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Structural transformation of eigenvalues for a perturbed algebraic soliton potential

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Abstract

We present a modified perturbation technique for the AKNS spectral system to evaluate eigenvalues supported by a perturbed algebraic soliton potential. The results of this technique are applied to the problem of the structural instability of algebraic solitons in the modified Korteweg-de Vries equation. It is shown analytically and numerically that the algebraic soliton is destroyed under the action of small initial perturbations and transforms either to a steady-state soliton with exponentially decaying tails or to a pulsating "breather"-type wave packet. © 1997 Published by Elsevier Science B.V.

1. Introduction

Solitary wave solutions (or solitons) have been found for various nonlinear evolution equations describing nonlinear waves (see, e.g., Ref. [1]). However, such solutions are important for wave dynamics only if (i) they are stable under small perturbations and (ii) they are not in resonance with small-amplitude quasilinear waves. Under these conditions, an initial localized pulse evolves into a sequence of solitons propagating with different but constant velocities separated from a wave train decaying due to linear dispersion [1]. In the opposite case, when the solitary waves are unstable or in resonance with linear waves, the initial pulse transforms to other, usually nonstationary, structures of the nonlinear wave field.

It has been discovered for various fields in contemporary physics (see, e.g., Refs. [2-10]) that, besides the conventional solitary wave solutions with strongly (exponentially) decaying tails at infinity, there exist also solitary waves whose fields fall off more weakly, in fact algebraically. In some problems, for example, for internal waves in a deep fluid governed by the Benjamin-Ono equation [2,3], these algebraic solitary waves play the same role as conventional solitons, i.e. they propagate with different velocities and are stable under the action of small perturbations [4]. However, in other cases [5-10], the algebraic solitons have been found to co-exist in parameter space with a family of exponentially decaying soliton solutions and are a special degenerate limit of this family.

This situation is rather typical for nonlinear dispersive wave systems where a balance between linear dispersion and nonlinearity can be achieved only when taking into account the strong ("higher-order") nonlinear properties of the wave field. For instance, it has recently been shown that the gap-solitons associated with wave resonances of different types [5,6], the quasi-parallel Alfvén solitary waves in a magnetized plasma [7,8], the self-guided waves in refractive optical materials with non-Kerr nonlinearities [9,10], all have an algebraic shape at the edge between the

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branches of the linear and nonlinear wave spectrums.

The prototype example for this phenomenon was found a long time ago [11,12] in the framework of the modified Korteweg-de Vries (mKdV) equation which governs the evolution of long waves in the critical case when the usual quadratic nonlinearity vanishes. For example, this happens for oblique ion-acoustic plasma waves at a critical angle to a magnetic field [13,14], for the wave propagation in nonlinear lattices with symmetrical potentials [15], and for propagation of small-amplitude dark solitons along the continuous wave background in defocusing optical materials [16]. In all these cases, the underlying model, providing a balance between linear dispersion, small quadratic and strong cubic nonlinearities, has the dimensionless form

$$u_t + 12quu_x + 6u^2u_x + u_{xxx} = 0, \tag{1}$$

where q is a parameter for the quadratic nonlinearity which here is supposed to be positive.

The mKdV equation (1) has two branches of soliton solutions (see Refs. [11,12]) given by

$$u = u_p = \frac{2\sigma p^2}{\sqrt{q^2+p^2}\cosh[2p(x-vt)] + \sigma q}, \tag{2}$$

where $\sigma = \pm 1$ and p is the soliton parameter determining the velocity, $v = 4p^2$. In the limit $p \gg q$ both branches transform to the sech-type soliton solution of the mKdV equation which are known to be stable with respect to small perturbations (see, e.g., Ref. [17]). Furthermore, for $\sigma = +1$ the soliton field decays exponentially as $|x| \to \infty$ for any p and approaches in the limit $p \ll q$ to the well-known sech^2-type profile which satisfies the KdV equation (see Ref. [1]). Stability of solitons within the KdV equation has also been proved in several works (see Ref. [17] and references therein). On the other hand, the branch of the exponentially decaying soliton solutions with $\sigma = -1$ tends, for $p \ll q$, to the algebraic soliton [11,12],

$$u \to u_0 = -\frac{4q}{1+4q^2x^2}, \tag{3}$$

while the velocity $v = 0$. The criterion of soliton stability (see Ref. [17]) when applied to the soliton solutions (2) reveals that the algebraic soliton (3) represents a critical case when the unstable mode of the linearized problem has a degenerate zero eigenvalue. In this case, effects induced by this weak instability result in a power-like growth for small perturbations around the soliton solution. Besides, the algebraic soliton (3), which is stationary in time, is located at the edge between the exponentially decaying solitons (2) which propagate with positive velocities v and small-amplitude dispersive wave packets propagating with negative group velocities, the smaller wavelength having the larger group velocity. Therefore the algebraic soliton (3) is in resonance with infinitely long linear waves. These two mechanisms must lead to a structural instability of the algebraic solitons (3) in the mKdV equation (1).

The phenomenon of structural instability of algebraic solitons has been predicted for several different models (see, e.g., Refs. [7,8,10]) but a correct description of the long-term soliton dynamics has not yet been done even for the case when the underlying evolution equation, i.e. the mKdV equation, can be solved by means of the inverse scattering transform (IST) technique (see, e.g., Refs. [18,19]). The difficulties arising here are caused by the weak localization of the algebraic soliton potentials which do not allow the application of the results of the IST technique directly.

In this paper we develop a modified perturbation technique to analyse the linear problem associated with an algebraically decaying potential and construct perturbed eigenvalues to this problem. The results of this analysis enable us to describe, for the first time to our knowledge, the long-term transformation of the algebraic soliton due to its structural instability and resonance with long linear waves described by the mKdV equation.

2. Regular perturbation technique

The linear spectral problem associated with the mKdV equation (1) can be written in the form

$$\varphi_{1x} = -(q+u)\varphi_2 + \lambda\varphi_1,$$
$$\varphi_{2x} = (q+u)\varphi_1 - \lambda\varphi_2, \tag{4}$$

where λ is parameterized in terms of the spectral parameter k according to $\lambda = -\sqrt{q^2-k^2}$. This problem is a simple reduction of the general AKNS system which was investigated in detail for $q = 0$ in Ref. [18] and for $q \neq 0$ in Ref. [19]. It was shown that the spectral parameter k in (4) does not depend on time t if the function $u(x,t)$ satisfies (1). Therefore, the initial data $u(x,0)$ completely determines the number and location of the discrete-spectrum modes for this linear system and also the continuous-spectrum eigenfunctions which are referred to as the Jost functions. The discrete-spectrum modes are related to the soliton solutions of the mKdV equation while the Jost functions describe the radiative waves decaying due to linear dispersion (see Ref. [18]).

The Jost functions for the single-soliton potential $u = u_p$ at $t = 0$ have the following explicit form,

$$\begin{aligned}\boldsymbol{\varphi} &= \boldsymbol{\Phi}^{\pm}(x;k,p)\\ &= \mathrm{e}^{\pm \mathrm{i}kx}\left[\left(\mathrm{i}k \pm \frac{\sigma\lambda_p}{2p}\sinh(2px)u_p\right)\boldsymbol{e}_1^{\pm} - \tfrac{1}{2}u_p\boldsymbol{e}_2^{\mp}\right],\end{aligned} \tag{5}$$

where

$$\boldsymbol{e}_1^{+} = \begin{pmatrix}\lambda + \mathrm{i}k\\ q\end{pmatrix},\quad \boldsymbol{e}_1^{-} = \begin{pmatrix}q\\ \lambda + \mathrm{i}k\end{pmatrix},$$

$$\boldsymbol{e}_2^{+} = \begin{pmatrix}-(\lambda + \mathrm{i}k)\\ q\end{pmatrix},\quad \boldsymbol{e}_2^{-} = \begin{pmatrix}q\\ -(\lambda + \mathrm{i}k)\end{pmatrix},$$

and $\lambda_p = -\sqrt{q^2 + p^2}$. The single-soliton solution (2) supports only one discrete-spectrum mode which arises from (5) for $k = \mathrm{i}p$ and $\lambda = \lambda_p$,

$$\begin{aligned}\boldsymbol{\varphi} = \boldsymbol{\Phi}_p &= \boldsymbol{\Phi}^{+}(x;\mathrm{i}p,p) = -\sigma\boldsymbol{\Phi}^{-}(x;\mathrm{i}p,p)\\ &= \frac{\sigma\lambda_p}{2p}(\mathrm{e}^{px}\boldsymbol{e}_p^{+} - \sigma\mathrm{e}^{-px}\boldsymbol{e}_p^{-})u_p,\end{aligned} \tag{6}$$

where the vectors $\boldsymbol{e}_p^{\pm}$ follow from $\boldsymbol{e}_1^{\pm}$ for $k = \mathrm{i}p$ and $\lambda = \lambda_p$.

To describe the evolution of a single-soliton solution (2) under the action of small perturbations we need to evaluate a perturbation-induced correction to the eigenvalue $k = \mathrm{i}p$ of the discrete-spectrum mode (6). Therefore, we present the initial data $u(x,0)$ as $u = u_p + \epsilon\delta u$, where δu is an initial perturbation to the soliton and ϵ is a formal small parameter. Under the condition for the discrete-spectrum mode $\boldsymbol{\varphi}$ to be localized, it follows from (4) that the perturbation to the eigenvalue satisfies the following integral relation,

$$\begin{aligned}(\lambda - \lambda_p)&\int_{-\infty}^{+\infty}(\varphi_1\Phi_{p2} + \varphi_2\Phi_{p1})\,\mathrm{d}x\\ &= \epsilon\int_{-\infty}^{+\infty}\delta u(\varphi_1\Phi_{p1} + \varphi_2\Phi_{p2})\,\mathrm{d}x.\end{aligned} \tag{7}$$

Now, using a regular perturbation theory we seek solutions of (4) in the form of the asymptotic series,

$$\begin{aligned}\boldsymbol{\varphi} &= \boldsymbol{\Phi}_p + \epsilon\boldsymbol{\phi}_1 + \mathrm{O}(\epsilon^2),\\ \lambda &= \lambda_p + \epsilon\lambda_p'\delta p + \mathrm{O}(\epsilon^2),\end{aligned} \tag{8}$$

where $\lambda_p' = \mathrm{d}\lambda_p/\mathrm{d}p$ and δp is a perturbation to the parameter p of the discrete-spectrum mode (6). Substituting the series (8) into (7) we find the value of this perturbation,

$$\delta p = \frac{q^2 + p^2}{2p^2}\delta P_p, \tag{9}$$

where δP_p is the perturbation-induced correction to the soliton momentum P_p defined by

$$P_p = \tfrac{1}{2}\int_{-\infty}^{-\infty}u_p^2\,\mathrm{d}x,\quad \delta P_p = \int_{-\infty}^{+\infty}u_p\delta u\,\mathrm{d}x. \tag{10}$$

We mention that the asymptotic equation (9) serves also as a solvability condition for the perturbation $\boldsymbol{\phi}_1$ to be localized at infinity.

It follows from (9) that a small initial perturbation to a single soliton (2) does not affect the existence of the discrete-spectrum mode (6) but changes its eigenvalue $k = \mathrm{i}p$ to $k = \mathrm{i}p'$, where $p' = p + \epsilon\delta p + \mathrm{O}(\epsilon^2)$. This implies that the initial condition $u(x,0) = u_p + \epsilon\delta u$ also generates a soliton (2) but with an effective parameter p'. Besides, the initial condition gives birth to a continuous-spectrum wave field decaying due to dispersion and, possibly also, to additional discrete-spectrum modes with small eigenvalues k. However, all these waves take away only a small, order $\mathrm{O}(\epsilon^2)$, part of the perturbation-induced momentum of the wave field. Indeed, using the explicit form (2) we find the soliton momentum P_p,

$$P_p = 2p + q[2\tan^{-1}(qp^{-1}) - \pi\sigma]. \tag{11}$$

A change of the soliton momentum induced by an initial perturbation can be defined as $\Delta P = P_{p'} - P_p$ and, by virtue of (9), we find that $\Delta P = \epsilon \delta P_p + O(\epsilon^2)$. Therefore, the whole perturbation-induced correction δP_p to the soliton momentum goes to the renormalization of the soliton parameter. This is a general result which is valid only if the basic soliton solution is stable with respect to small perturbations.

Now we consider the limiting transition $p \ll q$ for $\sigma = -1$ when the sech-type soliton (2) transforms to the algebraic soliton (3). In this case, the expression (9) diverges like p^{-2} unless $\delta P_0 = 0$. In general, this latter condition is not satisfied and thus the regular perturbation theory fails to describe the eigenvalues of a perturbed algebraic soliton potential. For comparison, the similar limit for $\sigma = +1$ leads to a regular expression for δp if $\delta u \sim O(p^2)$ so that $\delta P_p \sim O(p^3)$ as $p \to 0$. In the next section we consider the correct limiting transition to a perturbed algebraic soliton potential following from a modified perturbation theory and, therefore, specify $\sigma = -1$.

3. Modified perturbation technique

The regular perturbation analysis fails in the limit $p \ll q$ ($\sigma = -1$) because the discrete-spectrum mode $\boldsymbol{\Phi}_p$ becomes orthogonal to itself. Indeed, the algebraic soliton potential (3) supports the discrete-spectrum mode $\boldsymbol{\Phi}_0$ following from (6),

$$\boldsymbol{\Phi}_p \to \boldsymbol{\Phi}_0 = -\tfrac{1}{2} q(\boldsymbol{l} + 2\boldsymbol{e}_0 x) u_0, \quad \boldsymbol{l} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{e}_0 = \begin{pmatrix} q \\ -q \end{pmatrix}. \tag{12}$$

It is easy to check that $\int_{-\infty}^{+\infty} (\Phi_{01}\Phi_{02})\, dx = 0$ and, as a result, the leading order term of the left-hand-side of (7) vanishes in the limit $p \to 0$ for the regular asymptotic expansion (8). This result is a consequence of the fact that the algebraic soliton represents a critical case for the stability problem (see Ref. [17]) so that the derivative dP_p/dv obtained from (11) at $v = 4p^2$ vanishes for $p \to 0$.

To treat the special case of small p, we consider the perturbation $\lambda - \lambda_p$ to have a smaller order than $O(\epsilon)$ and modify the regular asymptotic series (8) by an asymptotic expansion,

$$\boldsymbol{\varphi} = \boldsymbol{\phi}_0 + \epsilon \boldsymbol{\phi}_1 + O(\epsilon^2),$$

$$\lambda = \lambda_p - \frac{1}{2q}\epsilon^{2/3}(p'^2 - p^2) + O(\epsilon). \tag{13}$$

The leading-order term $\boldsymbol{\phi}_0$ then satisfies the following linear problem,

$$\begin{aligned} \phi_{01x} &= -(q + u_p)\phi_{02} + \left(\lambda_p - \frac{1}{2q}\epsilon^{2/3}(p'^2 - p^2)\right)\phi_{01}, \\ \phi_{02x} &= (q + u_p)\phi_{01} - \left(\lambda_p - \frac{1}{2q}\epsilon^{2/3}(p'^2 - p^2)\right)\phi_{02}. \end{aligned} \tag{14}$$

Here we have used the same notations for u_p and λ_p as above but the parameter p should be now replaced by $\epsilon^{1/3} p$.

Analysis of the linear problem (14) reveals that a naive one-scale solution in the form $\boldsymbol{\phi}_0 = \boldsymbol{\Phi}_p + \epsilon^{2/3}\delta\boldsymbol{\Phi}_p + O(\epsilon^{4/3})$ fails because the perturbation term $\delta\boldsymbol{\Phi}_p$ which is referred to as an associated discrete-spectrum mode is secularly diverging as $|x| \to \infty$. Indeed, for the algebraic soliton ($p \to 0$), the associated mode has the form

$$\delta\boldsymbol{\Phi}_0 = \tfrac{1}{4} p'^2 (\boldsymbol{l} - 2\boldsymbol{e}_0 x), \tag{15}$$

and grows linearly in x. This difficulty is a consequence of another property of the algebraic soliton (3) which is a resonance with infinitely long linear waves. Because of this resonance, the discrete-spectrum mode $\boldsymbol{\Phi}_0$ and the associated mode $\delta\boldsymbol{\Phi}_0$ are limiting degenerations of the Jost functions (5) at $p = 0$ and $\sigma = -1$ in the asymptotic limit $k \to 0$.

To remove these secular divergences, we have to introduce the second (outer) scale of the asymptotic expansion (13), i.e. that described by the variable $X = \epsilon^{1/3} x$, and match the asymptotic solution at the outer interval with the naive (inner-scale) expansion described above. It is obvious from (14) that the function $\boldsymbol{\phi}_0$ at this outer scale coincides, on neglecting terms of $O(\epsilon^{4/3})$ and higher-order terms, with the Jost functions (5) extended to the complex value $k = i\epsilon^{1/3} p'$. Supposing that the parameter p' has a positive real part, the Jost functions $\boldsymbol{\Phi}^+$ and $\boldsymbol{\Phi}^-$ are exponentially localized in the limits $x \to +\infty$ and $x \to -\infty$, respectively. Therefore, matching these Jost functions with the inner asymptotic expansion we find the following uniform representation for the two-scale asymptotic solution of the linear problem (14),

$$\phi_0 = \Phi^+(x; i\epsilon^{1/3}p', \epsilon^{1/3}p) + O(\epsilon^{4/3}), \quad x > 0$$
$$= \Phi^-(x; i\epsilon^{1/3}p', \epsilon^{1/3}p) + O(\epsilon^{4/3}), \quad x < 0. \tag{16}$$

Finally, substituting the leading order of (16) into (7) and evaluating the integrals we find a modified asymptotic equation for the spectral parameter p',

$$p'\left(p'^2 - p^2\right) = q^2\delta P_0. \tag{17}$$

In the limit $|p' - p| \ll p$ the modified asymptotic equation produces

$$p' - p = \frac{q^2}{2p^2}\delta P_0,$$

which is consistent with Eq. (9) of the regular perturbation theory. In the opposite limit, $|p' - p| \gg p$, the formula (17) determines an eigenvalue of the discrete-spectrum mode supported by the perturbed algebraic soliton potential according to the simple equation, $p'^3 = q^2\delta P_0$.

4. Discussion

First, we analyse the results of the modified asymptotic theory applied to the case of the algebraic soliton. Then, the cubic equation (17) for $p = 0$ exhibits only one real positive solution p' if $\delta P_0 > 0$ and two complex conjugate solutions $p' = p_r \pm ip_i$ with positive real part p_r if $\delta P_0 < 0$. Thus, an initial perturbation which increases the soliton momentum, i.e. $\delta P_0 > 0$, leads to the formation of a steady-state exponentially decaying soliton (2) with effective parameter p' and also to the generation of a train of small-amplitude dispersive waves. Using the approximation of P_p (see (11)) in the limit $p \ll q$ for $\sigma = -1$ given by

$$P_p = 2\pi q + \frac{2}{3q^2}p^3 + O(p^5),$$

we find a change of the soliton momentum induced by an initial perturbation,

$$\Delta P = P_{p'} - P_0 = \tfrac{2}{3}\epsilon\delta P_0 + O(\epsilon^{5/3}).$$

Therefore, the train of radiative waves takes away a third part of the momentum of an initial perturbation.

On the other hand, if the perturbation decreases the momentum of the algebraic soliton, i.e. $\delta P_0 < 0$, then the steady-state propagation of a soliton-like initial perturbation is revealed to be impossible. It is well-known (see, e.g., Ref. [20]) that a complex conjugate pair of eigenvalues in spectral space corresponds to a nonstationary (pulsating) nonlinear wave packet which is referred to as a breather. This solution to the mKdV equation (1) is described by the function

$$u = u_b = 2\frac{\partial}{\partial x}\tan^{-1}\left(\frac{p_r\cos\theta\cosh\psi - p_i\sinh\eta\cos\phi}{p_r\sin\theta\sinh\psi + p_i\cosh\eta\sin\phi}\right), \tag{18}$$

where

$$\eta = 2p_r(x - vt) + \eta_0, \quad \theta = 2p_i(x - wt) + \theta_0,$$
$$v = 4(p_r^2 - 3p_i^2), \quad w = 4(3p_r^2 - p_i^2),$$
$$\phi + i\psi = \tanh^{-1}[q^{-1}(p_r + ip_i)],$$

and the parameters η_0 and θ_0 are arbitrary phases. Using the values of p_r and p_i predicted by the modified asymptotic theory we find that $v = 4p_r^{-1}q^2\delta P_0 < 0$ and $w = 0$. Therefore, the breather generated due to the algebraic soliton transformation propagates to the left as a small-amplitude dispersive wave. Furthermore, the momentum P_b of the breather solutions (18) is given by

$$P_b = 4p_r + 2q\left[\pi - \tan^{-1}\left(\frac{2qp_r}{q^2 - (p_r^2 + p_i^2)}\right)\right]. \tag{19}$$

Using the approximation

$$P_b = 2\pi q + \frac{4p_r}{3q^2}(p_i^2 - 3p_r^2) + O((p_r^2 + p_i^2)^{5/2}),$$

we find the change of the soliton momentum $\Delta P = P_b - P_0$ as

$$\Delta P = \tfrac{4}{3}\epsilon\delta P_0 + O(\epsilon^{5/3}).$$

Therefore, similarly to the case $\delta P_0 > 0$, the radiative waves take away a third part of the momentum induced by an initial perturbation. Thus, the algebraic soliton is revealed to be structurally unstable under a small perturbation and the dynamics of its transformation depends on the initial perturbation and is accompanied by rather strong, $O(\epsilon)$, generation of small-amplitude dispersive waves.

It is easy to generalize this analysis for the case of the exponentially decaying solitons (2) with a small but finite value of p. The modified asymptotic equation (17) also describes a unique real eigenvalue $p' > p$ for the case $\delta P_0 > 0$ and two eigenvalues with positive real parts for the case $\delta P_0 < 0$. However, for small initial perturbations, i.e. for $-P_{cr} < \delta P_0 < 0$, where

$$P_{cr} = \frac{2}{3\sqrt{3}q^2} p^3,$$

these two eigenvalues are real and correspond to two solitons (2), one of which with a greater value of p' corresponds to a soliton with $\sigma = -1$ and the other one with a smaller value of p' corresponds to a soliton with $\sigma = +1$. For initial perturbations with $\delta P_0 < -P_{cr}$ these two eigenvalues merge and go into the complex plane. Therefore, for such large perturbations the soliton-like initial perturbation transforms into the nonstationary breather (18). Location of the eigenvalues for a perturbed soliton (2) with $p = q = 1$ and $\sigma = -1$ in a complex k-plane is shown in Fig. 1 for varying perturbations with different values of δP_p.

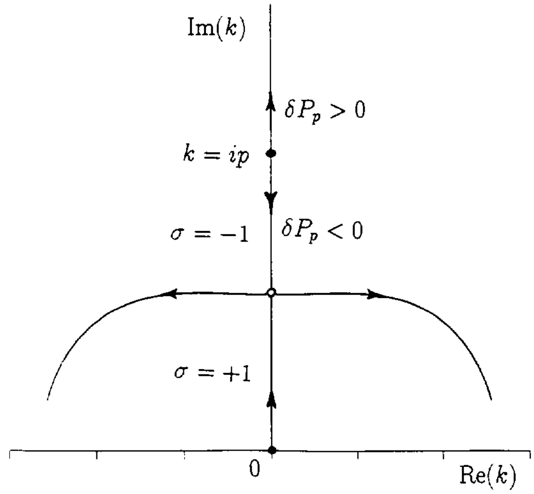

Fig. 1. Transformation of eigenvalues in a complex k-plane for a perturbed soliton (2) with $p = q = 1$ and $\sigma = -1$. δP_p denotes momentum introduced by the soliton perturbation. For $\delta P_p > 0$ there is only one eigenvalue, while for $\delta P_p < 0$ there are two eigenvalues which correspond to modes (6) with $\sigma = 1$ and $\sigma = -1$ merging at $\delta P_p = -P_{cr}$.

In both the cases, the transformation of an exponentially decaying soliton is accompanied by the generation of radiation which is small in the limit $|p' - p| \ll p$, where the regular perturbation theory can be applied, and is maximal in the limit $|p' - p| \gg p$. Thus, the phenomenon of structural instability of the algebraic solitons in the mKdV equation leads also to structural instability of the exponentially decaying solitons but with respect to finite-amplitude rather than to infinitesimal perturbations.

To confirm the results of the modified asymptotic theory we have applied the numerical scheme proposed by Boffetta and Osborne [21] for evaluation of the eigenvalues of the AKNS spectral problem (4). The perturbed algebraic soliton potential u was taken in the form $u = au_0(x)$, where u_0 is given by (3). The case $a = 1$ corresponds to the exact soliton solutions while an increase or decrease of the parameter a induces increasing or decreasing of the initial momentum of the nonlinear field. For the case $a > 1$ we have found a unique discrete-spectrum mode with the parameter $k = ip'$ while for $a < 1$ the spectral system (4) supports a complex-valued pair of the discrete-spectrum modes with the eigenvalues $k = i(p_r \pm ip_i)$. The dependence of p', p_r, and p_i as functions of a is shown in Fig. 2 by solid lines. The dashed lines display approximations of the modified asymptotic theory which are revealed to agree with the numerical results in a vicinity of the point $a = 1$.

Next, we have simulated the mKdV equation (1) numerically with the same initial condition $u(x,0)$ as above. For $a > 1$ we have observed the formation of an exponentially decaying soliton propagating with constant velocity to the right away from the dispersive wave tail (see Fig. 3a). On the other hand, for $a < 1$ the initial pulse generates waves propagating to the left (see Fig. 3b). These waves are presented by a superposition of a pulsating strongly-nonlinear wave packet (breather) and a dispersive wave background. We have used periodic boundary conditions for the numerical simulations modified by damping elements at the boundaries to suppress penetration of artificial ripples through a period of a computational window. However, these artificial ripples are still presented in our simulations (see Figs. 3a, 3b). Nevertheless, the small-amplitude ripples can only change fine details of the observed processes near a critical value of an initial amplitude, $a \sim 1$, as well as some quantitative

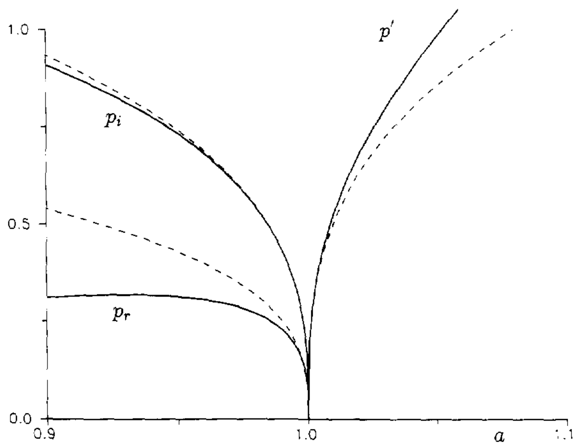

Fig. 2. The parameters of the eigenvalues of a perturbed algebraic soliton potential versus the amplitude a of the initial perturbation calculated numerically (solid) and analytically (dashed) within the linear problem (4) at $q = 1$.

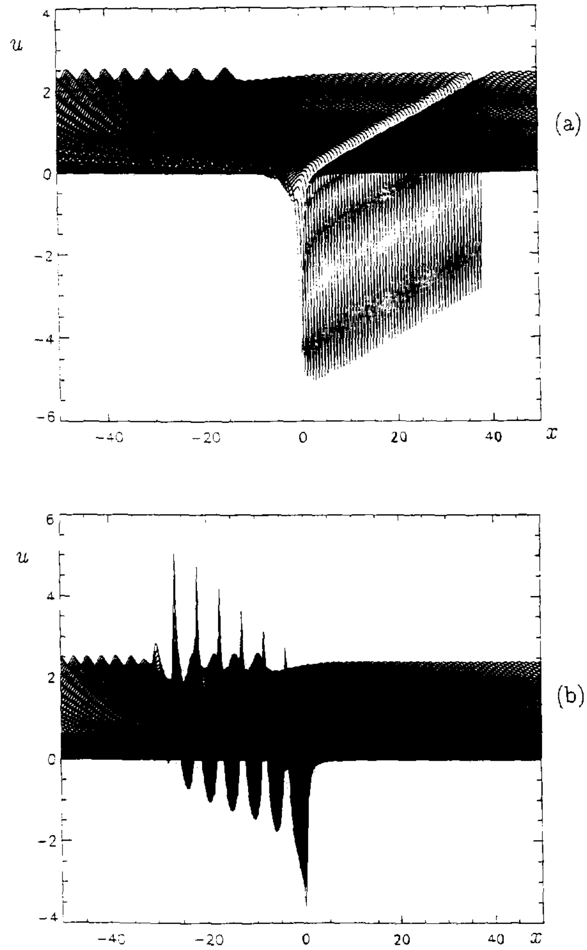

Fig. 3. Transformation of the algebraic soliton within the mKdV equation (1) at $q = 1$ for the initial value $u(x,0) = au_0(x)$, where (a) $a = 1.1$ and (b) $a = 0.9$.

characteristics of the nonlinear structures formed. A general qualitative picture of the structural instability of the algebraic soliton is not affected by this artificial effect and is clearly observed in direct numerical simulations.

Finally, we mention that a perturbation with $\delta P_0 = 0$ does not lead to a transformation of the algebraic soliton (3) at the first order of the modified perturbation theory. However, even in this case, there is an exact solution to the mKdV equation (1) which describes nontrivial dynamics of the algebraic soliton induced by its structural instability. This solution can be obtained from (18) in the limit $p_r^2 + p_i^2 \ll q^2$,

$$u = \frac{12q(3 - 24q^2x^2 + 384q^4xt - 16q^4x^4)}{4q^2(3x - 48q^2t - 4q^2x^3)^2 + 9(1 + 4q^2x^2)^2}. \tag{20}$$

As a matter of fact, this solution was first found in Ref. [22] but the authors did not analyse the corresponding soliton dynamics and referred to (20) as a new algebraic soliton. We present in Fig. 4 the wave field evolution in different time instants according to the exact solution (20) for $q = 1$. It is obvious that this solution describes a degenerate radiationless process of the resonant interaction of the algebraic soliton with an infinitely long linear wave (i.e. shelf) occurring when the initial perturbation does not change the soliton momentum, i.e. $\delta P_0 = 0$ (see (19) for $p_r = p_i = 0$). As a result of this interaction, the coordinate of the algebraic soliton propagates as $t^{1/3}$ and the soliton amplitude changes from its minimal value $u = -4q$ to the maximal value $u = +4q$ and back.

To conclude we have presented a new modification of the perturbation theory for linear spectral problems with algebraically decaying potentials and found that the perturbation to the algebraic soliton induces its structural instability and transformation either to a soliton with exponentially decaying tails or to pulsating wave packets. We believe that our results can be generalized for other physically relevant models which display similar phenomenon and which are solvable by means of the IST technique.

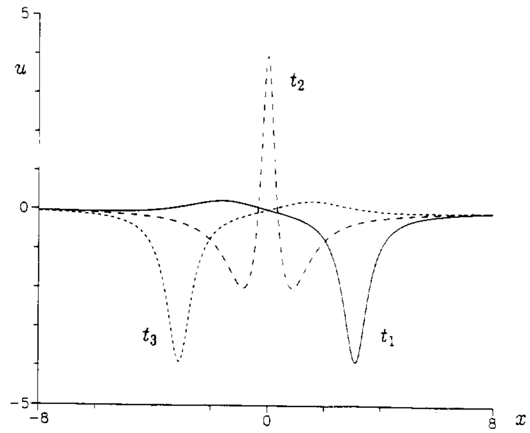

Fig. 4. Evolution of the exact rational solution (20) for $q = 1$ at different times, $t_1 = -2.5$, $t_2 = 0.0$ and $t_3 = 2.5$.

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