

## Self-focusing of plane dark solitons in nonlinear defocusing media

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We analyze a transverse instability of plane (quasi-one-dimensional) dark solitons in the framework of the two-dimensional nonlinear Schrödinger (NLS) equation for beam propagation in a defocusing nonlinear medium. We show that in the vicinity of the instability threshold the exponential growth of transverse perturbations is stabilized by nonlinearity and also by the radiation emitted from the plane dark soliton to the right and left. Dynamics of the transverse instability of the plane dark soliton of arbitrary amplitude is investigated analytically by means of the asymptotic technique, and also numerically by direct integration of the two-dimensional NLS equation. In particular we show that there exist generally three different scenarios of the instability dynamics, namely, (i) generation of a chain of two-dimensional "gray" solitons (anisotropic solitons of the Kadomtsev-Petviashvili, or KP1, equation) from the small-amplitude plane dark soliton, (ii) long-lived large-amplitude transverse oscillations of the plane dark soliton near the instability threshold, and finally, (iii) decay of the plane dark soliton into a chain of circular symmetric "black" solitons (optical vortices) of alternative topological charges. We estimate the region of the instability domain for the parameters of the soliton and perturbation where the instability of the plane dark soliton ends up in the formation of pairs of vortex and antivortex solitons.

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### I. INTRODUCTION

Dark spatial solitons are known to exist as low-intensity stationary dips on a background field that do not diffract as the beam propagates [1] (see also the review paper [2], and references therein). In the case of two transverse coordinates such solitons have been observed experimentally as dark stripes or grids with properties similar to those of one-dimensional dark solitons [3]. Dark solitons of circular symmetry (optical vortex solitons) have been predicted and shown to be stable [4,5] (see also [6,7] where a similar kind of vortex solitons was predicted in the theory of superfluidity), and they have also been observed experimentally in self-defocusing materials [8,9]. On the other hand, the linear analysis shows that a plane dark soliton is unstable to transverse long-wavelength modulations [10,11]. Numerical calculations show that due to that instability a dark stripe may decay into a sequence of optical vortex solitons of alternative polarities [12,13].

From a mathematical point of view all the problems mentioned above are described by the scalar wave equation in the so-called paraxial approximation. Let us consider propagation of the monochromatic (transverse) electric field  $E$  in a nonlinear self-defocusing medium with the intensity-dependent refractive index  $n = n_0 + n_2|E|^2$  ( $n_2 < 0$ ). Looking for solutions of Maxwell's equations in the form of a slowly varying envelope of a carrier

wave with the propagation constant  $\beta_0$  and frequency  $\omega$ , we derive the nonlinear Schrödinger (NLS) equation

$$2i\beta_0\partial_z\mathcal{E} + (\partial_x^2 + \partial_y^2)\mathcal{E} + \beta_0^2\left(\frac{n_2}{n_0}\right)|\mathcal{E}|^2\mathcal{E} = 0, \quad (1)$$

where the symbol  $\partial$  with a subindex is used for simplicity throughout this paper to denote the differentiation with respect to the corresponding variable,  $\mathcal{E}(x, y, z)$  is the slowly varying field envelope,  $z$  is the longitudinal coordinate, and  $x$  and  $y$  are two transverse coordinates. For the defocusing nonlinearity ( $n_2 < 0$ ) the nonlinear continuous wave (cw) described by Eq. (1) is modulationally stable, so that we consider nonlinear waves on the stable cw background. Making the transformations

$$z \rightarrow \beta_0 t \quad \text{and} \quad \mathcal{E}(x, y, z) \rightarrow \left(\frac{2n_0}{n_2|\beta_0^2}\right)^{1/2} \Psi(x, y, t)e^{-it}$$

we can reduce Eq. (1) to the canonical form

$$2i\partial_t\Psi + (\partial_x^2 + \partial_y^2)\Psi - 2(|\Psi|^2 - 1)\Psi = 0 \quad (2)$$

usually used in nonlinear dynamics in a different context (see, e.g., [14]).

As is well known, bright solitons do not exist in a defocusing nonlinear medium, so the pulse dynamics in this case is rather trivial: the pulse spreads due to dispersion

and develops a frequency chirp (see, e.g., [15]). However, more nontrivial dynamics may appear if we are interested in waves propagating on a modulationally stable cw background when the slowly varying field amplitude  $\Psi$  has nonzero asymptotics,  $|\Psi| \rightarrow 1$  for  $x, y \rightarrow \pm\infty$ . Such a situation corresponds to the case of dark solitons, localized waves of lower intensity (see, e.g., Ref. [2]). Equation (2) has exact solutions describing such one-dimensional dark solitons,

$$\Psi_0 = k \tanh(k\xi) + iv, \quad k^2 + v^2 = 1, \quad \xi = x - vt - x_0, \quad (3)$$

where  $k$  ( $0 < k < 1$ ) and  $x_0$  are arbitrary constants, and the parameter  $v$  ( $v^2 \leq 1$ ) is defined through  $k$ .

According to [10], the plane dark solitons (3) are unstable against transverse perturbation with the wave numbers  $p < p_c$  where

$$p_c^2 = k^2 - 2 + 2\sqrt{k^4 - k^2 + 1}. \quad (4)$$

The instability domain in the parameter plane ( $p, k$ ) is shown in Fig. 1(a). This domain is bounded by the curve

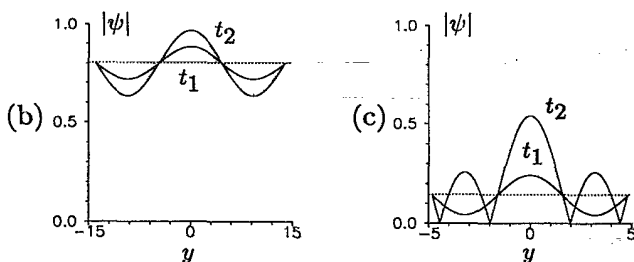
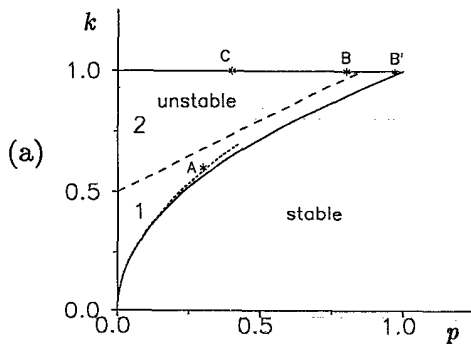


FIG. 1. (a) Instability domain in the parameter plane ( $k, p$ ) for the plane dark soliton (3) with respect to transverse perturbations with the wave number  $p$ . The dotted line is given by the small-amplitude approximation ( $k^2 \ll 1$ ) described by the Kadomtsev-Petviashvili equation with positive dispersion (10). The dashed line depicts the approximate boundary between two qualitatively different regimes of soliton decay, and the points  $A, B, B'$ , and  $C$  correspond to a particular selection of the initial data (see discussions in the text). (b) Schematic structure of the transverse perturbations of the plane dark soliton at  $\xi = 0$  for the initial stage of the instability dynamics at two different moments of time,  $t_2 > t_1$ . The parameters correspond to the point  $A$  in (a). (c) The same as in (b) but for the point  $B'$ .

$p = p_c(k)$  (solid line). The existence of the instability domain for the transverse perturbations means that, if we apply a periodic perturbation with the wave numbers  $p < p_c(k)$ , the amplitude of a plane dark soliton, i.e., a dark-soliton stripe, will grow in the transverse direction according to the linear stability analysis. Then a natural question arises: *What is the result of such an instability?* Particularly, this question was answered in Refs. [12,13] where it was demonstrated, experimentally and numerically, that a plane dark soliton may decay into a chain of optical vortices. The purpose of the present paper is to develop the theory of the soliton instabilities to describe qualitatively and quantitatively possible scenarios of the instability-induced long-time evolution of plane dark solitons of various amplitudes. In particular, we propose an asymptotic theory which allows us to describe a nonlinear regime of the instability and we show that there exist, generally speaking, three different scenarios of the instability dynamics, which include the creation of a chain of optical vortex solitons as a particular case.

The paper is organized as follows. In Sec. II we discuss the small-amplitude limit of the two-dimensional Schrödinger equation and derive the two-dimensional Boussinesq equation which allows for a reduction to two Kadomtsev-Petviashvili equations with positive dispersion for the waves propagating to the right and left. In this limit the problem becomes much simpler because the effect of the soliton instability and self-focusing can be described by an exact solution of the Kadomtsev-Petviashvili equation. For a more general case, in Sec. III we develop an asymptotic approach which can be applied for analysis of the dynamics of dark-soliton stripes of non-small amplitudes and it allows us to describe modulations of the front of the dark soliton taking into account nonlinear and radiation effects. The amplitude equation is analyzed in Sec. IV where different scenarios of the soliton instability are predicted. Numerical results which verify our analytical approach are presented in Sec. V. Finally, Sec. VI concludes the paper.

## II. SELF-FOCUSING OF SMALL-AMPLITUDE DARK SOLITONS

For slowly varying modulations of a small amplitude, when transverse variations are much longer than the longitudinal ones, the basic nonlinear equation (2) may be simplified to derive the Kadomtsev-Petviashvili equation with positive dispersion (the so-called KP1 equation) [10]. In this section we make a generalization of the results of Ref. [10] and derive the two-wave Boussinesq equation for two-dimensional small-amplitude waves propagating along the modulationally stable background  $|\Psi|^2 = 1$ .

Let us look for solutions of Eq. (2) in the form of the following asymptotic expansions:

$$\Psi = Q \exp[i\epsilon R(X, Y, T)], \quad (5)$$

$$Q \equiv |\Psi| = 1 + \sum_{n=1}^{\infty} \epsilon^{2n} u_n(X, Y, T),$$

where  $\epsilon$  is a small parameter ( $\epsilon \ll 1$ ) and  $R$ ,  $Q$ , and  $u_n$  are real functions of "slow" variables: the coordinate  $X = \epsilon x$ ,  $Y = \epsilon y$ , and time  $T = \epsilon t$ . Substituting the expansions (5) into Eq. (2) and equating the terms which have the same order in  $\epsilon$ , we obtain the relations for the coefficients  $u_n$ :

$$u_1 = -\frac{1}{2}\partial_T R, \quad (6)$$

$$u_2 = -\frac{1}{8}\partial_T \nabla^2 R - \frac{1}{8}(\partial_T R)^2 - \frac{1}{4}(\nabla R)^2, \quad (7)$$

and so on, where  $\nabla = (\partial_X, \partial_Y)$  is a vector operator. Additionally to Eqs. (6) and (7) from the imaginary part of Eq. (2) we derive the Boussinesq equation for the phase function  $R$ :

$$\partial_T^2 R - \nabla^2 R + \epsilon^2 \left[ \frac{1}{4} \nabla^4 R + \partial_T (\nabla R)^2 + \partial_T R \nabla^2 R \right] = 0. \quad (8)$$

Here we neglect terms of order of  $O(\epsilon^4)$ . Nevertheless, the linear part of Eq. (8) corresponds exactly to the dispersion relation for the small-amplitude linear waves of the model (2) propagating on the background  $|\Psi| = 1$ . It is interesting to note that Eq. (8) coincides with the corresponding Boussinesq equation for surface waves in hydrodynamics [15]. In the framework of this small-amplitude approximation the soliton (3) transforms into the solitary  $\text{sech}^2$ -type wave [15], so that the Boussinesq equation (8) may be considered as a model to analyze the transverse instability of one-dimensional solitons. Recently this problem has been solved in Ref. [16]. As has been shown in Ref. [16], the transverse instability of one-dimensional solitons in the model (8) may be explained by the decaying type of the dispersive surface for nonlinear solitary waves so that the resonant (decaying) interaction between plane (one-dimensional) and weakly transverse modulated nonlinear waves becomes available. It was also shown in Ref. [16] that instability may appear only for rather long two-dimensional perturbations with large modulations in the transverse direction, so that for such perturbations  $\partial_Y \sim \epsilon \partial_X$ . To describe the dynamics of such a transverse instability, it is useful to introduce a new, slowly varying transverse coordinate  $Y = \epsilon^2 y$  instead of the former one. Then, in the main order in  $\epsilon$  the corresponding equation for the phase  $R$  takes the form of the standard one-dimensional wave equation which has a solution in the form of a superposition of the waves propagating to the right and left,

$$R = R_r(X - T, Y, \tau) + R_l(X + T, Y, \tau) + O(\epsilon^2), \quad (9)$$

where  $\tau = \epsilon^3 t$ . After separation of slow and fast variables in the framework of the standard technique of multiscale asymptotic expansions and splitting the initial perturbations into two waves propagating into the opposite directions along  $x$ , we can consider each wave separately. The dynamics of weakly nonlinear dispersive waves on the modulationally stable background can be described

by two uncoupled KP1 equations [10], which straightforwardly follow from the Boussinesq equation (8):

$$\partial_X (\mp 8 \partial_\tau u_{r,l} - 24 u_{r,l} \partial_X u_{r,l} + \partial_X^3 u_{r,l}) = 4 \partial_Y^2 u_{r,l}, \quad (10)$$

where  $u_{r,l} \equiv (u_1)_{r,l} = \pm \frac{1}{2} \partial_X R_{r,l}$ . It is very important to point out here that in nonlinear *defocusing* media the evolution of small-amplitude waves on the finite-amplitude stable background is described by the Kadomtsev-Petviashvili equations with *positive* dispersion, and this equation is known to display the effect of self-focusing of quiplane waves which have been investigated by means of the exact solutions [17,18] as well as of numerical simulations [19] and asymptotic analysis [20].

The exact solution of the KP1 equation found in Ref. [18] by the inverse scattering method allows us to describe the plane dark-soliton instability in the case of small amplitudes, when Eqs. (10) are valid in the framework of the asymptotic technique, i.e., for  $k^2 \ll 1$ . For the function  $u_r$  which describes the waves propagating to the right with the velocity close to the limiting velocity of linear waves, the corresponding exact solution has the form

$$u_r = -\frac{1}{2} \partial_X^2 \ln \mathcal{F}, \quad (11)$$

$$\begin{aligned} \mathcal{F} = & 1 + \exp(2k\xi) + \exp(2k'\xi + 2\lambda\tau) \\ & + \frac{4\sqrt{kk'}}{(k+k')} \exp[(k+k')\xi + \lambda\tau] \cos(pY), \end{aligned} \quad (12)$$

where

$$\lambda = -\frac{2k'p}{\sqrt{3}}, \quad p = \frac{\sqrt{3}}{2}(k^2 - k'^2), \quad \xi = X - T + \frac{1}{2}k^2\tau. \quad (13)$$

Simple analysis of the solution (11)–(13) shows that in the region of instability bounded by the curve  $p_c = \sqrt{3}k^2/2$  [dotted line in Fig. 1(a)] the self-focusing of a plane soliton results in the generation of the two-dimensional wave which is periodic in the  $Y$  direction and localized in the  $X$  direction. Such a localized wave is given by the exact solution

$$\begin{aligned} u_r(\eta, Y; k, k') \\ = (k - k')^2 \frac{1 + 2\mu \cos(pY) \cosh[(k - k')\eta]}{\{\cosh[(k - k')\eta] + 2\mu \cos(pY)\}^2}, \end{aligned} \quad (14)$$

where

$$\mu = \frac{\sqrt{kk'}}{(k+k')}, \quad \eta = X - T + \frac{1}{2}(k^2 + k'^2 + kk')\tau. \quad (15)$$

Additionally to the two-dimensional wave (14) there appears a plane soliton with smaller intensity and larger velocity in comparison with the former plane soliton. Its amplitude and velocity are defined by the value of the parameter  $k'$ ,  $|k'| < |k|$ . According to Ref. [18], this

“secondary” plane soliton completely absorbs the energy which is released as the result of the self-focusing process so that the effect of the wave self-focusing in the exactly integrable KP1 equation is purely elastic in the sense that it takes place without radiation of linear waves. However, the secondary plane soliton, which formally goes to infinity, makes the process of the modulated soliton decay effectively irreversible and finally it looks dissipative.

Solution (14) describes a chain of two-dimensional solitons of the KP1 equation (see Refs. [17,18] for more details) which for the primary model (2) corresponds to the two-dimensional analog of the so-called “gray” solitons of the one-dimensional NLS equation [2]. In region 1 of the instability domain shown in Fig. 1(a) [the boundary of which is shown as a dashed line and it may be found approximately from Eq. (14)], such two-dimensional gray solitons are created at minimum points of the perturbed plane dark soliton [see Fig. 1(b)]. However, for the small-amplitude dark solitons the minimum intensity is stabilized at a nonzero level approximately corresponding to the values given by the exact solution (14). At the same time, the instability dynamics is quite different in region 2 [Fig. 1(a)]. In this region the approximation leading to the KP1 equation is no longer valid and the intensity of a perturbation can reach the minimal (zero) value. This finally leads to the creation of the zero-intensity points [see Fig. 1(c)] which give birth to pairs of optical vortices of different polarities on each period of the perturbation, i.e., a pair of vortex and antivortex solitons. Every vortex has the total phase jump  $2\pi$  whereas an antivortex has the opposite value,  $-2\pi$ . Exactly this regime has been previously reported in Refs. [12,13].

Here we would like to note that in the one-dimensional NLS equation both types of dark solitons, “gray” and “black” ones, are similar and they exist as two limiting cases of the same localized solution. In the two-dimensional case these solutions are very different. A small-amplitude KP-type soliton is highly anisotropic with the width much larger in the transverse direction than in the longitudinal one. On the other hand, a large-amplitude dark soliton can be regarded as a composite pair of vortex and antivortex solitons propagating with a small velocity as a whole and oriented transversally to the motion [7]. At last, the two-dimensional analog of the black soliton is an individual, immobile, radially symmetrical vortex with nontrivial phase structure [4–6]. Below we show that this difference in the soliton structure does define the difference in their evolution as well as their generation in the corresponding regions 1 and 2 of the instability domain shown in Fig. 1(a).

### III. ASYMPTOTIC ANALYSIS OF THE SELF-FOCUSING

Let us consider the evolution of small-amplitude perturbations along the plane dark solitons in the vicinity of the instability threshold curve [solid line in Fig. 1(a)] by means of the multiscale asymptotic expansion method. Such an approach has been shown to be useful for many problems (see, e.g., Refs. [20–22]) to describe the long-

scale dynamics of the unstable quasiplane solitons and generation of two-dimensional solitonic structures.

To apply this technique, we introduce again the slowly varying coordinates  $X = \epsilon x$ ,  $Y = \epsilon^2 y$ ,  $T = \epsilon t$ , and  $\tau = \epsilon^3 t$ . Now we choose the reference frame where the plane soliton is at rest and the coordinate of its center  $x_0$  is considered as a small, slowly varying function,  $x_0 = \epsilon s(Y, T)$ . Solution of Eq. (2) may be presented in the form of the asymptotic expansions in powers of  $\epsilon$ ,

$$\Psi = \Psi_0(\xi) + \sum_{n=1}^{\infty} \epsilon^n \Psi_n(\xi, y; X, Y, T, \tau). \quad (16)$$

Here the main term of the expansion  $\Psi_0(\xi)$  has a form of the one-dimensional dark soliton described by Eq. (3). The first-order correction is given by a general solution of the linear equation which is just a linearized form of Eq. (2) around the soliton  $\Psi_0$ ,

$$\underline{L}\Psi_1 = 0, \quad \underline{L} = \partial_\xi^2 + \partial_y^2 - 2iv\partial_\xi + 2[1 - 2|\Psi_0|^2 - \Psi_0^2(*)], \quad (17)$$

where (\*) is the operator of complex conjugation. The higher-order corrections  $\Psi_n$  are subsequently found as solutions of the linearized equations,

$$\underline{L}\Psi_n = H_n(\Psi_0, \Psi_1, \dots, \Psi_{n-1}), \quad n \geq 2 \quad (18)$$

where the right-hand side operators  $H_n$  can be calculated with the help of the corrections of lower orders, for example,

$$H_2 = 2(\Psi_1^2\Psi_0^* + 2\Psi_0|\Psi_1|^2) - 2i\partial_T\Psi_1 + 2i\partial_T s\partial_\xi\Psi_0, \quad (19)$$

$$H_3 = 2(|\Psi_1|^2\Psi_1 + 2\Psi_0\Psi_1\Psi_2^* + 2\Psi_0\Psi_1^*\Psi_2 + 2\Psi_0^*\Psi_1\Psi_2) - 2i\partial_T\Psi_2 + 2i\partial_T s\partial_\xi\Psi_1 - 2\partial_\xi\partial_X\Psi_2 - 2\partial_y\partial_Y\Psi_1. \quad (20)$$

Solutions of the linear equation (17) have been found in Ref. [10]. From general solutions we take the one which is localized in  $\xi$  and periodic in  $y$ . Such a solution corresponds to an eigenfunction of the discrete spectrum of the operator  $\underline{L}$  at the boundary of the instability domain (4),

$$\Psi_1 = [a(Y, T)e^{ip_c y} + \text{c.c.}] \Phi(\xi), \quad (21)$$

where

$$\Phi(\xi) = -\frac{3v \sinh(k\xi)}{2k \cosh^2(k\xi)} + i\frac{p_c^2 + 3k^2}{4k^2 \cosh(k\xi)}. \quad (22)$$

Here the function  $a(Y, T)$  describes the amplitude of the transverse perturbations along the plane dark soliton, and c.c. stands for the complex conjugate. To derive equations for the function  $a(Y, T)$  and also for  $s(Y, T)$  we use the orthogonality conditions which mean that the corrections  $\Psi_3$  and  $\Psi_4$  have no terms exponentially growing in  $\xi$  (i.e., secular terms). However, before this procedure we should consider powerlike divergent terms in the

asymptotic series (16) which appear as a result of solving Eqs. (18).

First of all, we write a formal solution for  $\Psi_2$  in the form

$$\Psi_2 = ir \Psi_0 + \partial_T s \partial_v \Psi_0 + |a|^2 \Psi_{20} + [\partial_T a e^{ipcy} + \text{c.c.}] \Psi_{21} + [a^2 e^{2ipcy} + \text{c.c.}] \Psi_{22}. \quad (23)$$

Here  $r(X, Y, T, \tau)$  is an arbitrary function of slow variables and the function  $\Psi_{20}(\xi)$  may be found in an explicit form. To find other functions  $\Psi_{21}(\xi)$  and  $\Psi_{22}(\xi)$  we have used a numerical technique of solving linear inhomogeneous equations based on the matrix expansion approach.

It is obvious that the coefficients of zero-order harmonics, which are proportional to  $i\Psi_0$  and  $\partial_v \Psi_0$ , do not vanish for  $\xi \rightarrow \pm\infty$ . A similar problem also appears in the linear analysis of the plane dark-soliton stability by means of the asymptotic expansions in the perturbation wave number  $p$  (see Ref. [10]). As was shown in [10], weak decay of the eigenfunction of the corresponding linearized

problem may be achieved in the asymptotic region (i.e., far from the soliton) by taking into account finite values of  $p$ . However, this method to remove singularities in the asymptotic expansions does not work for analysis of nonlinear nonstationary problems.

Let us consider this problem from the other point of view. Indeed, we note that, in spite of the fact that the second-order terms of the asymptotic expansion are non-localized functions, the asymptotic series (16) may be reordered to make it localized with the same boundary conditions ( $|\Psi| \rightarrow 1$  for  $\xi \rightarrow \pm\infty$ ) which are satisfied up to the first terms of the expansion. To do this, we should remove the terms  $\sim i\Psi_0$  by changing the phase of the complex function  $\Psi$  similarly to Eq. (5). The terms  $\sim \partial_v \Psi_0$  are excluded by a change of  $v$  and the corresponding renormalization of  $k$ , see Eq. (3). After such a simple procedure the corresponding correction (23) becomes a localized function of  $\xi$  which does not make the expansion (16) divergent. In this way the corrections which may change the asymptotic value of  $|\Psi|$  for  $\xi \rightarrow \pm\infty$  appear only for the terms of order of  $\epsilon^3$ ,  $\epsilon^4$  and they have the form

$$\begin{aligned} \frac{1}{\epsilon^2} \left( |\Psi| \Big|_{\xi \rightarrow \pm\infty} - 1 \right) &\equiv u^\pm(X, Y, T, \tau) \\ &= \frac{\epsilon}{2v} (vq - \partial_T r \pm 2vD) \\ &\quad + \frac{\epsilon^2 \xi}{2} \left[ -\frac{(k^2 + 2v^2)}{2vk^2} (\partial_T q + v\partial_X q) - \frac{1}{2} \partial_X q + \frac{1}{2k^2} (\partial_T^2 r - \partial_X^2 r) \pm \partial_T F \right] + O(\epsilon^3), \end{aligned} \quad (24)$$

where  $q(X, Y, T, \tau)$  is an arbitrary function which must be taken into account in solving Eq. (18) at  $n = 3$ , and the functions  $D$  and  $F$  have the form

$$D = \frac{(v^2 - k^2)}{2vk^3} \partial_T^2 s + \nu(k) \partial_T |a|^2, \quad (25)$$

$$F = -\frac{v^2}{k^5} \partial_T^2 s + \mu(k) \partial_T |a|^2.$$

Here parameters  $\nu(k)$  and  $\mu(k)$  are introduced to designate the coefficients which are found from numerical solution of Eq. (18) for  $n = 3, 4$ .

As follows from Sec. II, the radiation fields  $u^\pm$  far from the soliton satisfy the Boussinesq equation (8) and are given by a superposition of two waves propagating to the left and right similarly to Eq. (9). To complete the corresponding mathematical formulation of the problem for Eq. (8), we should formulate the boundary conditions at the soliton which is assumed to have the coordinate  $X_s(T)$ , i.e., we should determine the functions  $u^\pm|_{X=X_s}$  and  $\partial_X u^\pm|_{X=X_s}$ . These functions may be found considering the corresponding expansions of the first terms of Eq. (24) into the Taylor series in the parameter  $\epsilon\xi = X - X_s$ ,

$$u^\pm|_{X=X_s} = \frac{\epsilon}{2v} (vq_0 - \partial_T r_0 \pm 2vD), \quad (26)$$

$$\begin{aligned} \partial_X u^\pm|_{X=X_s} &= \frac{\epsilon}{2} \left[ -\frac{(k^2 + 2v^2)}{2vk^2} (\partial_T q_0 + v\partial_X q_0) \right. \\ &\quad \left. - \frac{1}{2v} \partial_X \partial_T r_0 \right. \\ &\quad \left. + \frac{1}{2k^2} (\partial_T^2 r_0 - \partial_X^2 r_0) \pm \partial_T F \right], \end{aligned} \quad (27)$$

where  $q_0 \equiv q|_{X=X_s}$ ,  $r_0 = r|_{X=X_s}$ .

In addition to the boundary condition at the moving soliton (26) and (27) for the radiation field we should also use two conditions at infinity, i.e., a boundary condition for the field  $u^+$  at  $X \rightarrow +\infty$  and for the field  $u^-$  at  $X \rightarrow -\infty$ . Let us assume that initially there is no radiation far away from the soliton. Then, every component of the radiation field consists only of one wave moving in one direction, in other words  $u^\pm = u_{r,l}(X \mp T, Y, \tau)$ . From Sec. II it follows that in such a case the evolution of the radiation far from the soliton center as a function of the slow time  $\tau$  is described by the KP1 equation (10).

To exclude the parameters  $q_0$  and  $r_0$  we use the relation

$$\partial_X u^\pm|_{X=X_s} = \frac{1}{(v \mp 1)} d_T u^\pm|_{X=X_s},$$

where  $d_T = (\partial_T + v\partial_X)|_{X=X_s}$  is the operator of the full derivative in the reference frame moving with the soliton. As a result, from the conditions (26) and (27) we obtain

the system of two coupled equations for two parameters  $q_0$  and  $r_0$  which after integration may be written in the following simple form:

$$\frac{D}{(v \mp 1)} = F + \frac{1}{2vk^2} [vq_0 - \partial_T r_0 \mp (q_0 + \partial_X r_0)]. \quad (28)$$

System (28) has the unique solution which allows us to exclude  $r_0$  and  $q_0$  from the subsequent equation.

Additionally to the power singularities, at the zero and first harmonics of the wave number  $p_c$  the general solution of Eq. (18) also displays exponentially growing terms. Such terms appear as a result of the existence of the localized eigenfunctions of the complex conjugated operator  $\underline{L}^*$ . To exclude the exponentially growing terms in the asymptotic series (16) we select the parameters  $a$  and  $s$  to make the right-hand sides  $H_n$  orthogonal to the corresponding eigenfunctions of the operator  $\underline{L}^*$  for zero ( $H_n^{(0)}$ ) and the first ( $H_n^{(1)}$ ) harmonics [14]. This leads to the relations

$$\int_{-\infty}^{+\infty} \frac{\text{Re } H_n^{(0)}}{\cosh^2(k\xi)} d\xi = 0, \quad (29)$$

$$\int_{-\infty}^{+\infty} (\text{Re } \Phi \text{ Re } H_n^{(1)} + \text{Im } \Phi \text{ Im } H_n^{(1)}) d\xi = 0. \quad (30)$$

With accuracy up to the terms of the order of  $O(\epsilon^5)$  the orthogonality conditions (29) and (30) for  $n = 3, 4$  lead to the system of two coupled equations for  $a$  and  $s$ ,

$$k^2 \partial_T^2 s + \alpha(k) \partial_T |a|^2 + \frac{\epsilon}{k^3} \delta_1(k) \partial_T^2 |a|^2 = 0 \quad (31)$$

and

$$-ik^2 p_c \partial_Y a + \beta(k) \partial_T^2 a + k^4 \gamma_1(k) a \partial_T s + k^2 \gamma_2(k) a |a|^2 + \frac{\epsilon}{k} \delta_2(k) a \partial_T |a|^2 = 0, \quad (32)$$

where  $\alpha, \beta, \gamma_1, \gamma_2, \delta_1$ , and  $\delta_2$  are numerical coefficients which depend in a nontrivial way on the parameter  $k$ . Equations (31) and (32) give rise to the final equation for the amplitude  $a$  of the transverse perturbation of the plane soliton,

$$-ik^2 p_c \partial_Y a + \beta(k) \partial_T^2 a + k^2 \gamma(k) a |a|^2 + \frac{\epsilon}{k} \delta(k) a \partial_T |a|^2 = 0, \quad (33)$$

where  $\gamma = \gamma_2 - \alpha \gamma_1$ ,  $\delta = \delta_2 - \gamma_1 \delta_1$ . Parameters  $\alpha, \beta, \gamma$ , and  $\delta$  as functions of  $k$  have been found numerically and are presented in Figs. 2(a) and 2(b). It is interesting to mention that Eq. (33) describing the nonlinear regime of the instability of the plane soliton is exactly the same, up to the notation, as the well-known equation governing dynamics of solitons in a nonlinear optical fiber in the presence of the intrapulse Raman effect (see, e.g., Refs. [23]), and this establishes a similarity between the insta-

bility analyzed here and that known for bright solitons in optical fibers.

Using Eqs. (28) and (31) it is possible to calculate the functions (26) for the radiation fields propagating to the right and left from the quasiplane soliton at the mean velocity close to the limiting velocity of linear waves. On the scales defined by the slow time  $\tau$  the radiation evolution is described by two KP1 equations (10) with the "initial" profiles as functions of the spatial variables  $X$  and  $Y$ :

$$u^\pm(X, Y, \tau = 0) = \frac{\epsilon}{k^3} \zeta^\pm(k) \partial_T |a|^2 \Big|_{T=T_s} + O(\epsilon^2), \quad (34)$$

where  $T_s(X)$  is a function inverse to the function  $X_s(T)$ . In the leading order of our asymptotic approach, the plane soliton moves straight forward so that we have the expressions  $X_s(T) = vT$  and  $T_s(X) = X/v$ . As to coefficients of the radiation waves  $\zeta^\pm$ , they have been found numerically and are shown in Fig. 2(c).

Thus Eqs. (33) and (34) describe, in the main order of the asymptotic expansion, the dynamics of the transverse perturbations of the plane dark soliton and the radiation fields it generates. We would like to mention again that

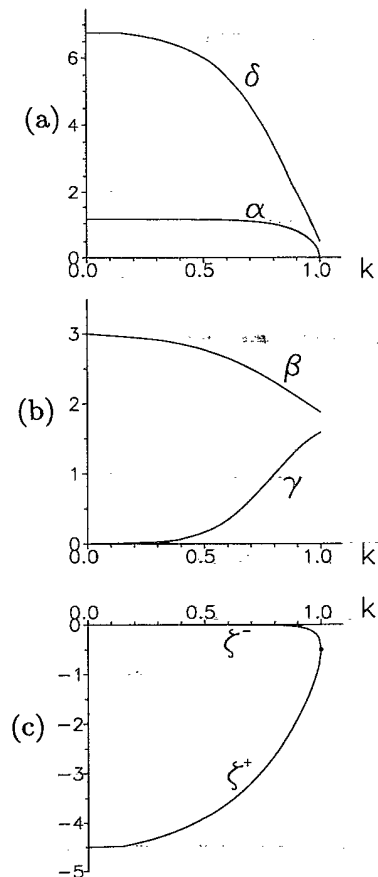


FIG. 2. The coefficients in Eq. (33) [(a) and (b)] and Eq. (34) (c) versus the soliton parameter  $k$ . The case  $k^2 \ll 1$  corresponds to small-amplitude (gray) solitons, whereas the limiting point  $k = 1$  corresponds to a black soliton.

such an asymptotic analysis is valid, provided the parameters are selected near the instability threshold, i.e.,  $p$  is close to  $p_c(k)$ . Such equations are very useful to describe different scenarios of the soliton instability in a nonlinear regime and we discuss these scenarios in detail in the next section.

#### IV. DIFFERENT SCENARIOS OF THE SOLITON SELF-FOCUSING

Let us consider now the dynamics of the dark-soliton instability for the case of a periodic transverse perturbation of the form  $a(Y, T) = A(T) \exp(i\Delta p Y)$ , where  $\Delta p = p - p_c$ . For this case the relation (33) transforms into a second-order ordinary differential equation

$$\beta \ddot{A} + k^2 p_c \Delta p A + k^2 \gamma A^3 + \frac{2\epsilon\delta}{k} A^2 \dot{A} = 0, \quad (35)$$

which may be easily investigated. Note that, according to Figs. 2(a) and 2(b), the parameters  $\beta$  and  $\delta$  are positive, and  $\gamma$  is not negative.

Equation (35) may be analyzed on the phase plane of the parameters  $A$  and  $\dot{A}$ . The critical point  $A = 0$ , which corresponds to the unperturbed plane dark soliton, is stable, provided  $\Delta p > 0$  (the corresponding critical point is a center) and it is unstable otherwise, i.e., for  $\Delta p < 0$  when the corresponding critical point is a saddle type. The instability growth rate is given by  $\lambda = k\sqrt{-p_c \Delta p / \beta}$ . However, the nonlinear terms ( $\sim A^3$  and  $\sim A^2 \dot{A}$ ) in Eq. (35) lead to a stabilization of the growing amplitude of perturbation and also to dissipation which describes the energy losses by the plane dark soliton for the radiation emitted to the right and left.

In the instability region (i.e., for  $\Delta p < 0$ ) there always exist two symmetric nonzero equilibrium states  $A = \pm A_0 = \pm\sqrt{-p_c \Delta p / \gamma}$  which correspond to a dark soliton periodically modulated in the transverse direction. Far from the threshold instability curve  $p_c(k)$  this steady-state periodic structure is nothing but a chain of two-dimensional solitons. In the small-amplitude limit, it is given by the exact solution (14) of the KP1 equation. Note also that, because the coefficient  $\alpha$  is not negative [see Fig. 2(a)], it follows from Eq. (31) that the modulated plane soliton moves slower than the nonmodulated one, i.e., such a modulation decreases the soliton velocity.

It is easy to verify that in the framework of Eq. (35) the stationary states  $A = \pm A_0$  are stable in a linear approximation. We may expect that an arbitrary periodic perturbation of the plane dark soliton tends to saturation and formation of a two-dimensional modulated structure. However, the relative contribution of conservative and dissipative nonlinear terms of Eq. (35) is different in region 1 or 2 of the instability domain shown in Fig. 1(a). As a result, the qualitative picture of the soliton instability strongly depends on the initial value of the soliton amplitude. Therefore below we consider the cases of small- and large-amplitude solitons (gray and black ones) separately.

#### A. Self-focusing of "gray" solitons

For gray solitons we have  $k < 0.5$  so that the nonlinear dissipation is much stronger than the cubic nonlinearity [see the dependence of  $\gamma(k)$  and  $\delta(k)$  presented in Figs. 2(a) and 2(b)]. Therefore because of the small coefficient  $\gamma$  the applicability region of Eq. (35) in this case is rather narrow, and we should take into account the nonlinear terms of a higher order, in fact, the fifth-order terms. This may be done by changing the asymptotic expansion scales to  $X \rightarrow \epsilon X$ ,  $Y \rightarrow \epsilon^2 Y$ ,  $T \rightarrow \epsilon T$ , and  $\tau \rightarrow \epsilon^3 \tau$ . Then the conservative and dissipative nonlinear terms will appear in the same order of  $\epsilon$  and instead of Eq. (35) we obtain the equation

$$\beta \ddot{A} + k^2 p_c \Delta p A + \frac{k^2 \gamma}{\epsilon^2} A^3 + \frac{\chi}{k^2} A^5 + \frac{2\delta}{k} A^2 \dot{A} = 0. \quad (36)$$

In the small-amplitude approximation ( $k^2 \ll 1$ ) the coefficients in Eq. (36) may be calculated in an explicit form:  $\beta = 3$ ,  $\gamma = 0$ ,  $\chi = 243/16$ , and  $\delta = 27/4$ . In this case it is possible to obtain a general solution of Eq. (36) [20] which describes the evolution of almost all phase trajectories on the plane  $(A, \dot{A})$  to the states  $A = \pm A_0$  which are characterized by degenerate stable critical points of the node type. The corresponding phase plane for Eq. (36) in the limit  $k^2 \ll 1$  is shown in Fig. 3(a). The soliton instability in this picture is described by separatrix trajectories  $A(T)$  which have the asymptotic values  $A(T \rightarrow -\infty) = 0$  and  $A(T \rightarrow +\infty) = \pm A_0$ .

For small but finite values of  $k$ , when the cubic nonlinearity is also included, the critical points  $A = \pm A_0$  become focuses. This simply means that for gray solitons

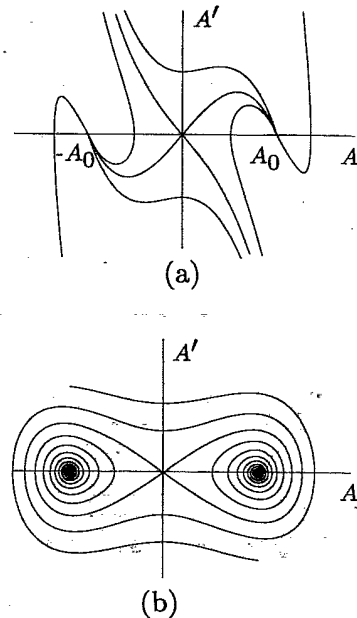


FIG. 3. Qualitative picture of the phase plane of Eq. (36) at  $k^2 \ll 1$  (a) and of Eq. (35) at  $k = 1$  (b). Separatrix trajectories correspond to the transition from the plane dark soliton to the nonlinear  $y$ -periodic wave.

which are far from the limit given by the KP1 approximation the dissipative effects are weaker than the conservative nonlinear ones. Such a difference between the solitons with  $k \rightarrow 0$  and small but finite values of  $k$  may also be noted in analyzing the amplitude of the radiation emitted due to the soliton instability [which are defined by Eq. (34) and are shown in Fig. 2(c)]. For a gray soliton the instability generates a small-amplitude wave which propagates in the same direction as the plane soliton. In the KP1 limit such a wave is simply the other plane soliton and the whole process is elastic in the sense that the linear dispersive waves are not excited while the soliton instability develops [18,20]. The increase of the soliton amplitude makes radiation of the linear waves possible and part of the energy of the primary plane soliton is lost due to wave dispersion.

### B. Self-focusing of “black” solitons

For larger  $k$  the difference between the soliton instability described by the NLS and KP1 equations becomes even more obvious. From Eq. (35) it follows that for rather large values of  $k$  the nonlinear dissipation is rather small. As a result, the transverse oscillations of the soliton profile, which are described by the oscillating trajectories around the points  $A = \pm A_0$ , do not decay rapidly. The corresponding phase plane for Eq. (35) is shown in Fig. 3(b) ( $k = 1$ ,  $\beta \approx 1.88$ ,  $\gamma \approx 1.59$ ,  $\delta = 0.5$ , and  $\epsilon = 1$ ).

If we neglect the last small dissipative term in Eq. (35) (which is proportional to the small parameter  $\epsilon$ ), such oscillations become periodic and describe periodic modulations of the plane dark soliton. A small but important dissipative contribution in Eq. (35) makes such oscillations quasiperiodic and irreversible due to the portion of the soliton energy lost for small radiation. Thus, as a result, we may expect that in the limit  $T \rightarrow +\infty$  some two-dimensional structure will be formed.

As follows from Fig. 2(c), for the values of  $k$  close to its limiting value,  $k = 1$ , the soliton oscillations generate waves propagating both to the right and to the left and at the point  $k = 1$  the amplitudes of these two radiation waves coincide. Additionally, the presence of the small parameter  $\epsilon$  in the initial profile of radiation fields (34) means that the nonlinear effects in Eqs. (10) become weaker than the dispersive ones. Therefore the emitted energy becomes more and more homogeneously distributed between linear dispersive waves emitted to the right and left.

In the limiting case  $k = 1$ , the primary soliton is at rest (i.e.,  $v = 0$  and  $s_T = 0$ ) and the radiation amplitudes are equal to each other,  $\zeta^+ = \zeta^- = 0.5$ . Because in this case the radiation emitted consists of regions of higher and lower intensities, we may expect that the plane black soliton also generates small-amplitude dark solitons of very small intensity. However, such solitons, if they are created, are very wide and the greatest part of the radiation is carried by linear dispersive waves.

In concluding of this section we briefly discuss a more general nonperiodical perturbation  $a(Y, T)$ . According to the well-known Lighthill criterion [14,15] the positive

relative sign of the coefficients  $\beta$  and  $\gamma$  in Eq. (33) leads to modulational instability of the two-dimensional transverse perturbation of the form  $a_0(Y) = A_0 \exp(i\Delta p Y)$  and two satellite waves  $a_+$  and  $a_-$  are excited with the wave numbers lying to the right and left of the wave number of the modulated wave,  $p_0 = p_c + \Delta p$ . Such an instability is well known in the theory of the KP1 equation (see Ref. [18]) and it has been already discussed in Ref. [20] in the framework of an effective amplitude equation similar to Eq. (33). For gray solitons such an instability transforms the primary wave  $a_0$  into a two-dimensionally modulated wave  $a_-$  with the transverse wave number  $p_- < p_0$ . However, for the black soliton the nonlinear dissipation is weak and modulational instability can be saturated by purely conservative effects, so that long-lived quasiperiodic oscillations may be observed near a new, two-dimensional wave  $a_-$ .

Thus the analytical results we presented above may be summarized as follows. First, in the small-amplitude limit the primary two-dimensional NLS equation is shown to reduce to the two-dimensional Boussinesq equation which allows for the subsequent reduction to the KP1 equation for which an exact solution describing the soliton self-focusing may be found in an explicit form. Second, in the case when the small-amplitude approximation is not valid, the asymptotic analysis developed above allows us to describe the nonlinear regime of the self-focusing when the exponential growth due to the instability is saturated by nonlinearity and radiation-induced effective damping.

### V. NUMERICAL SIMULATION RESULTS

To verify the results of the analytical approach we numerically integrate Eq. (2) with periodic boundary conditions in  $x$  and  $y$  by means of the known method of the operator exponent [24]. The soliton (3) was selected as the initial condition to Eq. (2) but in a perturbed form with a transverse modulation of the soliton coordinate:  $\Psi(x - x_0) = \Psi_0 + \epsilon \partial_x \Psi_0 \cos(py)$ , where  $\epsilon = 0.1$ . Since the complex phase of the solution  $\Psi_0$  gets a finite phase jump across the dark soliton the periodic boundary conditions in  $x$  were satisfied by placing the other (symmetric) dark soliton at the point  $x = -x_0$  with the opposite phase jump, i.e.,  $\Psi^*(x + x_0)$ .

For the gray solitons in region 1 we have selected the values of the parameters to be  $k = 0.6$  and  $p = 0.3$  [the point  $A$  in Fig. 1(a)]. Results of the corresponding numerical simulations are presented in Figs. 4(a)–4(c). As may be clearly seen from these figures, the soliton instability ends up in the formation of two-dimensional localized structures with smaller intensities and velocities than that of the primary plane dark soliton. Such a process is accompanied by the creation of the other small-amplitude plane dark soliton which moves in the same direction as the primary plane dark soliton and has greater velocity. We would like to note that this type of soliton instability is very close to the type which is described by the exact solution (11)–(13) obtained in the framework of the KP1 equation. Additionally, in Figs. 4(b)



and 4(c) we clearly observe a highly anisotropic profile of small-amplitude dark solitons which is similar to that of the two-dimensional "antilump" solitons of the KP1 equation.

In region 2 of the instability domain shown in Fig. 1(a) we investigate dynamics of a black plane soliton when the soliton parameter is selected to be  $k = 1$ . For short-wave instability (the point  $B$  at  $p = 0.8$ ) we observe periodic oscillations of the amplitude of the dark soliton modulated in the transverse direction which do not result in strong phase modulations [see Figs. 5(a)–5(d)]. In this case, a pair of vortex and antivortex solitons of circular symmetry appears only at the intermediate stage of the plane dark-soliton instability which is depicted in Fig. 5(c). As follows from our asymptotic approach, the radiation-induced damping is almost negligible for black solitons. As a result, the large-amplitude oscillations observed are weakly damped. In Fig. 6 we present an example of such a quasiperiodic dynamics displaying the dependence of the minimum value of the soliton intensity

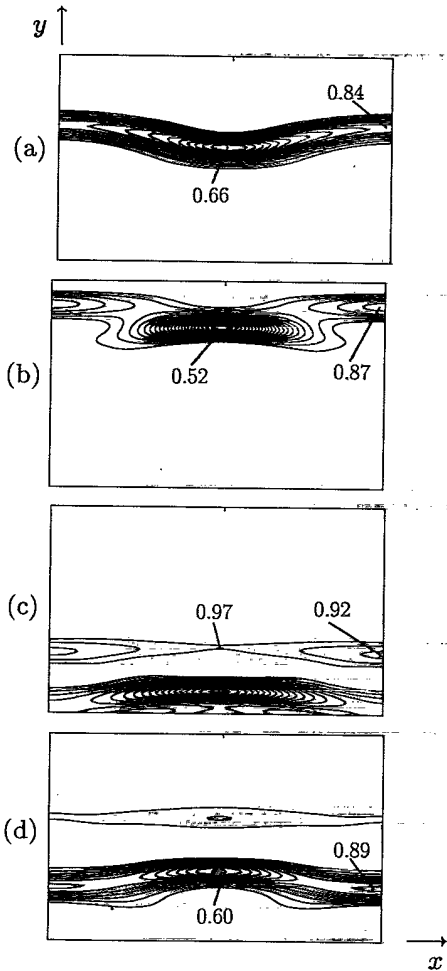


FIG. 4. Development of the transverse instability for a small-amplitude dark soliton at  $k = 0.6$ ,  $p = 0.3$  [the point  $A$  in Fig. 1(a)]: (a)  $t = 3$ , (b)  $t = 9$ , (c)  $t = 18$ , (d)  $t = 24$  (level lines of the function  $|\Psi|$  are depicted). Creation and stable propagation of two-dimensional solitonlike objects are observed.

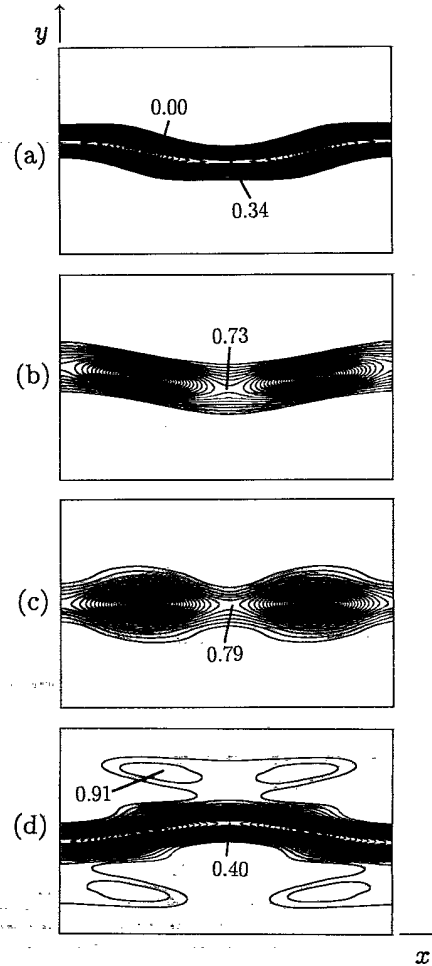


FIG. 5. Development of the transverse instability for the large-amplitude dark soliton at  $k = 1.0$ ,  $p = 0.8$  [point  $B$  in Fig. 1(a)]: (a)  $t = 1$ , (b)  $t = 2$ , (c)  $t = 2.75$ , (d)  $t = 5$ . A pair of optical vortex and antivortex solitons is observed at the intermediate stage of the plane dark-soliton instability.

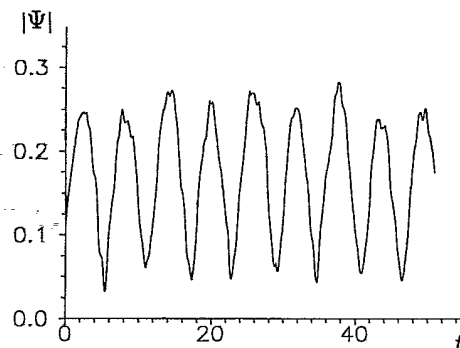


FIG. 6. Minimum value of the local field intensity  $|\Psi|$  vs  $t$  at the line  $y = 0$ . Parameters are  $k = 1.0$  and  $p = 0.9$ . Large-amplitude oscillations near a modulated state are observed.

$|\Psi|$  at the line  $y = 0$  vs time for the transverse perturbations at  $p = 0.9$ . At the same time, we reveal a small radiation escaping quasiplane dark soliton [see Fig. 5(d)]. Such energy losses must lead to formation of a chain of vortex and antivortex solitons at the long-time asymptotics  $t \rightarrow +\infty$ . Note that this picture fits well with the analytical results described in Sec. IV.

For the long-wave instability in region 2 [the point  $C$  at  $p = 0.4$  in Fig. 1(a)], where our asymptotic theory is no longer valid, we observe a qualitatively different scenario of the instability dynamics. In this case, the large-amplitude modulations of the plane dark soliton result in strong phase modulation [see Figs. 7(a)–7(d)]. Such an irreversible phase dynamics results, eventually, in a change of the constant-phase lines and creation of pairs of vortex and antivortex solitons already at the first stage of the dark-soliton instability. This difference from the short-wave instability dynamics is accounted for by a strong radiation in the form of two quasiplane dark-soliton structures which remove a considerable portion of the energy of the original dark soliton. As a matter of fact, this regime of the soliton decay was earlier observed numerically [12,13].

Thus Figs. 4, 5, and 7 present three qualitatively different scenarios of the soliton instability, namely, creation of two-dimensional gray solitons similar to the KP1 “antilump” solitons and emission of a quasiplane dark soliton of smaller intensity (Fig. 4), long-lived quasiperiodic oscillation of the plane soliton without nontrivial phase dynamics (Figs. 5 and 6), and finally the decay of the plane solitons into a chain of vortex and antivortex pairs (Fig. 7).

## VI. CONCLUSION

We have investigated analytically and numerically the transverse instabilities of the plane dark soliton in a nonlinear (cubic) defocusing medium. As follows from our analysis, the main features of such an instability have very much in common with those of the transverse instability of the plane solitons in the Kadomtsev-Petviashvili equation with positive dispersion, but there are also several qualitatively different features. First, the growth of the transverse perturbations in the vicinity of the instability threshold is replaced by their damping so that a chain of two-dimensional dark solitons is periodically formed at the intermediate stage of the plane dark-soliton instability. Second, the radiation energy is distributed between secondary small-amplitude dark solitons and linear dispersive waves emitted in both directions, so that a decay of a plane dark soliton of finite amplitude has no elastic character. Finally, as a result of radiation losses, we expect formation of a chain of two-dimensionally localized solitons at a final stage of the dark-soliton instability. It is important to emphasize that, the greater the radiation energy which escapes the unstable quasiplane dark soliton, the faster a chain of two-dimensional dark solitons can be formed.

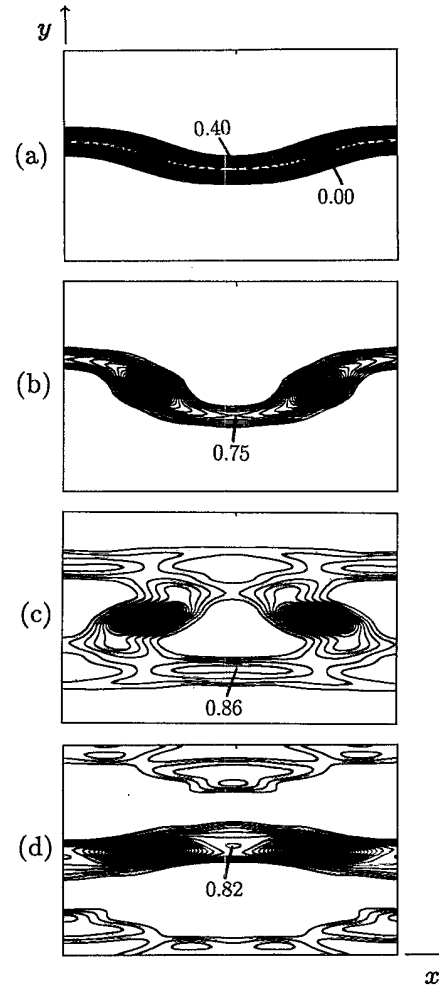


FIG. 7. Development of the transverse instability for the large-amplitude dark soliton at  $k = 1.0$ ,  $p = 0.4$  [point  $C$  in Fig. 1(a)]: (a)  $t = 3$ , (b)  $t = 6$ , (c)  $t = 9$ , and (d)  $t = 13$ . Creation of pairs of optical vortex and antivortex solitons is observed.

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