversally with nonempty intersection, we have $N!$ points of the intersection. Then, the ordering $\kappa_1 > \cdots > \kappa_N > 0$ chooses a unique point of the intersection. □

**Time evolution theorem.** *For all $\epsilon > 0$, there exists $\delta > 0$ such that if $\|u(x,0) - U_N(x)\|_{H^N(\mathbb{R})} \leqslant \delta$, then there exist $\theta_j^{\pm}(t) = c_j t + \delta_j^{\pm}$, where $t \equiv t_2$, such that*

$$\left\| u(x,t) - U_N(x - \theta_1^{\pm}(t), \ldots, x - \theta_N^{\pm}(t)) \right\|_{L^2(|x| > c_N|t|)} \leqslant \epsilon, \qquad \text{as} \quad t \to \pm\infty. \tag{4.17}$$

**Proof.** The proof relies on proposition 4.1 and the formalism above. The time evolution of $N$-solitons has been obtained in [AK82] by the inverse spectral method, see equations (2.15a) and (2.15b) in [AK82], where $\kappa_N t$ should be read as $4\kappa_N^2 t = c_N t$. We only note that the soliton speeds $c_j = 4\kappa_j^2$ and soliton phases $\delta_j$ must be obtained by the scattering problem (4.1) for the perturbed potential $u = U_N + \Delta U_N$. □

## 5. Summary

We have studied the spectral stability and time evolution of $N$-solitons in the mixed commuting flows of the integrable KdV hierarchy. We have proved that $N$-solitons of the KdV hierarchy are spectrally stable with respect to the time evolution. The analysis extends the recent results of [CPV05, KKS04, P05] to the KdV-type evolution equation (1.1). In particular, the proof of spectral stability is related to the absence of negative eigenvalues of linearized energy in constrained function spaces. We have also characterized time evolution of $N$-solitons and proved that the $N$-solitons are uniquely determined in a constrained variational problem. Further development of the spectral stability analysis to non-integrable KdV-type evolution equations, e.g. to the fifth-order KdV model, will be reported elsewhere.

## Acknowledgments

YK is partially supported by the NSF grant DMS-0404931. DP is partially supported by the NSERC Discovery grant.

## References

[AK82] Ablowitz M J and Kodama Y 1982 Note on asymptotic solutions of the Korteweg–de Vries equation with solitons *Stud. Appl. Math.* **66** 159–70

[B72] Benjamin T B 1972 The stability of solitary waves *Proc. R. Soc. Lond.* A **328** 153–83

[B75] Bona J L 1975 On the stability of solitary waves *Proc. R. Soc. Lond.* A **344** 363–74

[BSS87] Bona J L, Souganidis P E and Strauss W A 1987 Stability and instability of solitary waves of Korteweg–de Vries type *Proc. R. Soc. Lond.* A **411** 395–412

[CPV05] Cuccagna S, Pelinovsky D and Vougalter V 2005 Spectra of positive and negative energies in the linearized NLS problem *Commun. Pure Appl. Math.* **58** 1–29

[G90] Grillakis M 1990 Analysis of the linearization around a critical point of an infinite dimensional Hamiltonian system *Commun. Pure Appl. Math.* **43** 299–333

[GSS90] Grillakis M, Shatah J and Strauss W A 1990 Stability theory of solitary waves in the presence of symmetry II *J. Funct. Anal.* **94** 308–48

[HK02] Hiraoka Y and Kodama Y 2002 Normal form and solitons *Preprint* nlin.SI/0206021

[KKS04] Kapitula T, Kevrekidis P and Sandstede B 2004 Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems *Physica* D **195** 263–82

Kapitula T, Kevrekidis P and Sandstede B 2005 *Physica* D **201** 199–201 (addendum)

[KN78] Kaup D and Newell A C 1978 Solitons as particles, oscillators and in slowly changing media: a singular perturbation theory *Proc. R. Soc. Lond.* A **361** 413

[KT78] Kodama Y and Taniuti T 1978 Higher order approximation in the reductive perturbation method: I. The weakly dispersive system *J. Phys. Soc. Japan* **45** 298–310

[L02] Leblond H 2002 Higher-order terms in multiscale expansions: a linearized KdV hierarchy *J. Nonlinear Math. Phys.* **9** 325–46

[L03] Leblond H 2003 A new criterion for the existence of KdV solitons in ferromagnets *J. Phys. A: Math. Gen.* **36** 1855–68

[MS93] Maddocks J H and Sachs R L 1993 On the stability of KdV multi-solitons *Commun. Pure Appl. Math.* **46** 867–901

[MMT02] Martel Y, Merle F and Tsai T P 2002 Stability and asymptotic stability in the energy space of the sum of $N$ solitons for subcritical gKdV equation *Commun. Math. Phys.* **231** 347–73

[N85] Newell A C 1985 *Solitons in Mathematics and Physics (CBMS-NSF Regional Conf. Ser. in Applied Mathematics* vol 48*)* (Philadelphia, PA: SIAM)

[PW92] Pego R L and Weinstein M I 1992 Eigenvalues, and instabilities of solitary waves *Phil. Trans. R. Soc. Lond.* A **340** 47–94

[PW94] Pego R L and Weinstein M I 1994 Asymptotic stability of solitary waves *Commun. Math. Phys.* **164** 305–49

[P05] Pelinovsky D E 2005 Inertia law for spectral stability of solitary waves in coupled nonlinear Schrödinger equations *Proc. R. Soc. Lond.* A **481** 783–812

[SS90] Souganidis P E and Strauss W A 1990 Instability of a class of dispersive solitary waves *Proc. R. Soc. Edinb.* A **114** 195–212

[W87] Weinstein M I 1987 Existence and dynamic stability of solitary wave solutions of equations arising in long wave propagation *Commun. PDEs* **12** 1133–73