ELSEVIER
9 October 1995
PHYSICS LETTERS A
Physics Letters A 206 (1995) 195–202

Collapse transformation for self-focusing solitary waves in boundary-layer type shear flows

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Received 18 May 1995; revised manuscript received 28 July 1995; accepted for publication 2 August 1995
Communicated by C.R. Doering

Abstract

Evolution of long-wave weakly nonlinear two-dimensional perturbations in parallel boundary-layer type shear flows is considered within the model simplified by using the paraxial approximation. The transverse instability of 1D solitons is shown to result in formation of 2D collapsing clusters which evolve in a self-similar manner. The explicit "critical" collapse transformation of 2D solitons is found within the framework of the Whitham adiabatic approach.

1. It is well-known (see, e.g. Ref. [1]) that the dynamics of solitary waves in various physical problems being well described within the framework of approximate weakly nonlinear models often exhibits formation of amplitude singularities emerging in a certain finite time ("collapse"), the more probable the higher dimension of the problem is. In the case of collapse, one may expect formation of a number of drastic effects depending on the specific physics of the problem. Within the hydrodynamics context, it may mean formation of strongly nonlinear large-amplitude coherent structures from the small-amplitude solitary waves unstable with respect to self-focusing or wave breaking. The collapse is always characterized by very strong energy transfer from large to small scales. Therefore, all cases of collapse in hydrodynamical systems are of great interest and merit special study. The present work is aimed to study the specific collapse of boundary-layer type shear flows.

Among the various types of collapses there is an interesting physical and mathematical phenomenon, namely the so-called *weak* or *critical collapse*, which was discovered at the edge of the instability domain [1]. Physically, this phenomenon manifests itself by the existence of an energy threshold with respect to which an initial perturbation evolves in a qualitatively different manner, either decays or collapses. Mathematically, the models of critical collapse have a remarkable property: The Hamiltonian of the system is a homogeneous function under the scaling transformation retaining the other integral of motion (momentum or number of particles). Moreover, it was found for the most investigated model of this

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0375-9601/95/$09.50
SSDI 0375-9601(95)00608-7

phenomenon, namely the 2D nonlinear Schrödinger (NLS) equation [2–4], that such a nontrivial property results in the existence of an additional symmetry (and corresponding integral of motion) which generates an exact solution describing the radiationless self-similar collapse of 2D solitary waves with the critical (threshold) value of the energy.

Recently, a new approach to the problem of formation of "thorn"-type large-amplitude localized structures in boundary layers (some experiments were discussed in Refs. [5,6]) started to develop. Using an asymptotic procedure, Shira [7] derived a model for the description of essentially two-dimensional (2D) weakly nonlinear long-wave perturbations on the background of a boundary-layer type plane-parallel shear flow without inflection points. Shortly speaking, this model is valid for the boundary layers along an inviscid boundary, say for free surface flows, although there is a possibility of its extension. Within the model, the description of the field evolution reduces to a single equation for the amplitude A of the longitudinal velocity of the fluid. This governing equation has the form

$$A_T + cA_X + \alpha A A_X + BQ(A_X) = 0, \qquad (1)$$

where $Q(A)$ is the Cauchy–Hadamard integral transform given by

$$Q(A) = \frac{1}{2\pi} \int\int_{-\infty}^{\infty} \frac{A(X', Y')\, \mathrm{d}X'\, \mathrm{d}Y'}{\sqrt{\left[(X-X')^2 + (Y-Y')^2\right]^3}}$$

and c, α, and β are parameters expressed through a profile of the shear flow. This equation is a strongly anisotropic generalization of the Benjamin–Ono (BO) equation. The latter was used earlier for the description of 1D perturbations in boundary-layer type shear flows (see Ref. [5] and references therein). The BO solitons were found to be unstable with respect to transverse perturbations within the framework of model (1) [8,9]. On the other hand, the 2D solitons solutions were numerically constructed in Ref. [10], however their stability has not been investigated. Quite recently, D'yachenko and Kuznetsov [8] pointed out that model (1) has some properties of the critical collapse models and suggested that it might display the same scenarios of collapse formation as that in the 2D NLS equation. Although their numerical simulations seem to confirm the development of the initial stage of the collapse, the accurate mathematical description of this phenomenon, similar to that done for the 2D NLS equation [3,4] has not been elaborated for model (1).

In this paper we consider this problem for nearly one-dimensional wave perturbations when model (1) can be simplified in the framework of the paraxial approximation introduced originally by Kadomtsev and Petviashvili [11] for waves in isotropic weakly dispersive media. For such a simplified analog of Eq. (1) we apply the Whitham adiabatic approach [12,13] to describe the evolution of the modulated BO solitons and find an explicit collapse transformation for the soliton parameters.

It is worth noting that the Whitham approach usually gives a correct description only for perturbations sufficiently long compared to the soliton width but fails to describe shorter perturbations [14]. The main reason for this failure lies in the appearance of soliton non-decaying tails which are associated with the radiation field escaping the solitary wave (see Ref. [15]). However, in our case, the radiation field is proved to appear in the higher-order (cubic in the amplitude of the perturbations) approximations and, therefore, the Whitham equations give a good description not only of the linear instability of the BO solitons but also of the nonlinear evolution of 1D and 2D localized perturbations. We show that the nonlinear effects cannot suppress the transverse instability of the BO solitons and lead to the formation of self-similar collapsing structures having the shape of 2D solitary waves. Furthermore, we present approximate analytical expressions for the family of 2D soliton solutions of the model and show that the collapse transformation of the Whitham equations generates slow, power-like instability of 2D solitons which results in weak (critical) collapse.

2. First, we simplify Eq. (1), confining ourselves to considering nearly one-dimensional waves, i.e. we assume $|A_{YY}| \ll |A_{XX}|$. This paraxial approximation might be justified by the fact that the transverse instability of the 1D soliton is confined to the *large* scales only (see Ref. [9] and formula (5) below). Then, in the dimensionless form $u(x, y, t) = \alpha A(X - cT, \sqrt{2}\, Y, \beta T)/2\beta$ Eq. (1) reduces to the 2D BO equation,

$$u_t + 2uu_x + H(u_{xx}) + H(u_{yy}) = 0 \qquad (2)$$

where $H(u)$ is the Hilbert integral transform performed in the variable x,

$$H(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x', y, t)\, dx'}{x' - x}$$

The BO solitons of Eq. (2) are given by the function

$$u_0(x - \Theta, V) = \frac{2V}{1 + V^2(x - \Theta)^2}, \tag{3}$$

where $\Theta_t = V > 0$. They were found to be unstable with respect to transverse perturbations [8,9]. Moreover, Eq. (2) possesses an *exact discrete-spectrum solution* to the problem linearized with respect to the soliton background,

$$u = u_0 + \left(\lambda \partial_x u_0 - V^2 p^2 \partial_V u_0\right) \exp(\lambda t + ipy) + O(\exp(2\lambda t)). \tag{4}$$

Here t is assumed to be negative (and large) and the growth rate of the soliton instability λ ($\lambda < 0$) is related to the transverse wave number p ($|p| < V$) according to [9]

$$\lambda^2 = V^2 p^2 - p^4. \tag{5}$$

It follows from Eq. (4) that the linear unstable discrete-spectrum mode is presented by a simple adiabatic response of the BO soliton to the modulation of its parameters in the transverse direction. This fact enables us to apply the Whitman adiabatic approach [12,13] to extend the linear theory and describe the nonlinear evolution of a perturbed BO soliton in the framework of model (2).

We note that Eq. (2) can be rewritten in the variational form

$$\delta S = 0, \quad S = \int_0^t \int \int_{-\infty}^{\infty} dx\, dy\, L, \tag{6}$$

where S is the action and L is the Lagrangian,

$$L = \tfrac{1}{2}\varphi_t \varphi_x + \tfrac{1}{3}\varphi_x^3 + \tfrac{1}{2}\varphi_x H(\varphi_{xx}) + \tfrac{1}{2}\varphi_x H(\varphi_{yy}),$$
$$\varphi_x = u. \tag{7}$$

Now we use the soliton solution (3) with varying parameters $\Theta = \Theta(y, t)$, $V = V(y, t) > 0$ and the leading order term of the test function,

$$u(x, y, t) = u_0(x - \Theta, V) + u_1(x - \Theta, V; y, t) + \dots .$$

Here u_1 ($|u_1| \ll |u_0|$) is a nonadiabatic correction to the BO soliton u_0 induced due to transverse effects which are described by the last, perturbation-like term in Eq. (2). Thus, we substitute the soliton solution (3) into the variational problem (6), (7) and perform the integration along the soliton longitudinal coordinate x. After such an "averaging" procedure [12,13] we get the averaged Lagragian $\tilde{L} = \int_{-\infty}^{+\infty} dx\, L$,

$$\tilde{L} = \pi \left(-V\Theta_t + \tfrac{1}{2}V^2 - \tfrac{1}{2}V^2\Theta_y^2 + \frac{V_{yy}}{V} - \frac{3V_y^2}{2V^2} \right), \tag{8}$$

which depends now on the new variables, the soliton velocity (V) and soliton "center-mass" coordinate (Θ). Then, variations of action (6) with respect to the new variables reduce the original (2 + 1) problem (2) to the (1 + 1) Whitham equations,

$$V_t + \left(V^2 \Theta_y\right)_y = 0, \tag{9a}$$

$$\Theta_t - V + V\Theta_y^2 + \frac{V_y^2}{V^3} - \frac{V_{yy}}{V^2} = 0. \tag{9b}$$

Usually the Whitman equations are analyzed in the long-wave limit, that is in the present context the last two terms in Eq. (9b) are neglected. However, it is easy to check that the linearized analog of a full system (9a), (9b) allows us to reproduce the exact discrete-spectrum solution (4), (5) for an arbitrary value of a transverse wave number (p). It implies that the evolution of the modulated BO solitons occurs almost adiabatically and the nonlocalized soliton tail (radiation) is not excited within the framework of the linear approximation. We have calculated the first nonadiabatic correction u_1 to the BO soliton u_0 induced due to the transverse effects and found that it is also free of nonlocalized terms,

$$u_1 = \left(\Theta_y^2 - \frac{V_y^2}{V^4} \right)(u_0 - 2Vu_{0V}) + \frac{2V_y\Theta_y}{V^2} H(u_0). \tag{10}$$

Thus, the radiation field does not appear even in the quadratic nonlinear approximation and the evolution of the modulated BO solitons is described by the Whitham equations (9a), (9b) with good accuracy. However, it is necessary to mention that the last term in (10) decays like $\sim xu_0$ as $x \to \infty$ and, hence, the next-order (cubic) approximation generates nonlocalized terms describing a very small radiation field emitted by the BO soliton.

3. As it is clear from (4), the transverse instability of the BO solitons is of self-focusing type [16–18], i.e., the bending of the soliton front is enforced by the growth of the soliton amplitude (velocity) at the faster moving concave portions of the front. Here we investigate the problem of long-term development of this instability using the weakly nonlinear (quadratic) approximation to the Whitham equations (9a), (9b). To this end, we introduce a small-amplitude smoothly modulated correction to the coordinate of a nonperturbed BO soliton having the velocity V_0 ($\Theta = V_0 t + \epsilon S(\eta, \sigma)$, where the small parameter ϵ determines the slow variables $\eta = \epsilon y$ and $\sigma = \epsilon t$). Then, from Eq. (9b) we derive the following asymptotic expansion for the soliton velocity,

$$V = V_0 + \epsilon^2 S_\sigma + \epsilon^4 \left(V_0 S_\eta^2 - \frac{1}{V_0^2} S_{\sigma\eta\eta} \right) + \mathrm{O}(\epsilon^6).$$

Finally, substituting this expansion into Eq. (9a) gives a closed equation for $S(\eta, \sigma)$,

$$S_{\sigma\sigma} + V_0^2 S_{\eta\eta} + \epsilon^2 (S_{\eta\eta\eta\eta} + 4V_0 S_\eta S_{\sigma\eta} + 2V_0 S_\sigma S_{\eta\eta}) + \mathrm{O}(\epsilon^4) = 0. \tag{11}$$

Eq. (11) coincides with the Kaup equation [19] with the accuracy of the cubic nonlinear term having order $\mathrm{O}(\epsilon^4)$. On the other hand, the transformation of Eq. (11) to Lagrangian coordinates (see Ref. [19]),

$$\eta' = \eta, \quad \sigma' = \sigma - \frac{2}{V_0} \epsilon^2 S(\eta, \sigma),$$

$$R = S_\sigma - \frac{1}{V_0} \epsilon^2 (S_\sigma)^2,$$

allows us to reduce Eq. (11) to the *integrable elliptic Boussinesq equation* for $R(\eta', \sigma')$,

$$R_{\sigma\sigma} + V_0^2 R_{\eta\eta} + \epsilon^2 \left(R_{\eta\eta} + 3V_0 R^2 \right)_{\eta\eta} + \mathrm{O}(\epsilon^4) = 0. \tag{12}$$

Here we drop the prime of the new variables for simplicity. It is important to note that the elliptic Boussinesq equation (12) gives the same dispersion relation (5) for unstable periodic perturbations on the soliton background ($R \equiv 0$) as the original model (2). Using the Zakharov–Shabat dressing method, Breizman and Malkin [20] found the *exact solutions* to Eq. (12) describing the long-term dynamics of the unstable periodic perturbation. According to the approximation made above, we keep the leading order of these solutions and thus obtain *an approximate solution* to the Whitham equations (9a), (9b) in terms of the original variables Θ, V,

$$\Theta \approx V_0 t - \frac{2p^2 \sinh[\lambda(t-\tau)]}{V_0 \lambda \{\cosh[\lambda(t-\tau)] - \cos(py)\}}, \tag{13a}$$

$$V \approx V_0 + \frac{2p^2 \{\cosh[\lambda(t-\tau)] \cos(py) - 1\}}{V_0 \{\cosh[\lambda(t-\tau)] - \cos(py)\}^2}, \tag{13b}$$

where the growth rate λ is defined by formula (5) and τ is an arbitrary parameter. Let us analyze the general solution (13a), (13b) for a special limiting case of the transverse wave number, $p \to 0$. In this case, the solution describes self-focusing of the BO soliton subjected to an initially localized perturbation, it is given by rational functions of the form

$$\Theta \approx V_0 t - \frac{4(t-\tau)}{V_0 \left[y^2 + (t-\tau)^2 \right]}, \tag{14a}$$

$$V \approx V_0 + \frac{4\left[V_0^2 (t-\tau)^2 - y^2 \right]}{V_0 \left[y^2 + V_0^2 (t-\tau)^2 \right]^2}. \tag{14b}$$

Using formulae (3), (14a), (14b) we get an approximate picture of the dynamics of a perturbed BO soliton and present it in Figs. 1a, 1b. It can be clearly seen that the initial self-focusing of the soliton amplitude in a localized region is accompanied by its self-contraction and acceleration. As a result, a 2D localized perturbation with permanently increasing amplitude escapes the BO soliton leaving a hole in its profile. Formally speaking, an approximate solution (14a), (14b) displays the formation of a singularity at $y = 0$ in a finite time $t = \tau$. However, the applicability of this solution is limited by the time $t = t^* = \tau - 1/\sqrt{2}\, V_0^2$ when the velocity V turns to zero at the points $y = \pm y^* = \pm \sqrt{3}/\sqrt{2}\, V_0$. The breakdown time of our approximate solution (14a), (14b), t^*, corresponds to the time of spatial separation of the 2D localized perturbation and the original BO soliton. The further dynamics of such a collapsing 2D cluster should be investigated in the framework of the original Whitham equations (9a), (9b).

The general solution (13a), (13b) describes the same dynamics of a periodic transverse perturbation upon the BO soliton as the solution (14a), (14b)

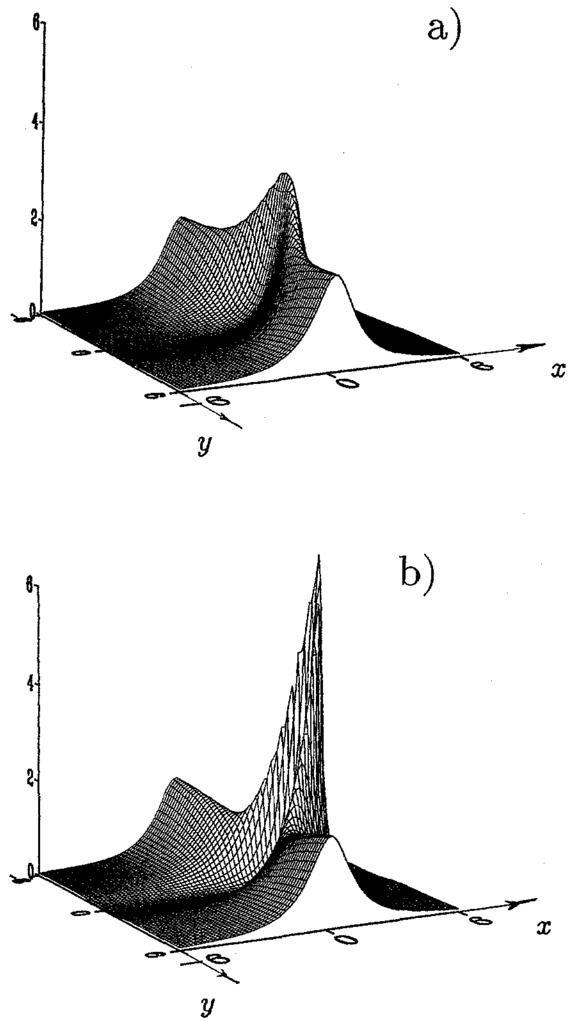

Fig. 1. Formation of a 2D collapsing cluster. Evolution of a perturbed BO soliton described by formulae (3), (14a), (14b) with $V_0 = 1$, $\tau = 0$: (a) $t = -2.5$; (b) $t = -1.3$.

does, the collapsing 2D clusters are formed at each period of the transverse perturbation. We would like to mention that our analytical results are in good agreement with the numerical simulations within the original model (1) reported in Ref. [8]. In this paper, the initial stage of the 2D collapse initiated by the self-focusing BO solitons was simulated.

4. In the previous section we found that the self-focusing instability of the BO solitons leads to formation of 2D soliton-like structures with increasing amplitudes escaping the perturbed BO soliton. Therefore, in order to describe the further dynamics of the structures it is necessary to consider essentially 2D soliton solutions to the Whitham equations (9a), (9b). Here we present the steady-state solutions which are prescribed by the functions $\Theta = \nu t + \Theta_0$, $V = W(y - y_0)$, where ν, Θ_0, and y_0 are arbitrary parameters and the function $W(y)$ satisfies the second-order ODE,

$$W\ddot{W} - \dot{W}^2 + W^3(W - \nu) = 0. \tag{15}$$

A general solution to Eq. (15) can be readily found and is given by the periodic function

$$W = \frac{p^2}{\nu - \sqrt{\nu^2 - p^2}\cos(py)}. \tag{16}$$

Here the parameter ν must be positive in order to satisfy the condition $W(y) > 0$, while the parameter p belongs to the interval $0 \leqslant p \leqslant \nu$. The functions (3), (16) describe a steady-state soliton with a transversely modulated front, and the parameters ν and p determine the soliton velocity and transverse wave number, respectively. In the limit $p \to \nu$ the amplitude of the transverse perturbations on the soliton background tends to zero and this point coincides with the instability cutoff for the BO solitons with velocity ν (see formula (5)). Thus, near the critical point $p = \nu$ the branch of the steady-state 2D waves bifurcates from the 1D soliton solutions according to the hypothesis of Ref. [9]. In the other limit ($p \to 0$) a transversely periodic solitary wave degenerates to the algebraic soliton,

$$W = \frac{2\nu}{1 + \nu^2 y^2}, \tag{17}$$

which corresponds to the 2D solitary wave in the original model (2). Thus, our results imply that the field of 2D solitons falls to zero like x^{-2}, y^{-2}, and, moreover, formulae (3), (17) give a good analytical approximation to the real soliton solution which was calculated numerically in Ref. [10] for model (1). However, surprisingly, the soliton solution (17) is not unique within the framework of the Whitham equations (9a), (9b). Extending the parameter p of function (16) into a complex plane ($p = \mathrm{i}q$) we can find another, exponentially localized solution

$$W = \frac{q^2}{\sqrt{\nu^2 + q^2}\cosh(qy) - \nu}. \tag{18}$$

The 2D solitary wave (18) has two parameters ν, q and in the limit $q \to 0$, $\nu > 0$ transforms into the algebraic soliton (17). It is interesting to note that the parameter ν for finite q may have an arbitrary sign because the function $W(y)$, (18), is positive for any ν. Therefore, 2D solitary waves of this type may propagate in an arbitrary direction with respect to the x-axis. It means that, in the original problem, such solitary waves correspond both to "subsonic" and "supersonic" waves, while the BO solitons and algebraic 2D solitons are always "subsonic" [8].

This new class of steady-state 2D soliton solutions to model (2) is essentially anisotropic as the soliton field falls to zero algebraically in the longitudinal (x) direction and exponentially in the transverse (y) direction. However, the supersonic solitary waves are in resonance with the linear dispersing perturbations and, hence, we can expect that the whole family of exponential solitons (18) for $\nu < 0$ is unstable due to such a resonant interaction. The problem of linear stability of new solitons for $\nu > 0$ and their role in the dynamics of localized perturbations remains open at the moment. However, we will show in the next section that all kinds of 2D solitary waves are unstable with respect to finite perturbations and that they transform into collapsing 2D clusters due to this nonlinear instability.

5. Now we consider the long-term dynamics of the 2D localized perturbations and show that the Whitham equations (9a), (9b) permit the formation of 2D singularities in finite time (collapse). Indeed, it is easy to check that the model (9a), (9b) inherits the critical collapse property of model (1). To this end, we write the conserved quantities for Eqs. (9a), (9b),

$$P_x = \pi \int_{-\infty}^{\infty} dy\, V, \tag{19a}$$

$$P_y = -\pi \int_{-\infty}^{\infty} dy\, V\Theta_y, \tag{19b}$$

$$H = \tfrac{1}{2}\pi \int_{-\infty}^{\infty} dy\, [h(y,t) - h(\infty,t)],$$

$$h(y,t) = V^2\Theta_y^2 + \frac{3V_y^2}{V^2} - \frac{2V_{yy}}{V} - V^2. \tag{19c}$$

Here we have subtracted the diverging component of H at infinity so that the integral (19c) is well defined both for algebraic (17) and exponential (18) soliton solutions to Eqs. (9a), (9b). The quantities (19a)–(19c) have in the original problem (2) the meanings of the x and y projections of the momentum and the energy (Hamiltonian) averaged on the BO soliton (3).

Now we make the scaling transformation $V(y) = aV'(ay)$, $\Theta_y(y) = \Theta_y'(ay)$ retaining the projections of the momentum $P_{x,y}$. Then, it follows from Eq. (19c) that the Hamiltonian H is a homogeneous function of the scaling parameter a,

$$H(\Theta, V) = aH(\Theta', V').$$

Therefore, Eqs. (9a), (9b) provide a model of the critical collapse. The critical value of the energy H occurs on the soliton solutions (17), (18) for which $H = 0$. Furthermore, in contrast to the original models (1), (2) we are able to find another remarkable property of the critical collapse, the *explicit collapse transformation* between the solutions to the Whitham equations (9a), (9b) (Θ, V) and their steady-state reduction (15) (W). This transformation has the simple form

$$\Theta = -\frac{1}{3\tau}\int_0^{\eta} \frac{\eta\, d\eta}{W(\eta)} + \frac{3}{2}\nu\tau\left[1 - \left(\frac{\tau - t}{\tau}\right)^{2/3}\right], \tag{20a}$$

$$V = \left(\frac{\tau}{\tau - t}\right)^{1/3} W(\eta), \quad \eta = \left(\frac{\tau}{\tau - t}\right)^{1/3} y. \tag{20b}$$

It describes the self-similar dynamics of perturbed 2D solitary waves given by Eqs. (17), (18) and the parameter τ determines the characteristic time of the soliton dynamics. For small values of t we can expand (20a), (20b) in power series in τ^{-1} and get a linear perturbation to the 2D solitary waves,

$$\Theta = \nu t + \frac{\nu t^2}{6\tau} - \frac{1}{3\tau}\int_0^{y} \frac{y\, dy}{W} + O(\tau^{-2}), \tag{21a}$$

$$V = W(y) + \frac{t}{3\tau}(W + y\dot{W}) + O(\tau^{-2}), \tag{21b}$$

The expansions (21a), (21b) determine the instability of the 2D soliton with respect to finite-amplitude perturbations and such an instability is relatively slow, power-like. It is obvious from (20) that the nonlinear stage of the 2D soliton instability is different for different signs of the parameter τ, either the self-similar collapse for $\tau > 0$ or the self-similar decay for $\tau < 0$. Thus, depending on a weak initial perturbation, the solitons with slightly *increased* amplitudes *collapse* whereas the solitons with slightly *decreased* amplitudes *spread out*.

It follows from (17), (20a), (20b) at $\tau > 0$ that near the singularity created by the collapse of the algebraic 2D soliton the amplitude u in the original model (2) is given by formula (3) with the parameters Θ, V in the form

$$\Theta \approx -\frac{\nu y^4}{24\tau}\left(\frac{\tau}{\tau - t}\right)^{4/3}, \tag{22a}$$

$$V \approx \frac{2}{\nu y^2}\left(\frac{\tau - t}{\tau}\right)^{1/3}. \tag{22b}$$

The collapse of the self-focusing 2D algebraic soliton is shown in Figs. 2a, 2b. It can be easily seen that the shape of the self-focusing 2D soliton (Fig. 2a) is similar to the shape of the soliton-like structure which is formed as a result of the BO soliton instability (Fig. 1b). Moreover, they have the same asymptotics of the field decay ($\sim x^{-2}$, y^{-2}) as follows from formulae (14b), (22b). We would like to emphasize that the collapsing structure remains self-similar for any time (Fig. 2b).

Note that our results are different from the self-similar substitution given in Ref. [8], $u = aU(ax, ay)$, where $a = (\tau - t)^{-1/2}$. However, in analogy with the 1D BO equation (see Ref. [21]), this substitution is not responsible for the collapse phenomenon, it describes the self-similar dynamics of linear dispersive waves. The correct structure of the collapsing clusters can be found only approximately and is presented by formulae (3) and (22). Furthermore, using (17), (19a) we can estimate the threshold value P_{cr} for the momentum of the 2D algebraic soliton, $P_{cr} \approx 2\pi^2$. According to the general theory of the critical collapse phenomenon (see, e.g. Ref. [1]), the localized initial distributions with $P_x < P_{cr}$ spread out and decay while those with $P_x > P_{cr}$ focus and collapse with permanent emission of radiation waves. The critical ($P_x = P_{cr}$) collapse or decay are radiationless, and they are approximately described by the transformation (20a), (20b).

Thus, the 2D solitary waves found in Ref. [10] cannot describe the *steady-state* localized perturbations in shear flows because of their slow power-like instability. However, the structure of self-focusing

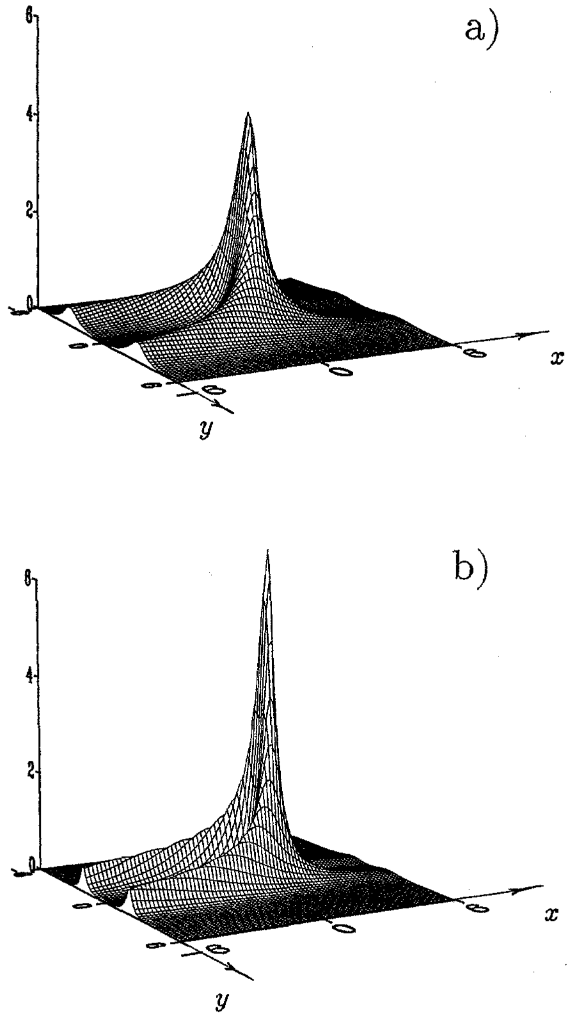

Fig. 2. Self-similar radiationless collapse. Evolution of a 2D algebraic soliton described by formulae (3), (17), (20a), (20b) with $\nu = 1$, $\tau = 1$: (a) $t = 0.2$; (b) $t = 0.8$.

2D perturbations remains similar to that of these 2D algebraic solitons. Therefore, we might expect that the large-amplitude perturbations slowly evolve into the "thorn"-like coherent structures similar to those observed in Ref. [6] in the boundary-layer type shear flows above the plate, having at each instant the same shape as the 2D soliton of model (1).

6. In this paper we have presented the explicit collapse transformation of the steady-state solutions to the Whitham equations which were obtained from the models (1), (2). This transformation is similar to that for the critical 2D NLS equation [2,3]. However, in the latter case it is associated with a nontrivial symmetry of solutions to the 2D NLS equation and with the virial theorem which is very useful for describing long-term dynamics of localized initial perturbations [4]. Unfortunately, in our case, such additional symmetry and the virial theorem do not exist for the Whitham equations (9a), (9b). Moreover, we were unable to generalize even the collapse transformation (20a), (20b) for the original models (1), (2). Apparently, they do not possess this remarkable mathematical property of the 2D NLS equation. Therefore, the Whitham adiabatic approach seems to be especially fitted for the analytical description of the critical 2D wave collapse. We note that we have checked that the Whitham equations for the 2D NLS equation (which were obtained in Ref. [13]) inherit all properties of the model including the general collapse transformation and the virial theorem. Thus, we may expect that the Whitham approach gives correct results in other problems where the radiation field is not essential for the soliton evolution.

The authors thank E.A. Kuznetzov and Yu.A. Stepanyants for useful discussions and remarks. This work was partially supported by INTAS (grant No. INTAS-93-1373) and by SIF (grants No. MMP00 and No. R8T000). The paper was completed when one of the authors (VIS) visited the Department of Mathematics at the Monash University (Australia) within the framework of the collaboration agreement between the Monash University and the Shirshov Institute.

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